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TWO-WAREHOUSE INVENTORY MODEL FOR DETERIORATING ITEMS WITH PARTIAL BACKLOGGING AND ADVANCE PAYMENT SCHEME

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Abstract. Advance payment has a great influence on making the optimal decision in an inventory system. Two-warehouse inventory system is another imperative factor in inventory analysis. Due to competitive marketing situation, the position of a warehouse performs a significant role in business strategy. Generally, retailers want to find a shop in a popular marketing place. So, they need an additional store room due to insufficient space in a popular marketplace. Also, we have considered the advance payment scheme which is made by equal installment up to n times before receiving the products. Using all of these concepts in together, we have developed a two-warehouse inventory model for deteriorating items with advanced payment scheme. Shortages are allowed with a constant partial backlogging rate. Demand of the product is dependent on selling price. We have presented this physical problem in mathematically and solved. Also, we proved the optimality mathematically as well as graphically and proposed one theorem in order to show the optimality in theoretically. We have supplied a numerical example to illustrate the proposed inventory model. To validate the numerical result of the proposed model, we have plotted 2D and 3D graphs by using MATLAB and observed these satisfy the numerical result. Finally, we have performed sensitivity analysis changing one parameter and keeping others the same.

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1. Introduction

Inventory problem is the daily life problem for mankind. The word inventory defines stock of useable goods. In the real world, there are different forms of goods coming from the raw materials, work-in-progress and finished goods in the inventory. Nowadays there is a big concern over the management of different forms of inventories. So, many business organizations are putting emphasis on proper management of inventory for running their

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business successfully. There are many factors which come under the management of inventory. Here we have discussed one of them *i.e.*, a two-warehouse inventory problem under an advance payment scheme.

Advance payment is one of the most popular business strategies in the real world. It has a huge impact on an inventory system. Basically, suppliers/retailers want to attract to his customers due to some facilities of advance payment. They want some money (partly or fully) before received the product. Due to this payment, they give some price discount on the product or other kinds of rebate on the purchased amount. These facilities attract the retailers to buy more and more products. Two-warehouse and advance payments are correlated with each other. If retailers buy more products, then they need to the additional storage space. In this work, we have considered these two factors together and in the next section, we described about two-warehouse system.

Special cases for business houses may suddenly arise very often, particularly while acting on seasonal demands, providing discounts for boosting up sales, importing essential intermediaries for meeting up to current technological and market-related challenges etc. In view of all these factors, companies often want to buy a large number of items. The large amounts of items cannot be stored in a single warehouse or owned warehouse due to limited capacity. So other storage facilities are essential to keep the extra or excess amount of items. Hence, the prosperity of business largely depends on storing the inventories efficiently. So, proper warehousing facilities are essential for excelling in business operations and a smooth and uninterrupted running of supplies of outputs. Generally, from the business perspective, the demand of customers is fulfilled from owned warehouses. But the holding cost of a rented warehouse is more than that of an owned warehouse. So, companies always want to vacate the rented warehouse to meet the customer demand by transferring stocks from the rented warehouse to the owned warehouse. Many researchers have worked on two-warehouse inventory system considering the deterioration effect. Deterioration means decay or damage, and it is necessary to introduce the concept of deterioration in the inventory models. Most of the products deteriorate during their time of stay in the warehouse. That's why so many researchers and academicians have taken interest to study the management of deteriorating products in warehousing system.

1.1. Literature review

In the inventory model demand depends upon many factors. Most prominent out of them being price of the product, income distribution, tastes and preferences which is mostly linked with lifestyle, prices of other related goods (*e.g.* substitute and compliment) and the number of relevant customers, credit availability, insurance facilities etc. In particular markets at a particular time, the price variable is most dominant in the model. Irrespective of level economic development and income distribution, it is often seen that price has been the most important single determinant of demand. Many researchers developed their inventory models taking a different type of demands, but nowadays demand dependent on price catches many eyes of the customers, because before buying products from the market, many factors running in the mind of the customers, the price of a product out of them.

Many inventory models were developed using price dependent demand. Maiti *et al.* [17] have derived an inventory model with time and price dependent demand and stochastic lead time while Sridevi *et al.* [24] have introduced the Weibull rate of replenishment and developed an inventory model taking selling price dependent demand. Sana [20] has developed an EOQ model for a perishable item with price-sensitive demand and Maihami and Kamalabadi [16] have developed a price dependent demand inventory model with non-instantaneous deteriorating items. On the other hand, many inventory models were derived taking price-dependent demand with different other aspects regarding inventory and also their effects in real life. These inventory models are by Avinadav *et al.* [2], Sarkar *et al.* [21], Saha and Goyal [19], Alfares and Ghaithan [1] and Feng *et al.* [10], among others.

Nowadays, consumer credit facilities in different forms are promoted by the business organization for easy product disposal and for boosting of demand sharply. This phenomenon is such a wide business practice today by lot majority of business organization that no business can be thought of prospering without consumer credit facilities and easy facilities for installment payments with the least possible interest rates [36]. The competition is so fierce that some companies are ready to forgo any interest or processing fees for the credit to be paid later by the customer as per his convenience. So, the point is credit facilities (advance payment systems) are highly popular and very common practice in modern businesses. Advance payment policy is one of them where a supplier asks retailers, to give an opportunity to the customers to pay a fraction of the purchasing cost after delivery of the ordered items. This advance payment policy encourages the customers to increase their orders by which supplier, retailer as well as customers get benefited respectively. It makes successful the whole business cycle *i.e.*, the supplier–retailer–customer business point of view.

Many research works have been carried out in this regards such as researchers like Tsao [35] and Thangam [30], who have developed inventory models using both trade credit and advance sales discount and advance payment scheme, respectively. Again, Thangam [31] developed an inventory model using two-echelon trade credits and advance payment scheme. On the other hand, Taleizadeh *et al.* [25] have derived an EOQ model with multiple partial prepayments and partial back ordering. Again, Taleizadeh [26, 27] has developed an inventory model for an evaporating item using advance payment schemes. Similarly, Zia and Taleizadeh [37] developed a lot sizing model with back ordering under advance payment and delay payments. Researchers like Lashgari *et al.* [14] introduce partial-up-stream and partial-down-stream in their model and developed the model in a three-level supply chain. Similarly, Teng *et al.* [29] have derived a deteriorating inventory model with expiration date and advance payments. Recently, Tavakoli and Taleizadeh [28] have discussed advance payment in their inventory model and developed an EOQ model.

The researcher Gayen and Pal [11] have discussed a two-warehouse inventory model for deterioration items with stock dependent demand and Panda *et al.* [18] have developed a two-warehouse inventory model taking price and stock dependent demand. Similarly, Singh *et al.* [23] have introduced a permissible delay in payments in their two-warehouse inventory model. On the other hand, there exist several two-warehouse inventory models that take into account deterioration and its effect and also taking a different type of demands. These inventory models were developed by Das *et al.* [9], Guchhait *et al.* [12], Bhunia *et al.* [5], Bhunia and Shaikh [3] and Bhunia *et al.* [6], among others. On the other hand, Bhunia *et al.* [7] have developed a two-warehouse partially backlogged deteriorating inventory model using particle swarm optimization, again Bhunia *et al.* [8] has introduced a permissible delay in payment in their two-warehouse inventory model via particle swarm optimization. On the other hand, Bhunia and Shaikh [4] has investigated a two-warehouse inventory problem in interval environment under inflation via particle swarm optimization. Recently, Shaikh [22] has derived a two-warehouse inventory model with variable demand taking alternative trade credit policy. We summarize the contribution in tabular form in Table 1.

1.2. Research gap and the contribution

To best of our knowledge, a few research works have been done in advance payment with single warehouse system. Still now none has introduced the advance payment facility in a two-warehouse inventory system. Two-warehouse systems have a great influence in inventory analysis. In a popular marketplace found a big showroom is very difficult. Due to insufficient space in a popular marketplace, retails need to hire an additional store room on rental basis. Due to the globalization of marketing policy, retailers cannot ignore this strategy. This is a real-life problem in the business world. So, for the first time, we have introduced advance payment facility in a two-warehouse inventory system. We are very much interested in filling this research gap in an advance payment system. In this paper, we have discussed the advance payment facility with equal installment up to n times before receiving the products. Introducing this concept in a two-warehouse system, we have discussed an inventory model for deteriorating product and constant partial backlogging rate. Here, we have considered the demand of the inventory system is dependent on the selling price of the product. We have represented this physical problem in mathematically and solved. During the proof of optimality, we have introduced two necessary theorems and solve the problem numerically. Also, we have proved the optimality graphically by considering a numerical example with the help of MATLAB software. The main contributions are being summarized below:

- Advance payment in **a** two-warehouse system
- n equal installment before received the product

Table 1. Major contribution of the proposed model.

Literature	Two-Warehouse/ Single-Warehouse	Payment	Demand rate	Deterioration	Shortage
Zhou and Liang [15]	Two	Delay in payment	Constant	Constant	No
Maiti *et al.* [17]	Single	Advance	Price-dependent (Non-linear function)	No	Completely backlogged
Taleizadeh *et al.* [26]	Single	Advance	Constant	Constant	Completely backlogged
Taleizadeh *et al.* [27]	Single	Advance	Constant	Constant	Partially backlogged
Lashgari *et al.* [14]	Single	Advance Delay	Constant	No	No shortages, Completely and Partially backlogged
Teng *et al.* [29]	Single	Advance	Constant	Time-Varying	Partially backlogged
Tiwari ei al. [33]	Two	No	Stock dependent	Constant	Fully backlogged
Jaggi *et al.* [13]	Two	No	Price dependent	Constant	Fully backlogged
Tiwari *et al.* [32]	Single	Delay	Price dependent	Expiration	Partially backlogged
Tiwari *et al.* [34]	Two	Delay	Price dependent	Constant	Fully backlogged
This paper	Two	Advance	Price dependent	Constant	Partially backlogged

- Demand of the product is dependent on the price
- Partially backlogged shortages with a constant rate.

The rest part of the paper is organized as follows: In Section 2, we have described assumptions and notation. In Section 3, we have formulated the problem mathematically. In Section 4, we have described the optimality of the proposed problem. Some special cases have been described in Section 5. In Section 6, numerical illustrations and concavity are supplied. Sensitivity analyses and observation regarding sensitivity analysis are performed in Section 7. In Section 8, we have made the conclusion and future scopes of research.

2. Assumptions and notation

To develop the inventory model, we have considered the following assumptions and notation:
Assumptions:

(1) We have considered the demand of this model, for a single item, depends on price *i.e.*, $D(p) = a - bp$.
(2) The deterioration rates for both warehouses (owned and rented) are constants whereas $\alpha(0 < \alpha < 1)$ is the deterioration rate in owned warehouse (OW) and $\beta(0 < \beta < 1)$ is the deterioration rate in rented warehouse (RW). Moreover, the RW offers better preserving facilities than the OW, consequently, we can assume that $\beta < \alpha$.
(3) No replacements or repairs for deteriorated products have been considered during this model.
(4) Inventory planning horizon is infinite.

(5) The enterprise pays a fraction k of the total purchasing cost with n equally spaced multiple installments within the lead time M and receives the lot by paying the remaining purchasing cost.
(6) The holding cost per unit, c_{hr}, in rented warehouse (RW) is greater than the holding cost per unit, c_{ho}, in owned warehouse (OW) due to the better facilities in the RW.
(7) Since $c_{hr} > c_{ho}$, the products in RW will be consumed first.
(8) The maximum deteriorating items in OW, αW_1, has been considered less than the demand rate, $D(p) = (a - bp)$, of the product in order to the existence of the optimal solution and accordingly, $\frac{\alpha W_1}{(a-bp)}$ is a small positive number less than 1.
(9) Shortages are allowed and during the stock out period, a fraction η of the demand $D(p) = (a - bp)$ will be backorder.

Notation:

Notations	Units	Description
A	\$/order	Ordering cost
a	Constant	Coefficient part of the demand rate ($a > 0$)
b	Constant	Constant of the price in the demand rate ($b > 0$)
p	\$/unit	Selling price per unit
c_l	\$/unit	Opportunity cost per unit
c_s	\$/unit	Shortage cost per unit
α	Constant	Deterioration rate at owned warehouse
β	Constant	Deterioration rate at rented warehouse
c_{ho}	\$/unit	Holding cost per unit for owned warehouse
c_{hr}	\$/unit	Holding cost per unit for rented warehouse
W_1	Units	Inventory level at rented warehouse
c_p	Units	Purchase cost per unit
η	Units	Backlogging unit ($0 < \eta < 1$)
M	yr	Length of the lead time during which the enterprise will pay the prepayments
n	Constant	Number of equally spaced prepayments during the lead time
k	Constant	Fraction of the purchasing cost that must be paid with multiple prepayments ($0 < k < 1$)
c_d	\$/unit	Deterioration cost per unit
		Dependent variable
t_2	yr	Time at which the stock reaches to zero at OW
S	Units	Total Inventory level
R	Units	Backlogged units
		Decision variable
t_1	yr	Time at which the stock reaches to zero at RW
T	yr	The length of the replenishment cycle

3. Problem definition

Let us assume that an enterprise makes an order of $(S+R)$ units of a product by prepaying a fraction k of the purchasing cost by n equal multiple installments at equal intervals within the lead time M and receives the lot by paying the remaining purchasing cost at time $t = 0$. Shortly after R units are utilized to fulfill the backlogged demand partially consequently the on hand inventory level becomes S. Out of which W_1 units are stored in OW and the remaining part $(S - W_1)$ are stored in RW. Since the RW offers better facilities apparently the holding cost in RW is greater than that of in OW and as such the products in RW will be consumed first. During the time interval $[0, t_1]$, the inventory level in RW depletes due to meet up the customers' demand $D(p)$ as well as constant deterioration rate β. At the time $t = t_1$ it becomes zero in RW. On the other hand inventory level in OW depletes due to constant deterioration rate α only during the period $[0, t_1]$. Shortly after, the inventory in

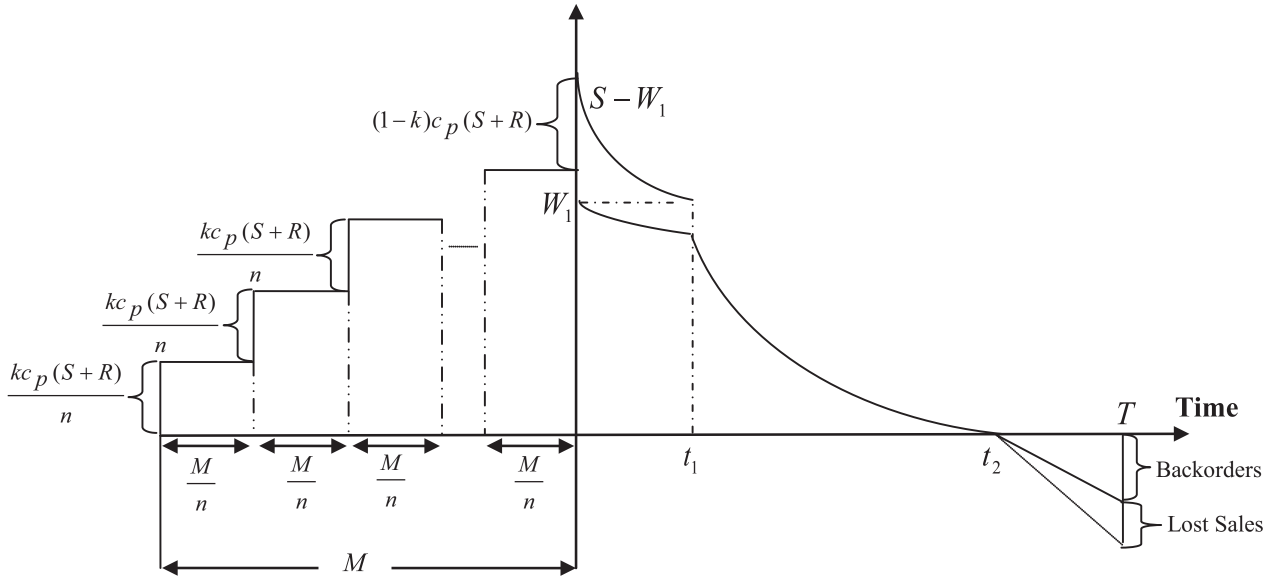

FIGURE 1. Graphical presentation of two-warehouse inventory system under prepayments with shortages.

OW is depleted due to the customers' demand $D(p)$ and deterioration as well during the time interval $[t_1, t_2]$. At time $t = t_2$, apparently, it also becomes zero. Thereafter, shortages are appeared which are accumulated with a constant rate η during the time interval $[t_2, T]$. The two-warehouse inventory level, using above assumptions, follows the pattern depicted in Figure 1.

So the inventory level $I_r(t)$ in RW at any instant t can be described by the following differential equation:

$$\frac{\mathrm{d}I_r(t)}{\mathrm{d}t} + \beta I_r(t) = -(a - bp), \quad 0 \le t \le t_1 \tag{3.1}$$

subject to the conditions:

$$I_r(t) = \begin{cases} S - W_1, & \text{at} \quad t = 0 \\ 0, & \text{at} \quad t = t_1 \end{cases}. \tag{3.2}$$

The solution of equation (3.1), with boundary condition equation (3.2), is given by:

$$I_r(t) = \frac{a - bp}{\beta} \left[\mathrm{e}^{\beta(t_1 - t)} - 1 \right], \quad 0 \le t \le t_1. \tag{3.3}$$

Using $I_r(0) = S - W_1$ in equation (3.3), one has

$$S = W_1 + \frac{a - bp}{\beta} \left(\mathrm{e}^{\beta t_1} - 1 \right). \tag{3.4}$$

Again, the inventory level $I_o(t)$ in OW at any instant t can be described by the following differential equations

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} + \alpha I_o(t) = 0, \qquad 0 \le t \le t_1 \tag{3.5}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} + \alpha I_o(t) = -(a - bp), \quad t_1 \le t \le t_2 \tag{3.6}$$

$$\frac{\mathrm{d}I_o(t)}{\mathrm{d}t} = -\eta(a - bp), \quad t_2 < t \le T \tag{3.7}$$

subject to the boundary conditions

$$I_o(t) = \begin{cases} W_1, & \text{at} \quad t = 0 \\ 0, & \text{at} \quad t = t_2 \\ -R, & \text{at} \quad t = T. \end{cases} \tag{3.8}$$

The solutions of the differential equations (3.5)–(3.7), with the help of the boundary conditions (3.8), can be written as:

$$I_o(t) = W_1 \mathrm{e}^{-\alpha t}, \qquad 0 \le t \le t_1 \tag{3.9}$$

$$I_o(t) = \frac{a-bp}{\alpha}\left[\mathrm{e}^{\alpha(t_2-t)} - 1\right], \quad t_1 < t \le t_2 \tag{3.10}$$

$$I_o(t) = \eta(a-bp)(T-t) - R, \quad t_2 < t \le T. \tag{3.11}$$

By considering the continuity at $t = t_1$ and $t = t_2$, we can write:

$$W_1 \mathrm{e}^{-\alpha t_1} = \frac{a-bp}{\alpha}\left[\mathrm{e}^{\alpha(t_2-t_1)} - 1\right] \tag{3.12}$$

and

$$R = \eta(a-bp)(T-t_2). \tag{3.13}$$

Here we have described inventory related cost for this model derived from the assumptions:

(a) Ordering cost: A

(b) Purchase cost: $c_p(S+R) = c_p\left[W_1 + \frac{a-bp}{\beta}\left(\mathrm{e}^{\beta t_1} - 1\right) + \eta(a-bp)(T-t_2)\right]$

(c) Holding cost: $c_{hr}\int_0^{t_1} I_r(t)\mathrm{d}t + c_{ho}\int_0^{t_1} I_o(t)\mathrm{d}t + c_{ho}\int_{t_1}^{t_2} I_o(t)\mathrm{d}t$

$$= \frac{c_{hr}(a-bp)}{\beta^2}\left(\mathrm{e}^{\beta t_1} - \beta t_1 - 1\right) + \frac{c_{ho}W_1}{\alpha}\left(1 - \mathrm{e}^{-\alpha t_1}\right) + \frac{c_{ho}(a-bp)}{\alpha^2}\left(\mathrm{e}^{\alpha(t_2-t_1)} - \alpha(t_2-t_1) - 1\right)$$

(d) Deterioration cost: $c_d\beta\int_0^{t_1} I_r(t)\mathrm{d}t + c_d\alpha\int_0^{t_1} I_o(t)\mathrm{d}t + c_d\alpha\int_{t_1}^{t_2} I_o(t)\mathrm{d}t$

$$= \frac{c_d(a-bp)}{\beta}\left(\mathrm{e}^{\beta t_1} - \beta t_1 - 1\right) + c_d W_1\left(1 - \mathrm{e}^{-\alpha t_1}\right) + \frac{c_d(a-bp)}{\alpha}\left(\mathrm{e}^{\alpha(t_2-t_1)} - \alpha(t_2-t_1) - 1\right)$$

(e) Shortage cost: $-c_s\int_{t_2}^{T} I_o(t)\mathrm{d}t = \frac{1}{2}c_s\eta(a-bp)(T-t_2)^2$

(f) Opportunity cost: $c_l(1-\eta)\int_{t_2}^{T} D\mathrm{d}t = c_l(1-\eta)(a-bp)(T-t_2)$

(g) Capital cost: The capital cost from Figure 1 or Taleizadeh [26] or Taleizadeh [27] is

$$I_c\left[\frac{kc_p(S+R)}{n}.\frac{M}{n}(1+2+3+\ldots+n)\right] = \frac{n+1}{2n}I_c M k c_p\left[W_1 + \frac{a-bp}{\beta}\left(\mathrm{e}^{\beta t_1} - 1\right) + \eta(a-bp)(T-t_2)\right]$$

Therefore, the total cyclic cost per unit time is

$$\mathrm{TC} = \frac{1}{T}\big[\langle\text{Ordering cost}\rangle + \langle\text{Purchase cost}\rangle + \langle\text{Holding cost}\rangle + \langle\text{Deterioration cost}\rangle + \langle\text{Shortage cost}\rangle + \langle\text{Opportunity cost}\rangle + \langle\text{Capital cost}\rangle\big]$$

i.e.,

$$\mathrm{TC} = \frac{1}{T}\left[\begin{array}{l} A + \frac{(a-bp)}{\beta^2}\left(c_{hr} + \beta c_d\right)\left(\mathrm{e}^{\beta t_1} - \beta t_1 - 1\right) + \frac{(a-bp)}{\alpha^2}\left(c_{ho} + \alpha c_d\right)\left(\mathrm{e}^{\alpha(t_2-t_1)} - \alpha(t_2-t_1) - 1\right) \\ +\frac{W_1}{\alpha}\left(c_{ho} + \alpha c_d\right)\left(1 - \mathrm{e}^{-\alpha t_1}\right) + \frac{1}{2}c_s\eta(a-bp)(T-t_2)^2 + c_l(1-\eta)(a-bp)(T-t_2) \\ +\left(1 + \frac{n+1}{2n}I_c M k\right)c_p\left\{W_1 + \frac{a-bp}{\beta}\left(\mathrm{e}^{\beta t_1} - 1\right) + \eta(a-bp)(T-t_2)\right\} \end{array}\right]. \tag{3.14}$$

Note that from equation (3.12) one can write $t_2 = t_1 + \frac{1}{\alpha}\log\left[1+\frac{\alpha W_1 \mathrm{e}^{-\alpha t_1}}{a-bp}\right]$. So if we substitute each t_2 with the help of this expression, the total cyclic cost per unit time TC contains only two independent variables namely t_1 and T. Consequently, there are only two decision variables in the proposed inventory model under which the total cyclic cost per unit time $\mathrm{TC}(t_1, T)$ has to be minimized.

4. Theoretical results

In this section, the proof of the optimality of the proposed problem has been presented mathematically by using theorem and Lemma. We have shown that the total cyclic cost per unit time TC contains the global minimum value at the optimal solution (t_1^*, T^*) with the help of Hessian matrix.

Theorem 4.1. *The total cost function per unit time* TC *attains its minimum value as the Hessian matrix is always positive definite at the optimal solution* (t_1^*, T^*).

Proof. To prove the theorem, firstly, we want to show that all the principal minors of the Hessian matrix at (t_1^*, T^*) are positive. The Hessian matrix of the total cyclic cost per unit time at the optimal values is given as

$$H_{ii} = \begin{bmatrix} \frac{\partial^2 \mathrm{TC}}{\partial t_1^{*2}} & \frac{\partial^2 \mathrm{TC}}{\partial t_1^* \partial T^*} \\ \frac{\partial^2 \mathrm{TC}}{\partial T^* \partial t_1^*} & \frac{\partial^2 \mathrm{TC}}{\partial T^{*2}} \end{bmatrix}.$$

For the convenience of calculation, with the help of equation (3.12), we can rewrite the equation (4.9) in the following way

$$\frac{\partial \mathrm{TC}}{\partial t_1} = \frac{1}{T}\begin{bmatrix} \frac{(a-bp)}{\beta}\left(c_{hr}+\beta c_d\right)\left(\mathrm{e}^{\beta t_1}-1\right) - \left(c_{ho}+\alpha c_d\right)\frac{\alpha W_1^2 \mathrm{e}^{-\alpha(t_1+t_2)}}{(a-bp)} + W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1} \\ -c_s\eta(a-bp)(T-t_2)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right) - c_l(1-\eta)(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right) \\ +\left(1+\frac{n+1}{2n}I_c M k\right)c_p(a-bp)\left(\mathrm{e}^{\beta t_1}-\eta\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)\right) \end{bmatrix}. \tag{4.1}$$

Now all the second order partial derivatives at the point (t_1^*, T^*) are given by:

$$\begin{aligned} \frac{\partial^2 \mathrm{TC}}{\partial {t_1}^{*2}} &= \frac{1}{T^*}\begin{bmatrix} (a-bp)\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t_1} + W_1\left(c_{ho}+\alpha c_d\right)\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha(t_1+t_2)}}{(a-bp)}\left(\frac{\mathrm{d}t_2}{\mathrm{d}t_1}+1\right) - \alpha W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1} \\ -c_s\eta(a-bp)\left\{-\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right) + (T-t_2)\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right\} \\ -c_l(1-\eta)(a-bp)\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\frac{\mathrm{d}t_2}{\mathrm{d}t_1} + \left(1+\frac{n+1}{2n}I_c M k\right)c_p(a-bp)\left(\beta \mathrm{e}^{\beta t_1} - \eta\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right) \end{bmatrix}_{(t_1^*, T^*)} \\ &= \frac{1}{T^*}\begin{bmatrix} (a-bp)\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t_1} + W_1\left(c_{ho}+\alpha c_d\right)\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha(t_1+t_2)}}{(a-bp)}\left(2-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right) \\ -\alpha W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1} - c_l(1-\eta)(a-bp)\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right) \\ +c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)\left\{\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right) - \frac{\alpha^2 W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}(T-t_2)\right\} \\ +\left(1+\frac{n+1}{2n}I_c M k\right)c_p(a-bp)\left\{\beta \mathrm{e}^{\beta t_1} - \eta\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)\right\} \end{bmatrix}_{(t_1^*, T^*)}. \end{aligned} \tag{4.2}$$

Based on the assumption that the demand rate $(a-bp)$ is greater than the maximum deteriorated products in OW *i.e.*, $(a-bp) > \alpha W_1$, so $\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}$ is a small positive number less than 1. Moreover, the value of α is quite small, consequently, $\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)} \to 0$ and $\frac{\alpha^2 W_1 \mathrm{e}^{-\alpha(t_1+t_2)}}{(a-bp)} \to 0$. Then equation (4.2) can be written as

$$\frac{\partial^2 \mathrm{TC}}{\partial t_1{}^{*2}} = \frac{1}{T^*}\left[\begin{array}{l}(a-bp)\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t_1^*} - \alpha W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1^*} + c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)^2 \\ +\left(1+\frac{n+1}{2n}I_c Mk\right)c_p(a-bp)\beta \mathrm{e}^{\beta t_1^*}\end{array}\right] \quad (4.3)$$

$$\begin{aligned}\frac{\partial^2 \mathrm{TC}}{\partial T^*\partial t_1^*} &= -\frac{1}{T^*}\left[\frac{\partial \mathrm{TC}}{\partial t_1}\right]_{(t_1^*,T^*)} - \frac{1}{T^*}\left[c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)\right]_{(t_1^*,T^*)} \\ &= -\frac{1}{T^*}\left[c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)\right] \quad (4.4)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \mathrm{TC}}{\partial t_1^*\partial T^*} &= -\frac{1}{T^*}\left[\frac{\partial \mathrm{TC}}{\partial t_1}\right]_{(t_1^*,T^*)} - \frac{1}{T^*}\left[c_s\eta(a-bp)\frac{\mathrm{d}t_2}{\mathrm{d}t_1})\right]_{(t_1^*,T^*)} \\ &= -\frac{1}{T^*}\left[c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)\right] \quad (4.5)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \mathrm{TC}}{\partial T^{*2}} &= -\frac{1}{T^*}\left[\frac{\partial \mathrm{TC}}{\partial T}\right]_{(t_1^*,T^*)} + \left[\frac{1}{T^2}\mathrm{TC} - \frac{1}{T^2}\{c_s\eta(a-bp)(T-t_2)+c_l(1-\eta)(a-bp)\right. \\ &\quad \left.+\left(1+\frac{n+1}{2n}I_c Mk\right)c_p\eta(a-bp)\}\right]_{(t_1^*,T^*)} + \frac{1}{T^*}c_s\eta(a-bp) = \frac{1}{T^*}c_s\eta(a-bp). \quad (4.6)\end{aligned}$$

So the first principal minor at (t_1^*, T^*) is

$$\begin{aligned}\det\left(H_{11}\right) &= \det\left(\frac{\partial^2 \mathrm{TC}}{\partial t_1^{*2}}\right) \\ &= \frac{1}{T^*}\left[\begin{array}{l}(a-bp)\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t_1^*} - \alpha W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1^*} + c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)^2 \\ +\left(1+\frac{n+1}{2n}I_c Mk\right)c_p(a-bp)\beta \mathrm{e}^{\beta t_1^*}\end{array}\right] \\ &> \frac{1}{T^*}\left[c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)^2 + \left(1+\frac{n+1}{2n}I_c Mk\right)c_p(a-bp)\beta \mathrm{e}^{\beta t_1^*}\right]. \text{(by Lemma 4.2)}\end{aligned}$$

Consequently, the first principal minor $\det(H_{11}) > 0$ as all the terms are positives.
Now the second principal minor at (t_1^*, T^*) is

$$\begin{aligned}\det\left(H_{22}\right) &= \begin{vmatrix}\frac{\partial^2 \mathrm{TC}}{\partial t_1^{*2}} & \frac{\partial^2 \mathrm{TC}}{\partial t_1^*\partial T^*} \\ \frac{\partial^2 \mathrm{TC}}{\partial T^*\partial t_1^*} & \frac{\partial^2 \mathrm{TC}}{\partial T^{*2}}\end{vmatrix} = \frac{\partial^2 \mathrm{TC}}{\partial t_1^{*2}}\frac{\partial^2 \mathrm{TC}}{\partial T^{*2}} - \frac{\partial^2 \mathrm{TC}}{\partial t_1^*\partial T^*}\frac{\partial^2 \mathrm{TC}}{\partial T^*\partial t_1^*} \\ &= \frac{1}{T^*}\left[\begin{array}{l}(a-bp)\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t_1^*} - \alpha W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1^*} \\ +c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)^2 \\ +\left(1+\frac{n+1}{2n}I_c Mk\right)c_p(a-bp)\beta \mathrm{e}^{\beta t_1^*}\end{array}\right] \\ &\quad \times \frac{1}{T^*}c_s\eta(a-bp) - \frac{1}{T^{*2}}\left[c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)\right]^2 \\ &> \frac{1}{T^{*2}}\left[c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)^2 + \left(1+\frac{n+1}{2n}I_c Mk\right)c_p(a-bp)\beta \mathrm{e}^{\beta t_1^*}\right]\cdot c_s\eta(a-bp) \\ &\quad -\frac{1}{T^{*2}}\left[c_s\eta(a-bp)\left(1-\frac{\alpha W_1 \mathrm{e}^{-\alpha t_2^*}}{(a-bp)}\right)\right]^2 \qquad \text{(by Lemma 4.2)} \\ &= \frac{1}{T^{*2}}\left(1+\frac{n+1}{2n}I_c Mk\right)c_p(a-bp)\beta \mathrm{e}^{\beta t_1^*} > 0.\end{aligned}$$

Therefore, $\det(H_{22}) > 0$. As the entire principal minors at (t_1^*, T^*) are positive, consequently, the Hessian matrix is positive definite. As a result, the total cyclic cost per unit time TC attains its minimum value at the optimal solution (t_1^*, T^*). □

For the proof of the Theorem 4.1, we need to discuss about Lemma 4.2 and which is given bellow.

Lemma 4.2. *If $(a-bp) > \alpha W_1$, for all $t>0$, the value of $(a-bp)\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t}$ is always strictly greater than the value of $\alpha W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t}$.*

Proof. For the convenient, let $\xi(t)=(a-bp)\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t}-\alpha W_1\left(c_{ho}+\beta c_d\right)\mathrm{e}^{-\alpha t}$, where $t>0$. The value of $\xi(t)$ at $t=0$ is

$$\begin{aligned}\xi(0)&=(a-bp)\left(c_{hr}+\beta c_d\right)-\alpha W_1\left(c_{ho}+\beta c_d\right)\\&>(a-bp)\left(c_{ho}+\beta c_d\right)-\alpha W_1\left(c_{ho}+\beta c_d\right)=\left\{(a-bp)-\alpha W_1\right\}\left(c_{ho}+\beta c_d\right).\end{aligned}$$

If the demand rate $(a-bp)$ is greater than the maximum deteriorated products in OW *i.e.*, $(a-bp)>\alpha W_1$, the value $\xi(0)$ is positive. Furthermore, the first order derivative of $\xi(t)$, $\xi'(t)=(a-bp)\beta\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t}+\alpha^2 W_1\left(c_{ho}+\beta c_d\right)\mathrm{e}^{-\alpha t}$ is positive for all $t>0$. Therefore, $\xi(t)$ is a positive valued and also increasing function in the time interval $[0,\infty)$. Hence we can write $(a-bp)\left(c_{hr}+\beta c_d\right)\mathrm{e}^{\beta t}>\alpha W_1\left(c_{ho}+\beta c_d\right)\mathrm{e}^{-\alpha t}>\alpha W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t}$. This completes the proof. □

For minimization the total cyclic cost per unit time equation (3.14), calculate the first order derivatives of equation (3.14) with respect to t_1 and T we have:

$$\frac{\partial \mathrm{TC}}{\partial t_1}=\frac{1}{T}\left[\begin{array}{l}\frac{(a-bp)}{\beta}\left(c_{hr}+\beta c_d\right)\left(\mathrm{e}^{\beta t_1}-1\right)+\frac{(a-bp)}{\alpha}\left(c_{ho}+\alpha c_d\right)\left(\frac{\mathrm{d}t_2}{\mathrm{d}t_1}-1\right)\left(\mathrm{e}^{\alpha(t_2-t_1)}-1\right)+W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1}\\-c_s\eta(a-bp)(T-t_2)\frac{\mathrm{d}t_2}{\mathrm{d}t_1}-c_l(1-\eta)(a-bp)\frac{\mathrm{d}t_2}{\mathrm{d}t_1}+\left(1+\frac{n+1}{2n}I_cMk\right)c_p(a-bp)\left(\mathrm{e}^{\beta t_1}-\eta\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right)\end{array}\right].\tag{4.7}$$

From equation (3.12), we have:

$$\begin{aligned}-\alpha W_1\mathrm{e}^{-\alpha t_1}&=(a-bp)\,\mathrm{e}^{\alpha(t_2-t_1)}\left(\frac{\mathrm{d}t_2}{\mathrm{d}t_1}-1\right)\\\frac{\mathrm{d}t_2}{\mathrm{d}t_1}&=1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_1}}{(a-bp)\,\mathrm{e}^{\alpha(t_2-t_1)}}\\\frac{\mathrm{d}t_2}{\mathrm{d}t_1}&=1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\cdot\end{aligned}\tag{4.8}$$

Using equation (4.8) in equation (4.7), one can get

$$\frac{\partial \mathrm{TC}}{\partial t_1}=\frac{1}{T}\left[\begin{array}{l}\frac{(a-bp)}{\beta}\left(c_{hr}+\beta c_d\right)\left(\mathrm{e}^{\beta t_1}-1\right)-W_1\mathrm{e}^{-\alpha t_2}\left(c_{ho}+\alpha c_d\right)\left(\mathrm{e}^{\alpha(t_2-t_1)}-1\right)+W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1}\\-c_s\eta(a-bp)(T-t_2)\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)-c_l(1-\eta)(a-bp)\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)\\+\left(1+\frac{n+1}{2n}I_cMk\right)c_p(a-bp)\left(\mathrm{e}^{\beta t_1}-\eta\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)\right)\end{array}\right]\tag{4.9}$$

$$\frac{\partial \mathrm{TC}}{\partial T}=-\frac{1}{T}\mathrm{TC}+\frac{1}{T}\left[c_s\eta(a-bp)(T-t_2)+c_l(1-\eta)(a-bp)+\left(1+\frac{n+1}{2n}I_cMk\right)c_p\eta(a-bp)\right].\tag{4.10}$$

To find the necessary condition of minimization of the total cyclic cost per unit time, we set all the first order derivatives with respect to the decision variables t_1 and T of TC are equal to zero.

$$\left[\begin{array}{l}\frac{(a-bp)}{\beta}\left(c_{hr}+\beta c_d\right)\left(\mathrm{e}^{\beta t_1}-1\right)-W_1\mathrm{e}^{-\alpha t_2}\left(c_{ho}+\alpha c_d\right)\left(\mathrm{e}^{\alpha(t_2-t_1)}-1\right)+W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1}\\-c_s\eta(a-bp)(T-t_2)\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)-c_l(1-\eta)(a-bp)\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)\\+\left(1+\frac{n+1}{2n}I_cMk\right)c_p(a-bp)\left(\mathrm{e}^{\beta t_1}-\eta\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)\right)\end{array}\right]=0\tag{4.11}$$

$$-\frac{1}{T}\mathrm{TC}+\frac{1}{T}\left[c_s\eta(a-bp)(T-t_2)+c_l(1-\eta)(a-bp)+\left(1+\frac{n+1}{2n}I_cMk\right)c_p\eta(a-bp)\right]=0.\tag{4.12}$$

On solving equations (4.11) and (4.12) we get the optimal values of t_1 and T (say t_1^* and T^*) which is unique.

5. Some special cases

(a) Model with complete backlogging

If $\eta = 1$, then the total cost function per unit time is given by

$$\mathrm{TC}(t_1,T) = \frac{1}{T}\begin{bmatrix} A + \frac{(a-bp)}{\beta^2}(c_{hr}+\beta c_d)\left(\mathrm{e}^{\beta t_1}-\beta t_1-1\right) + \frac{(a-bp)}{\alpha^2}(c_{ho}+\alpha c_d)\left(\mathrm{e}^{\alpha(t_2-t_1)}-\alpha(t_2-t_1)-1\right) \\ +\frac{W_1}{\alpha}(c_{ho}+\alpha c_d)\left(1-\mathrm{e}^{-\alpha t_1}\right)+\frac{1}{2}c_s(a-bp)(T-t_2)^2 \\ +\left(1+\frac{n+1}{2n}I_cMk\right)c_p\left\{W_1+\frac{a-bp}{\beta}\left(\mathrm{e}^{\beta t_1}-1\right)+(a-bp)(T-t_2)\right\} \end{bmatrix}.$$

Then the necessary conditions for $\mathrm{TC}(t_1,T)$ to be minimized are:

$$\begin{bmatrix} \frac{(a-bp)}{\beta}(c_{hr}+\beta c_d)\left(\mathrm{e}^{\beta t_1}-1\right) - W_1\mathrm{e}^{-\alpha t_2}(c_{ho}+\alpha c_d)\left(\mathrm{e}^{\alpha(t_2-t_1)}-1\right)+W_1(c_{ho}+\alpha c_d)\mathrm{e}^{-\alpha t_1} \\ -c_s(a-bp)(T-t_2)\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)+\left(1+\frac{n+1}{2n}I_cMk\right)c_p(a-bp)\left(\mathrm{e}^{\beta t_1}-1+\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right) \end{bmatrix} = 0$$

$$\left[c_s(a-bp)(T-t_2)+\left(1+\frac{n+1}{2n}I_cMk\right)c_p(a-bp)\right] = \mathrm{TC}.$$

(b) Model without Shortage

If $T \approx t_2$, *i.e.*, $R = 0$, then the total cost function per unit becomes

$$\mathrm{TC}(t_1) = \frac{1}{T}\begin{bmatrix} A + \frac{(a-bp)}{\beta^2}(c_{hr}+\beta c_d)\left(\mathrm{e}^{\beta t_1}-\beta t_1-1\right)+\frac{(a-bp)}{\alpha^2}(c_{ho}+\alpha c_d)\left(\mathrm{e}^{\alpha(T-t_1)}-\alpha(T-t_1)-1\right) \\ +\frac{W_1}{\alpha}(c_{ho}+\alpha c_d)\left(1-\mathrm{e}^{-\alpha t_1}\right)+\left(1+\frac{n+1}{2n}I_cMk\right)c_p\left\{W_1+\frac{a-bp}{\beta}\left(\mathrm{e}^{\beta t_1}-1\right)\right\} \end{bmatrix}.$$

Using continuity at $t = t_1$, we can get

$$W_1\mathrm{e}^{-\alpha t_1} = \frac{a-bp}{\alpha}\left[\mathrm{e}^{\alpha(T-t_1)}-1\right]$$

$$\frac{\mathrm{d}T}{\mathrm{d}t_1} = 1-\frac{\alpha W_1\mathrm{e}^{-\alpha T}}{(a-bp)}\cdot$$

The necessary condition for $\mathrm{TC}(t_1)$to be minimized is

$$\frac{\mathrm{d}TC(t_1)}{\mathrm{d}t_1} = 0$$

i.e.,

$$\begin{bmatrix} \frac{(a-bp)}{\beta}(c_{hr}+\beta c_d)\left(\mathrm{e}^{\beta t_1}-1\right)-W_1\mathrm{e}^{-\alpha T}(c_{ho}+\alpha c_d)\left(\mathrm{e}^{\alpha(T-t_1)}-1\right) \\ +W_1(c_{ho}+\alpha c_d)\mathrm{e}^{-\alpha t_1}+\left(1+\frac{n+1}{2n}I_cMk\right)c_p(a-bp)\mathrm{e}^{\beta t_1} \end{bmatrix} = \left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha T}}{(a-bp)}\right)\mathrm{TC}.$$

The second order derivative is

$$\frac{d^2\mathrm{TC}(t_1)}{\mathrm{d}t_1^2} = \frac{1}{T}\begin{bmatrix} -\frac{\alpha^2 W_1\mathrm{e}^{-\alpha T}}{a-bp}\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha T}}{a-bp}\right)\mathrm{TC}+(a-bp)(c_{hr}+\beta c_d)\mathrm{e}^{\beta t_1} \\ +\frac{\alpha^2 W_1^2\mathrm{e}^{-2\alpha T}}{a-bp}(c_{ho}+\alpha c_d)\mathrm{e}^{\alpha(T-t_1)}-\alpha W_1(c_{ho}+\alpha c_d)\mathrm{e}^{-\alpha t_1} \\ +\left(1+\frac{n+1}{2n}I_cMk\right)c_p\beta(a-bp)\mathrm{e}^{\beta t_1} \end{bmatrix}.$$

Based on the assumption that $\frac{\alpha W_1\mathrm{e}^{-\alpha T}}{(a-bp)}$ is a small positive number less than 1. Moreover, the value of α is quite small, consequently, $\frac{\alpha^2 W_1\mathrm{e}^{-\alpha T}}{(a-bp)} \to 0$. Then we can write

$$\frac{d^2\mathrm{TC}(t_1)}{\mathrm{d}t_1^2} = \frac{1}{T}\begin{bmatrix} (a-bp)(c_{hr}+\beta c_d)\mathrm{e}^{\beta t_1}+\frac{\alpha^2 W_1^2\mathrm{e}^{-2\alpha T}}{a-bp}(c_{ho}+\alpha c_d)\mathrm{e}^{\alpha(T-t_1)}-\alpha W_1(c_{ho}+\alpha c_d)\mathrm{e}^{-\alpha t_1} \\ +\left(1+\frac{n+1}{2n}I_cMk\right)c_p\beta(a-bp)\mathrm{e}^{\beta t_1} \end{bmatrix}$$

$$> \frac{1}{T}\left[\frac{\alpha^2 W_1^2\mathrm{e}^{-2\alpha T}}{a-bp}(c_{ho}+\alpha c_d)\mathrm{e}^{\alpha(T-t_1)}+\left(1+\frac{n+1}{2n}I_cMk\right)c_p\beta(a-bp)\mathrm{e}^{\beta t_1}\right] > 0.$$

(by Lemma 4.2)

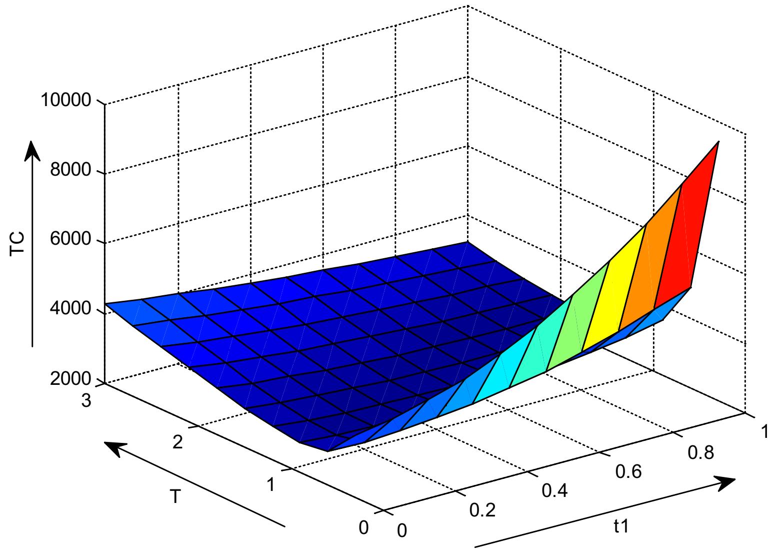

FIGURE 2. Total cyclic cost per unit time TC *versus* t_1 and T.

(c) Model without advance payment
If $M = 0$ *i.e.*, the purchasing cost will be paid at the receiving time of the lot, then the total cost function per unit time, $\mathrm{TC}(t_1, T)$, is given by

$$\mathrm{TC}(t_1,T) = \frac{1}{T}\begin{bmatrix} A + \frac{(a-bp)}{\beta^2}\left(c_{hr}+\beta c_d\right)\left(\mathrm{e}^{\beta t_1}-\beta t_1-1\right)+\frac{(a-bp)}{\alpha^2}\left(c_{ho}+\alpha c_d\right)\left(\mathrm{e}^{\alpha(t_2-t_1)}-\alpha(t_2-t_1)-1\right) \\ +\frac{W_1}{\alpha}\left(c_{ho}+\alpha c_d\right)\left(1-\mathrm{e}^{-\alpha t_1}\right)+\frac{1}{2}c_s\eta(a-bp)(T-t_2)^2+c_l(1-\eta)(a-bp)(T-t_2) \\ +c_p\left\{W_1+\frac{a-bp}{\beta}\left(\mathrm{e}^{\beta t_1}-1\right)+\eta(a-bp)(T-t_2)\right\} \end{bmatrix}.$$

Then the necessary conditions for $\mathrm{TC}(t_1, T)$ to be minimized are:

$$\begin{bmatrix} \frac{(a-bp)}{\beta}\left(c_{hr}+\beta c_d\right)\left(\mathrm{e}^{\beta t_1}-1\right)-W_1\mathrm{e}^{-\alpha t_2}\left(c_{ho}+\alpha c_d\right)\left(\mathrm{e}^{\alpha(t_2-t_1)}-1\right)+W_1\left(c_{ho}+\alpha c_d\right)\mathrm{e}^{-\alpha t_1} \\ -c_s(a-bp)(T-t_2)\left(1-\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right)+c_p(a-bp)\left(\mathrm{e}^{\beta t_1}-1+\frac{\alpha W_1\mathrm{e}^{-\alpha t_2}}{(a-bp)}\right) \end{bmatrix}=0$$

$$[c_s(a-bp)(T-t_2)+c_p(a-bp)] = \mathrm{TC}.$$

(d) If $S - W_1 = 0$, $t_1 = 0$, $\eta = 1$ and D is constant, then the proposed model is reduced to a single warehouse model and similar to Taleizadeh [26].
(e) If $S - W_1 = 0$, $t_1 = 0$ and D is constant, then the proposed model is reduced to a single warehouse model and similar to Taleizadeh [27].

6. Numerical illustration

According to the assumptions, we have developed a two-warehouse inventory model in the area of inventory control with advance payment. We already proved the optimality in the previous section. Now, we are going to validate the proposed model by considering a numerical example with the following values of different parameters. Also, we have validated the obtained result from lingo by using MATLAB 3D and 2D plot.

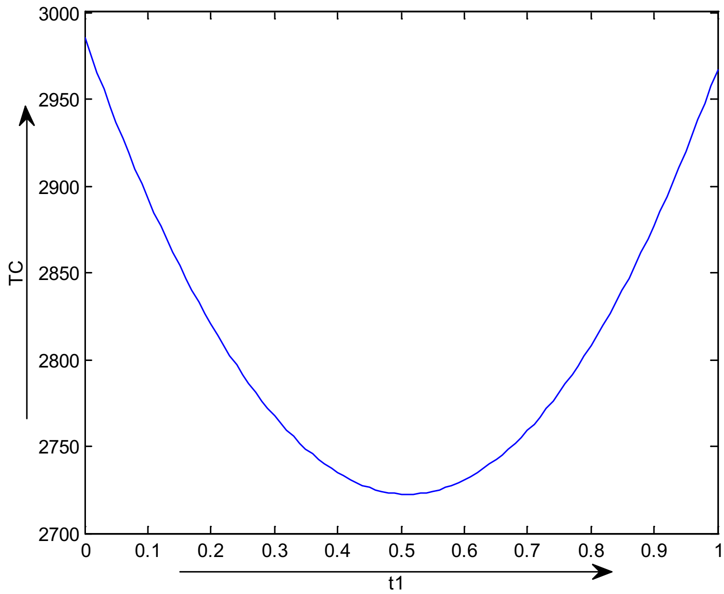

FIGURE 3. Line diagram of total cyclic cost per unit time TC *versus* t_1.

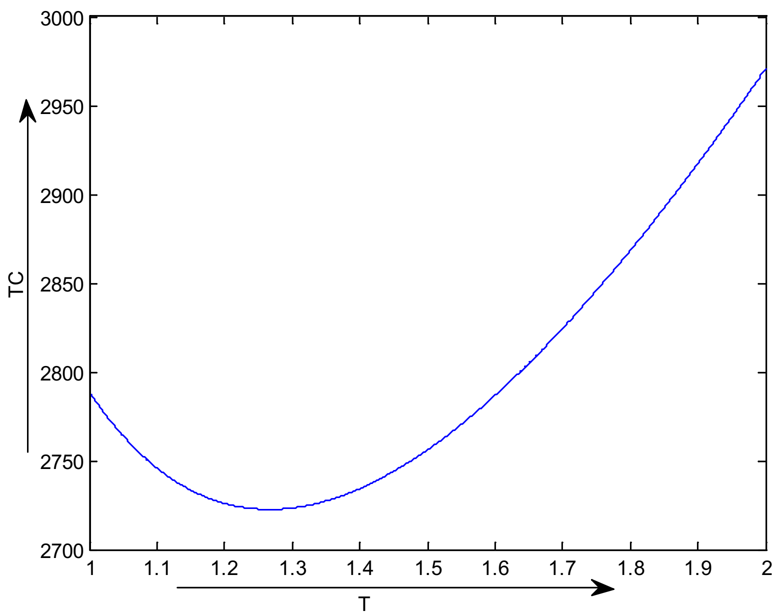

FIGURE 4. Line diagram of total cyclic cost per unit time TC *versus* T.

Example:

$A = \$500/\text{order}$, $a = 200\,\text{units}\,\text{yr}^{-1}$, $b = 0.5$, $p = \$15/\text{unit}$, $c_p = \$10/\text{unit}$, $c_{hr} = \$3/\text{unit}\,\text{yr}^{-1}$, $c_{ho} = \$1/\text{unit}\,\text{yr}^{-1}$, $c_s = \$12/\text{unit}\,\text{yr}^{-1}$, $c_l = \$17/\text{unit}\,\text{yr}^{-1}$, $c_d = \$10/\text{unit}\,\text{yr}^{-1}$, $M = 0.25\,\text{yr}$, $I_c = \$0.25/\text{yr}$, $W_1 = 100\,\text{units}$, $\alpha = 0.1$, $\beta = 0.08$, $n = 15$, $k = 0.4$ and $\eta = 0.8$.

TABLE 2. Sensitivity analysis with respect to different parameters.

Parameter	% Changes	% Changes in					
		t_1^*	t_2^*	T^*	S^*	R^*	TC*
	+20	15.67	7.69	9.24	8.04	14.86	2.77
	+10	8.02	3.93	4.73	4.11	7.6	1.42
A	−10	−8.45	−4.14	−4.97	−4.31	−7.97	−1.49
	−20	−17.39	−8.53	−10.23	−8.86	−16.39	−3.05
	+20	−1.46	−8.91	−9.55	9.5	6.45	18.11
	+10	−0.64	−4.78	−5.12	4.84	3.4	9.09
a	−10	0.33	5.64	6.02	−5.05	−3.78	−9.16
	−20	0.16	12.43	13.23	−10.35	−8	−18.4
	+20	0.04	0.39	0.42	−0.37	−0.27	−0.68
	+10	0.02	0.19	0.21	−0.19	−0.13	−0.34
b	−10	−0.02	−0.19	−0.21	0.19	0.13	0.34
	−20	−0.04	−0.39	−0.41	0.37	0.27	0.68
	+20	0.04	0.39	0.42	−0.37	−0.27	−0.68
	+10	0.02	0.19	0.21	−0.19	−0.13	−0.34
p	−10	−0.02	−0.19	−0.21	0.19	0.13	0.34
	−20	−0.04	−0.39	−0.41	0.37	0.27	0.68
	+20	−2.99	−2.16	−0.63	−1.53	4.92	0.92
	+10	−1.49	−1.08	−0.31	−0.76	2.48	0.46
α	−10	1.50	1.10	0.31	0.76	−2.51	−0.47
	−20	3.00	2.20	0.63	1.53	−5.07	−0.94
	+20	−7.05	−3.46	−2.37	−3.38	1.56	0.29
	+10	−2.95	−1.45	−0.99	−1.41	0.65	0.12
β	−10	3.14	1.54	1.06	1.5	−0.68	−0.13
	−20	6.5	3.19	2.19	3.09	−1.39	−0.26
	+20	−1.59	−0.78	−0.12	−0.81	2.24	0.42
	+10	−0.79	−0.39	−0.06	−0.4	1.12	0.21
c_{ho}	−10	0.78	0.38	0.06	0.4	−1.13	−0.21
	−20	1.56	0.77	0.11	0.8	−2.26	−0.42
	+20	−9.6	−4.71	−3.21	−4.9	2.18	0.41
	+10	−5.05	−2.48	−1.69	−2.58	1.14	0.21
c_{hr}	−10	5.64	2.77	1.9	2.89	−1.24	−0.23
	−20	11.99	5.88	4.04	6.15	−2.61	−0.49
	+20	−15.26	2.02	1.1	2.21	−2.25	−0.42
	+10	−7.68	0.98	0.51	1.07	−1.17	−0.22
W_1	−10	7.78	−0.91	−0.44	−1.01	1.27	0.24
	−20	15.67	−1.76	−0.81	−1.95	2.63	0.49
	+20	−8.11	−3.98	0.05	−4.14	14.61	14.19
	+10	−4.03	−1.98	0.06	−2.06	7.40	7.11
c_p	−10	3.97	1.95	−0.12	2.03	−7.57	−7.14
	−20	7.87	3.86	−0.30	4.03	−15.33	−14.32
	+20	1.97	0.97	−2.52	1.01	−15.11	0.35
	+10	1.06	0.52	−1.37	0.54	−8.18	0.19
c_s	−10	−1.25	−0.61	1.64	−0.64	9.8	−0.22
	−20	−2.74	−1.34	3.66	−1.4	21.76	−0.48
	+20	−4.32	−2.12	−1.04	−2.21	2.84	0.53
	+10	−2.18	−1.07	−0.53	−1.12	1.43	0.27
c_d	−10	2.23	1.1	0.54	1.14	−1.46	−0.27
	−20	4.52	2.22	1.1	2.31	−2.93	−0.55

Table 2. Continued.

Parameter	% Changes	% Changes in					
		t_1^*	t_2^*	T^*	S^*	R^*	TC*
c_l	+20	5.36	2.63	−2.43	2.74	−20.72	0.95
	+10	2.82	1.38	−1.13	1.44	−10.23	0.5
	−10	−3.09	−1.52	0.97	−1.58	9.98	−0.54
	−20	−6.45	−3.16	1.8	−3.29	19.71	−1.13
η	+20	−7.58	−3.72	−0.29	−3.87	34.51	−1.33
	+10	−3.72	−1.82	0.02	−1.9	17.33	−0.65
	−10	3.53	1.73	−0.45	1.81	−17.49	0.62
	−20	6.83	3.35	−1.5	3.5	−35.21	1.21
n	+20	0.0055	0.0027	−0.0001	0.0028	−0.0103	−0.0098
	+10	0.0039	0.0019	−0.0001	0.002	−0.0072	−0.0069
	−10	−0.0023	−0.0011	0.0001	−0.0012	0.0044	0.0042
	−20	−0.0082	−0.004	0.0002	−0.0042	0.0154	0.0147
M	+20	−0.11	−0.05	0.0023	−0.05	0.2	0.19
	+10	−0.05	−0.03	0.0012	−0.03	0.1	0.09
	−10	0.05	0.03	−0.0011	0.03	−0.1	−0.09
	−20	0.11	0.05	−0.0022	0.05	−0.2	−0.19
k	+20	−0.11	−0.05	0.0023	−0.05	0.2	0.19
	+10	−0.05	−0.03	0.0012	−0.03	0.1	0.09
	−10	0.05	0.03	−0.0011	0.03	−0.1	−0.09
	−20	0.11	0.05	−0.0022	0.05	−0.2	−0.19

To find out the optimal values of t_1, t_2, T, S and R along with the total cyclic cost per unit time TC of the system, we have used LINGO 10.0 for this example. The optimal values are: $t_1^* = 0.5107498$ yr, $t_2^* = 0.9925676$ yr, $T^* = 1.267193$ yr, $S^* = 200.3556$ units, $R^* = 42.29238$ units and TC = \$2722.542 (see Fig. 2).

The Figure 2 reveals that the total cost function is a convex function and it attains the global minimum value. The optimal solution, moreover, can be easily observed from the line diagrams of total cost per unit time *versus* t_1 and total cost per unit time *versus* cycle length T in Figures 3 and 4, respectively.

7. Sensitivity analysis

In this section, we have described sensitivity analysis. In the above described numerical example which mentioned earlier, sensitivity analysis has been investigated to study the effect of changes (under or over estimation) of different inventory parameters and the effect of the optimal solutions of different variables and total cost. This analysis has been performed by changing (increasing and decreasing) the parameters from –20% to +20%, considering one parameter at a time and making the other parameters at their original values. The numerical results of this analysis are presented in Table 2.

From Table 2, we can make the following observations:

– The cycle length of the system (T) is less sensitive with respect to parameters (n), (M) and (K). These mentioned parameters hardly have any effect in the optimal cycle length (T). Highly sensitive with respect to ordering cost (A) and demand parameter (a). So, these two have lots of impacts on cycle length of the inventory system. However, it is moderately sensitive with respect to the rest of the parameters.
– Maximum stock level (S) is highly sensitive with respect to ordering cost (A) and demand parameter (a). So, there is a huge impact with respect to the said parameters on maximum stock level. Also, it is less sensitive with respect to parameters (n), (M) and (K). There is little effect on initial stock with respect to the mentioned parameters. It is moderately sensitive with respect to the rest parameters.

- Highest shortages level (R) is highly sensitive with respect to parameters (A), (a), (c_p), (c_s), (c_l), (η).So, these mentioned parameters have much impact on the highest shortages level and the retailer (decision maker) can abate the highest shortages level(R)by abating the values of the parameters (A), (a), (c_p), (c_l), (η) but augmenting (c_s). It is less sensitive with respect to the parameter (n). Hence, advance installment has less impact on the highest shortages level(R). On the other hand, R**is** moderately sensitive with respect to the rest of the parameters.
- Time period of RW (t_1) is highly sensitive with respect to the parameters (A), (c_{hr}), (W_1), (c_p) and less sensitive with respect to the parameter (n). So, it has high impact with respect to the mentioned parameters and less impact on installment parameter. Moreover, the rest of the parameters are moderately sensitive for the time period of RW (t_1).
- Time period of OW (t_2) is highly sensitive with respect to the parameters (A), (a) and less sensitive with respect to the parameter (n). So, only two parameters, namely, ordering cost (A) and demand parameter (a) have great effect on the time frame (t_2) and the retailer can give his concentration either on abating ordering cost (A) or enhancing demand parameter (a) in order to curtail the time at which the stock reaches to zero at OW . Additionally, t_2 is moderately sensitive with respect to the rest of the parameters.
- The total cost of the system (TC) is highly sensitive with respect to the demand parameter (a), ordering cost (A) and purchase cost (c_p). It indicates that if the value of both parameters increase then total cost increases. Less sensitive with respect to the installment parameter (n) *i.e.*, it has less effect in the system whereas moderately sensitive the rest of the parameters. This reveals that the retailer should give much concentration on abating the ordering cost (A), purchase cost (c_p) and demand parameter (a) in order to abate the total cost (TC) instead of abating all other costs, abating or increasing selling price(p).

8. Conclusion

Two-warehouse system is a popular and interesting field in inventory analysis. Lots of research works have been done by several researchers. To best of our knowledge still now, anyone cannot do any research in a two-warehouse system by considering advance payment facility. After seeing this gap, we are highly motivated to introduce advance payment scheme in a two-warehouse system. In this work, we have introduced a two-warehouse inventory model by considering prepayment facility with equal installment. We also considered the prepayment must be given before received the product. Shortages are allowed with a constant rate of partial backlogging. For the first time, we have proposed advance payment in a two-warehouse system. The advance payment is made by equal installment up to n times before receiving the products where demand of the product is dependent on selling price. Here, we have optimized the total cost of the system. We have proved the optimality mathematically by using theorem and lemma. To validate the proposed model, we have solved a numerical example and presented managerial insights by performing sensitivity analysis. Based on the considered example, we exhorted to the retailer to give meticulous concentration on abating the ordering cost, purchase cost and demand parameter (a) in order to abate the total cost (TC). Also, we have shown the optimality graphically as 3D plot by using MATLAB software.

This proposed model is more practicable for highly demandable seasonal products as advance payment provides an assurance not only for the retailer to get on-time delivery of the ordered products but also for the supplier to mitigate the possibilities to cancel the orders. Here, shortages are allowed with a constant rate of partial backlogging, so variable backordering may create much opportunity for the retailer to lessen the total cost. The proposed inventory model can extend by considering several realistic features such as non-instantaneous deteriorating items, variable backlogged shortages and without ending inventory policies. One may extend this model by taking nonlinear demand with nonlinear holding cost. Also, anyone can introduce another realistic feature such as trade credit (single level, two level or partial), non-linear price dependent demand by taking price as a decision variable and extend this paper. Anyone may extend this model by considering the inventory costs are interval valued or fuzzy valued.

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