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APPROACHES TO MULTIPLE ATTRIBUTE DECISION MAKING BASED ON PICTURE 2-TUPLE LINGUISTIC POWER HAMY MEAN AGGREGATION OPERATORS

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Abstract. In this paper, the multiple attribute decision making (MADM) problems are investigated with picture 2-tuple linguistic information. Then, based on Hamy mean (HM) operator and dual Hamy mean (DHM) operator, the power average and power geometric operations are utilized to develop some picture 2-tuple linguistic power Hamy mean aggregation operators: picture 2-tuple linguistic power weighted Hamy mean (P2TLPWHM) operator, picture 2-tuple linguistic power weighted dual Hamy mean (P2TLPWDHM) operator, picture 2-tuple linguistic power ordered weighted Hamy mean (P2TLPOWHM) operator, picture 2-tuple linguistic power ordered weighted dual Hamy mean (P2TLPOWDHM) operator, picture 2-tuple linguistic power hybrid Hamy mean (P2TLPHHM) operator and picture 2-tuple linguistic power hybrid dual Hamy mean (P2TLPHDHM) operator. The prominent characteristic of these proposed operators are studied. Then, these operators are utilized to develop some approaches to solve the picture 2-tuple linguistic multiple attribute decision making problems. Finally, the proposed method is demonstrated through a practical example for enterprise resource planning (ERP) system selection of how the proposed methods help us and is effective in MADM problems.

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1. Introduction

Multiple attribute decision making (MADM) problems under linguistic environment are an interesting research topic having received more and more attention during the last several years [1–7]. One of the well-known linguistic information processing models is the 2-tuple linguistic computational model [8–14]. The fuzzy linguistic approach has been applied successfully to many problems. However, there is a limitation of this approach imposed by its information representation model and the computation methods used when fusion processes are performed on linguistic values. This limitation is the loss of information caused by the need to express the results in the initial expression domain that is discrete *via* an approximate process. Herrera and Martinez [10] presented tools for overcoming this limitation, in which the linguistic information can be expressed by means of 2-tuples, which are composed by a linguistic term and a numeric value assessed in $[-0.5, 0.5)$, this model allows a continuous representation of the linguistic information on its domain, therefore, it can represent any counting

Keywords. Multiple attribute decision making (MADM), picture 2-tuple linguistic set, power weighted Hamy mean (PWHM) operator, power weighted dual Hamy mean (PWDHM) operator, enterprise resource planning (ERP).

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of information obtained in a aggregation process. Herrera and Martinez [15] developed a procedure for combining numerical and linguistic information without loss of information in the transformation processes between numerical and linguistic information, taking as base for representing the information the a-tuple fuzzy linguistic representation model. Moreno *et al.* [16] presented a qualitative and user-oriented methodology for assessing quality of health-related websites based on a 2-tuple fuzzy linguistic approach. Park *et al.* [17] developed some new linguistic aggregation operators such as 2-tuple linguistic harmonic (2TLH) operator, 2-tuple linguistic weighted harmonic (2TLWH) operator, 2-tuple linguistic ordered weighted harmonic (2TLOWH) operator and 2-tuple linguistic hybrid harmonic (2TLHH) operator, which can be utilized to aggregate preference information taking the form of linguistic variables and then studied some desirable properties of the operators. Truck [18] dealt with linguistic models that may prove useful for representing the information during decision making. Qin and Liu [19] developed some 2-tuple linguistic aggregation operators based on Muirhead mean (MM) operator [20–23], which is combined with multiple attribute group decision making (MAGDM) and applied the proposed MAGDM model for supplier selection under 2-tuple linguistic environment. Ju *et al.* [24] proposed three 2-tuple linguistic aggregation operators called Shapley 2-tuple linguistic Choquet averaging operator, Shapley 2-tuple linguistic Choquet geometric operator and generalized Shapley 2-tuple linguistic Choquet averaging operator and discussed some properties of these operators, such as idempotency, monotonicity, boundary and commutativity. Santos *et al.* [25] proposed a segmentation model based on the relationship with suppliers capable of aggregating quantitative and qualitative criteria which Analytic Hierarchy Process (AHP) was used to determine the relative importance of each criteria. Zhang and Su [26] proposed an approach based on 2-tuple fuzzy linguistic method to recommend tasks to the workers who would be capable of completing and accept them. Ju *et al.* [27] extended the traditional MSM operator [28] to the single-valued neutrosophic interval 2-tuple linguistic environment, propose some novel aggregation operators, and develop a novel method to solve multiple attribute group decision making (MAGDM) problems.

Recently, Cuong and Kreinovich [29] proposed picture fuzzy set (PFS) and studied the some basic operations and properties of PFS. The PFS is characterized by three functions expressing the degree of membership, the degree of neutral membership and the degree of non-membership. The only constraint is that the sum of the three degrees must not exceed 1. Basically, PFS based models can be applied to situations requiring human opinions involving more answers of types: yes, abstain, no, refusal, which can't be accurately expressed in the traditional FS and IFS. Singh [30] proposed correlation coefficients for picture fuzzy sets which considered the degree of positive membership, degree of neutral membership, degree of negative membership and the degree of refusal membership. Son [31] presented a novel distributed picture fuzzy clustering method on picture fuzzy sets so-called DPFCM. Thong and Son [32] proposed a novel hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems for medical diagnosis so-called HIFCF (Hybrid Intuitionistic Fuzzy Collaborative Filtering). Son [33] proposed a generalized picture distance measure and integrated it to a novel hierarchical picture fuzzy clustering method called Hierarchical Picture Clustering (HPC). Thong and Son [34] defined a method called Automatic Picture Fuzzy Clustering (AFC-PFS) for determining the most suitable number of clusters for FC-PFS which is a hybrid method between Particle Swarm Optimization (PSO) and FC-PFS where combined solutions consisting of the number of clusters and equivalent clustering centers and membership matrices are packed and optimized in PSO. Son and Thong [35] proposed two novel hybrid forecast methods based on picture fuzzy clustering for weather nowcasting. Wei *et al.* [36] defined an extended bidirectional projection method in picture fuzzy MAGDM for safety assessment of construction project.

Although, picture fuzzy set theory has been successfully applied in some areas, but there are situations in real life which can't be represented by picture fuzzy sets. Voting can be a good example of such situation as the human voters may be divided into four groups of those who: vote for, abstain, refusal of voting. Basically, picture fuzzy sets [29] based models may be adequate in situations when we face human opinions involving more answers of the type: yes, abstain, no, refusal. However, all the above approaches are unsuitable to describe the degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership of an element to a linguistic label, which can reflect the decision maker's confidence level when they are making an evaluation. In order to overcome this limit, Wei [37] and Wei *et al.* [38] proposed the concept of picture 2-tuple linguistic set to solve this problem based on the picture fuzzy sets [29] and 2-tuple linguistic information processing model [39]. Thus, how to aggregate these picture 2-tuple linguistic numbers which take into account the information about the relationship between the values being fused is an interesting topic. To solve this issue, in this paper, we shall develop some picture 2-tuple linguistic power aggregation operators on the basis of the traditional Hamy mean operators [40–42]. In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce the concept of picture 2-tuple linguistic set on the basis of the picture fuzzy set and 2-tuple linguistic information processing model. In Section 3, we propose some picture 2-tuple linguistic power Hamy mean aggregation operators. In Section 4, we present some picture 2-tuple linguistic power dual Hamy mean aggregation operators. In Section 5, based on these operators, we present some approaches to MADM with picture 2-tuple linguistic information. In Section 6, we present a numerical example for enterprise resource planning (ERP) system selection with picture 2-tuple linguistic information in order to illustrate the method proposed in this paper. Section 7 concludes the paper with some remarks.

2. Preliminaries

In the following, we introduced some basic concepts related to 2-tuple linguistic term sets and picture fuzzy sets.

2.1. Two-tuple linguistic term sets

Let $S = \{s_i | i = 1, 2, \cdots, t\}$ be a linguistic term set with odd cardinality. Any label, s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics [43]:

(1) The set is ordered: $s_i > s_j$, if $i > j$; (2) Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$; (3) Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$. For example, S can be defined as

$$S = \{s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{medium},$$
$$s_5 = \text{good}, s_6 = \text{very good}, s_7 = \text{extremely good}\}.$$

Herrera and Martinez [10] developed the 2-tuple fuzzy linguistic representation model based on the concept of symbolic translation. It is used for representing the linguistic assessment information by means of a 2-tuple (s_i, α_i), where s_i is a linguistic label from predefined linguistic term set S and α_i is the value of symbolic translation, and $\alpha_i \in [-0.5, 0.5)$.

Definition 2.1 (Herrera & Martinez [10])**.** Let β be the result of an aggregation of the indices of a set of labels assessed in a linguistic term set S, *i.e.*, the result of a symbolic aggregation operation, $\beta \in [1, t]$, being t the cardinality of S. Let $i = \text{round}(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [1, t]$ and $\alpha \in [-0.5, 0.5)$ then α is called a symbolic translation.

Definition 2.2 (Herrera & Martinez [10])**.** Let $S = \{s_1, s_2, \cdots, s_t\}$ be a linguistic term set and $\beta \in [1, t]$ is a number value representing the aggregation result of linguistic symbolic. Then the function Δ used to obtain the 2-tuple linguistic information equivalent to β is defined as:

$$\Delta : [1, t] \rightarrow S \times [-0.5, 0.5), \tag{2.1}$$

$$\Delta(\beta) = \begin{cases} s_i, \ i = \text{round}(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5) \end{cases}, \tag{2.2}$$

where round (.) is the usual round operation, s_i has the closest index label to β and α is the value of the symbolic translation.

Definition 2.3 (Herrera & Martinez [10]). Let $S = \{s_1, s_2, \cdots, s_t\}$ be a linguistic term set and (s_i, α_i) be a 2-tuple. There is always a function Δ^{-1} can be defined, such that, from a 2-tuple (s_i, α_i) it return its equivalent numerical value $\beta \in [1, t] \subset R$, which is:

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [1, t], \tag{2.3}$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta. \tag{2.4}$$

From Definitions 2.1 to 2.2, we can conclude that the conversion of a linguistic term into a linguistic 2-tuple consists of adding a value 0 as symbolic translation:

$$\Delta(s_i) = (s_i, 0). \tag{2.5}$$

2.2. Picture fuzzy set

Although, intuitionistic fuzzy set theory [44–48] has been successfully applied in different areas, but there are situations in real life which can't be represented by intuitionistic fuzzy sets. Picture fuzzy sets [29] are extension of intuitionistic fuzzy sets. Picture fuzzy set based models may be adequate in situations when we face human opinions involving more answers of types: yes, abstain, no, refusal. It can be considered as a powerful tool represent the uncertain information in the process of patterns recognition and cluster analysis.

Definition 2.4 (Cuong & Kreinovich [29]). A picture fuzzy set (PFS) A on the universe X is an object of the form

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle \,|x \in X\}, \tag{2.6}$$

where $\mu_A(x) \in [0,1]$ is called the "degree of positive membership of A", $\eta_A(x) \in [0,1]$ is called the "degree of neutral membership of A" and $\nu_A(x) \in [0,1]$ is called the "degree of negative membership of A", and $\mu_A(x)$, $\eta_A(x)$, $\nu_A(x)$ satisfy the following condition: $0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1$, $\forall\ x \in X$. Then for $x \in X$, $\pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ could be called the degree of refusal membership of x in A.

Cuong and Kreinovich [29] also defined some operations as follows:

Definition 2.5 (Cuong & Kreinovich [29]). Given two PFEs represented by A and B on universe X, the inclusion, union, intersection and complement operations are defined as follows:

(1) $A \subseteq B$, if $\mu_A(x) \le \mu_B(x)$, $\eta_A(x) \le \eta_B(x)$ and $\nu_A(x) \ge \nu_B(x)$, $\forall x \in X$;
(2) $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x)))\,|x \in X\}$;
(3) $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x)))\,|x \in X\}$;
(4) $\bar{A} = \{(x, \nu_A(x), \eta_A(x), \mu_A(x))\,|x \in X\}$.

For convenience, we call $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha)$ a picture fuzzy number (PFN), where

$$\mu_\alpha \in [0,1], \eta_\alpha \in [0,1], \nu_\alpha \in [0,1], \mu_\alpha + \eta_\alpha + \nu_\alpha \le 1.$$

Motivated by the operations of the intuitionistic fuzzy number [44] and according to Definition 2.5, in the following, Wei [49] defined some operational laws of picture fuzzy number.

Definition 2.6 (Wei [49]). Let $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \eta_\beta, \nu_\beta)$ be two picture fuzzy numbers, then

$$\begin{aligned}
&\bar{\alpha} = \alpha{=}(\nu_\alpha, \eta_\alpha, \mu_\alpha);\\
&\alpha \wedge \beta = (\min\{\mu_\alpha, \mu_\beta\}, \max\{\eta_\alpha, \eta_\beta\}, \max\{\nu_\alpha, \nu_\beta\});\\
&\alpha \vee \beta = (\max\{\mu_\alpha, \mu_\beta\}, \min\{\eta_\alpha, \eta_\beta\}, \min\{\nu_\alpha, \nu_\beta\});\\
&\alpha \oplus \beta = (\mu_\alpha + \mu_\beta - \mu_\alpha\mu_\beta, \eta_\alpha\eta_\beta, \nu_\alpha\nu_\beta);\\
&\alpha \otimes \beta = (\mu_\alpha\mu_\beta, \eta_\alpha + \eta_\beta - \eta_\alpha\eta_\beta, \nu_\alpha + \nu_\beta - \nu_\alpha\nu_\beta);\\
&\lambda\alpha = \left(1 - (1-\mu_\alpha)^\lambda, \eta_\alpha^\lambda, \nu_\alpha^\lambda\right);\\
&\alpha^\lambda = \left(\mu_\alpha^\lambda, 1 - (1-\eta_\alpha)^\lambda, 1 - (1-\nu_\alpha)^\lambda\right).
\end{aligned}$$

Based on the Definition 2.6, Wei [49] derived the following properties easily.

Theorem 2.7. *Let* $\alpha = (\mu_\alpha, \eta_\alpha, \nu_\alpha)$ *and* $\beta = (\mu_\beta, \eta_\beta, \nu_\beta)$ *be two picture fuzzy numbers,* $\lambda, \lambda_1, \lambda_2 > 0$*, then*

(1) $\alpha \oplus \beta = \beta \oplus \alpha$;
(2) $\alpha \otimes \beta = \beta \otimes \alpha$;
(3) $\lambda (\alpha \oplus \beta) = \lambda\alpha \oplus \lambda\beta$;
(4) $(\alpha \otimes \beta)^\lambda = \alpha^\lambda \otimes \beta^\lambda$;
(5) $\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha$;
(6) $\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{(\lambda_1+\lambda_2)}$;
(7) $(\alpha^{\lambda_1})^{\lambda_2} = \alpha^{\lambda_1\lambda_2}$.

2.3. Picture 2-tuple linguistic sets

In the following, Wei [37] and Wei *et al.* [38] proposed the concepts and basic operations of the picture 2-tuple linguistic sets on the basis of the picture fuzzy sets [29] and 2-tuple linguistic information processing model [10, 15].

Definition 2.8 (Wei [37]; Wei *et al.* [38])**.** A picture 2-tuple linguistic sets A in X is given

$$A = \left\{ \left(s_{\theta(x)}, \rho\right), \left(\mu_A(x), \eta_A(x), \nu_A(x)\right), x \in X \right\}, \tag{2.7}$$

where $\left(s_{\theta(x)}, \rho\right) \in S, \rho \in [-0.5, 0.5)$, $u_A(x) \in [0,1], \eta_A(x) \in [0,1]$ and $v_A(x) \in [0,1]$, with the condition $0 \le u_A(x) + \eta_A(x) + v_A(x) \le 1$, $\forall x \in X$, $s_{\theta(a)} \in S$ and $\rho \in [-0.5, 0.5)$. The numbers $\mu_A(x), \eta_A(x), \nu_A(x)$ represent, respectively, the degree of positive membership, degree of negative membership and degree of negative membership of the element x to linguistic variable $\left(s_{\theta(x)}, \rho\right)$. Then for $x \in X$, $\pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ could be called the degree of refusal membership of the element x to linguistic variable $\left(s_{\theta(x)}, \rho\right)$.

For convenience, Wei [37] called $\tilde{a} = \left\langle \left(s_{\theta(a)}, \rho\right), (\mu_\alpha, \eta_\alpha, \nu_\alpha) \right\rangle$ a picture 2-tuple linguistic number (P2TLN), where $\mu_\alpha \in [0,1], \eta_\alpha \in [0,1], \nu_\alpha \in [0,1]$, $\mu_\alpha + \eta_\alpha + \nu_\alpha \le 1$, $s_{\theta(a)} \in S$ and $\rho \in [-0.5, 0.5)$.

Definition 2.9 (Wei [37]; Wei *et al.* [38])**.** Let $\tilde{a} = \left\langle \left(s_{\theta(a)}, \rho\right), (\mu_\alpha, \eta_\alpha, \nu_\alpha) \right\rangle$, a picture 2-tuple linguistic number (P2TLN), a score function $\tilde{a}$ of a picture 2-tuple linguistic number can be represented as follows:

$$S(\tilde{a}) = \Delta\left(\Delta^{-1}\left(s_{\theta(a)}, \rho\right) \cdot \frac{1 + \mu_\alpha - \nu_\alpha}{2}\right), \quad \Delta^{-1}(S(\tilde{a})) \in [1, t]. \tag{2.8}$$

Definition 2.10 (Wei [37]; Wei *et al.* [38])**.** Let $\tilde{a} = \left\langle \left(s_{\theta(a)}, \rho\right), (\mu_\alpha, \eta_\alpha, \nu_\alpha) \right\rangle$ a picture 2-tuple linguistic number (P2TLN), an accuracy function H of a picture 2-tuple linguistic number can be represented as follows:

$$H(\tilde{a}) = \Delta\left(\Delta^{-1}\left(s_{\theta(a)}, \rho\right) \cdot \frac{\mu_\alpha + \eta_\alpha + \nu_\alpha}{2}\right), \quad \Delta^{-1}(H(\tilde{a})) \in [1, t], \tag{2.9}$$

to evaluate the degree of accuracy of the picture 2-tuple linguistic number $\tilde{a} = \left\langle \left(s_{\theta(a)}, \rho\right), (\mu_\alpha, \eta_\alpha, \nu_\alpha) \right\rangle$, where $\Delta^{-1}(H(\tilde{a})) \in [1, t]$. The larger the value of $H(\tilde{a})$, the more the degree of accuracy of the picture 2-tuple linguistic number a.

Based on the score function S and the accuracy function H, in the following, the order relation between two picture 2-tuple linguistic numbers is defined as follows:

Definition 2.11 (Wei [37]; Wei *et al.* [38])**.** Let $\tilde{a}_1 = \left\langle \left(s_{\theta(a_1)}, \rho_1\right), (u_{a_1}, \eta_{a_1}, v_{a_1}) \right\rangle$ and $\tilde{a}_2 = \left\langle \left(s_{\theta(a_2)}, \rho_2\right), (u_{a_2}, \eta_{a_2}, v_{a_2}) \right\rangle$ be two picture 2-tuple linguistic numbers, $S(\tilde{a}_1) = \Delta\left(\Delta^{-1}\left(s_{\theta(a_1)}, \rho_1\right) \cdot \frac{1+\mu_{\alpha_1}-\nu_{\alpha_1}}{2}\right)$ and $S(\tilde{a}_2) = \Delta\left(\Delta^{-1}\left(s_{\theta(a_2)}, \rho_2\right) \cdot \frac{1+\mu_{\alpha_2}-\nu_{\alpha_2}}{2}\right)$ be the scores of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, and let $H(\tilde{a}_1) = \Delta\left(\Delta^{-1}\left(s_{\theta(a_1)}, \rho_1\right) \cdot \frac{\mu_{\alpha_1}+\eta_{\alpha_1}+\nu_{\alpha_1}}{2}\right)$ and $H(\tilde{a}_2) = \Delta\left(\Delta^{-1}\left(s_{\theta(a_2)}, \rho_2\right) \cdot \frac{\mu_{\alpha_2}+\eta_{\alpha_2}+\nu_{\alpha_2}}{2}\right)$ be the accuracy degrees of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, then if $S(\tilde{a}_1) < S(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$; if $S(\tilde{a}_1) = S(\tilde{a}_2)$, then

(1) if $H(\tilde{a}_1) = H(\tilde{a}_2)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, denoted by $\tilde{a}_1 = \tilde{a}_2$;
(2) if $H(\tilde{a}_1) < H(\tilde{a}_2)$, $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$.

Motivated by the operations of the 2-tuple linguistic information [10,43] and Definition 2.5, in the following, some operational laws of picture 2-tuple linguistic numbers are defined as follows.

Definition 2.12 (Wei [37]; Wei *et al.* [38])**.** Let $\tilde{a}_1 = \left\langle \left(s_{\theta(a_1)}, \rho_1\right), (u_{a_1}, \eta_{a_1}, v_{a_1})\right\rangle$ and $\tilde{a}_2 = \left\langle \left(s_{\theta(a_2)}, \rho_2\right), (u_{a_2}, \eta_{a_2}, v_{a_2})\right\rangle$ be two picture 2-tuple linguistic numbers, then

$$\begin{aligned}
\tilde{a}_1 \oplus \tilde{a}_2 &= \left\langle \Delta\left(\Delta^{-1}\left(s_{\theta(a_1)}, \rho_1\right) + \Delta^{-1}\left(s_{\theta(a_2)}, \rho_2\right)\right),\right. \\
&\quad \left.(u_{a_1} + u_{a_2} - u_{a_1}u_{a_2}, \eta_{a_1}\eta_{a_2}, \nu_{a_1}\nu_{a_2})\right\rangle; \\
\tilde{a}_1 \otimes \tilde{a}_2 &= \left\langle \Delta\left(\Delta^{-1}\left(s_{\theta(a_1)}, \rho_1\right) \cdot \Delta^{-1}\left(s_{\theta(a_2)}, \rho_2\right)\right),\right. \\
&\quad \left.(u_{a_1}u_{a_2}, \eta_{a_1} + \eta_{a_2} - \eta_{a_1}\eta_{a_2}, \nu_{a_1} + \nu_{a_2} - \nu_{a_1}\nu_{a_2})\right\rangle; \\
\lambda\tilde{a}_1 &= \left\langle \Delta\left(\lambda\Delta^{-1}\left(s_{\theta(a_1)}, \rho_1\right)\right), \left(1 - (1 - u_{a_1})^{\lambda}, \eta_{a_1}^{\lambda}, \nu_{a_1}^{\lambda}\right)\right\rangle; \\
(\tilde{a}_1)^{\lambda} &= \left\langle \Delta\left(\left(\Delta^{-1}\left(s_{\theta(a_1)}, \rho_1\right)\right)^{\lambda}\right), \left(u_{a_1}^{\lambda}, 1 - (1 - \eta_{a_1})^{\lambda}, 1 - (1 - \nu_{a_1})^{\lambda}\right)\right\rangle.
\end{aligned}$$

Based on the Definition 2.12, the following properties can be derived easily.

Theorem 2.13 (Wei [37]; Wei *et al.* [38])**.** *For any two picture 2-tuple linguistic numbers* $\tilde{a}_1 = \left\langle \left(s_{\theta(a_1)}, \rho_1\right), (u_{a_1}, \eta_{a_1}, v_{a_1})\right\rangle$ *and* $\tilde{a}_2 = \left\langle \left(s_{\theta(a_2)}, \rho_2\right), (u_{a_2}, \eta_{a_2}, v_{a_2})\right\rangle$*, it can be proved the calculation rules shown as follows:*

(1) $\tilde{a}_1 \oplus \tilde{a}_2 = \tilde{a}_2 \oplus \tilde{a}_1$
(2) $\tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_2 \otimes \tilde{a}_1$
(3) $\lambda(\tilde{a}_1 \oplus \tilde{a}_2) = \lambda\tilde{a}_1 \oplus \lambda\tilde{a}_2,\ 0 \le \lambda \le 1$
(4) $\lambda_1\tilde{a}_1 \oplus \lambda_2\tilde{a}_1 = (\lambda_1 \oplus \lambda_2)\tilde{a}_1,\ 0 \le \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \le 1$
(5) $\tilde{a}_1^{\lambda_1} \otimes \tilde{a}_1^{\lambda_2} = (\tilde{a}_1)^{\lambda_1+\lambda_2},\ 0 \le \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \le 1$
(6) $\tilde{a}_1^{\lambda_1} \otimes \tilde{a}_2^{\lambda_1} = (\tilde{a}_1 \otimes \tilde{a}_2)^{\lambda_1},\ \lambda_1 \ge 0.$
(7) $\left(\tilde{\alpha}^{\lambda_1}\right)^{\lambda_2} = \tilde{\alpha}^{\lambda_1\lambda_2}$.

2.4. HM operator

Definition 2.14 (Hara *et al.* [40])**.** The HM operator is defined as follows:

$$\mathrm{HM}^{(k)}\left(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n\right) = \frac{\sum\limits_{1 \le i_1 < \ldots < i_k \le n} \left(\prod\limits_{j=1}^{k} \tilde{p}_{i_j}\right)^{\frac{1}{k}}}{C_n^k}, \tag{2.10}$$

where k is a parameter and $k = 1, 2, \ldots, n$, $i_1, i_2, \ldots, i_k$ are k integer values taken from the set $\{1, 2, \ldots, n\}$ of k integer values, C_n^k denotes the binomial coefficient and $C_n^k = \frac{n!}{k!(n-k)!}$.

3. Picture 2-tuple linguistic power Hamy mean aggregation operators

Yager [50] developed a nonlinear weighted average aggregation operator called power average (PA) operator, which can be defined as follows:

$$\mathrm{PA}\left(a_1, a_2, \cdots, a_n\right) = \frac{\sum\limits_{i=1}^{n} \left(1 + T\left(a_i\right)\right) a_i}{\sum\limits_{i=1}^{n} \left(1 + T\left(a_i\right)\right)}, \tag{3.1}$$

where $T(a_i)=\sum_{\substack{j=1\\ j\neq i}}^{n} \mathrm{Sup}(a_i,a_j)$, and $\mathrm{Sup}(a,b)$ is the support for a from b, which satisfies the following three properties: (1) $\mathrm{Sup}(a,b)\in[0,1]$; (2) $\mathrm{Sup}(a,b)=\mathrm{Sup}(b,a)$; (3) $\mathrm{Sup}(a,b)\geq \mathrm{Sup}(x,y)$, if $|a-b|<|x-y|$. Obviously, the support (Sup) measure is essentially a similarity index. The more similar, the closer two values, and the more they support each other.

In this section, we shall develop some Hamy mean aggregation operators with picture 2-tuple linguistic information and power operation laws, such as picture 2-tuple linguistic power Hamy mean (P2TLPHM) operator, picture 2-tuple linguistic power weighted Hamy mean (P2TLPWHM) operator, picture 2-tuple linguistic power ordered weighted Hamy mean (P2TLPOWHM) operator and picture 2-tuple linguistic power hybrid Hamy mean (P2TLPHHM) operator.

Definition 3.1. Let $\tilde{p}_j=\langle(r_j,\alpha_j),(\mu_j,\eta_j,\nu_j)\rangle\,(j=1,2,\cdots,n)$ be a collection of picture 2-tuple linguistic numbers. The picture 2-tuple linguistic power Hamy mean (P2TLPHM) operator is a mapping $P^n\to P$ such that

$$\mathrm{P2TLPHM}(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)=\frac{\underset{1\leq i_1<\ldots<i_k\leq n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)\right)^{\frac{1}{k}}}{C_n^k}, \tag{3.2}$$

where

$$T\left(\tilde{p}_{i_j}\right)=\sum_{\substack{j=1\\ j\neq t}}^{n}\mathrm{Sup}\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right) \tag{3.3}$$

and $\mathrm{Sup}\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right)$ is the support for $\tilde{p}_{i_j}$ from $\tilde{p}_{i_t}$, with the conditions:

(1) $\mathrm{Sup}\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right)\in[0,1]$;
(2) $\mathrm{Sup}\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right)=\mathrm{Sup}\left(\tilde{p}_{i_t},\tilde{p}_{i_j}\right)$;
(3) $\mathrm{Sup}\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right)\geq \mathrm{Sup}(\tilde{p}_s,\tilde{p}_t)$, if $d\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right)<d(\tilde{p}_s,\tilde{p}_t)$, where d is a distance measure.

Based on the Definition 3.1 and Theorem 2.13, we can get the following result:

Theorem 3.2. *The aggregated value by using P2TLPHM operator is also a picture 2-tuple linguistic numbers, where*

$$\begin{aligned}
&P2TLPHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)=\frac{\underset{1\leq i_1<\ldots<i_k\leq n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)\right)^{\frac{1}{k}}}{C_n^k}\\
&=\left\langle\Delta\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\left(\prod_{j=1}^{k}\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)\Delta^{-1}(r_j,\alpha_j)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right),\right.\\
&\left(1-\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-(1-\mu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-(\eta_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\right.\\
&\left.\left.\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-(\nu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right)\right\rangle
\end{aligned} \tag{3.4}$$

where

$$T\left(\tilde{p}_{i_j}\right)=\sum_{\substack{j=1\\ j\neq t}}^{n} Sup\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right) \tag{3.5}$$

Proof.

$$\begin{aligned}\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum\limits_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}=&\left\langle\Delta\left(\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)\Delta^{-1}(r_j,\alpha_j)}{\sum\limits_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right),\right.\\&\left.\left(1-(1-\mu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}},\ (\eta_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}},(\nu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)\right\rangle.\end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}\mathop{\otimes}_{j=1}^{k}\left(\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum\limits_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)=&\left\langle\Delta\left(\prod_{j=1}^{k}\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)\Delta(r_j,\alpha_j)}{\sum\limits_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right),\right.\\&\left.\left(\prod_{j=1}^{k}\left(1-(1-\mu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right),1-\prod_{j=1}^{k}\left(1-(\eta_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right),\ 1-\prod_{j=1}^{k}\left(1-(\nu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)\right)\right\rangle.\end{aligned} \tag{3.7}$$

Thereafter,

$$\begin{aligned}\mathop{\otimes}_{j=1}^{k}\left(\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum\limits_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)^{\frac{1}{k}}=&\left\langle\Delta\left(\prod_{j=1}^{k}\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)\Delta^{-1}(r_j,\alpha_j)}{\sum\limits_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}},\right.\\&\left.\left(\prod_{j=1}^{k}\left(1-(1-\mu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}},1-\prod_{j=1}^{k}\left(1-(\eta_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}},\ 1-\prod_{j=1}^{k}\left(1-(\nu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)\right\rangle.\end{aligned} \tag{3.8}$$

Furthermore,

$$\begin{aligned}\mathop{\oplus}_{1\le i_1<\ldots<i_k\le n}\mathop{\otimes}_{j=1}^{k}\left(\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum\limits_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)^{\frac{1}{k}}=&\left\langle\Delta\left(\sum_{1\le i_1<\ldots<i_k\le n}\left(\prod_{j=1}^{k}\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)\Delta^{-1}(r_j,\alpha_j)}{\sum\limits_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right),\right.\\&\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-(1-\mu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right),\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-(\eta_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right),\right.\\&\left.\left.\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-(\nu_j)^{\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)\right)\right\rangle.\end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned}
&\mathrm{P2TLPHM}\left(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n\right)=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\bigoplus}\left(\overset{k}{\underset{j=1}{\bigotimes}}\left(\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)\right)^{\frac{1}{k}}}{C_n^k}\\
&=\left\langle\Delta\left(\frac{1}{C_n^k}\sum_{1\le i_1<\ldots<i_k\le n}\left(\prod_{j=1}^{k}\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)\Delta^{-1}\left(r_j,\alpha_j\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right),\right.\\
&\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\eta_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\right.\\
&\left.\left.\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\nu_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right)\right\rangle.
\end{aligned}\tag{3.10}$$

Hence, (3.4) is kept. □

Definition 3.3. Let $\tilde{p}_j=\langle(r_j,\alpha_j),(\mu_j,\eta_j,\nu_j)\rangle\,(j=1,2,\cdots,n)$ be a collection of picture 2-tuple linguistic numbers, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $\tilde{p}_j\,(j=1,2,\cdots,n)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$. The picture 2-tuple linguistic power weighted Hamy mean (P2TLPWHM) operator is a mapping $P^n\to P$ such that

$$\mathrm{P2TLPWHM}\left(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n\right)=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\bigoplus}\left(\overset{k}{\underset{j=1}{\bigotimes}}\left(\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)\right)^{\frac{1}{k}}}{C_n^k}\ ,\tag{3.11}$$

where

$$T\left(\tilde{p}_{i_j}\right)=\omega_i\sum_{\substack{j=1\\ j\ne t}}^{n}\mathrm{Sup}\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right).\tag{3.12}$$

Based on the Definition 3.3, Theorem 2.13 and mathematical induction on n, we can get the following result:

Theorem 3.4. *The aggregated value by using P2TLPWHM operator is also a picture 2-tuple linguistic numbers, where*

$$\begin{aligned}
&P2TLPWHM\left(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n\right)=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\bigoplus}\left(\overset{k}{\underset{j=1}{\bigotimes}}\left(\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)\right)^{\frac{1}{k}}}{C_n^k}\\
&=\left\langle\Delta\left(\frac{1}{C_n^k}\sum_{1\le i_1<\ldots<i_k\le n}\left(\prod_{j=1}^{k}\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)\Delta^{-1}\left(r_j,\alpha_j\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right),\right.\\
&\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_j\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\eta_j\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\right.\\
&\left.\left.\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\nu_j\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right)\right\rangle
\end{aligned}\tag{3.13}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ *be the weight vector of* $\tilde{p}_j\ (j=1,2,\cdots,n)$, *and* $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$.

It can be easily proved that the P2TLPWHM operator has the following properties.

Theorem 3.5 (Idempotency)**.** *If all* $\tilde{p}_j\ (j=1,2,\cdots,n)$ *are equal, i.e.* $\tilde{p}_j=\tilde{p}$ *for all* j, *then*

$$P2TLPWHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)=\tilde{p}. \tag{3.14}$$

Theorem 3.6 (Boundedness)**.** *Let* $\tilde{p}_j\ (j=1,2,\cdots,n)$ *be a collection of P2TLNs, and let*

$$\tilde{p}^-=\min_j\tilde{p}_j,\quad \tilde{p}^+=\max_j\tilde{p}_j.$$

Then

$$\tilde{p}^-\le P2TLPWHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)\le\tilde{p}^+. \tag{3.15}$$

Theorem 3.7 (Monotonicity)**.** *Let* $\tilde{p}_j\ (j=1,2,\cdots,n)$ *and* $\tilde{p}'_j\ (j=1,2,\cdots,n)$ *be two set of P2TLNs, if* $\tilde{p}_j\le\tilde{p}'_j$, *for all* j, *then*

$$P2TLPWHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)\le P2TLPWHM(\tilde{p}'_1,\tilde{p}'_2,\cdots,\tilde{p}'_n). \tag{3.16}$$

Further, we give a picture 2-tuple linguistic power ordered weighted Hamy mean (P2TLPOWHM) operator below.

Definition 3.8. Let $\tilde{p}_j=\langle(r_j,\alpha_j),(\mu_j,\eta_j,\nu_j)\rangle\ (j=1,2,\cdots,n)$ be a collection of P2TLNs, the picture 2-tuple linguistic power ordered weighted Hamy mean (P2TLPOWHM) operator of dimension n is a mapping P2TLPOWHM: $P^n\to P$, that has an associated weight vector $w=(w_1,w_2,\cdots,w_n)^T$ such that $w_j>0$ and $\sum_{j=1}^{n}w_j=1$. Furthermore,

$$\begin{aligned}
&\mathrm{P2TLPOWHM}(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)=\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(\frac{nw_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}\tilde{p}_{\sigma(i_j)}\right)\right)^{\frac{1}{k}}}{C_n^k}\\
&=\left\langle\Delta\left(\frac{1}{C_n^k}\sum_{1\le i_1<\ldots<i_k\le n}\left(\prod_{j=1}^{k}\frac{nw_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)\Delta^{-1}\left(r_{\sigma(j)},\alpha_{\sigma(j)}\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}\right)^{\frac{1}{k}}\right),\right.\\
&\left(1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_{\sigma(j)}\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right),\\
&\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\eta_{\sigma(j)}\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\\
&\left.\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\nu_{\sigma(j)}\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right)\right\rangle
\end{aligned} \tag{3.17}$$

where $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $\tilde{p}_{\sigma(i_j-1)}\ge\tilde{p}_{\sigma(i_j)}$ for all $j=2,\cdots,n$, $w_j\ (j=1,2,\cdots,n)$ is collection of weights such that

$$w_j=g\left(\frac{R_j}{TV}\right)-g\left(\frac{R_{j-1}}{TV}\right),R_j=\sum_{i=1}^{j}V_{\sigma(i)},\ TV=\sum_{i=1}^{n}V_{\sigma(i)},V_{\sigma(i)}=1+T\left(\tilde{p}_{\sigma(i_j)}\right) \tag{3.18}$$

and $T\left(\tilde{p}_{\sigma(i_j)}\right)$ denotes the support of the jth largest picture 2-tuple linguistic numbers $T\left(\tilde{p}_{\sigma(i_j)}\right)$ by all the other picture 2-tuple linguistic numbers, *i.e.*,

$$T\left(\tilde{p}_{\sigma(i_j)}\right) = \sum_{\substack{j=1\\ j\neq t}}^{n} \mathrm{Sup}\left(\tilde{p}_{\sigma(i_j)}, \tilde{p}_{\sigma(i_t)}\right), \tag{3.19}$$

where $\sum_{\substack{j=1\\ j\neq t}}^{n} \mathrm{Sup}\left(\tilde{p}_{\sigma(i_j)}, \tilde{p}_{\sigma(i_t)}\right)$ indicates the support of jth largest picture 2-tuple linguistic number $\tilde{p}_{\sigma(i_j)}$ for the tth largest picture 2-tuple linguistic number $\tilde{p}_{\sigma(i_t)}$, and g: $[0,1] \to [0,1]$ is a basic unit-interval monotonic (BUM) function, having the properties: $g(0) = 0$, $g(1) = 1$, and $g(x) \geq g(y)$, if $x > y$.

It can be easily proved that the P2TLPOWHM operator has the following properties.

Theorem 3.9 (Idempotency)**.** *If all $\tilde{p}_j$ $(j = 1, 2, \cdots, n)$ are equal, i.e. $\tilde{p}_j = \tilde{p}$ for all j, then*

$$\mathit{P2TLPOWHM}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \tilde{p}. \tag{3.20}$$

Theorem 3.10 (Boundedness)**.** *Let $\tilde{p}_j$ $(j = 1, 2, \cdots, n)$ be a collection of P2TLNs, and let*

$$\tilde{p}^- = \min_j \tilde{p}_j, \quad \tilde{p}^+ = \max_j \tilde{p}_j.$$

Then

$$\tilde{p}^- \leq \mathit{P2TLPOWHM}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \leq \tilde{p}^+. \tag{3.21}$$

Theorem 3.11 (Monotonicity)**.** *Let $\tilde{p}_j$ $(j = 1, 2, \cdots, n)$ and $\tilde{p}'_j$ $(j = 1, 2, \cdots, n)$ be two set of P2TLNs, if $\tilde{p}_j \leq \tilde{p}'_j$, for all j, then*

$$\mathit{P2TLPOWHM}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \leq \mathit{P2TLPOWHM}(\tilde{p}'_1, \tilde{p}'_2, \cdots, \tilde{p}'_n). \tag{3.22}$$

Theorem 3.12 (Commutativity)**.** *Let $\tilde{p}_j$ $(j = 1, 2, \cdots, n)$ and $\tilde{p}'_j$ $(j = 1, 2, \cdots, n)$ be two set of P2TLNs, for all j, then*

$$\mathit{P2TLOWHM}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \mathit{P2TLOWHM}(\tilde{p}'_1, \tilde{p}'_2, \cdots, \tilde{p}'_n), \tag{3.23}$$

where $\tilde{p}'_j$ $(j = 1, 2, \cdots, n)$ is any permutation of $\tilde{p}_j$ $(j = 1, 2, \cdots, n)$.

From Definitions 3.3 to 3.8, we know that the P2TLPWHM operators only weights the picture 2-tuple linguistic number itself, while the P2TLPOWHM operators weights the ordered positions of the picture 2-tuple linguistic number instead of weighting the arguments itself. Therefore, the weights represent two different aspects in both the P2TLPWHM and P2TLPOWHM operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the picture 2-tuple linguistic power hybrid Hamy mean (P2TLPHHM) operator.

Definition 3.13. Let $\tilde{p}_j = \langle (r_j, \alpha_j), (\mu_j, \eta_j, \nu_j) \rangle \; (j = 1, 2, \cdots, n)$ be a collection of P2TLNs. A picture 2-tuple linguistic power hybrid Hamy mean (P2TLPHHM) operator is a mapping P2TLPHHM: $P^n \rightarrow P$, such that

$$
\begin{aligned}
&\mathrm{P2TLPHHM}\left(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n\right) = \frac{\underset{1 \le i_1 < \ldots < i_k \le n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(\frac{n w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)} \dot{\tilde{p}}_{\sigma(i_j)}\right)\right)^{\frac{1}{k}}}{C_n^k} \\
&= \left\langle \Delta\left(\frac{1}{C_n^k} \sum_{1 \le i_1 < \ldots < i_k \le n}\left(\prod_{j=1}^{k} \frac{n w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right) \Delta^{-1}\left(\dot{r}_{\sigma(j)}, \dot{\alpha}_{\sigma(j)}\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}\right)^{\frac{1}{k}}\right), \right. \\
&\left(1 - \prod_{1 \le i_1 < \ldots < i_k \le n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(1 - \dot{\mu}_{\sigma(j)}\right)^{\frac{n w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}}\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{C_n^k}}, \\
&\prod_{1 \le i_1 < \ldots < i_k \le n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(\dot{\eta}_{\sigma(j)}\right)^{\frac{n w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}, \\
&\left. \prod_{1 \le i_1 < \ldots < i_k \le n}\left(1 - \prod_{j=1}^{k}\left(1 - \left(\dot{\nu}_{\sigma(j)}\right)^{\frac{n w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}} \right\rangle
\end{aligned}
\tag{3.24}
$$

where $w = (w_1, w_2, \cdots, w_n)$ is the associated weighting vector, with $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, and $\dot{\tilde{p}}_{\sigma(i_j)}$ is the jth largest element of the picture 2-tuple linguistic arguments $\dot{\tilde{p}}_{i_j}$ $\left(\dot{\tilde{p}}_{i_j} = (n\omega_j)\, \tilde{p}_{i_j},\ j = 1, 2, \cdots, n\right)$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weighting vector of picture 2-tuple linguistic arguments $\tilde{p}_{i_j}\,(j = 1, 2, \cdots, n)$, with $\omega_i \in [0,1]$, $\sum_{i=1}^{n} \omega_i = 1$, and n is the balancing coefficient. And $w_j\,(j = 1, 2, \cdots, n)$ is collection of weights such that

$$
w_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), R_j = \sum_{i=1}^{j} V_{\sigma(i)},\ TV = \sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)} = 1 + T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right) \tag{3.25}
$$

and $T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)$ denotes the support of the jth largest picture 2-tuple linguistic numbers $T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)$ by all the other picture 2-tuple linguistic numbers, *i.e.*,

$$
T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right) = \sum_{\substack{j=1 \\ i \neq t}}^{n} \mathrm{Sup}\left(\dot{\tilde{p}}_{\sigma(i_j)}, \dot{\tilde{p}}_{\sigma(i_t)}\right), \tag{3.26}
$$

where $\sum_{\substack{j=1 \\ i \neq t}}^{n} \mathrm{Sup}\left(\dot{\tilde{p}}_{\sigma(i_j)}, \dot{\tilde{p}}_{\sigma(i_t)}\right)$ indicates the support of jth largest picture 2-tuple linguistic number $\dot{\tilde{p}}_{\sigma(i_j)}$ for the jth largest picture 2-tuple linguistic number $\dot{\tilde{p}}_{\sigma(i_j)}$, and g: $[0,1] \rightarrow [0,1]$ is a basic unit-interval monotonic (BUM) function, having the properties: $g(0) = 0$, $g(1) = 1$, and $g(x) \geq g(y)$, if $x > y$. Especially, if $w = (1/n, 1/n, \cdots, 1/n)$, then P2TLPHHM is reduced to the picture 2-tuple linguistic power weighted Hamy mean (P2TLPWHM) operator; if $\omega = (1/n, 1/n, \cdots, 1/n)^T$, then P2TLPHHM is reduced to the picture 2-tuple linguistic power ordered weighted Hamy mean (P2TLPOWHM) operator.

4. Picture 2-tuple linguistic power geometric aggregation operators

Based on the PA operator [50] and geometric mean, in the following, Xu and Yager [51] further defined a power geometric (PG) operator:

$$\mathrm{PG}\left(a_1, a_2, \cdots, a_n\right)=\prod_{i=1}^n a_i^{\frac{1+T\left(a_i\right)}{\sum_{i=1}^n\left(1+T\left(a_i\right)\right)}}. \tag{4.1}$$

Obviously, the PA and PG operators are two nonlinear weighted aggregation tools, whose weighting vectors depend upon the input values and allow values being aggregated to support and reinforce each other, that's to say, the closer a_i and a_j, the more similar they are, and the more they support each other.

Wu *et al.* [52] proposed the dual Hamy mean (DHM) operator.

Definition 4.1. The DHM operator is defined as follows:

$$\mathrm{DHM}^{(k)}\left(\delta_1, \delta_2, \cdots, \delta_n\right)=\left(\prod_{1 \leq i_1<\ldots<i_k \leq n}\left(\frac{\sum_{j=1}^k \delta_{i_j}}{k}\right)\right)^{\frac{1}{C_n^k}}, \tag{4.2}$$

where k is a parameter and $k=1,2,\ldots,n$, $i_1,i_2,\ldots,i_k$ are k integer values taken from the set $\{1,2,\ldots,n\}$ of k integer values, C_n^k denotes the binomial coefficient and $C_n^k=\frac{n!}{k!(n-k)!}$.

In this section, we shall develop some dual Hamy mean aggregation operators with picture 2-tuple linguistic information and power operation laws, such as picture 2-tuple linguistic power dual Hamy mean (P2TLPDHM) operator, picture 2-tuple linguistic power weighted dual Hamy mean (P2TLPWDHM)) operator, picture 2-tuple linguistic power ordered weighted dual Hamy mean (P2TLPOWDHM) operator and picture 2-tuple linguistic power hybrid dual Hamy mean (P2TLPHDHM) operator.

Definition 4.2. Let $\tilde{p}_j=\langle(r_j,\alpha_j),(\mu_j,\eta_j,\nu_j)\rangle\ (j=1,2,\cdots,n)$ be a collection of P2TLNs. The picture 2-tuple linguistic power dual Hamy mean (P2TLPDHM) operator is a mapping $P^n \rightarrow P$ such that

$$\mathrm{P2TLPDHM}^{(k)}\left(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n\right)=\left(\underset{1 \leq i_1<\ldots<i_k \leq n}{\otimes}\left(\frac{1}{k}\left(\overset{k}{\underset{j=1}{\oplus}}\left(\left(\tilde{p}_{i_j}\right) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)\right)\right)\right)^{\frac{1}{C_n^k}}, \tag{4.3}$$

where

$$T\left(\tilde{p}_{i_j}\right)=\sum_{\substack{j=1 \\ j \neq t}}^n \operatorname{Sup}\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right) \tag{4.4}$$

and $\operatorname{Sup}\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right)$ is the support for $\tilde{p}_{i_j}$ from $\tilde{p}_{i_t}$, with the conditions:

(1) $\operatorname{Sup}\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right) \in [0,1]$;
(2) $\operatorname{Sup}\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right)=\operatorname{Sup}\left(\tilde{p}_{i_t}, \tilde{p}_{i_j}\right)$;
(3) $\operatorname{Sup}\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right) \geq \operatorname{Sup}\left(\tilde{p}_s, \tilde{p}_t\right)$, if $d\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right)<d\left(\tilde{p}_s, \tilde{p}_t\right)$, where d is a distance measure.

Based on Definition 4.2 and Theorem 2.13, we can get the following result:

Theorem 4.3. *The aggregated value by using P2TLPDHM operator is also a picture 2-tuple linguistic numbers, where*

$$\begin{aligned}
&P2TLPDHM(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \left(\underset{1 \le i_1 < \ldots < i_k \le n}{\otimes} \left(\frac{1}{k} \left(\overset{k}{\underset{j=1}{\oplus}} \left((\tilde{p}_{i_j}) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right) \right) \right) \right)^{\frac{1}{C_n^k}} \\
&= \left\langle \Delta \left(\prod_{1 \le i_1 < \ldots < i_k \le n} \left(\frac{1}{k} \sum_{j=1}^{k} \left(\Delta^{-1}(r_j, \alpha_j)\right) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right)^{\frac{1}{C_n^k}} \right), \right. \\
&\prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - (\mu_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right)^{\frac{1}{k}} \right)^{\frac{1}{C_n^k}}, 1 - \prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - (1-\eta_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right)^{\frac{1}{k}} \right)^{\frac{1}{C_n^k}}, \\
&\left. 1 - \prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - (1-\nu_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right)^{\frac{1}{k}} \right)^{\frac{1}{C_n^k}} \right\rangle
\end{aligned} \tag{4.5}$$

where

$$T\left(\tilde{p}_{i_j}\right) = \sum_{\substack{j=1 \\ j \ne t}}^{n} \operatorname{Sup}\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right). \tag{4.6}$$

Proof.

$$\begin{aligned}
(\tilde{p}_{i_j}) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} = \left\langle \Delta \left(\left(\Delta^{-1}(r_j, \alpha_j)\right) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right), \right. \\
\left. \left((\mu_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)},\ 1 - (1-\eta_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}, 1 - (1-\nu_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right) \right\rangle.
\end{aligned} \tag{4.7}$$

Thus,

$$\begin{aligned}
&\overset{k}{\underset{j=1}{\oplus}} \left((\tilde{p}_{i_j}) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right) = \left\langle \Delta \left(\sum_{j=1}^{k} \left(\Delta^{-1}(r_j, \alpha_j)\right) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right), \right. \\
&\left. \left(1 - \prod_{j=1}^{k} \left(1 - (\mu_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right), \prod_{j=1}^{k} \left(1 - (1-\eta_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right), \prod_{j=1}^{k} \left(1 - (1-\nu_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right) \right) \right\rangle.
\end{aligned} \tag{4.8}$$

Thereafter,

$$\begin{aligned}
&\frac{1}{k} \left(\overset{k}{\underset{j=1}{\oplus}} \left((\tilde{p}_{i_j}) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right) \right) = \left\langle \Delta \left(\frac{1}{k} \sum_{j=1}^{k} \left(\Delta^{-1}(r_j, \alpha_j)\right) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right), \right. \\
&\left. \left(1 - \prod_{j=1}^{k} \left(1 - (\mu_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right)^{\frac{1}{k}}, \prod_{j=1}^{k} \left(1 - (1-\eta_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right)^{\frac{1}{k}}, \prod_{j=1}^{k} \left(1 - (1-\nu_j) \frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)} \right)^{\frac{1}{k}} \right) \right\rangle.
\end{aligned} \tag{4.9}$$

Furthermore,

$$\underset{1\le i_1<\ldots<i_k\le n}{\otimes}\left(\frac{1}{k}\left(\overset{k}{\underset{j=1}{\oplus}}\left(\left(\tilde{p}_{i_j}\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)\right)\right)=\left\langle\Delta\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(\frac{1}{k}\sum_{j=1}^{k}\left(\Delta^{-1}\left(r_j,\alpha_j\right)\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)\right),\right.$$
$$\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\mu_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right),1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right),$$
$$\left.1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\nu_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)\right\rangle \tag{4.10}$$

Therefore,

$$\text{P2TLPDHM}\left(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n\right)=\left(\underset{1\le i_1<\ldots<i_k\le n}{\otimes}\left(\frac{1}{k}\left(\overset{k}{\underset{j=1}{\oplus}}\left(\left(\tilde{p}_{i_j}\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)\right)\right)\right)^{\frac{1}{C_n^k}}$$
$$=\left\langle\Delta\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(\frac{1}{k}\sum_{j=1}^{k}\left(\Delta^{-1}\left(r_j,\alpha_j\right)\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{C_n^k}}\right),\right.$$
$$\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\mu_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},$$
$$\left.1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\nu_j\right)\frac{n\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right)\right\rangle. \tag{4.11}$$

Hence, (41) is kept. □

Definition 4.4. Let $\tilde{p}_j = \langle(r_j,\alpha_j),(\mu_j,\eta_j,\nu_j)\rangle\,(j=1,2,\cdots,n)$ be a collection of P2TLNs, $\omega = (\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $\tilde{p}_j\,(j=1,2,\cdots,n)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$. The picture 2-tuple linguistic power weighted dual Hamy mean (P2TLPWDHM) operator is a mapping $P^n\to P$ such that

$$\text{P2TLPWDHM}^{(k)}\left(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n\right)=\left(\underset{1\le i_1<\ldots<i_k\le n}{\otimes}\left(\frac{1}{k}\left(\overset{k}{\underset{j=1}{\oplus}}\left(\left(\tilde{p}_{i_j}\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)\right)\right)\right)^{\frac{1}{C_n^k}}, \tag{4.12}$$

where

$$T\left(\tilde{p}_{i_j}\right)=\omega_j\sum_{\substack{j=1\\ j\neq t}}^{n}\text{Sup}\left(\tilde{p}_{i_j},\tilde{p}_{i_t}\right). \tag{4.13}$$

Based on Definition 4.4, Theorem 2.13 and mathematical induction on n, we can get the following result:

Theorem 4.5. *The aggregated value by using P2TLPWDHM operator is also a picture 2-tuple linguistic numbers, where*

$$\begin{aligned}
&P2TLPWDHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)=\left(\underset{1\le i_1<\ldots<i_k\le n}{\otimes}\left(\frac{1}{k}\left(\overset{k}{\underset{j=1}{\oplus}}\left(\left(\tilde{p}_{i_j}\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)\right)\right)\right)^{\frac{1}{C_n^k}}\\
&\left\langle\Delta\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(\frac{1}{k}\sum_{j=1}^{k}\left(\Delta^{-1}\left(r_j,\alpha_j\right)\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{C_n^k}}\right),\right.\\
&\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(\mu_j\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\\
&1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_j\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}},\\
&\left.1-\prod_{1\le i_1<\ldots<i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\nu_j\right)\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right)\right\rangle
\end{aligned}\tag{4.14}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ *be the weight vector of* $\tilde{p}_{i_j}$ $(j=1,2,\cdots,n)$, *and* $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$.

It can be easily proved that the P2TLPWDHM operator has the following properties.

Theorem 4.6 (Idempotency)**.** *If all* $\tilde{p}_j$ $(j=1,2,\cdots,n)$ *are equal, i.e.* $\tilde{p}_j=\tilde{p}$ *for all* j, *then*

$$P2TLPWDHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)=\tilde{p}. \tag{4.15}$$

Theorem 4.7 (Boundedness)**.** *Let* $\tilde{p}_j$ $(j=1,2,\cdots,n)$ *be a collection of P2TLNs, and let*

$$\tilde{p}^-=\min_j \tilde{p}_j,\quad \tilde{p}^+=\max_j \tilde{p}_j.$$

Then

$$\tilde{p}^-\le P2TLPWDHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)\le\tilde{p}^+. \tag{4.16}$$

Theorem 4.8 (Monotonicity)**.** *Let* $\tilde{p}_j$ $(j=1,2,\cdots,n)$ *and* $\tilde{p}'_j$ $(j=1,2,\cdots,n)$ *be two set of P2TLNs, if* $\tilde{p}_j\le\tilde{p}'_j$, *for all* j, *then*

$$P2TLPWDHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n)\le P2TLPWDHM(\tilde{p}'_1,\tilde{p}'_2,\cdots,\tilde{p}'_n). \tag{4.17}$$

Further, we give a picture 2-tuple linguistic power ordered weighted dual Hamy mean (P2TLPOWDHM) operator below:

Definition 4.9. Let $\tilde{p}_j=\langle(r_j,\alpha_j),(\mu_j,\eta_j,\nu_j)\rangle$ $(j=1,2,\cdots,n)$ be a collection of P2TLNs, the picture 2-tuple linguistic power ordered weighted dual Hamy mean (P2TLPOWDHM) operator of dimension n is a mapping P2TLPOWDHM: $P^n\to P$, that has an associated weight vector $w=(w_1,w_2,\cdots,w_n)^T$ such that $w_j>0$ and

$\sum_{j=1}^{n} w_j = 1$. Furthermore,

$$\begin{aligned}
&\text{P2TLPOWDHM}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \left(\underset{1 \le i_1 < \ldots < i_k \le n}{\otimes} \left(\frac{1}{k} \left(\overset{k}{\underset{j=1}{\oplus}} \left(\left(\tilde{p}_{\sigma(i_j)}\right) \frac{n w_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum\limits_{j=1}^{n} \omega_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)} \right) \right) \right) \right)^{\frac{1}{C_n^k}} \\
&= \left\langle \Delta \left(\prod_{1 \le i_1 < \ldots < i_k \le n} \left(\frac{1}{k} \sum_{j=1}^{k} \left(\Delta^{-1}\left(r_{\sigma(j)}, \alpha_{\sigma(j)} \right) \right) \frac{n w_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum\limits_{j=1}^{n} \omega_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)} \right)^{\frac{1}{C_n^k}} \right), \right. \\
&\prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(\mu_{\sigma(j)}\right) \frac{n w_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum\limits_{j=1}^{n} \omega_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)} \right)^{\frac{1}{k}} \right)^{\frac{1}{C_n^k}}, \qquad (4.18) \\
&1 - \prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \eta_{\sigma(j)}\right) \frac{n w_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum\limits_{j=1}^{n} \omega_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)} \right)^{\frac{1}{k}} \right)^{\frac{1}{C_n^k}}, \\
&\left. 1 - \prod_{1 \le i_1 < \ldots < i_k \le n} \left(1 - \prod_{j=1}^{k} \left(1 - \left(1 - \nu_{\sigma(j)}\right) \frac{n w_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)}{\sum\limits_{j=1}^{n} \omega_j \left(1+T\left(\tilde{p}_{\sigma(i_j)}\right)\right)} \right)^{\frac{1}{k}} \right)^{\frac{1}{C_n^k}} \right) \right\rangle
\end{aligned}$$

where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{p}_{\sigma(i_{j-1})} \ge \tilde{p}_{\sigma(i_j)}$ for all $j = 2, \cdots, n$, $w_j\ (j = 1, 2, \cdots, n)$ is collection of weights such that

$$w_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), R_j = \sum_{i=1}^{j} V_{\sigma(i)},\ TV = \sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)} = 1 + T\left(\tilde{p}_{\sigma(i_j)}\right) \qquad (4.19)$$

and $T\left(\tilde{p}_{\sigma(i_j)}\right)$ denotes the support of the jth largest picture 2-tuple linguistic numbers $T\left(\tilde{p}_{\sigma(i_j)}\right)$ by all the other picture 2-tuple linguistic numbers, *i.e.*,

$$T\left(\tilde{p}_{i_j}\right) = \sum_{\substack{j=1 \\ j \ne t}}^{n} \text{Sup}\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right) \qquad (4.20)$$

where $\sum_{\substack{j=1 \\ j \ne t}}^{n} \text{Sup}\left(\tilde{p}_{i_j}, \tilde{p}_{i_t}\right)$ indicates the support of jth largest picture 2-tuple linguistic number $\tilde{p}_{\sigma(i_j)}$ for the tth largest picture 2-tuple linguistic number $\tilde{p}_{\sigma(i_t)}$, and g: $[0,1] \to [0,1]$ is a basic unit-interval monotonic (BUM) function, having the properties: $g(0) = 0$, $g(1) = 1$, and $g(x) \ge g(y)$, if $x > y$.

It can be easily proved that the P2TLPOWDHM operator has the following properties.

Theorem 4.10 (Idempotency)**.** *If all $\tilde{p}_j\ (j = 1, 2, \cdots, n)$ are equal, i.e. $\tilde{p}_j = \tilde{p}$ for all j, then*

$$P2TLPOWDHM(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \tilde{p}. \qquad (4.21)$$

Theorem 4.11 (Boundedness)**.** *Let $\tilde{p}_j\ (j = 1, 2, \cdots, n)$ be a collection of P2TLNs, and let*

$$\tilde{p}^- = \min_j \tilde{p}_j, \quad \tilde{p}^+ = \max_j \tilde{p}_j.$$

Then

$$\tilde{p}^- \le P2TLPOWDHM(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) \le \tilde{p}^+. \qquad (4.22)$$

Theorem 4.12 (Monotonicity). *Let $\tilde{p}_j\,(j=1,2,\cdots,n)$ and $\tilde{p}'_j\,(j=1,2,\cdots,n)$ be two set of P2TLNs, if $\tilde{p}_j \leq \tilde{p}'_j$, for all j, then*

$$P2TLPOWDHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n) \leq P2TLPOWDHM(\tilde{p}'_1,\tilde{p}'_2,\cdots,\tilde{p}'_n)\,. \tag{4.23}$$

Theorem 4.13 (Commutativity). *Let $\tilde{p}_j\,(j=1,2,\cdots,n)$ and $\tilde{p}'_j\,(j=1,2,\cdots,n)$ be two set of P2TLNs, for all j, then*

$$P2TLPOWDHM(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n) = P2TLPOWDHM(\tilde{p}'_1,\tilde{p}'_2,\cdots,\tilde{p}'_n)\,, \tag{4.24}$$

where $\tilde{p}'_j\,(j=1,2,\cdots,n)$is any permutation of $\tilde{p}_j\,(j=1,2,\cdots,n)$.

From Definitions 4.4 to 4.9, we know that the P2TLPWDHM operators only weights the picture 2-tuple linguistic number itself, while the P2TLPOWDHM operators weights the ordered positions of the picture 2-tuple linguistic number instead of weighting the arguments itself. Therefore, the weights represent two different aspects in both the P2TLPWDHM and P2TLPOWDHM operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the picture 2-tuple linguistic power hybrid dual Hamy mean (P2TLPHDHM) operator.

Definition 4.14. A picture 2-tuple linguistic power hybrid dual Hamy mean (P2TLPHDHM) operator is a mapping P2TLPHDHM: $P^n \to P$, such that

$$\begin{aligned}
&\text{P2TLPHDHM}(\tilde{p}_1,\tilde{p}_2,\cdots,\tilde{p}_n) = \left(\underset{1\leq i_1<\ldots<i_k\leq n}{\otimes}\left(\frac{1}{k}\left(\overset{k}{\underset{j=1}{\oplus}}\left(\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\frac{nw_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}\right)\right)\right)\right)^{\frac{1}{C_n^k}} \\
&= \left\langle \Delta\left(\prod_{1\leq i_1<\ldots<i_k\leq n}\left(\frac{1}{k}\sum_{j=1}^{k}\left(\Delta^{-1}\left(\dot{r}_{\sigma(j)},\dot{\alpha}_{\sigma(j)}\right)\right)\frac{nw_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}\right)^{\frac{1}{C_n^k}}\right),\right. \\
&\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(\dot{\mu}_{\sigma(j)}\right)\frac{nw_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}, \\
&1-\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\dot{\eta}_{\sigma(j)}\right)\frac{nw_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}, \\
&\left.1-\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\dot{\nu}_{\sigma(j)}\right)\frac{nw_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}{\sum_{j=1}^{n} w_j\left(1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)\right)}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right\rangle
\end{aligned} \tag{4.25}$$

where $w=(w_1,w_2,\cdots,w_n)$ is the associated weighting vector, with $w_j\in[0,1]$, $\sum_{j=1}^{n} w_j=1$, and $\dot{\tilde{p}}_{\sigma(i_j)}$ is the jth largest element of the picture 2-tuple linguistic arguments $\dot{\tilde{p}}_{i_j}\left(\dot{\tilde{p}}_{i_j}=\left(\tilde{p}_{i_j}\right)^{n\omega_j},\ j=1,2,\cdots,n\right)$, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ is the weighting vector of picture 2-tuple linguistic arguments $\tilde{p}_{i_j}\,(j=1,2,\cdots,n)$, with $\omega_j\in[0,1]$, $\sum_{j=1}^{n}\omega_j=1$, and n is the balancing coefficient. And $w_j\,(j=1,2,\cdots,n)$ is collection of weights such that

$$w_j = g\left(\frac{R_j}{TV}\right) - g\left(\frac{R_{j-1}}{TV}\right), R_j=\sum_{i=1}^{j} V_{\sigma(i)},\ TV=\sum_{i=1}^{n} V_{\sigma(i)}, V_{\sigma(i)}=1+T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right) \tag{4.26}$$

and $T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)$ denotes the support of the jth largest picture 2-tuple linguistic numbers $T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right)$ by all the other picture 2-tuple linguistic numbers, *i.e.*,

$$T\left(\dot{\tilde{p}}_{\sigma(i_j)}\right) = \sum_{\substack{j=1\\ j\neq t}}^{n} \mathrm{Sup}\left(\dot{\tilde{p}}_{\sigma(i_j)}, \dot{\tilde{p}}_{\sigma(i_t)}\right) \tag{4.27}$$

where $\sum_{\substack{j=1\\ j\neq t}}^{n} \mathrm{Sup}\left(\dot{\tilde{p}}_{\sigma(i_j)}, \dot{\tilde{p}}_{\sigma(i_t)}\right)$ indicates the support of jth largest picture 2-tuple linguistic number $\dot{\tilde{p}}_{\sigma(i_j)}$ for the tth largest picture 2-tuple linguistic number $\dot{\tilde{p}}_{\sigma(i_t)}$, and g: $[0,1] \to [0,1]$ is a basic unit-interval monotonic (BUM) function, having the properties: $g(0) = 0$, $g(1) = 1$, and $g(x) \geq g(y)$, if $x > y$. Especially, if $w = (1/n, 1/n, \cdots, 1/n)^T$, then P2TLPHDHM is reduced to the picture 2-tuple linguistic power weighted dual Hamy mean (P2TLPWDHM) operator; if $\omega = (1/n, 1/n, \cdots, 1/n)$, then P2TLPHDHM is reduced to the picture 2-tuple linguistic power ordered weighted dual Hamy mean (P2TLPOWDHM) operator.

5. Approaches to multiple attribute decision making with picture 2-tuple linguistic information

Based the P2TLPWHM (P2TLPWDHM) operators, in this section, we shall propose the MADM model with picture 2-tuple linguistic information. Let $A = \{A_1, A_2, \cdots, A_m\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, \cdots, G_n\}$ be the set of attributes, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weighting vector of the attribute $G_j\ (j = 1, 2, \cdots, n)$, where $\omega_j \in [0,1]$, $\sum_{j=1}^{n} \omega_j = 1$. Suppose that $\tilde{P} = (\tilde{p}_{ij})_{m\times n} = \langle (s_{ij}, \rho_{ij}), (\mu_{ij}, \eta_{ij}, \nu_{ij})\rangle_{m\times n}$ is the picture 2-tuple linguistic decision matrix, where $\tilde{r}_{ij}$ take the form of the picture 2-tuple linguistic numbers, and μ_{ij} indicates the degree of positive membership that the alternative A_i satisfies the attribute G_j given by the decision maker, η_{ij} indicates the degree of neutral membership that the alternative A_i doesn't satisfy the attribute G_j, ν_{ij} indicates the degree that the alternative A_i doesn't satisfy the attribute G_j given by the decision maker, $\mu_{ij} \in [0,1]$, $\eta_{ij} \in [0,1]$ $\quad \nu_{ij} \in [0,1]$, $\mu_{ij} + \eta_{ij} + \nu_{ij} \leq 1$, $\pi_{ij} = 1 - (\mu_{ij} + \eta_{ij} + \nu_{ij})$, $s_{ij} \in S$, $\rho_{ij} \in [-0.5, 0.5)$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

In the following, we apply the P2TLPWHM (P2TLPWDHM) operator to the MADM problems with hesitant fuzzy information.

Step 1. Calculate the supports:

$$\mathrm{Sup}\left(\tilde{p}_{ij}, \tilde{p}_{ik}\right) = 1 - d\left(\tilde{p}_{ij}, \tilde{p}_{ik}\right), \quad j, k = 1, 2, \cdots, n. \tag{5.1}$$

which satisfies the support conditions (1)–(3) in Section 3. Here, without loss of generality, we calculate $d\left(\tilde{p}_{ij}, \tilde{p}_{ik}\right)$ with the normalized Hamming distance [9]:

$$d\left(\tilde{p}_{ij}, \tilde{p}_{ik}\right) = \frac{\left|\Delta^{-1}\left(s_{ij}, \rho_{ij}\right) - \Delta^{-1}\left(s_{ik}, \rho_{ik}\right)\right|}{t} \cdot \frac{\left(\left|\mu_{ij} - \mu_{ik}\right| + \left|\eta_{ij} - \eta_{ik}\right| + \left|\nu_{ij} - \nu_{ik}\right|\right)}{2} \quad j, k = 1, 2, \cdots, n. \tag{5.2}$$

Step 2. Utilize the weights $\omega_j\ (j = 1, 2, \cdots, n)$ of the attribute $G_j\ (j = 1, 2, \cdots, n)$ to calculate the weighted support $T\left(\tilde{p}_{ij}\right)$ of the P2TLN $\tilde{p}_{ij}$ by the other P2TLN $\tilde{p}_{ik}\ (k = 1, 2, \cdots, n,\ k \neq j)$:

$$T\left(\tilde{p}_{ij}\right) = \sum_{\substack{k=1\\ k\neq j}}^{n} \omega_j \mathrm{Sup}\left(\tilde{p}_{ij}, \tilde{p}_{ik}\right) \tag{5.3}$$

and calculate the weight $\xi_{ij}\ (j = 1, 2, \cdots, n)$ associated with the P2TLN $\tilde{p}_{ij}\ (i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$:

$$\xi_{ij} = \frac{n\omega_j\left(1 + T\left(\tilde{p}_{ij}\right)\right)}{\sum_{j=1}^{n} \omega_j\left(1 + T\left(\tilde{p}_{ij}\right)\right)}, \quad i = 1, 2, \cdots, m, j = 1, 2, \cdots, n. \tag{5.4}$$

where $\xi_{ij} \geq 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} \xi_{ij} = 1, i = 1, 2, \cdots, m$.
Step 3. We utilize the decision information given in matrix $\tilde{P}$, and the P2TLPWHM operator

$$\begin{aligned}
&\text{P2TLPWHM}\left(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n\right) \\
&= \frac{\underset{1\leq i_1<\ldots<i_k\leq n}{\oplus}\left(\overset{k}{\underset{j=1}{\otimes}}\left(\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\tilde{p}_{i_j}\right)\right)^{\frac{1}{k}}}{C_n^k} \\
&= \left\langle \Delta\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\left(\prod_{j=1}^{k}\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)\Delta^{-1}\left(r_j,\alpha_j\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}\right)^{\frac{1}{k}}\right),\right. \\
&\left(1-\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\mu_j\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right), \\
&\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(\eta_j\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}, \\
&\left.\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(\nu_j\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right\rangle
\end{aligned} \tag{5.5}$$

or

$$\begin{aligned}
&\text{P2TLPWDHM}\left(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n\right) \\
&= \left(\underset{1\leq i_1<\ldots<i_k\leq n}{\otimes}\left(\frac{1}{k}\left(\overset{k}{\underset{j=1}{\oplus}}\left(\left(\tilde{p}_{i_j}\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)\right)\right)\right)^{\frac{1}{C_n^k}} \\
&= \left\langle \Delta\left(\prod_{1\leq i_1<\ldots<i_k\leq n}\left(\frac{1}{k}\sum_{j=1}^{k}\left(\Delta^{-1}\left(r_j,\alpha_j\right)\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{C_n^k}}\right),\right. \\
&\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(\mu_j\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}, \\
&1-\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\eta_j\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}, \\
&\left.1-\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-\left(1-\nu_j\right)^{\frac{n\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\tilde{p}_{i_j}\right)\right)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right\rangle
\end{aligned} \tag{5.6}$$

to derive the overall preference values $\tilde{p}_i\,(i = 1, 2, \cdots, m)$ of the alternative A_i.
Step 4. Calculate the scores $S(\tilde{p}_i)$ $(i = 1, 2, \cdots, m)$ of the overall picture 2-tuple linguistic numbers $\tilde{p}_i\,(i = 1, 2, \cdots, m)$to rank all the alternatives $A_i\,(i = 1, 2, \cdots, m)$ and then to select the best one(s). If there is no difference between two scores $S(\tilde{p}_i)$ and $S(\tilde{p}_j)$, then we need to calculate the accuracy degrees $H(\tilde{p}_i)$ and $H(\tilde{p}_j)$ of the overall picture 2-tuple linguistic numbers $\tilde{p}_i$ and $\tilde{p}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(\tilde{p}_i)$ and $H(\tilde{p}_j)$.

TABLE 1. The picture 2-tuple linguistic decision matrix.

	G_1	G_2
A_1	<(S_3, 0), (0.43, 0.23, 0.19)>	<(S_2, 0), (0.35, 0.12, 0.17)>
A_2	<(S_4, 0), (0.63, 0.12, 0.18)>	<(S_3, 0), (0.46, 0.24, 0.12)>
A_3	<(S_2, 0), (0.71, 0.13, 0.12)>	<(S_1, 0), (0.57, 0.12, 0.15)>
A_4	<(S_5, 0), (0.65, 0.12, 0.15)>	<(S_4, 0), (0.74, 0.07, 0.12)>
A_5	<(S_1, 0), (0.70, 0.15, 0.12)>	<(S_2, 0), (0.55, 0.12, 0.13)>
	G3	G4
A_1	<(S1, 0), (0.42, 0.35, 0.18)>	<(S4, 0), (0.37, 0.12, 0.16)>
A_2	<(S2, 0), (0.53, 0.12, 0.14)>	<(S2, 0), (0.54, 0.16, 0.18)>
A_3	<(S3, 0), (0.64, 0.15, 0.16)>	<(S1, 0), (0.48, 0.13, 0.16)>
A_4	<(S5, 0), (0.67, 0.09, 0.15)>	<(S5, 0), (0.78, 0.04, 0.05)>
A_5	<(S4, 0), (0.47, 0.22, 0.13)>	<(S3, 0), (0.67, 0.15, 0.17)>

Step 5. Rank all the alternatives $A_i\,(i = 1, 2, \cdots, m)$ and select the best one(s) in accordance with $S\left(\tilde{p}_i\right)\ (i = 1, 2, \cdots, m)$.
Step 6. End.

6. Numerical example and comparative analysis

6.1. Numerical example

In this section, we utilize a practical MADM problem to illustrate the application of the developed approaches. Suppose an organization plans to implement enterprise resource planning (ERP) system. The first step is to form a project team that consists of chief information officer (CIO) and two senior representatives from user departments. By collecting all possible information about ERP vendors and systems, project term choose five potential ERP systems $A_i\,(i = 1, 2, \cdots, 5)$ as candidates. The company employs some external professional organizations (or experts) to aid this decision-making. The project team selects four attributes to evaluate the alternatives: (1) function and technology G_1, (2) strategic fitness G_2, (3) vendor's ability G_3, (4) vendor's reputation G_4.

The five possible ERP systems $A_i\,(i = 1, 2, \cdots, 5)$ are to be evaluated using the picture 2-tuple linguistic numbers by the decision makers under the above four attributes (whose weighting vector is $\omega = (0.2, 0.1, 0.3, 0.4)$), and construct the following matrix $\tilde{R} = \left(\tilde{r}_{ij}\right)_{5\times 4}$ is shown in Table 1.

In the following, in order to select the most desirable ERP systems, we utilize the P2TLPWA (P2TLPWG) operator to develop an approach to MADM problems with picture 2-tuple linguistic information, which can be described as following.
Step 1. Utilize (5.1)–(5.4) to calculate the weight $\xi_{ij}\,(i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$ associated with the P2TLN $\tilde{p}_{ij}\,(i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$, which are contained in the matrix $\tilde{R} = \left(\tilde{r}_{ij}\right)_{5\times 4}$ which is shown in Table 1.

In the following, in order to select the most desirable ERP systems, we utilize the P2TLPWA (P2TLPWG) operator to develop an approach to MADM problems with picture 2-tuple linguistic information, which can be described as following.

Step 1 utilize (5.1)–(5.4) to calculate the weight $\xi_{ij}\,(i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$ associated with the P2TLN $\tilde{p}_{ij}\,(i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$, which are contained in the matrix $\tilde{R} = \left(\tilde{r}_{ij}\right)_{5\times 4}$ which is shown as following:

$$\xi = \begin{bmatrix} 0.8570 & 0.4534 & 1.1864 & 1.5032 \\ 0.8409 & 0.4445 & 1.2043 & 1.5103 \\ 0.8483 & 0.4498 & 1.1971 & 1.5047 \\ 0.8475 & 0.4443 & 1.2011 & 1.5071 \\ 0.8413 & 0.4489 & 1.1902 & 1.5195 \end{bmatrix}$$

TABLE 2. The aggregating results of the ERP systems by the P2TLPWHM (P2TLPWDHM) operators.

	P2TLPWHM	P2TLPWDHM
A_1	$\{(S_2, 0.21), (0.885, 0.082, 0.059)\}$	$\{(S_3, -0.21), (0.121, 0.927, 0.895)\}$
A_2	$\{(S_2, 0.44), (0.904, 0.038, 0.057)\}$	$\{(S_2, 0.47), (0.152, 0.860, 0.909)\}$
A_3	$\{(S_2, -0.44), (0.912, 0.047, 0.047)\}$	$\{(S_2, -0.23), (0.155, 0.907, 0.890)\}$
A_4	$\{(S_5, -0.48), (0.902, 0.029, 0.039)\}$	$\{(S_6, -0.39), (0.106, 0.945, 0.928)\}$
A_5	$\{(S_2, 0.26), (0.904, 0.060, 0.045)\}$	$\{(S_3, -0.04), (0.140, 0.909, 0.896)\}$

TABLE 3. The score functions of the ERP systems.

	P2TLPWHM	P2TLPWDHM
A_1	$(S_2, 0.02)$	$(S_0,0.31)$
A_2	$(S_2, 0.26)$	$(S_0, 0.30)$
A_3	$(S_1, 0.45)$	$(S_0, 0.23)$
A_4	$(S_4, 0.21)$	$(S_0, 0.49)$
A_5	$(S_2, 0.10)$	$(S_0, 0.36)$

TABLE 4. Ordering of the ERP systems.

	Ordering
P2TLPWHM	$A_4 > A_2 > A_5 > A_1 > A_3$
P2TLPWDHM	$A_4 > A_5 > A_1 > A_2 > A_3$

Step 2. According to ξ and Table 1, aggregate all picture 2-tuple linguistic numbers $\tilde{r}_{ij}\,(j = 1, 2, \cdots, n)$ by using the P2TLPWHM (P2TLPWDHM) operator to derive the overall picture 2-tuple linguistic numbers $\tilde{p}_i\,(i = 1, 2, 3, 4, 5)$ of the alternative A_i. The aggregating results are shown in Table 2 (Let $k = 2$).
Step 3. According to the aggregating results shown in Table 2 and the score functions of the ERP systems are shown in Table 3.
Step 4. According to the score functions shown in Table 3 and the comparison formula of score functions, the ordering of the ERP systems are shown in Table 4. Note that ">" means "preferred to". As we can see, depending on the aggregation operators used, the ordering of the ERP systems is the same, and the best ERP system is A_4.

6.2. Sensitive Analysis

In order to show the effects on the ranking results by changing parameters of k in the P2TLPWHM (P2TLPWDHM) operators, the results are shown in Tables 5 and 6. It can be seen that: (1) different aggregation operators can result in different results; (2) different values of parameter have some influence on the results. In real MADM problems, the decision makers can choose the different aggregation operators and parameter according to decision makers' personal preference.

6.3. Comparative analysis

Then, we compare the proposed method with picture 2-tuple linguistic weighted average (P2TLWA) operator and picture 2-tuple linguistic weighted geometric (P2TLWG) operator [38]. The comparative results are listed in Table 7.

TABLE 5. Ranking results for different parameters values with P2TLPWHM operator.

	$s(A_1)$	$s(A_2)$	$s(A_3)$	$s(A_4)$	$s(A_5)$	Ordering
$k = 1$	$(S_2, -0.37)$	$(S_2, -0.24)$	$(S_1, 0.31)$	$(S_4, -0.03)$	$(S_2, 0.04)$	$A_4 > A_5 > A_2 > A_1 > A_3$
$k = 2$	$(S_1, 0.28)$	$(S_2, -0.38)$	$(S_1, 0.08)$	$(S_3, 0.41)$	$(S_2, -0.42)$	$A_4 > A_2 > A_5 > A_1 > A_3$
$k = 3$	$(S_1, 0.18)$	$(S_2, -0.44)$	$(S_1, 0.00)$	$(S_3, 0.22)$	$(S_1, 0.42)$	$A_4 > A_2 > A_5 > A_1 > A_3$
$k = 4$	$(S_1, 0.13)$	$(S_2, -0.47)$	$(S_1, -0.04)$	$(S_3, 0.13)$	$(S_1, 0.35)$	$A_4 > A_2 > A_5 > A_1 > A_3$

TABLE 6. Ranking results for different parameters values with P2TLPWDHM operator.

	$s(A_1)$	$s(A_2)$	$s(A_3)$	$s(A_4)$	$s(A_5)$	Ordering
$k = 1$	$(S_1, 0.41)$	$(S_2, -0.33)$	$(S_1, 0.15)$	$(S_4, -0.09)$	$(S_2, -0.2)$	$A_4 > A_5 > A_2 > A_1 > A_3$
$k = 2$	$(S_2, -0.24)$	$(S_2, -0.24)$	$(S_1, 0.31)$	$(S_5, -0.44)$	$(S_2, 0.21)$	$A_4 > A_5 > A_2 > A_1 > A_3$
$k = 3$	$(S_2, -0.05)$	$(S_2, -0.21)$	$(S_1, 0.38)$	$(S_5, -0.20)$	$(S_2, 0.38)$	$A_4 > A_5 > A_1 > A_2 > A_3$
$k = 4$	$(S_1, 0.43)$	$(S_1, -0.10)$	$(S_1, -0.38)$	$(S_2, -0.38)$	$(S_1, 0.06)$	$A_4 > A_1 > A_5 > A_2 > A_3$

TABLE 7. Ordering of the ERP systems.

	Ordering
P2TLWA	$A_4 > A_5 > A_2 > A_1 > A_3$
P2TLWG	$A_4 > A_5 > A_2 > A_1 > A_3$

From above, we can get the same best alternative to show the practicality and effectiveness of the proposed methods. However, P2TLWA operator and P2TLWG operator don't consider the information about the relationship between arguments being aggregated, and thus cannot eliminate the influence of unfair arguments on decision result. Our proposed P2TLPWHM and P2TLPWDHM operators consider the relationship among the arguments.

Deng *et al.* [53] proposed some Hamy mean operators with 2-tuple linguistic Pythagorean fuzzy numbers. But, these operators can only deal with linguistic membership, but these operators can't deal with the picture 2-tuple linguistic numbers (P2TLNs); however, the P2TLPWHM (P2TLPWDHM) operators which are proposed can deal with the picture 2-tuple linguistic numbers (P2TLNs).

And Li *et al.* [54] proposed some Hamy mean operators for Pythagorean Fuzzy numbers and these operators can only deal with both membership and non-membership are non-negative real numbers, but these operators can't deal with the picture 2-tuple linguistic numbers (P2TLNs), however, the P2TLPWHM (P2TLPWDHM) operators which are proposed can deal with the picture 2-tuple linguistic numbers (P2TLNs).

7. Conclusion

This paper conducts a research on investigating the MADM problems with P2TLNs and performs a practical example of application. The strengths of the approach lie in the following points.

(1) Based on Hamy mean (HM) and dual Hamy mean (DHM) operator, we utilize power average and power geometric operations to develop some picture 2-tuple linguistic power Hamy mean aggregation operators: picture 2-tuple linguistic power weighted Hamy mean (P2TLPWHM) operator, picture 2-tuple linguistic power weighted dual Hamy mean (P2TLPWDHM) operator, picture 2-tuple linguistic power ordered weighted Hamy mean (P2TLPOWHM) operator, picture 2-tuple linguistic power ordered weighted dual Hamy mean (P2TLPOWDHM) operator, picture 2-tuple linguistic power hybrid Hamy mean (P2TLPHHM) operator

and picture 2-tuple linguistic power hybrid dual Hamy mean (P2TLPHDHM) operator. The prominent characteristic of these proposed operators are studied.

(2) Then, we have utilized these operators to develop some approaches to solve the picture 2-tuple linguistic MADM problems.
(3) A practical example for enterprise resource planning (ERP) system selection is given to verify the developed approach and a comparative analysis is applied to demonstrate its practicality and effectiveness.
(4) It is totally possible for industry managers to employ the proposed approach to solve other similar evaluation problems or other investment decision making problems.

Finally, we assure several directions for future studies as follows.

(1) The application of the proposed power aggregating operators of P2TLSs needs to be explored in the other uncertain decision making [55–61] and many other fields domains' applications [62–69].
(2) In this work, we ignore the interrelationships among the attributes. Researchers and scholars can devote themselves to improve these methods and models in future work to overcome this defect.
(3) The proposed methods and models don't take into account the influence of DMs' psychological behavioral characteristics. In the future, Researchers and scholars can introduce the psychological behavior of DMs into the proposed methods [70, 71].

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