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SOME PROGRESS ON THE MIXED ROMAN DOMINATION IN GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. A mixed Roman dominating function of G is a function $f : V \cup E \to \{0, 1, 2\}$ satisfying the condition that every element $x \in V \cup E$ for which $f(x) = 0$ is adjacent or incident to at least one element $y \in V \cup E$ for which $f(y) = 2$. The weight of a mixed Roman dominating function f is $\omega(f) = \sum_{x\in V\cup E} f(x)$. The mixed Roman domination number $\gamma^*_R(G)$ of G is the minimum weight of a mixed Roman dominating function of G. We first show that the problem of computing $\gamma^*_R(G)$ is NP-complete for bipartite graphs and then we present upper and lower bounds on the mixed Roman domination number, some of them are for the class of trees.

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1. Introduction

For terminology and notation on graph theory not given here, the reader is referred to [19]. In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$ and the size $|E|$ of G is denoted by $m = m(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighborhood* is $N(S) = \cup_{v\in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. A vertex of degree one is called a *leaf* and its neighbor is called a *stem*. A double star is a tree with exactly two non-leaf vertices. A *strong stem* is a stem adjacent to at least two leaves. An *end-stem* is a stem having at most one non-leaf neighbor. An edge incident with a leaf is called a *pendant edge*. We denote the set of leaves and stems of a graph G by $L(G)$ and $S(G)$, respectively. Further we let $|L(G)| = \ell(G)$ and $|S(G)| = s(G)$. A graph G is said to be a *generalized corona* if $V(G) = L(G) \cup S(G)$.

Keywords. Roman dominating function, Roman domination number, mixed Roman dominating function, mixed Roman domination number.

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We write $\overline{G}$ for the *complement* of a graph G, K_n for the *complete graph* of order n, C_n for a *cycle* of length n and P_n for a *path* of length $n-1$.

An element of a graph is either a vertex or an edge. For any $x \in V \cup E$, we denote by $N_m[x] = \{x\} \cup \{y \in V \cup E : y \text{ is either adjacent or incident with } x\}$. For a set $S \subseteq V$, the *open mixed neighborhood* is $N_m(S) = \cup_{v\in S} N_m(v)$ and the *closed mixed neighborhood* is $N_m[S] = N_m(S) \cup S$.

A set $S \subseteq V$ is *independent* if no two vertices in S are adjacent. A *matching* in a graph G is a set of pairwise non-adjacent edges. The *matching number* $\alpha'(G)$ (α' for short) is the size of a largest matching in G. A *perfect matching* M of G is a matching with $V(M) = V(G)$.

A set $S \subseteq V$ is a *dominating set* of a graph G if $N[S] = V$, that is, every vertex in $V \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $S \subseteq V \cup E$ is a *mixed dominating set* if for every element $x \in V \cup E$ we have $N_m[x] \cap S \neq \emptyset$. The *mixed domination number* $\gamma^*(G)$ is the minimum cardinality of a mixed dominating set of G. Mixed domination was introduced by Sampathkumar and Kamath [23]. Note that a mixed dominating set is also called *total cover* in [6,7,15,18].

A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne *et al.* [13] in 2004 and was inspired by the work of ReVelle and Rosing [22] and Stewart [24]. It is worth mentioning that since 2004, more than 150 papers were published on the Roman domination and its several variants [1–3, 5, 8, 10–12, 14, 16, 20, 25].

In this paper we are interested in mixed Roman domination which was recently introduced in [4]. A *mixed Roman dominating function* (MRDF) of a graph G is a function $f : V \cup E \to \{0, 1, 2\}$ such that every element $x \in V \cup E$ for which $f(x) = 0$, is adjacent or incident to at least one element $y \in V \cup E$ with $f(y) = 2$. In other words, we say that an element x for which $f(x) \in \{1, 2\}$ dominates itself, while an element x with $f(x) = 0$ is *dominated* by the mixed Roman function f if it is adjacent or incident to at least one element y for which $f(y) = 2$. The minimum weight, $w(f) = \sum_{x\in V\cup E} f(x)$, of a MRDF is the *mixed Roman domination number* $\gamma_R^*(G)$. A MRDF with minimum weight is called a $\gamma_R^*(G)$-function. Each MRDF determines a partition of the set $V \cup E = (V_0 \cup E_0) \cup (V_1 \cup E_1) \cup (V_2 \cup E_2)$, where $V_i \cup E_i = \{x \in V \cup E : f(x) = i\}$. For the sake of simplicity, we will denote by $f[x] = f(N_m[x]) = \sum_{v\in N_m[x]} f(v)$, for all $x \in V \cup E$.

To clarify this definition of MRDF, we consider the example given in [4]. Let C_n be the cycle graph with $n \equiv 2, 3, 4 \pmod 5$. Suppose $V(C_n) = \left\{v_1, v_2, \ldots, v_{5\lfloor \frac{n}{5} \rfloor + j}\right\}$, where $j = 2, 3, 4$ and consider the function in G defined as follows: $f(v_{5k-3}) = 2$ and $f(v_{5k-1}v_{5k}) = 2$ for $1 \le k \le \lfloor \frac{n}{5} \rfloor$, and $f(x) = 0$ for every $x \in V \cup E - \left\{v_{5k-3}, v_{5k-1}v_{5k} : 1 \le k \le \lfloor \frac{n}{5} \rfloor\right\}$ except for $f\left(v_{\lfloor \frac{n}{5} \rfloor + 2}\right) = 2$ if $n \equiv 2, 3, 4 \pmod 5$, $f(v_n v_1) = 1$ if $n \equiv 3 \pmod 5$, and $f(v_n) = 2$ if $n \equiv 4 \pmod 5$. Clearly, f is an MRDF with $\omega(f) \le \lceil \frac{4n}{5} \rceil$. Now using the Proposition 3.2 [4] implies that $\omega(f) \ge \lceil \frac{4n}{5} \rceil$, and so $\omega(f) = \lceil \frac{4n}{5} \rceil$.

For Roman domination, each vertex in the graph model corresponds to a location (city) in the Roman Empire, and for protection, legions (armies) are stationed at various locations. A location is protected if a legion is located at it and unprotected, otherwise. A location having no legion can be protected by a legion sent from a neighboring location. In order to prevent the problem of leaving a location unprotected when its legion is dispatched to a neighboring location, Emperor Constantine the Great [13] decreed that a legion cannot be sent to a neighboring location if it leaves its original station unprotected. Hence, every location with no legion must be adjacent to a location that has at least two legions. The protection of both vertices (locations) and edges (roads) has motivated the authors [4] to introduce mixed Roman domination. Thereby, for mixed Roman domination, legions can be placed at a camp on a road as well as stationed in a location, and both locations and roads must be protected. Any unprotected edge must be adjacent to a protected edge with two legions or incident to a protected vertex with two legions. Further, any vertex or edge with two legions can dispatch a legion to secure any unsecured element (vertex or edge) adjacent or incident to it.

Before presenting our results, we list below some known results that will be useful here.

Proposition 1.1 ([4]). *For any graph G,*

$$\gamma^*(G) \leq \gamma_R^*(G) \leq 2\gamma^*(G). \tag{1.1}$$

We note that graphs G with $\gamma_R^*(G) = 2\gamma^*(G)$ are called *mixed Roman graphs*.

Proposition 1.2 ([4]). *For any graph G of order n, $\gamma_R^*(G) \leq n$. Moreover, if $G \in \{K_n, \overline{K}_n\}$, then $\gamma_R^*(G) = n$.*

Proposition 1.3 ([4]). *Let $n \geq 3$ be a positive integer. Then*

$$\gamma_R^*(P_n) = \begin{cases} \left\lceil \dfrac{4n-2}{5} \right\rceil & \text{if} \quad n \equiv 0,1,2,3 \pmod 5 \\ \left\lceil \dfrac{4n-2}{5} \right\rceil + 1 & \text{if} \quad n \equiv 4 \pmod 5). \end{cases}$$

Theorem 1.4. *Let G be a connected graph of order n.*

(i) *[6] Then $\gamma^*(G) \leq \lceil n/2 \rceil$.*
(ii) *[7] If $\gamma^*(G) = n/2$, then G has a perfect matching.*

This paper is organized as follows: in Section 2, we show that the problem of computing the mixed Roman domination number is in the NP-complete class even when restricted to bipartite graphs. In Section 3, we characterize all graphs of odd order attaining equality in the upper bound of Proposition 1.2, and we give a necessary condition for such graphs when n is even. Finally we present in Section 4 some upper and lower bounds on the mixed Roman domination number. More precisely, if T is a tree, then we show that $\gamma_R^*(T)$ is bounded below by $3\alpha'(T)/2$ and above by $3\gamma(T) - 1$. In addition, if T has order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ stems, then $\gamma_R^*(T) \leq n - \ell(T) + s(T)$, and we characterize all extremal trees. For arbitrary connected graphs G, we show that $\gamma_R^*(G) \geq \alpha'(G) + 1$.

2. Complexity result

Our aim in this section is to study the complexity of the following decision problem, to which we shall refer as MIXED ROMAN DOMINATION:

MIXED ROMAN DOMINATION (MRD)
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Does G have a mixed Roman dominating function of weight at most k?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, Exact-3-Cover (X3C), to MRD.

EXACT 3-COVER (X3C)
Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
Question: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

Theorem 2.1. *Problem MRD is NP-Complete for bipartite graphs.*

Proof. MRD is a member of $\mathcal{NP}$, since we can check in polynomial time that a function $f : V \cup E \to \{0, 1, 2\}$ has weight at most k and is a mixed Roman dominating function. Now let us show how to transform any instance of X3C into an instance G of MRD so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.

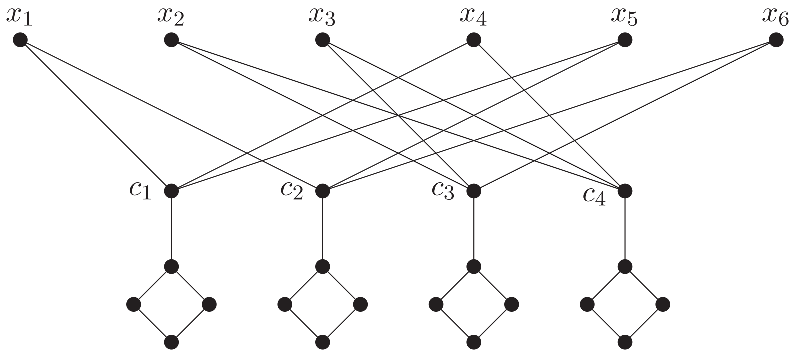

FIGURE 1. NP-completeness for bipartite graphs.

For each $x_i \in X$, we create a vertex w_i. Let $W = \{w_1, w_2, \ldots, w_{3q}\}$. For each $C_j \in C$ we build a graph H_j of order 5 obtained from a cycle $C_4 : a_j$–b_j–r_j–d_j–a_j and a new vertex c_j by adding the edge $c_j r_j$. Let $Y = \{c_1, c_2, \ldots, c_t\}$. Now to obtain a graph G, we add edges $c_j w_i$ if $x_i \in C_j$. Clearly G is a bipartite graph (see Fig. 1). Set $k = 4t + 2q$, and let H be the subgraph of G induced by all $V(H_j)$. Observe that for every mixed Roman dominating function f on G, each H_j has weight at least 4, and so $f(H) \geq 4t$. We also note that if $f(c_j) = 2$ for some vertex c_j, then $f(H_j) \geq 6$.

Suppose that the instance X, C of X3C has a solution C'. We construct a mixed Roman dominating function f on G of weight k. We assign the value 0 to every w_i and to every edge incident with w_i. For every $C_j \in C'$, assign the value 2 to c_j, r_j and a_j, and 0 to the remaining elements of H_j. Also for every $C_j \notin C'$, assign the value 2 to edge $c_j r_j$ and vertex a_j, and 0 to the remaining elements of H_j. Note that since C' exists, its cardinality is precisely q, and so the number of c_j's with weight 2 is q, having disjoint neighborhoods in $\{x_1, x_2, \ldots, x_{3q}\}$, Since C' is a solution for X3C, every vertex in W is adjacent to a vertex assigned a 2. Moreover, every edge incident with a vertex of Y is adjacent to an element assigned 2 under f. Hence, it is straightforward to see that f is a mixed Roman dominating function with weight $f(V) = 6q + 4(t - q) = k$.

Conversely, suppose that G has a mixed Roman dominating function with weight at most k. Among all such functions, let $g = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ with a fewest elements assigned the value 1. According to our choice of g, we claim that every vertex of Y is assigned either 0 or 2. Indeed, suppose that $g(c_j) = 1$ for some j. Then it is easy to see that $g(H_j) \geq 5$. In this case, let g' be the function defined on G by $g'(c_j r_j) = 2$, $g'(a_j) = 2$ and $g'(y) = 0$ for any other element y of H_j, and $g'(x) = g(x)$ for every element x of G not in H_j. Clearly g' is a MRDF on G with weight at most k but with fewer vertices assigned the value 1 than those under g, contradicting our choice of g, which proves the claim. Now, on the basis of the previous fact and since $g(H) \geq 4t$, we may assume that if $g(c_j) = 0$, then $g(c_j r_j) = 2$. Hence regardless of the value assigned to every c_j; all edges of the form $w_i c_j$ are assigned the value 0. Now since $g(H) \geq 4t$, it follows that $g(W) \leq 2q$. Also since $|W| = 3q$, we have $W \cap V_0 \neq \emptyset$ which implies that $Y \cap V_2 \neq \emptyset$. Let $y = |Y \cap V_2|$. Recall our earlier fact that if $f(c_j) = 2$ for some c_j, then $f(H_j) \geq 6$. This fact implies that $y \leq q$ (for otherwise $g(H) > 4t + 2q$). On the other hand, using the fact that each c_j has exactly three neighbors in W we deduce that $g(W) \geq 3q - 3y$. Note that $g(H) \geq 4t + 2y$. Now combining all these facts with $g(G) \leq k = 4t + 2q$ we obtain $y \geq q$. Hence $y = q$ and so $W \subset V_0$. Consequently, $C' = \{C_j : g(c_j) = 2\}$ is an exact cover for C. □

3. GRAPHS WITH LARGE MIXED ROMAN DOMINATION NUMBER

The aim of this section is to characterize connected graphs attaining equality in the upper bound of Proposition 1.2, that is graphs G of order n for which $\gamma_R^*(G) = n$. We provide a complete characterization of such graphs when n is odd and we give a necessary condition when n is even.

Theorem 3.1. *Let G be a connected graph of odd order n. Then $\gamma_R^*(G) = n$ if and only if $G = K_n$.*

Proof. Let G be a connected graph of odd order n such that $\gamma_R^*(G) = n$. Let $M = \{u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}\}$ be a maximum matching of G and X the independent set of M-unsaturated vertices. Since n is odd, $|X| \geq 1$. Note that if $y \in X$ and $yu_i \in E(G)$ for some i, then for every $z \in X - \{y\}$ we have $zv_i \notin E(G)$, for otherwise $(M - \{u_iv_i\}) \cup \{yu_i, zv_i\}$ is a matching larger than M, a contradiction.

Claim 1. If $u \in X$, then $uu_i \in E(G)$ if and only if $uv_i \in E(G)$.

Proof of Claim 1. Let $uu_i \in E(G)$ for some i, say $i = 1$, and assume, to the contrary, that $uv_1 \notin E(G)$. Then $zv_1 \notin E(G)$ for each $z \in X - \{u\}$. Define $g : V(G) \cup E(G) \to \{0, 1, 2\}$ by $g(u_1) = 2, g(u_iv_i) = 2$ for $2 \leq i \leq \alpha'(G)$ and $g(x) = 0$ otherwise. Clearly g is a MRDF on G of weight at most $n-1$ which is a contradiction. This proves Claim 1.

According to Claim 1, for every two vertices x and y of X we have $N(x) \cap N(y) = \emptyset$.

Claim 2. $|X| = 1$.

Proof of Claim 2. Suppose that $|X| \geq 2$ and let $X = \{x_1, x_2, \ldots, x_r\}$. By the connectedness of G and Claim 1, we may assume that $N(x_1) = \{u_i, v_i \mid 1 \leq i \leq t_1\}$, $N(x_2) = \{u_i, v_i \mid t_1 + 1 \leq i \leq t_2\}, \ldots, N(x_r) = \{u_i, v_i \mid t_{r-1} + 1 \leq i \leq \alpha'(G)\}$. Since G is connected, we may also assume, without loss of generality, that $u_{t_1}u_{t_2} \in E(G)$. But then $(M - \{u_{t_1}v_{t_1}, u_{t_2}v_{t_2}\}) \cup \{u_{t_1}u_{t_2}, x_1v_{t_1}, x_2v_{t_2}\}$ is a matching of G larger than M, a contradiction. This proves Claim 2.

Let $X = \{x\}$.

Claim 3. $N(x) = \{u_i, v_i \mid 1 \leq i \leq \alpha'(G)\}$.

Proof of Claim 3. Suppose to the contrary that $u_i \notin N(x)$ for some i. By Claim 1, we may assume that $N(x) = \{u_i, v_i \mid 1 \leq i \leq t\}$, where $t < \alpha'(G)$. Since G is connected, we may assume, without loss of generality, that u_t is adjacent to u_{t+1}. Then $M_1 = (M - \{u_tv_t\}) \cup \{xv_t\}$ is a maximum matching of G, where u_t is an M_1-unsaturated vertex. Since $u_tu_{t+1} \in E(G)$, Claim 1 implies that $u_tv_{t+1} \in E(G)$. Now if $v_tu_{t+1}, v_tv_{t+1} \notin E(G)$, then the function $g : V(G) \cup E(G) \to \{0, 1, 2\}$ defined by $g(xv_t) = g(u_{t+1}v_{t+1}) = 1, g(u_t) = 2, g(u_iv_i) = 2$ for $i \in \{1, \ldots, \alpha'(G)\} \setminus \{t, t+1\}$ and $g(y) = 0$ otherwise, is a MRDF of G of weight at most $n - 1$ which is a contradiction. Hence we assume that $v_tu_{t+1} \in E(G)$ (the case $v_tv_{t+1} \in E(G)$ is similar). Clearly $M_2 = (M - \{u_tv_t, u_{t+1}v_{t+1}\}) \cup \{xu_t, u_{t+1}v_t\}$ is a maximum matching of G, where v_{t+1} is an M_2-unsaturated vertex. As seen above, Claim 1 implies that $xv_{t+1} \in E(G)$, a contradiction. This proves Claim 3.

Thus x is adjacent to all vertices of G. For each $1 \leq i \leq \alpha'(G)$, $M' = (M - \{u_iv_i\}) \cup \{xv_i\}$ is a maximum matching of G and u_i is a M'-unsaturated vertex. Using the same argument as above, we conclude that u_i is adjacent to all vertices of G. Likewise, v_i is adjacent to all vertices of G. Now since u_i and v_i are arbitrary vertices, we deduce that G is a complete graph.

The converse follows from Proposition 1.2. ☐

Proposition 3.2. *Let G be a connected graph of even order n. If $\gamma_R^*(G) = n$, then G is a mixed Roman graph having a perfect matching.*

Proof. By (1.1) and Theorem 1.4(i) we have $\gamma_R^*(G) = n = 2\lceil n/2 \rceil \geq 2\gamma^*(G) \geq \gamma_R^*(G)$. Therefore $\gamma_R^*(G) = 2\gamma^*(G)$ and $\gamma^*(G) = n/2$. The first equality implies that G is mixed Roman and the second one implies that G has a perfect matching (by Theorem 1.4(ii)). ☐

4. Bounds on the mixed Roman domination number

We saw in Section 2 that computing $\gamma_R^*(G)$ is NP-complete even for bipartite graphs G. It is therefore natural to look for new upper and lower bounds on the mixed Roman domination number. In this section, we present two upper bounds as well as two lower bounds on this parameter, three of which are in the class of trees.

4.1. Upper bounds

Our first upper bound relates the mixed Roman domination number to the domination number of any tree.

Proposition 4.1. *For any nontrivial tree T, $\gamma_R^*(T) \leq 3\gamma(T) - 1$.*

Proof. Let $D = \{v_1, \ldots, v_k\}$ be a minimum dominating set of T, and let E_D denote the set of edges that don't belong to $N_m[D]$. Note that every edge of E_D is incident with only vertices of $V - D$. Let $A_i = N(v_i) \setminus D$ for $i \in \{1, \ldots, k\}$. Since T is a tree, A_i is independent for each i. Let G be the graph with vertex set $\{x_1, \ldots, x_k\}$ such that x_i and x_j are adjacent if and only if there is edge in E_D between A_i and A_j. If $(x_{i_1} x_{i_2} \ldots x_{i_s} x_{i_1})$ is a cycle in G and $a_{i_j} b_{i_{j+1}} \in E_D$ is the corresponding edge of $x_{i_j} x_{i_{j+1}}$ for each j (possibly $a_{i_j} = b_{i_j}$), then $\left(v_{i_1} a_{i_1} b_{i_2} v_{i_2} \ldots v_{i_{s-1}} a_{i_{s-1}} b_{i_s} v_{i_s} a_{i_s} b_{i_1} v_{i_1}\right)$ contains a cycle in T which is a contradiction. Thus G is a forest and so $|E_D| = |E(G)| \leq |D| - 1$. Now define a MRDF g on T by $g(x) = 2$ for every vertex $x \in D$, $g(x) = 0$ for every element x which is either adjacent or incident with a vertex of D, and $g(e) = 1$ for every edge $e \in E_D$. Then g has weight $2|D| + |E_D| \leq 3|D| - 1$, implying that $\gamma_R^*(T) \leq 3\gamma(T) - 1$. □

In what follows, we give a necessary condition for trees T with $\gamma_R^*(T) = 3\gamma(T) - 1$. We first need to recall a couple of definitions and result on trees with a unique minimum dominating set due to Gunther *et al.* [17]. A set S of vertices in a graph $G = (V, E)$ is a *packing* if the vertices in S are pairwise at distance at least 3 apart in G, or equivalently, for every vertex $v \in V$, $|N[v] \cap S| \leq 1$. As defined in Bange *et al.* [9], a dominating set S for which $|N[v] \cap S| = 1$ for all $v \in V$ is an *efficient dominating set*. Equivalently, a set S is an efficient dominating set if S is both a dominating set and a packing of G.

Theorem 4.2 ([17])**.** *Let T be a tree of order at least 3. Then T has a unique $\gamma(T)$-set D if and only if every vertex of D has at least two private neighbors in $V - D$.*

Proposition 4.3. *Let T be a tree of order $n \geq 3$ such that $\gamma_R^*(T) = 3\gamma(T) - 1$. Then T has a unique $\gamma(T)$-set D such that D is efficient and every vertex in $V - D$ has degree at most three.*

Proof. Assume now that $\gamma_R^*(T) = 3\gamma(T) - 1$, and let D and E_D be two sets as defined in the proof of Proposition 4.1. By using the same argument to that used in the proof of Proposition 4.1, we obtain $3\gamma(T) - 1 = \gamma_R^*(T) \leq 2|D| + |E_D| \leq 3\gamma(T) - 1$. Hence $\gamma_R^*(T) = 2|D| + |E_D|$ and $|E_D| = |D| - 1$. Observe that the removal of E_D from T yields to a forest with $|D|$ components. Now if two vertices of D are adjacent or have a common neighbor in $V - D$, then clearly $|E_D| < |D| - 1$, which implies that $\gamma_R^*(T) \leq 2|D| + |E_D| < 3\gamma(T) - 1$, a contradiction. We deduce that D is a packing set and so D is an efficient dominating set of T. Hence every vertex of D has at least one private neighbor in $V - D$. Suppose to the contrary that some vertex x of D has exactly one private neighbor in $V - D$, say y. Then the set $D' = (D - \{x\}) \cup \{y\}$ is a $\gamma(T)$-set having $|E_{D'}| < |D| - 1 = |E_D|$. But then $\gamma_R^*(T) \leq 2|D'| + |E_{D'}| < 3|D| - 1$, a contradiction. Hence every vertex of D has at least two private neighbors in $V - D$. By Theorem 4.2, D is a unique $\gamma(T)$-set. Finally, let us suppose that some vertex $u \in V - D$ has degree at least four. Since D is efficient, u has at least three neighbors in $V - D$. Let E_u be the set of all edges incident with u belonging to the subgraph induced by $V - D$. Thus $|E_u| \geq 3$. Define a function h on T by $h(x) = 2$ for every vertex $x \in D \cup \{u\}$, $h(y) = 0$ for every element $y \neq u$ which is either adjacent or incident with a vertex of $D \cup \{u\}$, and $h(e) = 1$ for every edge $e \in E_D - E_u$. Clearly, h is MRDF on T, and so

$$\begin{aligned} w(h) &= \sum_{x \in V \cup E} h(x) = 2(|D| + 1) + |E_D - E_u| \\ &= 2|D| + |E_D| - (|E_u| - 2) = 3|D| - 1 - (|E_u| - 2) \\ &< 3|D| - 1, \end{aligned}$$

a contradiction. Hence every vertex of $V - D$ has degree at most three. □

We note that converse of Proposition 4.3 is not true. To see consider the tree T obtained from a star $K_{1,3}$ by subdividing two edges twice and the remaining edge four times. One can easily see that T has a unique $\gamma(T)$-set D of size 4 such that D is efficient and every vertex in $V-D$ has degree at most three but $\gamma_R^*(T) < 3\gamma(T)-1 = 11$.

Theorem 4.4. *If T is a tree of order $n \geq 3$ with $\ell(T)$ leaves and $s(T)$ stems, then*

$$\gamma_R^*(T) \leq n - \ell(T) + s(T)$$

with equality if and only if T is a generalized corona.

Proof. Let T' be the forest obtained from T by deleting all leaves and stems. By Proposition 1.2, $\gamma_R^*(T') \leq n - \ell(T) - s(T)$. Suppose f is a $\gamma_R^*(T')$-function and define $g : V(T) \cup E(T) \to \{0,1,2\}$ by $g(x) = f(x)$ for $x \in V(T') \cup E(T')$, $g(x) = 2$ for $x \in S(T)$ and $g(x) = 0$ otherwise. Clearly g is a MRDF of T and hence

$$\gamma_R^*(T) \leq \omega(g) = \gamma_R^*(T') + 2s(T) \leq (n - \ell(T) - s(T)) + 2s(T) = n - \ell(T) + s(T).$$

Assume T is a generalized corona. We shall show that T has a $\gamma_R^*(T)$-function f assigning the value 2 to each stem and 0 to the remaining elements of T. Clearly, in this case we have $\gamma_R^*(T) \geq 2s(T) = n - \ell(T) + s(T)$ as desired. Assume, to the contrary, that $|V_2^g \cap S(T)| < s(T)$ for every $\gamma_R^*(T)$-function $g = (V_0^g \cup E_0^g, V_1^g \cup E_1^g, V_2^g \cup E_2^g)$ of T. Among all such functions, let $f = (V_0 \cup E_0, V_1 \cup E_1, V_2 \cup E_2)$ be one for which $|V_2^f \cap S(T)|$ is as large as possible. Let v be a stem such that $f(v) \leq 1$, and assume that $L(v) = \{v_1, \ldots, v_k\}$. If $\sum_{i=1}^k (f(v_i) + f(vv_i)) \geq 2$, then the function h defined by $h(v) = 2, h(v_i) = h(vv_i) = 0$ for $1 \leq i \leq k$ and $h(x) = f(x)$ otherwise, is a $\gamma_R^*(T)$-function with $|V_2^h \cap S(T)| > |V_2^f \cap S(T)|$, contradicting the choice of f. Hence $\sum_{i=1}^k (f(v_i) + f(vv_i)) \leq 1$. Since $f(v_i) + f(vv_i) \geq 1$ for each i, we deduce that $k = 1$, $f(v_1) = 1$ and $f(vv_1) = 0$. From $f(v_1) + f(vv_1) \leq 1$ and $f(v) \leq 1$ we deduce that $f(vu) = 2$ for some $u \in N(v) - L(v)$, thus $f(v) = 0$, $f(vv_1) = 0$ and $f(v_1) = 1$. If $f(u) = 2$, then the function h_1 defined by $h_1(v) = 2, h_1(v_1) = h_1(vv_1) = h(uv) = 0$ and $h_1(x) = f(x)$ otherwise, is a MRDF of T of weight less than $\gamma_R^*(T)$, a contradiction. Thus $f(u) \leq 1$. Recall that, by assumption, u is a stem. Let $w \in L(u)$. Since $f(u) \leq 1$, we obtain that $f(w) + f(uw) \geq 1$. Then the function h_2 defined by $h_2(u) = h_2(v) = 2, h_2(w) = h_2(uw) = h_2(uv) = h_2(v_1) = h_2(vv_1) = 0$ and $h_2(x) = f(x)$ otherwise, is a $\gamma_R^*(T)$-function that contradicts the choice of f. We conclude that T has a $\gamma_R^*(T)$-function f assigning the value 2 to each stem and this proves the claim.

Conversely, let T be a tree of order $n \geq 3$ such that $\gamma_R^*(T) = n - \ell(T) + s(T)$. Assume that T contains at least one vertex which is neither a stem nor a leaf. Let T' be the forest obtained from T by removing all stems and leaves. We consider the following cases.

Case 1. T' has an isolated vertex v.

Let $T'' = T' - v$ and f be a $\gamma_R^*(T'')$-function. Define $g : V(T) \cup E(T) \to \{0,1,2\}$ by $g(x) = 2$ for $x \in S(T)$, $g(x) = f(x)$ for $x \in V(T'') \cup E(T'')$ and $g(x) = 0$ otherwise. Clearly g is a MRDF of T. By Proposition 1.2,

$$\gamma_R^*(T) \leq \omega(f) + 2s(T) \leq (n - \ell(T) - s(T) - 1) + 2s(T) \leq n - \ell(T) + s(T) - 1,$$

a contradiction.

Case 2. T' has a component of order 2.

Let v and w be the vertices of such a component of T'. Let $T'' = T' - \{v, w\}$ and f be a $\gamma_R^*(T'')$-function. Define $g : V(T) \cup E(T) \to \{0,1,2\}$ by $g(x) = 2$ for $x \in S(T)$, $g(vw) = 1, g(x) = f(x)$ for $x \in V(T'') \cup E(T'')$ and $g(x) = 0$ otherwise. Obviously g is a MRDF of T and by Proposition 1.2, we obtain

$$\gamma_R^*(T) \leq \omega(f) + 2s(T) + 1 \leq (n - \ell(T) - s(T) - 2) + 2s(T) + 1 \leq n - \ell(T) + s(T) - 1,$$

a contradiction.

Case 3. T' has a strong stem.

Let v be a strong stem in T'. Assume that $T'' = T' - (\{v\} \cup L_{T'}(v))$ and let f be a $\gamma_R^*(T'')$-function. Define $g : V(T) \cup E(T) \to \{0,1,2\}$ by $g(x) = 2$ for $x \in S(T) \cup \{v\}$, $g(x) = f(x)$ for $x \in V(T'') \cup E(T'')$ and $g(x) = 0$ otherwise. Clearly, g is a MRDF of T. By Proposition 1.2, we have

$$\gamma_R^*(T) \le \omega(f) + 2s(T) + 2 \le (n - \ell(T) - s(T) - 1 - |L_{T'}(v)|) + 2s(T) + 2 \le n - \ell(T) + s(T) - 1,$$

a contradiction.

According to Cases 1–3, we may assume that T' has no strong stem and every component of T' has order at least four. Note that a component of order three implies that T' has a strong stem. Let v_1 and v_2 be two stems of T' such that the distance between v_1 and v_2 in T', $d_{T'}(v_1, v_2)$, is as small as possible and let z_i be the leaf adjacent to v_i, for $i \in \{1,2\}$. Note that each z_i is adjacent to a vertex of $S(T)$.

Case 4. $d_{T'}(v_1, v_2) = 1$.

Let $T'' = T' - \{v_1, v_2, z_1, z_2\}$ and let f be a $\gamma_R^*(T'')$-function. Define $g : V(T) \cup E(T) \to \{0,1,2\}$ by $g(x) = 2$ for $x \in S(T) \cup \{v_1v_2\}$, $g(x) = f(x)$ for $x \in V(T'') \cup E(T'')$ and $g(x) = 0$ otherwise. Clearly g is a MRDF of T and by Proposition 1.2

$$\gamma_R^*(T) \le (n - \ell(T) - s(T) - 4) + 2s(T) + 2 < n - \ell(T) + s(T),$$

which is a contradiction.

Case 5. $d_{T'}(v_1, v_2) = 2$.

Let w be the common vertex adjacent to v_1 and v_2. We deduce from the choice of v_1 and v_2 that w is not a stem in T'. Suppose $T'' = T' - \{w, v_1, v_2, z_1, z_2\}$ and let f be a $\gamma_R^*(T'')$-function. Then the function $g : V(T) \cup E(T) \to \{0,1,2\}$ defined by $g(x) = 2$ for $x \in S(T) \cup \{v_1w, v_2w\}$, $g(x) = f(x)$ for $x \in V(T'') \cup E(T'')$ and $g(x) = 0$ otherwise, is a MRDF of T. It follows from Proposition 1.2 that

$$\gamma_R^*(T) \le (n - \ell(T) - s(T) - 5) + 2s(T) + 4 < n - \ell(T) + s(T),$$

a contradiction.

Case 6. $d_{T'}(v_1, v_2) = 3$.

Let $v_1w_1w_2v_2$ be the (v_1, v_2)-path. By the choice of v_1 and v_2, we conclude that w_1 and w_2 are not stems in T'. Suppose $T'' = T' - \{w_1, w_2, v_1, v_2, z_1, z_2\}$ and let f be a $\gamma_R^*(T'')$-function. Then the function $g : V(T) \cup E(T) \to \{0,1,2\}$ defined by $g(x) = 2$ for $x \in S(T) \cup \{v_1w_1, v_2w_2\}$, $g(x) = f(x)$ for $x \in V(T'') \cup E(T'')$ and $g(x) = 0$ otherwise, is a MRDF of T. By Proposition 1.2, we obtain

$$\gamma_R^*(T) \le (n - \ell(T) - s(T) - 6) + 2s(T) + 4 < n - \ell(T) + s(T),$$

a contradiction.

Case 7. $d_{T'}(v_1, v_2) \ge 4$.

Suppose $v_1w_1 \dots w_kv_2$ is the (v_1, v_2)-path. As above, each w_i is not a stem in T'.

Let $T'' = T' - \{w_1, w_k, v_1, v_2, z_1, z_2\}$ and f be a $\gamma_R^*(T'')$-function. Then the function $g : V(T) \cup E(T) \to \{0,1,2\}$ defined by $g(x) = 2$ for $x \in S(T) \cup \{v_1w_1, v_2w_k\}$, $g(x) = f(x)$ for $x \in V(T'') \cup E(T'')$ and $g(x) = 0$ otherwise, is a MRDF of T of weight less than $n - \ell(T) + s(T)$, a contradiction.

Therefore, every vertex of T is either a stem or a leaf, and the proof is complete. □

The next result is an immediate consequence of Theorem 4.4. The *corona* of a graph $G = (V, E)$ is the graph formed from a copy of G by attaching for each $v \in V$, a new vertex v' and edge vv'.

Corollary 4.5. *Let T be a tree of order n. Then $\gamma_R^*(T) = n$ if and only if T is a corona of some tree T'.*

It is worth mentioning that the bounds in Proposition 4.1 and Theorem 4.4 are incomparable and the difference $(3\gamma(T) - 1) - (n - \ell(T) + s(T))$ as well as $(n - \ell(T) + s(T)) - (3\gamma(T) - 1)$ can be arbitrarily large. Indeed, for the corona tree H of high order we have $(3\gamma(H) - 1) = 3n/2 - 1$ while $(n - \ell(H) + s(H)) = n = \gamma_R^*(H)$. On the other hand, for the tree F_k ($k \ge 2$) obtained from a star $K_{1,k}$ by subdividing each edge of the star three times we have $(n - \ell(F) + s(F)) = n$ while $(3\gamma(F) - 1) = 3(k+1) - 1 = \gamma_R^*(F)$.

4.2. Lower bounds

The two lower bounds presented in this subsection relate the mixed Roman domination number to the matching number.

Theorem 4.6. *For every connected graph G of order $n \geq 2$,*

$$\gamma_R^*(G) \geq \alpha'(G) + 1.$$

Moreover, this bound is sharp for stars.

Proof. Let $M = \{u_1v_1, \ldots, u_{\alpha'}v_{\alpha'}\}$ be a maximum matching of G and let X be the set of vertices not saturated by M. Recall that X is independent. If y and z are vertices of X and $yu_i \in E(G)$, then since the matching M is maximum, $zv_i \notin E(G)$. Therefore, for all $i \in \{1, 2, \ldots, \alpha'\}$ there are at most two edges between the sets $\{u_i, v_i\}$ and $\{y, z\}$. Let $Z = V(G) \cup E(G)$ and $Z^* = \{u_i, v_i, u_iv_i \mid 1 \leq i \leq \alpha'\}$. Obviously, $|Z^*| = 3\alpha'$.

Assume, to the contrary, that $\gamma_R^*(G) \leq \alpha'(G)$ and let $f = (Z_0, Z_1, Z_2)$ be a $\gamma_R^*(G)$-function, where $Z_0 \cup Z_1 \cup Z_2 = Z$. Clearly $|Z_2| \geq 1$ and $|Z_1| + 2|Z_2| \leq \alpha'(G)$. Hence $|Z_2| \leq \left\lfloor \frac{\alpha'(G)}{2} \right\rfloor$. Suppose $|Z_2 \cap E(G)| = s$ and $|Z_2 \cap V(G)| = r$. It is worth noting that an edge of $Z_2 \cap E(G)$ can dominate 2 edges as well as 2 vertices of Z^* while a vertex of $Z_2 \cap V(G)$ dominates at most one edge of Z^* and possibly all vertices of Z^*. Based on these facts, let us consider the following two cases.

Case 1. $s = |Z_2|$. Hence $r = 0$. Clearly, at most $4s$ elements of Z^* are dominated by Z_2. Using the fact that $|Z_2| \leq \left\lfloor \frac{\alpha'(G)}{2} \right\rfloor$ we obtain $4s = 4|Z_2| \leq 4 \left\lfloor \frac{\alpha'(G)}{2} \right\rfloor \leq 2\alpha'(G)$. This implies that $|Z_1 \cap Z^*| \geq 3\alpha' - 2\alpha' = \alpha'$. Therefore, $|Z_1| + 2|Z_2| \geq \alpha' + 2s > \alpha' + 1$ which is a contradiction.

Case 2. $s < |Z_2|$. Hence $r \geq 1$. Since set Z_2 dominates at most $2\alpha' + r + 2s$ elements of Z^*, we obtain that $|Z_1 \cap Z^*| \geq 3\alpha' - (2\alpha' + r + 2s)$. Therefore, $|Z_1| + 2|Z_2| \geq 3\alpha' - (2\alpha' + r + 2s) + 2(r + s) = \alpha' + r \geq \alpha' + 1$, a contradiction.

We conclude that $\gamma_R^*(G) \geq \alpha'(G) + 1$, and this completes the proof. $\square$

Our next result improves the previous lower bound for the class of trees.

Observation 4.7. If T is a tree and $v_3v_2v_1$ is a path in T such that $\deg(v_1) = 1$ and $\deg(v_2) = 2$, then for any $\gamma_R^*(T)$-function f, $f(v_1) + f(v_2) + f(v_1v_2) + f(v_2v_3) \geq 2$.

Proof. If $f(v_1) = 0$, then to Roman dominate v_1 we must have $f(v_2) = 2$ or $f(v_1v_2) = 2$ implying that $f(v_1) + f(v_2) + f(v_1v_2) + f(v_2v_3) \geq 2$. If $f(v_1v_2) = 0$, then to Roman dominate v_1v_2 we must have $f(v_1) = 0$ or $f(v_2) = 2$ or $f(v_2v_3) = 2$ yielding $f(v_1) + f(v_2) + f(v_1v_2) + f(v_2v_3) \geq 2$. Hence we may assume that $f(v_1) \geq 1$ and $f(v_1v_2) \geq 1$ and so $f(v_1) + f(v_2) + f(v_1v_2) + f(v_2v_3) \geq 2$. $\square$

Theorem 4.8. *For any tree T,*

$$\gamma_R^*(T) \geq \left\lceil \frac{4\alpha'(T)}{3} \right\rceil.$$

Furthermore, this bound is sharp for the stars $K_{1,t}$ $(t \geq 1)$ and the path P_8.

Proof. Since $\gamma_R^*(T)$ is integer, it is enough to prove that $\gamma_R^*(T) \geq \frac{4\alpha'(T)}{3}$. We use an induction on the order n of the tree T. The statement is clearly true for all trees of order $n \leq 4$. Let $n \geq 5$ and assume that for every tree T' of order n' less than n the result is true. Let T be a tree of order n and M a maximum matching of T. If $\operatorname{diam}(T) = 2$, then T is a star which yields $\gamma_R^*(T) = 2 \geq 4\alpha'(T)/3$. If $\operatorname{diam}(T) = 3$, then T is a double star and we have $\alpha'(T) = 2$ and $\gamma_R^*(T) = 4$ implying that $\gamma_R^*(T) > 4\alpha'(T)/3$. Thus, we may assume that $\operatorname{diam}(T) \geq 4$. If T has a pendant edge uv with $\deg(u) = 1$ such that $uv \notin M$, then for any $\gamma_R^*(T)$-function f, the function

$g : V(T-u) \cup E(T-uv) \to \{0,1,2\}$ defined by $f(v) = \min\{2, f(v)+f(u)+f(uv)\}$ and $g(x) = f(x)$ otherwise, is a MRDF of $T-u$ and by the induction hypothesis we have $\gamma_R^*(T) \geq \gamma_R^*(T-u) \geq \frac{4}{3}\alpha'(T-u) = \frac{4}{3}\alpha'(T)$. Henceforth, we may assume all pendant edges of T belong to each maximum matching. It follows that all end-stems of T have degree 2. If $\Delta(T) = 2$, then T is a path and the result follows by Proposition 1.3. Assume that $\Delta(T) \geq 3$. Let $P = v_1v_2\dots v_k$ be a longest path in T and root T at v_k. By assumption all children of v_3 with depth one have degree two. In particular $\deg(v_2) = 2$. Suppose f is a $\gamma_R^*(T)$-function such that $f(v_3)$ is as large as possible. By Observation 4.7, we have $f(v_1)+f(v_2)+f(v_1v_2)+f(v_2v_3) \geq 2$. We consider the following cases.

Case 1. $\deg(v_3) \geq 3$.

First suppose there is a path $v_3w_2w_1$ in T such that $\deg(w_1) = 1$ and $w_2 \notin \{v_2, v_4\}$. Then $\deg(w_2) = 2$ and by Observation 4.7 we have $f(w_1)+f(w_2)+f(w_1w_2)+f(w_2v_3) \geq 2$. We may assume without loss of generality that $f(v_3w_2) \geq f(v_3v_2)$. Let $T' = T - \{v_1, v_2\}$. Clearly $\alpha'(T') = \alpha'(T) - 1$ and the function f restricted to T' is a MRDF of T' of weight at most $\omega(f) - 2$ and by the induction hypothesis we have

$$\gamma_R^*(T) \geq \gamma_R^*(T') + 2 \geq \frac{4(\alpha'(T)-1)}{3} + 2 > \frac{4\alpha'(T)}{3}.$$

Now let all children of v_3 but v_2 be leaves. Suppose w is a leaf adjacent to v_3. If $f(w)+f(v_3)+f(v_3w) \geq 2$, then we may assume that $f(v_3) = 2$ and then the function f, restricted to $T' = T - \{v_1, v_2\}$ is a mixed Roman dominating function of T' of weight at most $\omega(f) - 2$ and the result follows as above. Let $f(w)+f(v_3)+f(v_3w) \leq 1$. Then we must have $f(w) = 1$ and $f(v_3w) = f(v_3) = 0$. To Roman dominate v_3w, some edges at v_3 must assigned 2 under f. If $f(v_3v_2) = 2$, then clearly $f(v_1) = 1$ and it is easy to see that the function $g : V(T) \cup E(T) \to \{0,1,2\}$ defined by $g(v_3) = g(v_1v_2) = 2, g(w) = g(v_3w) = g(v_1) = g(v_2) = g(v_2v_3) = 0$ and $g(x) = f(x)$ otherwise, is a $\gamma_R^*(T)$-function with $g(v_3) > f(v_3)$, which leads to a contradiction to the choice of f. Let $f(v_3u) = 2$ for some $u \in N(v_3) - \{v_2, w\}$. Then the function f, restricted to $T' = T - \{v_1, v_2\}$ is a mixed Roman dominating function of T' of weight at most $\omega(f) - 2$, and the result follows as above.

Case 2. $\deg(v_3) = 2$ and $\deg(v_4) \geq 3$.

Since M is a maximum matching and all pendant edges must belong to M, we deduce that $v_1v_2, v_4v_3 \in M$. It follows that v_4 is not a stem. On the other hand, if there is a path $v_4w_3w_2w_1$ in T such that $w_3 \notin \{v_3, v_5\}$ and $\deg(w_1) = 1$, then by the assumption and Case 1, we may assume that $\deg(w_2) = \deg(w_3) = 2$. By the assumption we have $w_1w_2 \in M$ that implies $w_2w_3, w_3v_4 \notin M$ because of $v_4v_3 \in M$. But then $(M - \{w_1w_2\}) \cup \{w_2w_3\}$ is a maximum matching of T not containing a pendant edge which is a contradiction. It follows from $\deg(v_4) \geq 3$ and above argument that there is a path $v_4w_2w_1$ with $\deg(w_2) = 2$ and $\deg(w_1) = 1$.

By Observation 4.7, we have $f(w_1)+f(w_2)+f(w_1w_2)+f(w_2v_4) \geq 2$ and $f(v_1)+f(v_2)+f(v_1v_2)+f(v_2v_3) \geq 2$, and so $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4) \geq 2$. If $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4) \geq 4$, then the function $g : V(T) \cup E(T) \to \{0,1,2\}$ defined by $g(v_4) = g(v_2) = 2, g(v_1) = g(v_3) = g(v_1v_2) = g(v_2v_3) = g(v_3v_4) = 0$ and $g(x) = f(x)$ otherwise, is a $\gamma_R^*(T)$-function such that its restriction on $T - \{w_1, w_2\}$ is a mixed Roman dominating function of weight at most $\omega(g) - 2$ and using the induction hypothesis on $T - \{w_1, w_2\}$ we get the result. Assume that $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4) \leq 3$. Similarly, we may assume that $f(w_1)+f(w_2)+f(w_1w_2)+f(w_2v_4)+f(v_4) \leq 3$. This implies that $f(v_4) \leq 1$. We claim that $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4) < 3$ or $f(w_1)+f(w_2)+f(w_1w_2)+f(w_2v_3)+f(v_4) < 3$. Suppose, to the contrary, that $f(w_1)+f(w_2)+f(w_1w_2)+f(w_2v_3)+f(v_4) = 3$ and $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4) = 3$. Then the function $g : V(T) \cup E(T) \to \{0,1,2\}$ defined by $g(w_1) = 1, g(v_2) = g(w_2v_4) = 2, g(w_2) = g(w_1w_2) = g(v_1) = g(v_3) = g(v_1v_2) = g(v_2v_3) = g(v_3v_4) = 0$ and $g(x) = f(x)$ otherwise, is a MRDF of T of weight less than $\omega(f)$, a contradiction. This proves the claim. Consider the following subcases.

Subcase 2.1. $f(w_1)+f(w_2)+f(w_1w_2)+f(w_2v_3)+f(v_4) = 3$ and $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4) = 2$.

Then we must have $f(v_2)=2$ and $f(v_1)=f(v_3)=f(v_1v_2)=f(v_2v_3)=f(v_4v_3)=0$. On the other hand, we may assume without loss of generality that $f(w_1)=1, f(w_2v_4)=2$ and $f(w_2)=f(w_1w_2)=0$. Let $T'=T-\{v_1,v_2,w_1\}$ and define $g:V(T')\cup E(T')\rightarrow\{0,1,2\}$ by $g(v_4)=2, g(w_2)=g(v_3)=g(v_4v_3)=g(v_4w_2)=0$ and $g(x)=f(x)$ otherwise. Clearly $\alpha'(T)=\alpha'(T)-2$ and g is a mixed Roman dominating function of weight at most $\omega(f)-3$ and by the induction hypothesis on T' we obtain the result.

Subcase 2.2. $f(w_1)+f(w_2)+f(w_1w_2)+f(w_2v_3)+f(v_4)=2$ and $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4)=3$.

Then we must have $f(v_4)=0$ and $f(v_4v_3)\leq 1$. On the other hand, we may assume that $f(w_2)=2$. Let $T'=T-\{v_1,v_2,v_3\}$. Then $\alpha'(T)=\alpha'(T)-2$ and the function f restricted to T' is a mixed Roman dominating function of weight at most $\omega(f)-3$ and the result follows from the induction hypothesis on T'.

Subcase 2.3. $f(w_1)+f(w_2)+f(w_1w_2)+f(w_2v_3)+f(v_4)=2$ and $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4)=2$.

Then we must have $f(v_2)=2$, $f(v_4)=f(w_2v_4)=f(v_3v_4)=0$. To Roman dominate the edge v_3v_4, there is an edge incident to v_4, say v_4u such that $f(v_4u)=2$. Let $T'=T-\{w_1,w_2\}$. Then $\alpha'(T')=\alpha'(T)-1$ and the function f, restricted to T' is a mixed Roman dominating function of weight at most $\omega(f)-2$ and by the induction hypothesis on T' we get the result.

Case 3. $\deg(v_3)=2$ and $\deg(v_4)=2$.

By symmetry, we may assume that $\deg(v_{k-1})=\deg(v_{k-2})=\deg(v_{k-3})=2$. First let $f(v_5)=2$. It is easy to see that $f(v_1)+f(v_2)+f(v_3)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)=3$. Then $\alpha'(T-\{v_1,v_2,v_3\})=\alpha'(T)-2$ and the function f, restricted to $T-\{v_1,v_2,v_3\}$ is a mixed Roman dominating function of weight at most $\omega(f)-3$ and the result follows by the induction hypothesis on T'. Now let $f(v_5)\leq 1$. It is easy to verify that $f(v_1)+f(v_2)+f(v_3)+f(v_4)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4v_5)\geq 4$. Consider the following subcases.

Subcase 3.1. $\deg(v_5)=2$.

Since $f(v_1)+f(v_2)+f(v_3)+f(v_4)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4v_5)\geq 4$, we may assume that $f(v_2)=f(v_4v_5)=2$. Let $T'=T-\{v_1,v_2,v_3,v_4,v_5\}$. Since the edge v_1v_2 belongs to every maximum matching, we deduce that $\alpha'(T')=\alpha'(T)-3$. Now the function $g:V(T')\cup E(T')\rightarrow\{0,1,2\}$ defined by $g(v_6)=\min\{2, f(v_6)+f(v_5v_6)\}$ and $g(x)=f(x)$ otherwise, is a mixed Roman dominating function of T' of weight at most $\omega(f)-4$. Now the result follows by the induction hypothesis on T'.

Subcase 3.2. $\deg(v_5)\geq 3$ and T has a path $v_5w_4w_3w_2w_1$ where $w_4\notin\{v_4,v_6\}$.

Then $\deg(w_1)=1$ and by the above cases and subcases, we may assume that $\deg(w_4)=\deg(w_3)=\deg(w_2)=2$. Since $f(v_5)\leq 1$, we have $f(w_1)+f(w_2)+f(w_3)+f(w_4)+f(w_1w_2)+f(w_2w_3)+f(w_3w_4)+f(w_4v_5)\geq 4$. Let $r=\max\{f(w_1)+f(w_2)+f(w_3)+f(w_4)+f(w_1w_2)+f(w_2w_3)+f(w_3w_4)+f(w_4v_5)-4, f(v_1)+f(v_2)+f(v_3)+f(v_4)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4v_5)-4\}$ and define $g:V(T)\cup E(T)\rightarrow\{0,1,2\}$ by $g(v_5v_4)=g(v_2)=g(v_5w_4)=g(w_2)=2, g(v_5)=f(v_5)+r$, $g(x)=0$ for $x\in(V(T_{v_4}\cup T_{w_4})\cup E(T_{v_4}\cup T_{w_4}))-\{v_2,w_2\}$ and $g(x)=f(x)$ otherwise. Clearly g is a $\gamma_R^*(T)$-function. Then the function g, restricted to $T-T_{v_4}$ is a mixed Roman dominating function of weight at most $\omega(f)-4$ and the result follows from the induction hypothesis.

Subcase 3.3. $\deg(v_5)\geq 3$ and v_5 is a stem.

Let u be a leaf adjacent to v_5. Since $f(v_5)\leq 1$, we have $f(u)+f(uv_5)\geq 1$ and $f(v_1)+f(v_2)+f(v_3)+f(v_4)+f(v_1v_2)+f(v_2v_3)+f(v_3v_4)+f(v_4v_5)\geq 4$. Let $T'=T-\{v_1,v_2,v_3\}$ and define $g:V(T')\cup E(T')\rightarrow\{0,1,2\}$ by $g(v_5)=2, g(v_4)=g(v_5v_4)=g(u)=g(uv_5)=0$ and $g(x)=f(x)$ otherwise. Clearly g is a $\gamma_R^*(T')$-function of weight at most $\omega(f)-3$. As $\alpha'(T')=\alpha'(T)-2$, we deduce from induction hypothesis on T' that

$$\gamma_R^*(T)\geq\omega(g)+3\geq\frac{4(\alpha'(T')-2)}{3}+3>\frac{4\alpha'(T)}{3}.$$

Subcase 3.4. $\deg(v_5) \geq 3$ and v_5 has a neighbor w_2 with depth 1.

Let w_1 be a leaf adjacent to w_2. Since all end-stems of T have degree two, we have $\deg(w_2) = 2$. Since $f(v_5) \leq 1$, we must have $f(w_1) + f(w_2) + f(w_1w_2) + f(v_5w_2) \geq 2$ and $f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_1v_2) + f(v_2v_3) + f(v_3v_4) + f(v_4v_5) \geq 4$. We may assume without loss of generality that $f(v_2) = f(v_4v_5) = 2$. Let $T' = T - \{w_1, w_2\}$. Then the function f restricted to T' is a MRDF of T' of weight at most $\omega(f) - 2$ and it follows from the induction hypothesis on T' and the fact $\alpha'(T') = \alpha'(T) - 1$ that

$$\gamma_R^*(T) \geq \omega(f|_{T'}) + 2 \geq \frac{4(\alpha'(T') - 1)}{3} + 2 > \frac{4\alpha'(T)}{3}.$$

Subcase 3.5. $\deg(v_5) \geq 3$ and v_5 has a neighbor w_3 with depth 2.

Let $v_5w_3w_2w_1$ be a path in T. Since all end-stems of T have degree two, we have $\deg(w_2) = 2$. By Observation 4.7, $f(w_1) + f(w_2) + f(w_1w_2) + f(w_3w_2) \geq 2$. If $\deg(w_3) \geq 3$, then using an argument similar to that described in Case 1, the result follows. Assume that $\deg(w_3) = 2$. Similarly we may assume that all children of v_5 with depth 2 have degree 2. Considering above subcases, we may assume that all children of v_5 have depth two. Since every pendant edge of T belongs to any maximum matching, we conclude that w_3 is the only child of v_5 with depth two. Let $T' = T - \{v_1, v_2, v_3, v_4, v_5, w_1, w_2, w_3\}$. Clearly the function f restricted to T' is a MRDF of T' and using the induction hypothesis on T' and the fact $\alpha'(T') = \alpha'(T) - 4$, we obtain

$$\gamma_R^*(T) \geq \omega(f|_{T'}) + 6 \geq \frac{4(\alpha'(T') - 4)}{3} + 6 > \frac{4\alpha'(T)}{3}$$

and the proof is complete. □

We conclude this section with an open problem.

Problem. For any tree T, $\gamma_R^*(T) \geq \frac{3\alpha'(T)}{2}$.

5. Conclusion

In this paper, we continued the study of the mixed Roman domination in graphs introduced by Abdollahzadeh Ahangar *et al.* [4]. We first showed that computing the mixed Roman domination number in a graph is NP-complete even for bipartite graphs and we characterized graphs of odd order n for which the mixed Roman domination number equals n. We also focused on providing various bounds for general graphs and for trees. For arbitrary connected graphs, two lower bounds in terms of the matching number have been established, while for trees we have given two upper bounds in terms of the domination number and the number of leaves and stems.

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References

[1] H. Abdollahzadeh Ahangar, M. Chellali and S.M. Sheikholeslami, Signed double Roman domination in graphs. *Disc. Appl. Math.* **257** (2019) 1–11.

[2] H. Abdollahzadeh Ahangar, M. Chellali and S.M. Sheikholeslami, Outer independent double Roman domination. *Appl. Math. Comput.* **364** (2020) 124617.

[3] H. Abdollahzadeh Ahangar, M. Chellali, S.M. Sheikholeslami and J.C. Valenzuela-Tripodoro, Total Roman {2}-domination in graphs. *Discuss. Math. Graph Theory* (to appear) DOI: https://doi.org/10.7151/dmgt.2316 (2020).

[4] H. Abdollahzadeh Ahangar, T.W. Haynes and J.C. Valenzuela-Tripodoro, Mixed Roman domination in graphs. *Bull. Malays. Math. Sci. Soc.* **40** (2017) 1443–1454.

[5] M. Adabi, E. Ebrahimi Targhi, N. Jafari Rad and M.S. Moradi, Properties of independent Roman domination in graphs. *Australas. J. Comb.* **52** (2012) 11–18.

[6] Y. Alavi, M. Behzad, L. Lesniak and E.A. Nordhaus, Total matchings and total coverings of graphs. *J. Graph Theory* **1** (1977) 135–140.
[7] Y. Alavi, J.Q. Liu, J.F. Wang and Z.F. Zhang, On total covers of graphs. *Disc. Math.* **100** (1992) 229–233.
[8] J. Amjadi, Total Roman domination subdivision number in graphs. *Commun. Comb. Optim.* **5** (2020) 157–168.
[9] D. Bange, A.E. Barkauskas and P.J. Slater, Efficient dominating sets in graphs, edited by R.D. Ringeisen and F.S. Roberts. In: Applications of Discrete Math. SIAM, Philadelphia, PA (1988) 189–199.
[10] S. Bermudo, H. Fernau and J.M. Sigarreta, The differential and the Roman domination number of a graph. *Appl. Anal. Disc. Math.* **8** (2014) 155–171.
[11] M. Chellali and N. Jafari Rad, A note on the independent Roman domination in unicyclic graphs. *Opuscula Math.* **32** (2012) 715–718.
[12] M. Chellali and N. Jafari Rad, Strong equality between the Roman domination and independent Roman domination numbers in trees. *Discuss. Math. Graph Theory* **33** (2013) 337–346.
[13] E.J. Cockayne, P.M. Dreyer, Sr., S.M. Hedetniemi and S.T. Hedetniemi, Roman domination in graphs. *Disc. Math.* **278** (2004) 11–22.
[14] E. Ebrahimi Targhi, N. Jafari Rad, C. M. Mynhardt and Y. Wu, Bounds for the independent Roman domination number in graphs. *J. Comb. Math. Comb. Comput.* **80** (2012) 351–365.
[15] P. Erdös and A. Meir, On total matching numbers and total covering numbers of complementary graphs. *Disc. Math.* **19** (1977) 229–233.
[16] G. Hao, L. Volkmann and D.A. Mojdeh, Total double Roman domination in graphs. *Commun. Comb. Optim.* **5** (2020) 27–39.
[17] G. Gunther, B. Hartnell, L.R. Markus and D. Rall, Graphs with unique minimum dominating sets. *Congr. Numer.* **101** (1994) 55–63.
[18] P. Hatami, An approximation algorithm for the total covering problem. *Discuss. Math. Graph Theory* **27** (2007) 553–558.
[19] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs. Marcel Dekker, New York, NY (1998).
[20] S.T. Hedetniemi, R.R. Rubalcaba, P.J. Slater and M. Walsh, Few compare to the great Roman empire. *Congr. Numer.* **217** (2013) 129–136.
[21] J.K. Lan and G.J. Chang, On the mixed domination problem in graphs. *Theor. Comput. Sci.* **476** (2013) 84–93.
[22] C.S. ReVelle and K.E. Rosing, Defendens imperium romanum: a classical problem in military strategy. *Am. Math. Mon.* **107** (2000) 585–594.
[23] E. Sampathkumar and S.S. Kamath, Mixed domination in graphs. *Sankayā: Indian J. Stat.* **54** (1992) 399–402.
[24] I. Stewart, Defend the Roman empire! *Sci. Am.* **281** (1999) 136–139.
[25] L. Volkmann, Weak signed Roman domination in graphs. *Commun. Comb. Optim.* **5** (2020) 111–123.