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EFFICIENCY AND SUPER-EFFICIENCY UNDER INTER-TEMPORAL DEPENDENCE

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Abstract. In this paper, a linear programming (LP) model for measuring the relative efficiency of a decision-making unit (DMU) under inter-temporal dependence of data is proposed. Necessary and sufficient conditions are derived for identification of dynamically efficient paths. Furthermore, an LP model is proposed to estimate the super-efficiency of the dynamically efficient paths using an extended version of the modified MAJ model (Saati *et al.*, *Ric. Oper.* **31** (2001) 47–59). To evaluate the applicability of the proposed method in a banking sector example, this method is employed for ranking some branches of the Iranian commercial bank.

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1. Introduction

Data Envelopment Analysis (DEA) is a nonparametric method based on mathematical programming for evaluating the efficiency of decision making units (DMUs). This method has been introduced by Charnes *et al.* [5] (CCR model) and developed by many researchers working on the operational research and the management science area [6, 7, 11]. In fact, this technique could be considered as one of the main fields of the current and future studies in the operational research and the management sciences. Emrouznejad and Yang [14] studies, confirm this issue. The DEA models lead to efficiency score one for efficient DMUs and efficiency score less than one for inefficient DMUs. In a recent and different work, Saati *et al.* [34] proposed a model to evaluate of the efficiency of a set of DMUs. This model projects the unit under assessment on the efficiency frontier by the simultaneous decreasing inputs and increasing outputs with equal sizes. In this paper, we propose a new model to estimate of the efficiency of a set of DMUs based on the expansion of the model addressed in the reference [34].

Ranking of DMUs could be considered as an interesting and notable topic in the DEA research area from both theoretical and practical points of view. It is obvious that inefficient DMUs could be ranked *via* the efficiency score while the efficient DMUs could not be ranked through that. Various published works have been devoted to ranking of efficient DMUs [1, 3, 24]. Guo *et al.* [22] proposed a unified model based on the additive DEA to compute the efficiency scores of the inefficient units and the super-efficiency scores of the efficient units. The proposed model guarantees that the projections identified are strongly efficient. Tran *et al.* [41] proposed a novel one-stage method based on the slacks-based measure (SBM) to obtain the efficiency scores of the inefficient units and the super-efficiency scores of the efficient units. The proposed method is important from computational

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time viewpoint. Accordingly, this method could be applied in the applications with the large number of units. According to the efficiency score of each unit and its effects on other units, Ekiza and Sakara [9] proposed a method for ranking DMUs. Hatami-Marbini [23] proposed an imprecise network benchmarking for the goal of reflecting the human judgments. As a result of this paper, a classification method has provided for classifying organizations. Toloo [38] proposed a mixed integer DEA model to find the most BCC-efficient DMU. However, the proposed model may be infeasible for some data, and when the model is feasible, it may fail to recognize the most efficient unit. To overcome these drawbacks, a model presented to find and rank the BCC-efficient units by Ebrahimi and Rahmani [8]. It has been proven that the proposed model is always possible.

According to Hosseinzadeh Lotfi *et al.* [24] studies, the developed methods for ranking of efficient DMUs could be divided into the following seven groups: (i) cross-efficiency; (ii) finding optimal weights; (iii) super-efficiency; (iv) benchmarking; (v) multivariate statistics in DEA; (vi) multi-criteria decision-making (MCDM) methodologies and DEA; and (vii) other techniques including Monte Carlo method [26]. According to the DEA literature, ranking DMUs has been studied from both theoretical and practical aspects by many scholars, including [25, 27, 42].

As we know, the most popular and important approach for ranking DMUs is the super-efficiency method. The main idea in this approach is based on of leaving out one DMU and assessing this DMU by the remaining DMUs. The idea of super-efficiency for the first time was introduced by Andersen and Petersen (AP) [2]. They proposed a useful model for ranking efficient DMUs. However, the proposed model may be infeasible for some data [4] and unstable when some inputs are close to zero [37]. To overcome the instability of the AP method, Mehrabian *et al.* [31] presented a model based on the idea of excluding the unit under assessment and analyzing the changes of the efficiency frontier, called MAJ-model. Nevertheless, this model may be infeasible in some cases [31]. To overcome infeasibility of AP and MAJ models, a non-radial and always feasible model is proposed by Saati *et al.* [34], called modified MAJ model. In this paper, a super-efficiency model is proposed for ranking efficient DMUs based on the expansion of the modified MAJ model.

Dynamic DEA could be considered as an important nonparametric technique for evaluating the performance of DMUs in the presence of time factor and inter-temporal dependence of input and output levels. This technique was initially introduced by Sengupta [35]. In this study, a dynamic extension of the inter-temporal cost frontier was proposed in the presence of stochastic input prices. Afterwards, some of multi-period linear programming problems were introduced by Färe and Grosskopf [16] in order to present some of dynamic aspects of production to their DEA models. Using the production possibility set (PPS), some models provided to estimate of the overall efficiency measure a DMU in the dynamic DEA framework by Nemoto and Goto [32] and Sueyoshi and Sekitani [36]. Tone and Tsutsui [39] provided a slack-based dynamic DEA model to evaluate of the performance of DMUs with various carry-overs (such as desirable and undesirable) during different time periods. Also, Tone and Tsutsui [40] proposed a dynamic DEA model with network structure to evaluate the overall efficiency over the entire period. The proposed model can assess dynamic change of period and divisional efficiencies. Ghobadi *et al.* [21] provided a linear programming (LP) model to estimate the dynamic cost efficiency measure. The proposed cost efficiency is a convex combination of cost efficiencies of DMU in the available periods of the assessment window.

According to the dynamic DEA literature, some methods to ranking DMUs in the traditional DEA have been developed to the dynamic DEA framework. Pourmahmoud [33] provided a method for ranking DMUs in the dynamic DEA framework. In this work, according to optimistic and pessimistic viewpoints, the minimum and maximum efficiency values of each DMU are estimated. Then, the rank of each DMU is obtained through the combination of its maximum and minimum efficiency values. Ling *et al.* [30] proposed a method to estimate of the overall dynamic efficiency and super-efficiency of units over the entire observed period with interval data. Gharakhani *et al.* [17] proposed a method based on a common set of weights generated by goal programming (GP) for ranking DMUs in the dynamic network-DEA framework. This method makes it possible to monitor dynamic change of the period efficiency. To the best of author's knowledge, ranking efficient DMUs when the inter-temporal dependence occurs by influencing the stock capital on output levels in the various production periods has not been studied in the literature. The changes in stock studied as a particular cause of inter-temporal dependence by Emrouznejad [10] and Emrouznejad and Thanassoulis [12]. According to this studies, the traditional DEA models could not be used to evaluate the DMUs efficiency in the presence of the inter-temporal dependency data. Because after installing a physical asset, it influences over unit outputs during future years. This implies that investments made in previous years must be considered as an input factor for measuring the efficiency and productivity for each year. Emrouznejad and Thanassoulis [12] proposed an envelopment dynamic DEA model to estimate the technical dynamic efficiency and recognize efficient assessment paths. The proposed model by Emrouznejad and Thanassoulis [12] was modified by Jahanshahloo *et al.* [29]. After introducing dynamic DEA by Sengupta [35], this topic has been continuously enriched by new applications and a variety of dynamic DEA problems in combinatorial DEA (such as dynamic network DEA) have been studied by researchers in the operations research community. According to Fallah-Fini *et al.* [15] studies, these new applications could be divided into the following five groups based on the type of the inter-temporal dependence that may occur: (i) capital or generally quasi-fixed factors associated with embodied technological change, and vintage specific capital; (ii) production delays; (iii) inventories (inventories of exogenous inputs or inventories of intermediate and final products); (iv) incremental improvement and learning models (disembodied technological changes); and (v) adjustment costs. However, there are few applications of dynamic DEA model addressed in Emrouznejad and Thanassoulis [12] that are reported in the literature such as applications in inverse DEA suggested in the references [20,28]. Emrouznejad and Thanassoulis [13] proposed a dynamic model to measuring Malmquist productivity index under the inter-temporal dependence introduced in [12]. This model has been employed to evaluate of the productivity of the Organization for Economic Cooperation and Development (OECD) countries. The results showed that the dynamic Malmquist reflects the better reality of units than static DEA models. More recently, Ghobadi [18] presented a generalized DEA model for inputs/outputs estimation according to the proposed inter-temporal dependence in [12]. In addition, a method proposed for identification of the inherited input/output levels of the merged DMU from merging units [43]. Using to multiple-objective programming tools, a method provided to simultaneous estimation of input and output levels when the capital stock influences output levels over various production periods [19].

In this paper, the relative efficiency of a DMU under the inter-temporal dependence of input and output levels is measured using an appropriate LP model. The dynamically efficient paths could be indicated *via* this model. In addition, the modified MAJ model proposed by Saati *et al.* [34] is extended to measure the super-efficiency of the dynamically efficient paths. To the best of authors knowledge, ranking efficient DMUs under the inter-temporal dependent assumption proposed by Emrouznejad and Thanassoulis [12] has not been studied in the literature. The applicability of the proposed models are illustrated through a real example.

The paper is organized as follows. In Section 2, the DEA and inter-temporal dependence concepts are introduced. The main results of this paper are presented in Section 3. In this section, two LP models are presented to estimate the efficiency and super-efficiency of the assessment path of the DMUs. The applicability of the proposed methods are illustrated through a bank ranking example in Section 4. Section 5 gives a brief conclusion and some future research directions.

2. Preliminaries

2.1. Ranking DMUs by the static DEA

Let us to consider a set of n DMUs, $\{\mathrm{DMU}_j : j = 1, \ldots, n\}$, in which DMU_j consumes multiple positive inputs $x_{ij} (i = 1, \ldots, m)$ to produce multiple positive outputs $y_{rj} (r = 1, \ldots, s)$. Suppose that input and output vectors for DMU_j are denoted by $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^t$ and $Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^t$, respectively. We consider the following mix-oriented DEA model, in which DMU_o, $o \in \{1, 2, ..., n\}$, is the unit under assessment:

$$\begin{aligned} \omega_o = \min\ & 1 + \omega \\ \text{s.t.}\ & \sum_{j=1}^{n} \lambda_j \frac{x_{ij}}{R_i} \leq \frac{x_{io}}{R_i} + \omega, \quad i = 1, 2, \ldots, m, \\ & \sum_{j=1}^{n} \lambda_j \frac{y_{rj}}{R_r} \geq \frac{y_{ro}}{R_r} - \omega, \quad r = 1, 2, \ldots, s, \\ & \lambda_j \geq 0, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{2.1}$$

where

$$R_i = \max\{x_{ij} \mid j = 1, 2, \ldots, n\}, \quad i = 1, 2, \ldots, m,$$
$$R_r = \max\{y_{rj} \mid j = 1, 2, \ldots, n\}, \quad r = 1, 2, \ldots, s.$$

This model is proposed by Saati *et al.* [34]. It measures the efficiency under the constant returns to scale (CRS) assumption of the production technology. Since the inputs and outputs are not homogeneous and the objective function scaling depends on the measurement units of input and output data, the inputs and outputs in the above model are normalized. Here, ω_o is called the mix-oriented efficiency score of DMU_o. It is not difficult to see that $0 < \omega_o \leq 1$. DMU_o is called weakly efficient if $\omega_o = 1$.

Ranking of efficient DMUs is one of the most important subjects in the management and decision making. This problem has been studied in many theoretical and practical works. Different ranking methods to a set of DMUs are proposed [1, 24]. Super efficiency models could be considered as an important method in this area. These methods are based on the idea of excluding the under evaluation unit and analyzing the frontier changes.

The modified MAJ super-efficiency model [34] for ranking efficient DMUs is given as follows:

$$\begin{aligned} \bar{\omega}_o = \min\ & 1 + \omega \\ \text{s.t.} & \sum_{j=1, j\neq o}^{n} \lambda_j \frac{x_{ij}}{R_i} \leq \frac{x_{io}}{R_i} + \omega, \quad i = 1, 2, \ldots, m, \\ & \sum_{j=1, j\neq o}^{n} \lambda_j \frac{y_{rj}}{R_r} \geq \frac{y_{ro}}{R_r} - \omega, \quad r = 1, 2, \ldots, s, \\ & \lambda_j \geq 0, \quad j = 1, 2, \ldots, n,\ j \neq o. \end{aligned} \tag{2.2}$$

In the above model, $(\lambda, \omega) \in \mathbb{R}^{(n-1)} \times \mathbb{R}$ is the variable vector. This model does not have difficulties of AP [2] and MAJ [31] models. Saati *et al.* [34] established the following theorem.

Theorem 2.1. *Model 2.2 always is feasible.*

2.2. Inter-temporal dependence

In this section, an special case of the inter-temporal dependence that may occur in the production processes between input and output levels is presented. This case happens when the capital stock influences output levels over various production periods. This case has been studied by Emrouznejad and Thanassoulis [12], Jahanshahloo *et al.* [28, 29], and Ghobadi [18].

Suppose that there exist a set of n observations of DMUs, whose performance is determined in a window of periods $w = \{t \mid t = \tau, \tau + 1, \ldots, \tau + T\}$ as the assessment window. Each DMU has two kinds of inputs in each time period including period-specific inputs (x) and capital inputs (z) to produce a kind of output (y). The initial stock inputs provide the capital inputs in each time period of the assessment window. Since τ is considered as the initial time period in the assessment window, the initial stock input is represented by $Z^{\tau-1}$. A portion of the initial capital that is not employed in the assessment window by the DMU, is called as the final capital. Having terminal capital stocks is necessary to keep the DMU after the terminal time period in the assessing window. Since $\tau + T$ is the terminal time period in the assessment window, the terminal capital stock is denoted by $Z^{\tau+T}$.

To clarify the above discussion, suppose that the performance evaluation of a set of universities (as DMUs) is under study. The number of publications, the number of PhD awards, and the number of inventions could be considered as the main products of a university. These products are generated by different inputs including the tuition fee, the educational spaces, and the resources annually taken from the government. Apart from usual inputs (resources), some capital grants may be paid to university to compensate for the unpredictable costs imposed on the managers. These extra inputs are called capital inputs and denoted by z^t.

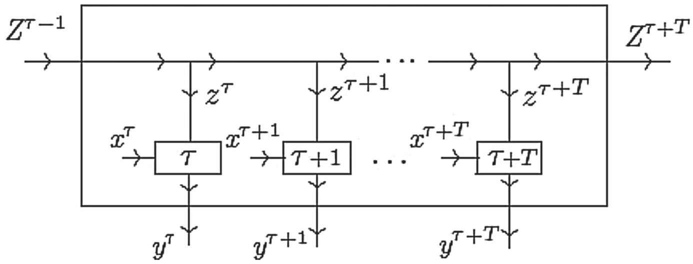

FIGURE 1. Production flow.

Suppose that there is a set of n DMUs, $\{\text{DMU}_j : j \in J = \{j = 1, \ldots, n\}\}$ for efficiency evaluation in the assessment window, $w = \{t \mid t = \tau, \tau+1, \ldots, \tau+T\}$. In each time period, $t \in w$, each DMU produces one kind of output $y_j^t = (y_{rj}^t : \forall r \in O = \{1, 2, \ldots, s\})$ utilizing two kinds of multiple positive inputs $(x_j^t, z_j^t) = (x_{ij}^t : \forall i \in I_1,\ z_{ij}^t : \forall i \in I_2)$, where $I_1, I_2 \subset I$, $I_1 \cup I_2 = I = \{1, 2, \ldots, m\}$, and $I_1 \cap I_2 = \emptyset$. In addition, suppose that $Z_j^{\tau-1} = (Z_{ij}^{\tau-1} : \forall i \in I_2)$ and $Z_j^{\tau+T} = (Z_{ij}^{\tau+T} : \forall i \in I_2)$ are the initial stock input and the terminal capital stock of DMU_j, respectively. It is clear that

$$Z_{ij}^{\tau+T} = Z_{ij}^{\tau-1} - \sum\nolimits_{t \in w} z_{ij}^t \qquad \forall i \in I_2. \tag{2.3}$$

Figure 1 indicates a production flow in the assessment window:

As we know, the inter-temporal dependencies of input-output levels can be occur for various reasons in the production process [15]. Here, the inter-temporal dependence has occurred due to influencing the stock capital on output levels in various production periods. In fact, apart from usual resources, the considered initial stock is applied to produce outputs for each period according to the required amount by DMU. A portion of the initial stock that is not used in during the assessment window by the DMU is considered to be the terminal capital stock which can be used to produce outputs in future.

Suppose that $(x_j^w, z_j^w, y_j^w, Z_j^{\tau-1}, Z_j^{\tau+T})$ denotes the assessment path of DMU_j; $j = \{1, 2, \ldots, n\}$ in the assessment window, w. The production possibility set (PPS) concept has been extended from traditional DEA to the dynamic DEA by Emrouznejad and Thanassoulis [12]. They have constructed some PPS within the assessment window, considering different postulates from the observed assessment paths $(x_j^w, z_j^w, y_j^w, Z_j^{\tau-1}, Z_j^{\tau+T})$, $j \in J$, as follows:

$$
\begin{aligned}
P_{\text{DDEA}} = \Big\{ \big(x^w, z^w, y^w, Z^{\tau+T}, Z^{\tau+T}\big) \mid & \\
\sum_{j \in J} \lambda_j x_{ij}^t \le x_i^t, &\qquad i \in I_1,\ \forall t \in w, \\
\sum_{j \in J} \lambda_j z_{ij}^t \le z_i^t, &\qquad i \in I_2,\ \forall t \in w, \\
\sum_{j \in J} \lambda_j y_{rj}^t \ge y_r^t, &\qquad r \in O,\ \forall t \in w, \\
\sum_{j \in J} \lambda_j Z_{ij}^{\tau+T} \ge Z_i^{\tau+T}, &\qquad i \in I_2, \\
\sum_{j \in J} \lambda_j Z_{ij}^{\tau-1} \le Z_i^{\tau-1}, &\qquad i \in I_2, \\
\lambda_j \ge 0, &\qquad j \in J \Big\}.
\end{aligned} \tag{2.4}
$$

In addition, the efficient paths have been introduced by Emrouznejad and Thanassoulis [12] as follows:

Definition 2.2. (Emrouznejad and Thanassoulis [12]) The assessment path of DMU_o is called Pareto efficient or dynamically efficient if all the assessment paths or any combinations of them do not give more than the DMU_o path on at least one period, without generating less in some other outputs in some periods or requiring more of at least one input in at least one period.

3. Efficiency and Super-Efficiency under inter-temporal dependence

In this section, the efficiency and super-efficiency measures provided by Saati *et al.* [34] are extended to the dynamic framework. It is proved that the proposed model could estimate the efficiency measure. Accordingly, the dynamically efficient paths could be distinguished.

According to the dynamic PPS (2.4) and along the lines of Saati *et al.* [34], the following LP model is considered to estimate the efficiency measure of DMU_o, $o \in J$ in the assessment window, w.

$$\begin{aligned}
\omega_o^* = \ & \min \frac{\sum_{t\in w} 1+\mu^t}{T+1} \\
& \text{s.t.} \sum_{j\in J} \lambda_j \left(\frac{x_{ij}^t}{R_i^t}\right) + s_i^{t-} = \frac{x_{io}^t}{R_i^t} + \mu^t, && i \in I_1,\ t\in w \\
& \sum_{j\in J} \lambda_j \left(\frac{z_{ij}^t}{R_i^t}\right) + \delta_i^{t-} = \frac{z_{io}^t}{R_i^t} + \mu^t, && i \in I_2,\ t\in w \\
& \sum_{j\in J} \lambda_j \left(\frac{y_{rj}^t}{R_r^t}\right) - s_r^{t+} = \frac{y_{ro}^t}{R_r^t} - \mu^t, && r \in O,\ t\in w \\
& \sum_{j\in J} \lambda_j \left(\frac{Z_{ij}^{\tau+T}}{R_i^{\tau+T}}\right) - \gamma_i^{+} = \frac{Z_{io}^{\tau+T}}{R_i^{\tau+T}}, && i \in I_2 \\
& \sum_{j\in J} \lambda_j \left(\frac{Z_{ij}^{\tau-1}}{R_i^{\tau-1}}\right) + \gamma_i^{-} = \frac{Z_{io}^{\tau-1}}{R_i^{\tau-1}}, && i \in I_2 \\
& \lambda_j \geq 0, && j\in J, \mu^t \text{ is free } \forall t\in w, \\
& s_i^{t-}, \delta_i^{t-}, s_r^{t+}, \gamma_i^{+}, \gamma_i^{-} \geq 0, && \text{for all indices},
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
R_i^t &= \max\{x_{ij}^t \mid j\in J\}, && i\in I_1,\ t\in w, \\
R_i^t &= \max\{x_{ij}^t \mid j\in J\}, && i\in I_2,\ t\in w, \\
R_r^t &= \max\{y_{rj}^t \mid j\in J\}, && r\in O,\ t\in w, \\
R_i^{\tau+T} &= \max\{Z_{ij}^{\tau+T} \mid j\in J\}, && i\in I_1, \\
R_i^{\tau-1} &= \max\{Z_{ij}^{\tau-1} \mid j\in J\}, && i\in I_2.
\end{aligned} \tag{3.2}$$

In the above model, μ^t, $t \in w$ and λ_j, $j \in J$ are considered as essential variables while $s_i^{t-}, \delta_i^{t-}, s_r^{t+}, \gamma_i^{+}, \gamma_i^{-}$ are considered as slack variables. This model projects the unit under assessment on the efficiency frontier. This could be achieved by decreasing inputs and increasing the outputs with equal sizes in each period of the assessment window. The initial and terminal stocks of the unit assessment during the periods have been employed to decided which inputs (both period-specific and stock) and outputs should be reduced and enlarged further, respectively. This means that model (3.1) is non-radial. By using the fourth and fifth groups of constraints in the model (3.1), the initial and terminal stock of the capital will be distinguished as an exogenously fixed input and output, respectively.

In the following, some of the characteristics of the proposed model are demonstrated from theoretical point of view. Theorem 3.1 shows that model (3.1) is always feasible and its optimal value is in the interval $(0, 1)$.

Theorem 3.1. *Model (3.1) is always feasible and* $0 < \omega_o^* \leq 1$.

Proof. Obviously, ($\lambda_j = 0, j \in J - \{o\}, \lambda_o = 1, \mu^t = 0; \forall t \in w; s_i^{t-} = \delta_i^{t-} = s_r^{t+} = \gamma_i^+ = \gamma_i^- = 0$, for all indices) is a feasible solution to model (3.1). Therefore, $\omega_o^* \leq 1$ (* indicates the optimality). By contradiction assume that there exists at least one $p \in w$ such that $\mu^{p*} \leq -1$. Then, $\frac{x_{io}^p}{R_i^p} + \mu^{p*} \leq 0$; $\forall i \in I_1$, and $\frac{z_{io}^p}{R_i^p} + \mu^{p*} \leq 0$; $\forall i \in I_2$. This is a contradiction, since it was shown that the proposed model is feasible. Therefore, $\mu^{t*} > -1$, for all $t \in w$ and $\omega_o^* \in (0, 1)$. □

Theorem 3.2. *Model (3.1) is units invariant.*

Proof. According to the definition of $R_i^t, R_r^t, R_i^{\tau+T}$, and $R_i^{\tau-1}$ in model (3.1), the efficiency score will remain valid if one replaces the $(x_{ij}^t, z_i^t, y_{rj}^t, Z_{ij}^{\tau-1}, Z_{ij}^{\tau+T})$ in (3.1) with $(\alpha_i^t x_{ij}^t, \eta_i^t z_i^t, \beta_r^t y_{rj}^t, \nu_i^{\tau-1} Z_{ij}^{\tau-1}, \nu_i^{\tau+T} Z_{ij}^{\tau+T})$ such that $\alpha_i^t, \eta_i^t, \beta_r^t, \nu_i^{\tau-1}$, and $\nu_i^{\tau+T}$ are positive values. This completes the proof. □

The following theorem is established to characterizing dynamically efficient paths using model (3.1):

Theorem 3.3. *The assessment path* $(x_o^w, z_o^w, y_o^w, Z_o^{\tau-1}, Z_o^{\tau+T})$ *of DMU*$_o$ *is efficient, if and only if*

$$\omega_o^* = 1,\ s_i^{t-} = \delta_i^{t-} = \gamma_i^+ = \gamma_i^- = 0;\ \forall i, t,\ s_r^{t+} = 0;\ \forall r, t,$$

for all alternative optimal solutions of problem (3.1).

Proof. By contradiction assume that the assessment path of DMU$_o$ is efficient and $\omega_o^* = \frac{\sum_{t\in w} 1+\mu^{t*}}{T+1} < 1$ (* indicates the optimality). Since $-1 < \mu^{t*} \leq 0$; for all t, then there exists at least one $h \in w$, such that $\mu^{h*} < 0$. Therefore, regarding constraints model (3.1), we have

$$\sum_{j\in J} \lambda_j^* \left(\frac{x_{ij}^h}{R_i^h}\right) \leq \frac{x_{io}^h}{R_i^h} + \mu^{h*} < \frac{x_{io}^h}{R_i^h}, \quad i \in I_1 \tag{3.3}$$

$$\sum_{j\in J} \lambda_j^* \left(\frac{z_{ij}^h}{R_i^h}\right) \leq \frac{z_{io}^h}{R_i^h} + \mu^{h*} < \frac{z_{io}^h}{R_i^h}, \quad i \in I_2 \tag{3.4}$$

$$\sum_{j\in J} \lambda_j^* \left(\frac{y_{rj}^h}{R_r^h}\right) \geq \frac{y_{ro}^h}{R_r^h} - \mu^{h*} > \frac{y_{ro}^h}{R_r^h}, \quad r \in O. \tag{3.5}$$

According to the constraints of model (3.1) we get

$$\left(\sum_{j\in J} \lambda_j^* x_j^w, \sum_{j\in J} \lambda_j^* z_j^w, \sum_{j\in J} \lambda_j^* Z_j^{\tau-1}\right) \leq \left(x_o^w, z_o^w, Z_o^{\tau-1}\right) \tag{3.6}$$

$$\left(\sum_{j\in J} \lambda_j^* y_j^w, \sum_{j\in J} \lambda_j^* Z_j^{\tau+T}\right) \geq \left(y_o^w, Z_o^{\tau+T}\right) \tag{3.7}$$

such that at least one of the above inequalities strictly holds, because of inequality (3.3)–(3.5). This contradicts the efficiency of the assessment path of DMU$_o$. Therefore, $\omega_o^* = 1$ and $\mu^{t*} = 0$ for all $t \in w$. Now, if there exists an optimal solution for model (3.1) such that some of the slack variables are non-zero. Therefore, considering the constraints of model (3.1) and using $\mu^{t*} = 1$ for all t, relations (3.6) and (3.7) are obtained and at least one of the inequalities holds strictly. This contradicts the efficiency of the assessment path of DMU$_o$. Therefore, $\omega_o^* = 1$ and $s_i^{t-} = \delta_i^{t-} = \gamma_i^+ = \gamma_i^- = 0$; $\forall i, t$, $s_r^{t+} = 0$; $\forall r, t$, for any optimal solution of model (3.1).

Now, suppose that the assessment path of DMU$_o$ is inefficient, then according to Definition 2.2 there exists a non-negative linear combination of assessment paths that dominates the assessment path of DMU$_o$. Then, there exists a $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \gneqq 0$ where

$$\left(\sum_{j\in J}\lambda_j x_j^w, \sum_{j\in J}\lambda_j z_j^w, \sum_{j\in J}\lambda_j Z_j^{\tau-1}\right) \leq \left(x_o^w, z_o^w, Z_o^{\tau-1}\right) \tag{3.8}$$

$$\left(\sum_{j\in J}\lambda_j y_j^w, \sum_{j\in J}\lambda_j Z_j^{\tau+1}\right) \geq \left(y_o^w, Z_o^{\tau+1}\right), \tag{3.9}$$

and at least one of these inequalities holds strictly. Accordingly, $\omega_o^* < 1$ or $s_i^{t-}, s_r^{t+}, \delta_i^{t-}, \gamma_i^+$, and γ_i^- are non-zero for some i, r, t. This contradicts the assumption. □

Suppose DMU$_o$ is efficient. To obtain the super-efficiency of DMU$_o$ under model (3.1), DMU$_o$ is omitted of dynamic PPS and assessing this unit through the remaining units. Therefore, we consider the following LP model to estimate the super-efficiency of DMU$_o$ in the assessment window, w.

$$\begin{aligned}
\bar{\omega}_o = \min\ & \frac{\sum_{t\in w} 1 + \mu^t}{T+1} \\
\text{s.t.}\ & \sum_{j\in J-\{o\}} \lambda_j \frac{x_{ij}^t}{R_i^t} \leq \frac{x_{io}^t}{R_i^t} + \mu^t, && i \in I_1,\quad t \in w \\
& \sum_{j\in J-\{o\}} \lambda_j \frac{z_{ij}^t}{R_i^t} \leq \frac{z_{io}^t}{R_i^t} + \mu^t, && i \in I_2,\quad t \in w \\
& \sum_{j\in J-\{o\}} \lambda_j \frac{y_{rj}^t}{R_r^t} \geq \frac{y_{ro}^t}{R_r^t} - \mu^t, && r \in O,\quad t \in w \\
& \sum_{j\in J-\{o\}} \lambda_j \frac{Z_{ij}^{\tau+T}}{R_i^{\tau+T}} \geq \frac{Z_{io}^{\tau+T}}{R_i^{\tau+T}}, && i \in I_2 \\
& \sum_{j\in J-\{o\}} \lambda_j \frac{Z_{ij}^{\tau-1}}{R_i^{\tau-1}} \leq \frac{Z_{io}^{\tau-1}}{R_i^{\tau-1}}, && i \in I_2 \\
& \lambda_j \geq 0, && j \in J-\{o\},\ \mu^t \geq 0,\ \forall t \in w.
\end{aligned} \tag{3.10}$$

Note that, after removing DMU$_o$ from the reference set of model (3.1), to obtain the super-efficiency, we need to increase of the inputs and to decrease of the outputs from DMU$_o$ to reach the constructed efficiency frontier by the remaining DMUs. Therefor, we consider $\mu^t \geq 0$, for all $t \in w$.

Model (3.10) projects the DMU$_o$ unit on the efficiency frontier by increasing the inputs and decreasing the outputs with equal sizes in each period of the assessment window. The proposed model provides a super-efficiency in the assessment window that not only ranks dynamic efficient units under inter-temporal dependence data, but also determines the status of super-efficiency for each unit in each period of the assessment window. In fact, this model not only measures the unit distance to the efficiency frontier in the assessment window, but also determines the distance in each period of the assessment window. Knowing the minimum distance of the unit to the efficiency frontier in each period has a high informative value for the decision-maker deliberating about sustainability of this unit compared with other units. This information sheds light on the highest stability that this unit has in each period compared with other units.

Some of the characteristics of the proposed model are demonstrated from theoretical point of view by the following theorems.

Theorem 3.4. *Let $\bar{\Delta}_o = (\alpha^w x_o^w, z_o^w, \beta^w y_o^w, Z_o^{\tau-1}, Z_o^{\tau+T})$ with $0 < \alpha^t \le 1$ and $\beta^t \ge 1$, for all $t \in w$ be a assessment path of a DMU with reduced inputs (period-specific) and enlarged outputs than $\Delta_o = (x_o^w, z_o^w, y_o^w, Z_o^{\tau-1}, Z_o^{\tau+T})$. Then, the super-efficiency score of assessment path $\bar{\Delta}_o(\bar{\omega}_o(\bar{\Delta}_o))$ is not less than that of $\Delta_o(\bar{\omega}_o(\Delta_o))$.*

Proof. Considering the assessment path of $\bar{\Delta}_o$, suppose that $\Lambda = (\lambda_j^*,\ j \in J-\{o\},\ \mu^{t*}; \forall t \in w)$ is an optimal solution to model (3.10) with optimal value of $\bar{\omega}_o$. Feasibility of Λ for LP (3.10), $0 < \alpha^t \le 1$, and $\beta^t \ge 1$, for all $t \in w$, implies

$$\begin{aligned}
&\sum_{j\in J-\{o\}} \lambda_j^* \left(\frac{x_{ij}^t}{R_i^t}\right) \le \frac{\alpha^t x_{io}^t}{R_i^t} + \mu^{t*} \le \frac{x_{io}^t}{R_i^t} + \mu^{t*}, && i \in I_1,\quad t \in w\\
&\sum_{j\in J-\{o\}} \lambda_j^* \left(\frac{z_{ij}^t}{R_i^t}\right) \le \frac{z_{io}^t}{R_i^t} + \mu^{t*}, && i \in I_2,\quad t \in w\\
&\sum_{j\in J-\{o\}} \lambda_j^* \left(\frac{y_{rj}^t}{R_r^t}\right) \ge \frac{\beta^t y_{ro}^t}{R_r^t} - \mu^{t*} \ge \frac{y_{ro}^t}{R_r^t} - \mu^{t*}, && r \in O,\quad t \in w\\
&\sum_{j\in J-\{o\}} \lambda_j^* \left(\frac{Z_{ij}^{\tau+T}}{R_i^{\tau+T}}\right) \ge \frac{Z_{io}^{\tau+T}}{R_i^{\tau+T}}, && i \in I_2\\
&\sum_{j\in J-\{o\}} \lambda_j^* \left(\frac{Z_{ij}^{\tau-1}}{R_i^{\tau-1}}\right) \le \frac{Z_{io}^{\tau-1}}{R_i^{\tau-1}}, && i \in I_2\\
&\lambda_j^* \ge 0, && j \in J-\{o\},
\end{aligned} \tag{3.11}$$

Corresponding to assessment path $\bar{\Delta}_o$, Λ is a feasible solution to model (3.10) using inequalities (3.11). Therefore, $\bar{\omega}_o(\Delta_o) \le \bar{\omega}_o(\bar{\Delta}_o)$. □

We close this section with a discussion about feasibility of model (3.10). In other words, necessary and sufficient conditions for feasibility of model (3.10) are established in Lemma 3.5 and Theorem 3.6.

Lemma 3.5. *System*

$$\begin{cases}
\sum_{j\in J-\{o\}} \lambda_j x_{ij}^t \le x_{io}^t + \mu^t, & i \in I_1,\quad t \in w,\\
\sum_{j\in J-\{o\}} \lambda_j z_{ij}^t \le z_{io}^t + \mu^t, & i \in I_2,\quad t \in w,\\
\sum_{j\in J-\{o\}} \lambda_j y_{rj}^t \ge y_{ro}^t - \mu^t, & r \in O,\quad t \in w,\\
\lambda_j \ge 0, & j \in J-\{O\}.
\end{cases} \tag{3.12}$$

is always feasible.

Proof. For any non-negative set of $\bar{\lambda}_j \ge 0,\ j \in J-\{o\}$, we can define

$$\bar{\mu}^t = \max\left\{x_{io}^t, \sum_{j\in J-\{o\}} \bar{\lambda}_j x_{ij}^t\right\} - x_{io}^t, \quad \forall i \in I_1\ \ \forall t \in w, \tag{3.13}$$

$$\bar{\mu}^t = \max\left\{z_{io}^t, \sum_{j\in J-\{o\}} \bar{\lambda}_j z_{ij}^t\right\} - z_{io}^t, \quad \forall i \in I_2 \ \ \forall t \in w, \tag{3.14}$$

$$\bar{\mu}^t = y_{ro}^t - \min\left\{y_{ro}^t, \sum_{j\in J-\{o\}} \bar{\lambda}_j y_{rj}^t\right\}, \quad \forall i \in I_1 \ \ \forall t \in w. \tag{3.15}$$

We then have

$$\bar{\mu}^t + x_{io}^t = \max\left\{x_{io}^t, \sum_{j\in J-\{o\}} \bar{\lambda}_j x_{ij}^t\right\} \geq \sum_{j\in J-\{o\}} \bar{\lambda}_j x_{ij}^t, \quad \forall i \in I_1, \ \ \forall t \in w, \tag{3.16}$$

$$\bar{\mu}^t + z_{io}^t = \max\left\{z_{io}^t, \sum_{j\in J-\{o\}} \lambda_j z_{ij}^t\right\} \geq \sum_{j\in J-\{o\}} \bar{\lambda}_j z_{ij}^t, \quad \forall i \in I_2, \ \ \forall t \in w, \tag{3.17}$$

$$\sum_{j\in J-\{o\}} \bar{\lambda}_j y_{rj}^t \geq \min\left\{y_{ro}^t, \sum_{j\in J-\{o\}} \bar{\lambda}_j y_{rj}^t\right\} = y_{ro}^t - \bar{\mu}^t, \quad \forall i \in I_1, \ \ \forall t \in w. \tag{3.18}$$

Therefore, $\bar{\lambda}_j \geq 0$ and (3.16)–(3.18) imply that $(\bar{\lambda}_j,\, j \in J - \{o\};\, \bar{\mu}^t,\, \forall t \in w)$ is a feasible solution to system (3.12), and the proof is completed. □

Theorem 3.6. *The model (3.10) is feasible, if and only if*

$$\begin{cases} \sum_{j\in J-\{o\}} \lambda_j Z_{ij}^{\tau+T} \geq Z_{io}^{\tau+T}, & i \in I_2 \\ \sum_{j\in J-\{o\}} \lambda_j Z_{ij}^{\tau-1} \leq Z_{io}^{\tau-1}, & i \in I_2 \\ \lambda_j \geq 0, & j \in J - \{o\}. \end{cases} \tag{3.19}$$

is feasible.

Proof. Considering the Lemma (3.5), it is straightforward. □

Remark 3.1. According to the above theorem, if there is no assessment path (virtual or observed) in the dynamic PPS with the initial stock of less than or equal to $Z_o^{\tau-1}$ and the terminal capital stock of more than or equal to $Z_o^{\tau+T}$, then the model (3.10) is infeasible.

4. Numerical illustrations

In this section two numerical examples are presented. In the first example, the proposed models are employed for ranking a set of branches of the Iranian commercial bank. The second example shows that the super-efficiency estimation *via* the proposed model may be infeasible for some specific data.

Example 4.1. The proposed ranking approach is applied to some commercial bank branches in Iran. There are 20 branches in this area. As we know, the production and the intermediation approaches are two approaches for defining the inputs and outputs of a bank. In this study, inputs and outputs are determined based on the intermediation approach.

Moreover, the inter-temporal dependence is occurred in the three-month period ($w = \{1, 2, 3\}$). According to the above discussion, two period-specific inputs and a capital input are utilized by each branch to produce three outputs. The labels of the period-specific inputs and outputs are presented in Table 1.

In addition, each branch receives the financial assistance from the central branch in each period of the evaluation window. These grants are considered as the capital stock. According to the level and rank of the branch, the maximum amount of the donation is determined in a given time period. Then, this financial assistance is considered as an initial capital or overall fund ($Z^{\tau-1}$). During this assessment window, the initial capital is

TABLE 1. The labels of data in each time period.

Input factors	Output factors
Employees score (x_1)	Loans (y_1)
Deferred claims (x_2)	Deposit (y_2)
	Profit (y_3)

TABLE 2. The efficiency score of 20 bank branches in the three-month period.

DMUs	B01	B02	B03	B04	B05	B06	B07	B08	B09	B10
Efficiency in 1th period (μ^{1*})	-0.0478	0.0000	-0.0040	0.0000	0.0000	0.0016	0.0000	0.0000	0.0000	-0.0036
Efficiency in 2th period (μ^{2*})	-0.0659	0.0000	-0.0108	0.0000	0.0000	-0.0342	0.0000	0.0000	0.0000	-0.0861
Efficiency in 3th period (μ^{3*})	-0.0776	0.0000	-0.0109	0.0000	0.0000	-0.0373	0.0000	0.0000	0.0000	-0.0937
Efficiency Score (ω_o^*)	0.9363	1.0000	0.9914	1.0000	1.0000	0.9767	1.0000	1.0000	1.0000	0.9389
DMUs	B11	B12	B13	B14	B15	B16	B17	B18	B19	B20
Efficiency in 1th period (μ^{1*})	0.0000	0.0000	-0.0040	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0192	0.0000
Efficiency in 2th period (μ^{2*})	0.0000	0.0000	-0.0062	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Efficiency in 3th period (μ^{3*})	0.0000	0.0000	-0.0132	0.0000	0.0000	0.0000	0.0000	0.0000	0.0004	0.0000
Efficiency Score (ω_o^*)	1.0000	1.0000	0.9922	1.0000	1.0000	1.0000	1.0000	1.0000	0.9937	1.0000

divided between branches for different periods. The remaining amount is considered as the terminal-stock and is denoted by $Z^{T+\tau}$. The data used in this section are reported in Table A.1 of the appendix and are reproduced from Ghobadi [18]. The efficiency score for each bank branch is obtained by the model (3.1) (see Tab. 2).

Table 2 shows that the B01 branch is inefficient ($\omega^*_{B01} = 0.9363$). Consider that the proposed model (3.1) corresponds to evaluate of the B01 branch in the assessment window, $w = \{1, 2, 3\}$. This model projects the B01 branch on the efficiency frontier. This will be achieved by decreasing the inputs and increasing the outputs with equal sizes in each period of the assessment window. Table 2 shows that in period of $t = 1$, the input and output levels of the B01 branch with equal size of 0.0478 decreases and increases, respectively. Also, the input levels of this branch with equal size of 0.0659 and 0.0776 decreases in the periods of $t = 2$ and $t = 3$, respectively, while the output levels of the B01 branch with equal size of 0.0659 and 0.0776 increases in the periods of $t = 2$ and $t = 3$, respectively. Table 2 demonstrates that the five other branches are inefficient ($B03$, $B06$, $B10$, $B13$, and $B19$). In other words, the project of these units on the efficiency frontier can be achieved by decreasing the inputs and increasing the outputs with non-zero equal sizes in each period of the assessment window. Then, these inefficient units can be ranked as follows:

$$B19 \succ B13 \succ B03 \succ B06 \succ B10 \succ B01.$$

Table 2 demonstrates that the B02 branch is dynamically DEA efficient using the corresponding model (3.1). In fact, it's not possible to decrease the inputs and increase the outputs by nonzero equal sizes in none of the assessment window periods ($\mu^{1*} = \mu^{2*} = \mu^{3*} = 0$). Therefore, the efficiency score for B02 branch is obtained equal to $\omega^*_{B02} = 1$. Then, this unit is on the efficiency frontier, though it may be the weak frontier. Table 2 shows that thirteen of the other bank branches are dynamically DEA efficient ($B04$, $B05$, $B07$, $B08$, $B09$, $B11$, $B12$, $B14$, $B15$, $B16$, $B17$, $B18$, and $B20$). However, their differences could not be distinguished *via* the efficiency score. Then, the proposed super-efficiency model (3.10) is employed for all dynamically efficient branches and the results are reported in Table 3.

Table 3 shows that the super-efficiency of the B02 branch is $\bar{\omega}_{B02} = 1.0433$. Consider that the super-efficiency proposed model (3.10) corresponds to evaluate of the B02 branch in the assessment window, $w = \{1, 2, 3\}$. This model refers to this basic concept that the B02 unit is removed from the dynamic PPS, and this branch is under assessment through the remaining branches. In fact, this model projects the B02 branch on the efficiency frontier by increasing the inputs and decreasing the outputs with equal sizes in each period of the assessment

TABLE 3. The super-efficiency of 14 efficient bank branches in the three-month period.

DMUs	B02	B04	B05	B07	B08	B09	B11
μ^{1*}	0.0047	0.0939	0.0194	0.9667	0.0422	0.0141	0.0716
μ^{2*}	0.1160	0.0696	0.0063	1.5682	0.0369	0.0000	0.0064
μ^{3*}	0.0063	0.0523	0.0102	1.5665	0.0358	0.0240	0.0033
Super-efficiency ($\bar{\omega}_o$)	1.0433	1.0719	1.0120	2.3671	1.0383	1.0127	1.0271
DMUs	B12	B14	B15	B16	B17	B18	B20
μ^{1*}	0.2671	0.0082	0.0410	0.0037	0.2484	0.0266	0.0000
μ^{2*}	0.2665	0.0000	0.0635	0.0000	0.2893	0.0064	0.0022
μ^{3*}	0.2673	0.0000	0.0597	0.0012	0.2825	0.0061	0.0000
Super-efficiency ($\bar{\omega}_o$)	1.2669	1.0027	1.0547	1.0016	1.2734	1.0130	1.0007

window. In other words, the proposed model not only measures the unit distance to the efficiency frontier in the assessment window, but also determines the distance in each period of the assessment window. Table 3 demonstrates that in the period of $t = 1$, the input and output levels of the B02 branch with equal size of 0.0047 increases and decreases, respectively. As the size increases, then the distance of this branch from the efficiency frontier also increases. Therefore, the efficiency of the B02 branch is more stable during this period. Also, the output levels of this branch with equal size of 0.1160 and 0.0063 decreases in the periods of $t = 2$ and $t = 3$, respectively while the input levels of B01 branch with equal size of 0.1160 and 0.0063 increases in the periods of $t = 2$ and $t = 3$, respectively. Therefore, during time period $t = 2$ of the assessment window, the efficiency of the B02 branch is more stable compared with other two periods.

According to the Table 3, the maximum super-efficiency is obtained for the $B07$ branch ($\bar{\omega}_{B07} = 2.3671$). Table 3 shows that the B07 branch projected on the efficiency frontier by increasing the inputs and decreasing the outputs in the time periods $t = 1$, $t = 2$, and $t = 3$ with equal sizes 0.9667, 1.5682, and 1.5665, respectively. It is obvious that the minimum distance of this branch to the efficiency frontier is more than the corresponding one for other branches in all three periods. Therefore, the B07 branch not only has the highest super-efficiency in the assessment window, but also has the maximum super-efficiency in each of three periods. Accordingly, the B07 branch has the best performance among the efficient branches in the three-month period. Moreover, the efficiency of the B07 branch in the period $t = 2$ of the assessment window is more stable compared with other two periods.

Table 3 shows that the minimum super-efficiency is obtained for the $B20$ branch $\bar{\omega}_{B20} = 1.0007$. This table demonstrates that the B20 branch projected on the efficiency frontier by increasing the inputs and decreasing the outputs in the time periods $t = 1$, $t = 2$, and $t = 3$ with equal sizes 0.0000, 0.0022, and 0.0000, respectively. Therefore, during the period $t = 2$ of the assessment window, the efficiency of the B20 branch is more stable compared with other two periods. Table 3 shows that the minimum distance of the B20 branch to the efficiency frontier is less than of the minimum distance of other branches in the periods $t = 1$ and $t = 3$, while the performance of this branch is superior to the branches of B09, B14, and B16 in the second period. However, the performance of the B20 branch is worse than the branches B09, B14, and B16 in the other two periods. Accordingly, the B20 branch has the worst performance among the efficient branches in a three-month period. According to Table 3, the efficient assessment paths can be ranked as follows:

$$B07 \succ B17 \succ B12 \succ B04 \succ B15 \succ B02 \succ B08 \\ \succ B11 \succ B18 \succ B09 \succ B05 \succ B14 \succ B16 \succ B20.$$

According to the Tables 2 and 3, the obtained ciency and super-efficienefficy for each bank branch based on the models (3.1) and (3.10), we can rank of all 20 bank branches in the assessment window. The rank of all 20 bank branches are presented in Table 4.

TABLE 4. The rank of all 20 bank branches in the three-month period.

DMUs	B01	B02	B03	B04	B05	B06	B07	B08	B09	B10
Efficiency score(ω_o^*)	0.9363	1.0000	0.9914	1.0000	1.0000	0.9767	1.0000	1.0000	1.0000	0.9389
Super-efficiency ($\bar{\omega}_o$)	–	1.0433	–	1.0719	1.0120	–	2.3671	1.0383	1.0127	–
Ranking	20	6	17	4	11	18	1	7	10	19
DMUs	B11	B12	B13	B14	B15	B16	B17	B18	B19	B20
Efficiency score (ω_o^*)	1.0000	1.0000	0.9922	1.0000	1.0000	1.0000	1.0000	1.0000	0.9937	1.0000
Super-efficiency ($\bar{\omega}_o$)	1.0271	1.2669	–	1.0027	1.0547	1.0016	1.2734	1.0130	–	1.0007
Ranking	8	3	16	12	5	13	2	9	15	14

TABLE 5. Inputs and outputs.

Periods	t_1			t_2				
DMUs	x	z	y	x	z	y	Z^0	Z^2
A	21	15	2	18	20	4	90	55
B	21	15	4	18	20	6	60	25
C	40	30	8	36	40	12	90	20

TABLE 6. The efficiency and super-efficiency of DMUs.

DMUs	μ^{1*}	μ^{2*}	Efficiency	μ^{1*}	μ^{2*}	Super-efficiency
A	0.0000	0.0000	1.0000	–	–	Infeasible
B	0.0000	0.0000	1.0000	0.0833	0.0556	1.0694
C	0.0000	0.0000	1.0000	0.2500	0.2500	1.2500

To the best of our knowledge, there is no approach for comparison of the results of our proposed method. In other words, ranking efficient DMUs under the inter-temporal dependent assumption addressed in reference [12] has not been studied in the literature.

The following example shows that the super-efficiency estimation by the proposed model may be infeasible for some specific data.

Example 4.2. Consider three DMUs (A, B, and C), which consume a period-specific input, x, and a capital input, z, to produce an output, y. The performance is evaluated in a window of two periods, $\{t_1, t_2\}$. The data have been shown in Table 5. Using models (3.1) and (3.10), the efficiency and super-efficiency for each of the DMUs are obtained and reported in Table 6.

Table 6 shows all three DMUs are dynamically efficient. The result of super-efficiency model (3.10) shows that this model corresponding to DMUs B and C are feasible while the model corresponding to DMU A is infeasible. According to Theorem 3.6, infeasibility the model corresponding to DMU A could be due to the fact that the following system is infeasible.

$$\begin{cases} 60\lambda_B + 90\lambda_C \leq 90, \\ 25\lambda_B + 20\lambda_C \geq 55, \\ \lambda_B \geq 0, \ \lambda_C \geq 0. \end{cases} \tag{4.1}$$

In fact, there is no non-zero linear combination of the initial and final stock values of DMUs B and C producing values less than or equal of the initial value and values more than or equal to the final value of unit A. In other words, there is no assessment path (virtual or observed) in the dynamic PPS with the initial stock less than or equal to 90 and the terminal capital stock more than or equal to 50.

5. Conclusions and future research directions

In this paper, firstly, an LP model is proposed to estimate the efficiency measure of the DMU in an assessment window. The proposed model projects the unit under evaluation on the efficiency frontier by decreasing the inputs and increasing the outputs with equal sizes in each period of the assessment window. Therefore, this model measures the technical efficiency by output expansion and input contraction in each period, simultaneously. It is worth noting that the simultaneous changes in the input and output levels (mix-oriented) for the decision makers provide an analytical tool superior to the changes in only inputs (input-oriented) or outputs (output-oriented). It is proved that the proposed model is units invariant. As other advantage, this model not only measures the efficiency score in the assessment window, but also determines the efficiency measure in each period of the assessment window. Knowing the efficiency measure in each period has a high informative value for the decision-maker for identification of inefficient factors. This information sheds light on the highest inefficiency of DMU in each period, compared with other periods and even other units. Moreover, necessary and sufficient conditions were established for characterizing dynamically efficient paths using our presented model.

Secondly, for the first time, an LP model is provided to estimate the super-efficiency measure of the DMU under inter-temporal dependence of data. The introduced model is based upon the idea of excluding the unit under evaluation and assessment this unit through the remaining units. This model projects the unit under evaluation on the efficiency frontier by increasing the inputs and decreasing the outputs with equal sizes in each period of the assessment window. Therefore, projecting the DMUs on the efficiency frontier in the mixed orientation is based on input expansion and output contraction in each period, simultaneously. The proposed model provides a super-efficiency in the assessment window that not only ranks dynamic efficient units under inter-temporal dependence data, but also determines the super-efficiency status of each unit in each period of the assessment window. In fact, this model measures not only the unit distance to the efficiency frontier in the assessment window, but also determines the distance in each periods of the assessment window. Knowing the minimum distance of the unit under evaluation to the efficiency frontier in each period has a high informative value for the decision-maker deliberating about sustainability of this unit compared with other DMUs. This information sheds light on the highest stability that this unit has in each period, compared with other DMUs.

It is worth noting that except the current study, there is no approach that it can be used to ranking efficient DMUs under inter-temporal dependent data addressed in Emrouznejad and Thanassoulis [12]. Although the dynamic super-efficiency model is proposed for ranking the efficient assessment paths, the proposed model may be infeasible for some special data. Therefore, proposing an always feasible model could be considered as a future work. In addition, the proposed models are based on simultaneous changes in the input and output levels with equal sizes in each period of the assessment window. However, obtaining similar models based on simultaneous changes in the input and output levels with different sizes could be considered as further research direction.

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Appendix A.

Table A.1. The data of 20 bank branches in the three-month period.

Period	Bank	B01	B02	B03	B04	B05	B06	B07	B08	B09	B10
	Employees score (x_1)	19.83	7.08	4.01	14.91	5.33	12.84	15.72	10.94	13.08	15.97
	Deferred claims (x_2)	4603910	9547	136115	1035215	1030194	2664633	1086083	225665	7480348	3486536
$t = 1$	Change stock (z)	25415944	631223	39350793	2823106	3536597	56976524	260385226	22036925	310092	163475721
	Loans (y_1)	75097467	25258238	60530507	65413851	45667593	157015854	462186659	105618280	54863663	274163028
	Deposit (y_2)	80023776	40413775	48420589	101707340	67411796	163070472	397280289	130611124	90586108	229181190
	Profit (y_3)	2211465	767801	1770276	3898218	1291753	4334957	16849485	4072227	2335849	5869352
	Employees score (x_1)	9.38	5.66	4.76	7.69	3.67	11	11.04	7.7	10.44	12.24
	Deferred claims (x_2)	4258676	9547	136115	1035215	837568	1217053	406113	22915	7479093	3674116
$t = 2$	Change stock (z)	33423425	2635237	42532282	6186940	7640404	73818634	237525938	21522051	3651731	193337240
	Loans (y_1)	77540480	28099162	63386263	69347301	51678971	165665373	457271922	111799242	63460385	300159210
	Deposit (y_2)	75205729	40501342	49871113	100617551	69103984	158442184	408482255	141070488	96511315	241263111
	Profit (y_3)	2372091	767801	1779213	3917817	1387823	4786804	17214726	4089487	2467889	6667268
	Employees score (x_1)	9.39	5.67	4.76	7.7	3.68	11.01	11.06	7.71	10.46	12.25
	Deferred claims (x_2)	4164067	9547	136115	1033215	786287	1267053	395522	225915	7477837	3657016
$t = 3$	Change stock (z)	34302366	6014493	39988455	9851458	13105122	70323601	229207401	17014036	10980081	213492432
	Loans (y_1)	79100108	31026240	64986239	73820219	58194638	174946921	465979022	116557642	79367959	308891608
	Deposit (y_2)	76524475	39083163	53471773	100647643	70522315	174602767	426766238	156249562	107494554	239236387
	Profit (y_3)	2608693	767801	1936626	3924818	1400253	4898347	17976626	4304301	2469807	7726739
$t = 0$	Overall fund (Z^0)	117244609	26566357	158665116	29209796	39252881	267062245	1037209839	87622951	30043956	729364697
$t = 3$	Terminal stock (Z^3)	24102874	17285404	36793586	10348292	14970758	65943486	310091274	27049939	15102052	159059304
	Employees score (x_1)	5.66	3.88	14.6	12.19	9.93	3.2	19.25	6.18	6	7.86
	Deferred claims (x_2)	9547	106162	430201	288733	79572	23274	655170	70840	5771009	604842
$t = 1$	Change stock (z)	2630237	2484718	55740403	45231323	3673025	10685654	45300766	63930945	25973395	5532201
	Loans (y_1)	25258238	32767317	164983122	142754970	48940586	30547469	178762561	88958994	49562263	40493880
	Deposit (y_2)	40413775	48484615	175915348	155702255	72499127	32161835	209467807	67461848	44092348	54972684
	Profit (y_3)	767801	1044161	6675662	4650278	2145795	923161	9683107	4552123	695511	1539726
	Employees score (x_1)	7.08	4.07	11.19	9.09	5.14	3.37	12.02	6.33	5.64	4.1
	Deferred claims (x_2)	9547	106162	430201	288733	79572	23274	654770	68785	152093	604842
$t = 2$	Change stock (z)	2630237	3853539	50458213	63179340	2663053	16251577	36316407	61785788	29507223	2316070
	Loans (y_1)	31026240	37246308	174059672	147933524	56425243	32096496	186258838	92482607	51036143	44447422
	Deposit (y_2)	39083163	52947228	195596771	144033211	87184148	29011554	234294279	72931204	43318395	68159592
	Profit (y_3)	767801	1059440	6683103	5174718	2518669	925370	10570787	4761568	695511	1563397
	Employees score (x_1)	5.67	4.07	11.2	9.1	5.15	3.37	12.04	6.33	5.65	4.1
	Deferred claims (x_2)	9547	106162	430201	277182	79572	23274	654770	68211	148329	604842
$t = 3$	Change stock (z)	6009493	5806457	60083361	91013826	6205102	14326858	39962610	61511021	31628113	6803359
	Loans (y_1)	28099162	40643483	175858151	153347359	60613709	33842858	215702229	93642456	54119292	47543624
	Deposit (y_2)	40501342	54711936	186753718	128483906	86298673	33071151	274686310	74504425	45692855	63980358
	Profit (y_3)	767801	1069796	6685865	5174718	2764479	966104	10576311	5066401	698963	1571893
$t = 0$	Overall fund (Z^0)	26566357	34626288	218101708	282355290	27056255	51708898	234220827	254370473	134860848	25822016
$t = 3$	Terminal stock (Z^3)	17285404	22481574	51819731	82930801	14515075	10444809	112641044	67142719	47752117	11170386

References

[1] N. Adler, L. Friedman and Z. Sinuany-Stern, Review of ranking methods in the data envelopment analysis context. *Eur. J. Oper. Res.* **140** (2002) 249–265.

[2] P. Andersen and N.C. Petersen, A procedure for ranking efficient units in data envelopment analysis. *Manag. Sci.* **39** (1993) 1261–1294.

[3] R. Azizi, R. Kazemi Matin and R. Farzipoor Saen, Ranking units and determining dominance relations in the cost efficiency analysis. *RAIRO: OR* **49** (2015) 879–896.

[4] R.D. Banker and J.L. Giffort, "Relative Efficiency Analysis", Unpublished Manuscript (A 1987 Version Appeared as a Working Paper, Schools of Urban and Public Affairs, Carnegie-millon University) (1987).

[5] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **2** (1978) 429–444.

[6] W.D. Cook and L.M. Seiford, Data envelopment analysis (DEA)-thirty years on. *Eur. J. Oper. Res.* **192** (2009) 1–17.

[7] W.W. Cooper, L.M. Seiford and K. Tone, Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software. Kluwer Academic Publisher (1999).

[8] B. Ebrahimi and M. Rahmani, An improved approach to find and rank BCC-efficient DMUs in data envelopment analysis (DEA). *J. Ind. Syst. Eng.* **10** (2017) 25–34.

[9] M.K. Ekiza and C.T. Sakara, A new DEA approach to fully rank DMUs with an application to MBA programs. *Int. Trans. Oper. Res.* **27** (2020) 1886–1910.

[10] A. Emrouznejad, An alternative DEA measure: A case of OECD countries. *Appl. Econ. Lett.* **10** (2003) 779–782.

[11] A. Emrouznejad and M.M. Tavana, Performance Measurement with Fuzzy Data Envelopment Analysis. Springer (2014).

[12] A. Emrouznejad and E. Thanassoulis, A mathematical model for dynamic efficiency using data envelopment analysis. *Appl. Math. Comput.* **160** (2005) 363–378.

[13] A. Emrouznejad and E. Thanassoulis, Measurement of productivity index with dynamic DEA. *Int. J. Oper. Res.* **8** (2010) 247–260.

[14] A. Emrouznejad and G. Yang, A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016. *J. Socio-Econ. Plan. Sci.* **61** (2018) 4–8.

[15] S. Fallah-Fini, K. Triantis and A.L. Johnson, Reviewing the literature on non-parametric dynamic efficiency measurement: State-of-the-art. *J. Product. Anal.* **41** (2014) 51–67.

[16] R. Färe and S. Grosskopf, Intertemporal Production Frontiers: With Dynamic DEA. Kluwer Academic Publishers, Dordrecht, Netherlands (1996).

[17] D. Gharakhani, A. Toloie Eshlaghy, K. Fathi Hafshejani, R. Kiani Mavi and F. Hosseinzadeh Lotfi, Common weights in dynamic network DEA with goal programming approach for performance assessment of insurance companies in iran. *Manag. Res. Rev.* **41** (2018) 920–938.

[18] S. Ghobadi, A generalized DEA model for inputs (outputs) estimation under inter-temporal dependence. *RAIRO: OR* **53** (2019) 1791–1805.

[19] S. Ghobadi, A dynamic DEA model for resource allocation. *Int. J. Math. Oper. Res.* (in press).

[20] S. Ghobadi, G.R. Jahanshahloo, F. Hoseinzadeh Lotfi and M. Rostami-malkhalifeh, Dynamic inverse DEA in the presence of fuzzy data. *Adv. Environ. Biol.* **8** (2014) 139–151.

[21] S. Ghobadi, G.R. Jahanshahloo, F. Hoseinzadeh Lotfi and M. Rostami-malkhalifeh, Efficiency measure under inter-temporal dependence. *Int. J. Technol. Decis. Mak.* **17** (2018) 657–675.

[22] I. Guo, H. Lee and D. Lee, An integrated model for slack-based measure of super-efficiency in additive DEA. *Omega* **67** (2016) 160–167.

[23] A. Hatami-Marbini, Benchmarking with network DEA in a fuzzy environment. *RAIRO: OR* **53** (2019) 687–703.

[24] F. Hoseinzadeh Lotfi, G.R. Jahanshahloo, M. Khodabakhshi, M. Rostami-malkhalifeh, Z. Moghaddas and M. Vaez-Ghasemi, A review of ranking models in data envelopment analysis. *J. Appl. Math.* **2013** (2013) 1–20.

[25] J. Jablonsky, Multicriteria approaches for ranking of efficient units in DEA models. *Cent. Eur. J. Oper. Res.* **20** (2012) 435–449.

[26] G.R. Jahanshahloo, F. Hosseinzadeh Lotfi, H. Zhiani Rezai and F. Rezai Balf, Using monte carlo method for ranking efficient DMUs. *Appl. Math. Comput.* **162** (2005) 371–379.

[27] G.R. Jahanshahloo, J. Sadeghi and M. Khodabakhshi, Fair ranking of the decision making units using optimistic and pessimistic weights in data envelopment analysis. *RAIRO: OR* **51** (2017) 253–260.

[28] G.R. Jahanshahloo, M. Soleimani-damaneh and S. Ghobadi, Inverse DEA under inter-temporal dependence using multiple-objective programming. *Eur. J. Oper. Res.* **240** (2015) 447–456.

[29] G.R. Jahanshahloo, M. Soleimani-damaneh and M. Reshadi, On the pareto (dynamically) efficient paths. *Int. J. Comput. Math.* **83** (2006) 629–633.

[30] L. Li, X. Lv, W. Xu, Z. Zhang and X. Rong, Dynamic super-efficiency interval data envelopment analysis. In: *10th International Conference on Computer Science & Education (ICCSE)* (2015) 213–218.

[31] S. Mehrabian, M. Alirezaee and G. R. Jahanshahloo, A complete efficiency ranking of decision making units in data envelopment analysis. *Comput. Optim. Appl.* **14** (1999) 261–266.

[32] J. Nemoto and M. Goto, Dynamic data envelopment analysis: Modeling intertemporal behavior of a firm in the presence of productive inefficiencies. *Econ. Lett.* **64** (1999) 51–56.

[33] J. Pourmahmoud, New model for ranking DMUs in DDEA as a special case. *Int. J. Ind. Math.* **7** (2015) 1–6.

[34] S. Saati, M.L. Zarafat Angiz, A. Memariani and G.R. Jahanshahloo, A model for ranking decision making units in data envelopment analysis. *Ric. Oper.* **31** (2001) 47–59.

[35] K. Sengupta, Dynamics of Data Envelopment Analysis: Theory of System Efficiency. Kluwer Academic Publishers, Dordrecht, London (1995).

[36] T. Sueyoshi and K. Sekitani, Returns to scale in dynamic DEA. *Eur. J. Oper. Res.* **16** (2005) 536–544.

[37] R.M. Thrall, Duality classification and slacks in data envelopment analysis. *Ann. Oper. Res.* **66** (1996) 109–138.

[38] M. Toloo, On finding the most BCC-efficient DMU: A new integrated MIP–DEA model. *Appl. Math. Model.* **36** (2012) 5515–5520.

[39] K. Tone and M. Tsutsui, Dynamic DEA: A slacks-based measure approach. *Omega* **38** (2010) 145–156.

[40] K. Tone and M. Tsutsui, Dynamic DEA with network structure: A slacksbased measure approach. *Omega* **42** (2014) 124–131.

[41] T.H. Tran, Y. Mao, P. Nathanail, P. Siebers and D. Robinson, Integrating slacks-based measure of efficiency and super-efficiency in data envelopment analysis. *Omega* **85** (2019) 156–165.

[42] Y.M. Wang, Y. Luo and Y.X. Lan, Commonweights for fully ranking decision making units by regression analysis. *Expert Syst. Appl.* **38** (2011) 9122–9128.

[43] E. Zeinodin and S. Ghobadi, Merging decision-making units under inter-temporal dependence. *IMA J. Manag. Math.* **31** (2020) 139–166.