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NECESSARY OPTIMALITY CONDITIONS FOR A FRACTIONAL MULTIOBJECTIVE OPTIMIZATION PROBLEM

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Abstract. In this paper, we are concerned with a fractional multiobjective optimization problem (P). Using support functions together with a generalized Guignard constraint qualification, we give necessary optimality conditions in terms of convexificators and the Karush–Kuhn–Tucker multipliers. Several intermediate optimization problems have been introduced to help us in our investigation.

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1. Introduction

In the last years, set-valued optimization problems have been considered by many researchers [2,3,5,11,12,16,19,25]. This is due to the fact that many optimization problems encountered in economics and other fields involve set-valued mappings constraints and set-valued mappings objectives. Corley [5] formulated optimality conditions for convex and nonconvex multiobjective problems in terms of Clarke derivative. Bao and Mordukhovich [3] introduced the normal subdifferential of set-valued mappings and deduced existence and necessary optimality conditions with respect to a generalized weak Pareto preference for set-valued optimization problems involving equilibrium constraints in terms of coderivatives. Gadhi [11], using the convex separation principle, suggested optimality conditions for a D.C. (Difference of Convex) set-valued optimization problem in terms of the weak and strong subdifferentials of set-valued mappings introduced by Sawaragi and Tanino [24] and Baier and Jahn [2]. Gadhi and Jawhar [12] used the extremal principle developed by Mordukhovich [21] to get necessary optimality conditions for a fractional multiobjective optimization problem.

Let $n, p, q, m \in \mathbb{N}$. Let $\varphi_j : \mathbb{R}^p \to \mathbb{R}$ and $\phi_t : \mathbb{R}^p \to \mathbb{R}, j \in J = \{1, \dots, q\}, t \in T = \{1, \dots, m\}$, be given lower semicontinuous functions and let $F_i : \mathbb{R}^p \rightrightarrows \mathbb{R}$ and $G_i : \mathbb{R}^p \rightrightarrows \mathbb{R}, i \in I = \{1, \dots, n\}$, be given locally Lipschitz set-valued mappings such that

$$F_i(C) = \underset{x\in C}{\cup} F_i(x), \quad G_i(C) = \underset{x\in C}{\cup} G_i(x)$$

and

$$y_i \geq 0, \quad z_i > 0 \quad \text{for all} \quad i \quad \text{and all} \quad y_i \in F_i(x), \quad z_i \in G_i(x), \quad x \in C, \tag{1.1}$$

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where

$$C = \{x \in \mathbb{R}^p : \varphi_j(x) \le 0, \quad \phi_t(x) = 0, \quad j \in J, \quad t \in T\}.$$

In this paper, we are concerned with the following fractional multiobjective optimization problem

$$(P) : \begin{cases} \min\limits_x H(x) \\ \text{subject to: } x \in C \end{cases}$$

where

$$h = (h_1, \dots, h_n) \in H(x) \Leftrightarrow \forall i \in I, \quad \exists y_i \in F_i(x), \quad z_i \in G_i(x) \quad \text{such that} \quad h_i = \frac{y_i}{z_i}\cdot$$

This problem can be seen either as a fractional multiobjective optimization problem or as a set-valued optimization problem. Due to this structure, it brings together several other problems previously studied by several authors. In it's framework, (P) includes convex and D.C. set-valued optimization problems, vector fractional optimization problems, mathematical programming problems, etc. Our intention is focused on finding optimality conditions for (P).

Let $\overline{h} \in H(\overline{x})$. The point $(\overline{x}, \overline{h})$ is said to be a weak local Pareto minimal point with respect to $\mathbb{R}^n_+$ of the problem (P) if there exists a neighborhood V of $\overline{x}$ such that $H(V \cap C) \subset \overline{h} + \mathbb{R}^n \setminus \left(-int\mathbb{R}^n_+\right)$; *i.e.*

$$h - \overline{h} \notin -int\mathbb{R}^n_+ \quad \forall h \in H(x), \quad \forall x \in V \cap C. \tag{1.2}$$

Here, $int\mathbb{R}^n_+$ denotes the interior of the nonnegative orthant $\mathbb{R}^n_+$ of the n-dimensional space $\mathbb{R}^n$.

Recently, the idea of convexificators has been used to extend, unify and sharpen the results in various aspects of optimization [7, 10, 17, 18, 20]. In [17], Jeyakumar and Luc gave a revised version of convexificators by introducing the notion of a convexificator which is a closed set but is not necessarily bounded or convex. In [10], the concepts of lower and upper semiregular convexificators were used to obtain the necessary optimality conditions for an inequality constrained mathematical programming problem.

In order to get necessary optimality conditions, we use support functions of the set-valued mappings together with a generalized Guignard constraint qualification. The obtained results are given in terms of convexificators and the Karush–Kuhn–Tucker multipliers.

The outline of the paper is as follows: preliminaries and basic definitions are given in Section 2; the main results are established in Section 3; a conclusion is given in Section 4.

2. Preliminaries

In this section, we give some definitions, notations and results, which will be used in the sequel. For a subset D of $\mathbb{R}^p$, the notations $cl\ D, conv\ D, \overline{co}\ D(= cl\ conv\ D)$, $cone\ D$, $\overline{cone}\ D(= cl\ cone\ D)$ and D^- stand for the closure of D, the convex hull of D, the closed convex hull of D, the convex cone generated by D, the closed convex cone generated by D and the negative polar cone of D, respectively.

Let D be a subset of $\mathbb{R}^p$ and $x \in cl\ D$. The contingent cone $T(D, x)$ to D at x is defined by

$$T(D, x) = \{v \in \mathbb{R}^p : \exists t_n \searrow 0 \quad \text{and} \quad \exists v_n \to v \quad \text{such that} \quad x + t_n v_n \in D\}.$$

Let $\Psi : \mathbb{R}^p \rightrightarrows \mathbb{R}$ be a set-valued mapping from $\mathbb{R}^p$ into $\mathbb{R}$. In the sequel, we denote the domain and the graph of Ψ respectively by

$$dom(\Psi) := \{x \in \mathbb{R}^p : \Psi(x) \neq \emptyset\} \quad \text{and} \quad gr(\Psi) := \{(x, y) \in \mathbb{R}^p \times \mathbb{R} : y \in \Psi(x)\}.$$

The set-valued mapping Ψ is said to be locally Lipschitz at $x \in \mathbb{R}^p$ if there exists a neighborhood U of x, such that for some constant k, we have

$$\Psi(x_1) \subset \Psi(x_2) + k \left\|x_1 - x_2\right\| \mathbb{B}_{\mathbb{R}}, \quad \forall x_1, x_2 \in U.$$

Here $\mathbb{B}_{\mathbb{R}}$ indicates the unite ball of $\mathbb{R}$. The number k is called a Lipschitz-constant for Ψ at x. Since the convexity plays an important role in the following investigations, recall the definition of the cone-convex mappings.

Definition 2.1 ([5])**.** Let $\Omega \subset \mathbb{R}^p$ be a convex set. The set-valued mapping $F : \Omega \rightrightarrows \mathbb{R}$ is said to be $\mathbb{R}^+$-convex on Ω, if for all $x_1, x_2 \in \Omega$ and all $\lambda \in [0,1]$, it holds

$$\lambda\Psi(x_1) + (1-\lambda)\Psi(x_2) \subset \Psi\left(\lambda x_1 + (1-\lambda)\, x_2\right) + \mathbb{R}^+.$$

Let $x \in \mathbb{R}^p$. Denoting by

$$\sigma(x) = \inf_{y\in\Psi(x)} y \quad \text{and} \quad \xi(x) = \sup_{y\in\Psi(x)} y$$

the lower and upper support functions of Ψ at x, we have the following result.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^p$ be a convex set. The function σ is convex on Ω if the set-valued mapping Ψ is $\mathbb{R}^+$-convex on Ω.*

Proof. Let $x_1, x_2 \in \Omega, y_1 \in \Psi(x_1), y_2 \in \Psi(x_2), \lambda \in [0,1], x_\lambda = \lambda x_1 + (1-\lambda)\, x_2$. From the $\mathbb{R}^+$-convexity assumption of Ψ, there exist $\overline{y}_\lambda \in \Psi\left(x_\lambda\right)$ and $p \in \mathbb{R}^+$ such that $\lambda y_1 + (1-\lambda)\, y_2 = \overline{y}_\lambda + p$. Since $p \in \mathbb{R}^+$, one has

$$\lambda y_1 + (1-\lambda)\, y_2 \geq \overline{y}_\lambda, \quad \text{for all} \quad y_1 \in \Psi(x_1) \quad \text{and} \quad y_2 \in \Psi(x_2).$$

Then,

$$\lambda\sigma(x_1) + (1-\lambda)\,\sigma(x_2) \geq \overline{y}_\lambda.$$

Since

$$\sigma\left(x_\lambda\right) = \inf_{y\in\Psi(x_\lambda)} y \quad \text{and} \quad \overline{y}_\lambda \in \Psi\left(x_\lambda\right)$$

we deduce

$$\lambda\sigma(x_1) + (1-\lambda)\,\sigma(x_2) \geq \sigma(x_\lambda),$$

which means that σ is convex on Ω. □

Remark 2.3. The function ξ is concave on Ω if the set-valued mapping Ψ is $\mathbb{R}^+$-concave on Ω.

Remark 2.4. According to Property 1.5 of [8], the functions σ and ξ are locally Lipschitz in x, and k is a Lipschitz-constant for σ and ξ at x if $k \in \mathbb{R}$ is a Lipschitz-constant for Ψ at x. See also [9].

Now, we recall the definitions related to convexificators given by Jeyakumar and Luc [17] and Dutta and Chandra [10]. Let $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ be a given function and let $x \in \mathbb{R}^p$ where $f(x)$ is finite. The expressions

$$f_d^-(x,v) = \liminf_{t\searrow 0} \left[f(x+tv) - f(x)\right]/t$$

and

$$f_d^+(x,v) = \limsup_{t\searrow 0} \left[f(x+tv) - f(x)\right]/t$$

denote respectively, the lower and upper Dini directional derivatives of f at x in the direction v.

Definition 2.5. The function $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ is said to have an upper convexificator (UCF) $\partial^* f(x)$ at x if $\partial^* f(x) \subset \mathbb{R}^p$ is closed and, for each $v \in \mathbb{R}^p$,

$$f_d^-(x,v) \leq \sup_{x^*\in\partial^* f(x)} \left\langle x^*, v\right\rangle.$$

Definition 2.6. The function $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ is said to have a lower convexificator (LCF) $\partial_* f(x)$ at x if $\partial_* f(x) \subset \mathbb{R}^p$ is closed and, for each $v \in \mathbb{R}^p$,

$$f_d^+(x,v) \geq \inf_{x^*\in\partial_* f(x)} \left\langle x^*, v\right\rangle.$$

A closed set $\partial^* f(x) \subset \mathbb{R}^n$ is said to be a convexificator of f at x if it is both an upper and lower convexificator of f at x.

Remark 2.7. The convexificators are neither necessarily compact nor convex [7]. These relaxations allow applications to a large class of nonsmooth continuous functions. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = - \mid x \mid$, admits a non-convex convexificator $\partial^* f(0) = \{-1, 1\}$ at 0.

Remark 2.8. Let $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ be finite at a point $x \in \mathbb{R}^p$. If f is locally Lipschitz at x, then $\partial^\circ f(x)$ is a convexificator of f at x. However, the convex hull of a convexificator of a locally Lipschitz function may be strictly contained in the Clarke subdifferential. Here, $\partial^\circ f(x)$ designs the Clarke generalized subdifferential of f at x defined by

$$\partial^\circ f(x) := \left\{ \eta \in \mathbb{R}^p : \limsup_{y \to x, t \searrow 0} \frac{f(y+tv) - f(y)}{t} \geq \langle \eta, v \rangle \quad \forall v \in \mathbb{R}^p \right\}.$$

Example 2.9 ([17])**.** Let $f : \mathbb{R}^p \to \mathbb{R}$ be a function defined by $f(x, y) = |x| - |y|$. It can easily be verified that

$$\partial^* f(0) = \{(1, -1), (-1, 1)\}$$

is a convexificator of f at 0, whereas

$$\partial^\circ f(0) = conv(\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}).$$

It is clear that

$$conv\,(\partial^* f(0)) \subset \partial^\circ f(0).$$

Clearly, this example shows that certain results such as the necessary optimality conditions that are expressed in terms of $\partial^* f(x)$ may provide sharp conditions even for locally Lipschitz functions.

The following definition has been proposed by Dutta and Chandra. For more details, see [10].

Definition 2.10 ([10])**.** The function $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ is said to have an upper semi-regular convexificator (USRCF) $\partial^* f(x)$ at x if $\partial^* f(x)$ is an upper convexificator at x and, for each $v \in \mathbb{R}^p$,

$$f_d^+(x, v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v \rangle .$$

Remark 2.11. The Clarke [4], Penot [23] and Mordukhovich [22] subdifferentials are upper semi-regular convexificators of f when f is a locally Lipschitz function.

In Proposition 2.12, we recall the chain rule for composite functions in terms of convexificators established by Jeyakumar and Luc [17].

Proposition 2.12 ([17])**.** *Let $f = (f_1, \ldots, f_n)$ be a continuous function from $\mathbb{R}^p$ to $\mathbb{R}^n$, and g be a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose that, for each $i = 1, \ldots, n$, f_i admits a bounded convexificator $\partial^* f_i(\bar{x})$ at $\bar{x}$ and that g admits a bounded convexificator $\partial^* g(f(\bar{x}))$ at $f(\bar{x})$. For each $i = 1, \ldots, n$, if $\partial^* f_i$ is upper semicontinuous at $\bar{x}$ and $\partial^* g$ is upper semicontinuous at $f(\bar{x})$, then the set*

$$\begin{aligned}\partial^*(g \circ f)(\bar{x}) &= \partial^* g(f(\bar{x}))\,(\partial^* f_1(\bar{x}), \ldots, \partial^* f_n(\bar{x})) \\ &= \left\{ \sum_{i=1}^n a_i \partial^* f_i(\bar{x}) : (a_1, \ldots, a_n) \in \partial^* g\,(f\,(\bar{x})) \right\}\end{aligned}$$

is a convexificator of $g \circ f$ at $\bar{x}$.

The following result for calculating a convexifactor for a max function has been given by Dutta and Chandra [10].

Proposition 2.13 ([10])**.** *Consider the function $f(x) = \max\{f_i(x) : i = 1, \dots, m\}$, where each $f_i : \mathbb{R}^p$ to $\mathbb{R}$ is continuous for $i = 1, \dots, m$. Assume that each f_i admits a bounded convexifactor $\partial^* f_i(\bar{x})$ at $\bar{x} \in \mathbb{R}^p$ and that $\partial^* f_i$ is upper semicontinuous at $\overline{x}$. Then, f admits a convexificator which is convex, compact, and is given as*

$$\partial^* f(\bar{x}) = conv\left\{ \bigcup_{i \in I(\bar{x})} \partial^* f_i(\bar{x}) \right\},$$

where

$$I(\bar{x}) = \{i : f_i(\bar{x}) = f(\bar{x})\}.$$

The proposition below, which is a variant of Theorem 2.8.2 from [4] has been proven by Dempe and Gadhi [6].

Proposition 2.14 ([6])**.** *Let $\mathbb{T}$ be a sequentially compact space, $\bar{x} \in \mathbb{R}^p$, $f_t : \mathbb{R}^p \to \overline{\mathbb{R}}$, and*

$$h(x) = \sup_{t \in \mathbb{T}} \{f_t(x)\} \quad \text{and} \quad J(\bar{x}) = \{t \in \mathbb{T} : f_t(\bar{x}) = h(\bar{x})\}.$$

Suppose that there exists a neighborhood U of $\bar{x}$ in $\mathbb{R}^p$ such that for each $t \in \mathbb{T}$, the function f_t is finite on U and admits a bounded convexificator on U. If in addition $t \mapsto f_t$ is upper semicontinuous then, $cl\Big(conv\,\{\partial^* f_t\,(\bar{x}) : t \in J(\bar{x})\}\Big)$ is a convexificator of h at $\bar{x}$.

Lemma 2.15 ([13])**.** *Let A and B be two non empty subsets of $\mathbb{R}^n$. Then,*

$$conv\,(A \cup B) = \underset{0 \le \alpha \le 1}{\cup} (\alpha\, conv(A) + (1-\alpha) conv(B)).$$

3. Necessary optimality conditions

In this section, we maintain the notations given in the previous section and we give necessary optimality conditions for the fractional multiobjective optimization problem (P). In the sequel $F : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ will be the set-valued mappings defined by

$$F(x) = (F_1(x), \dots, F_n(x)) = F_1(x) \times F_2(x) \times \dots \times F_n(x) \quad \text{for all} \quad x \in dom\,(F) = \bigcap_{i \in I} dom\,(F_i)$$

and

$$G(x) = (G_1(x), \dots, G_n(x)) = G_1(x) \times G_2(x) \times \dots \times G_n(x) \quad \text{for all} \quad x \in dom(G) = \bigcap_{i \in I} dom(G_i).$$

Let $\overline{x} \in C, \overline{y} = (\overline{y}_1, \dots, \overline{y}_n), \overline{z} = (\overline{z}_1, \dots, \overline{z}_n)$ and $\overline{h} = \left(\overline{h}_1, \dots, \overline{h}_n\right)$ such that

$$\overline{y}_i \in F_i(\overline{x}), \overline{z}_i \in G_i(\overline{x}) \quad \text{and} \quad \overline{h}_i = \frac{\overline{y}_i}{\overline{z}_i}, \quad \forall i \in I.$$

The following constraint qualification will be used in Theorem 3.4.

Definition 3.1. We say that the nonsmooth Guignard constraint qualification (NGCQ) holds at $\overline{x} \in C$ with respect to $\partial^*\varphi_j(\overline{x})$ and $\partial^*\phi_t(\overline{x})$, $j \in J, t \in T$ if

$$[T(C,\overline{x})]^- \subseteq \overline{cone}\ \Lambda(\overline{x})$$

where

$$\Lambda(\overline{x}) = \left(\bigcup_{j\in J_0(\overline{x})} \partial^*\varphi_j\left(\overline{x}\right)\right) \cup \left(\bigcup_{t\in T} \partial^*\phi_t(\overline{x})\right) \cup \left(\bigcup_{t\in T} \partial^*\left(-\phi_t\right)(\overline{x})\right)$$

and

$$J_0(\overline{x}) = \{j \in J : \varphi_j(\overline{x}) = 0\}.$$

Comparing the above constraint qualification with that of Abadie (ACQ) [14], which requires equality between tangent cone $T(C,\overline{x})$ and the linearized cone $L\left(C,\overline{x}\right) = [\Lambda(\overline{x})]^-$, one can conclude that (NGCQ) is weaker than (ACQ). Indeed, a necessary condition for the Abadie constraint qualification (ACQ) to be satisfied is that $T(C,\overline{x})$ is a polyhedral convex cone. This condition is not sufficient since it is known, see [1] for a simple standard optimization example, that the tangent cone $T(C,\overline{x})$ might be polyhedral without being equal to its linearized cone $L(C,\overline{x})$.

Example 3.2. Consider the problem

$$\min\ f(x,y) = x^2 + y^2 \quad \text{subject to} \quad x,y \geq 0 \quad \text{and} \quad xy = 0.$$

The global minimizer is $(0,0)$. It is easy to see that

$$T\left(C,(0,0)\right) = \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha,\beta \geq 0 \quad \text{and} \quad \alpha\beta = 0\}, \quad L\left(C,(0,0)\right) = \left\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha,\beta \geq 0\right\}$$

$$\overline{cone}\ \Lambda(\overline{x}) = \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha,\beta \leq 0\}.$$

Then,

$$[T\left(C,(0,0)\right)]^- = \overline{cone}\ \Lambda(\overline{x}) \quad \text{and} \quad L\left(C,(0,0)\right) \subsetneq T(C,(0,0)).$$

Hence, the Guignard constraint qualification holds at $(0,0)$ unlike that of Abadie.

Lemma 3.3. *Let $\overline{x} \in C$. Suppose that $\left(\overline{x},\overline{h}\right)$ is a weak local Pareto minimal point with respect to $\mathbb{R}^n_+$ of the problem (P). Then, there exist $\overline{y} \in F(\overline{x}), \overline{z} \in G(\overline{x})$ such that $((\overline{x},\overline{y},\overline{z}),0)$ is a weak local Pareto minimal point with respect to $\mathbb{R}^n_+$ of the multiobjective optimization problem*

$$(P_1): \quad \begin{cases} \text{Minimize}\, m(x,y,z) = (m_1(x,y,z),\ldots,m_n(x,y,z)) \\ \text{subject to:}\, (x,y,z) \in S \end{cases}$$

where

$$S = \left\{ \begin{array}{l} (x,y,z) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n : \\ x \in C, y \in F(x), z \in G(x) \end{array} \right\}$$

and

$$m_i(x,y,z) := y_i - \overline{h}_i z_i, \quad \forall i \in I.$$

Proof. By contrary, suppose that, for any neighborhood U of $(\overline{x},\overline{y},\overline{z})$, there exists $u^0 = \left(x^0,y^0,z^0\right) \in U \cap S$ such that

$$m\left(x^0,y^0,z^0\right) - m\left(\overline{x},\overline{y},\overline{z}\right) \in -int\mathbb{R}^n_+.$$

Since $\overline{y}_i - \overline{h}_i\overline{z}_i = 0$, we have

$$y_i^0 - \overline{h}_i z_i^0 < 0 \qquad \forall i \in I.$$

Consequently,

$$\frac{y_i^0}{z_i^0} - \frac{\overline{y}_i}{\overline{z}_i} < 0 \qquad \forall i \in I;$$

a contradiction with the fact that $(\overline{x}, \overline{h})$ be a weak local Pareto minimal point with respect to $\mathbb{R}_+^n$ of the problem (P). ☐

Theorem 3.4 provides necessary optimality conditions for the multiobjective optimization problem (P).

Theorem 3.4. *Let $\overline{x} \in C$. Suppose that $\varphi_j, \phi_t, \sigma_i$ and ξ_i admit bounded upper semi-regular convexificator $\partial^*\varphi_j(\overline{x}), \partial^*\phi_t(\overline{x})$, $\partial^*\sigma_i(\overline{x})$ and $\partial^*(-\xi_i), i \in I, j \in J, t \in T$. Let $(\overline{x}, \overline{h})$ be a weak local Pareto minimal point with respect to $\mathbb{R}_+^n$ of the problem (P) such that the nonsmooth Guignard constraint qualification* (NGCQ) *holds at $\overline{x}$. Then, there exist scalars $\alpha_i \geq 0$ such that $\sum_{i\in I}\alpha_i = 1$,*

$$\begin{aligned} 0 \in &\sum_{i\in I}\alpha_i co\left[\partial^*\sigma_i(\overline{x}) + \overline{h}_i\partial^*\left(-\xi_i\right)(\overline{x})\right] \\ &+ cl\left(\sum_{j\in J_0(\overline{x})} cone\ \partial^*\varphi_j(\overline{x}) + \sum_{t\in T} cone\ \partial^*\phi_t(\overline{x}) + \sum_{t\in T} cone\ \partial^*\left(-\phi_t\right)(\overline{x})\right) \end{aligned} \tag{3.1}$$

$$\overline{h}_i\overline{z}_i = \overline{h}_i\xi_i\left(\overline{x}\right) \quad \text{and} \quad \overline{y}_i = \sigma_i(\overline{x})$$

where

$$\sigma_i(x) = \inf_{y\in F_i(x)} y \quad \text{and} \quad \xi_i(x) = \sup_{z\in G_i(x)} z.$$

Proof. Since $(\overline{x}, \overline{h})$ is a weak local Pareto minimal point with respect to $\mathbb{R}_+^n$ of the problem (P), by Lemma 3.3, it is also a weak local Pareto minimal point with respect to $\mathbb{R}_+^n$ of the problem (P_1). Thus, there exists a neighborhood U of $(\overline{x}, \overline{y}, \overline{z})$ such that

$$m(x, y, z) - m\left(\overline{x}, \overline{y}, \overline{z}\right) \notin -int\mathbb{R}_+^n, \quad \forall (x, y, z) \in U \cap S.$$

Then, we can find $i \in I$ satisfying

$$m_i(x, y, z) - m_i\left(\overline{x}, \overline{y}, \overline{z}\right) \geq 0.$$

Thus, there exists a neighborhood V of $\overline{x}$ such that

$$\left(y_i - \overline{h}_i z_i\right) - \left(\overline{y}_i - \overline{h}_i\overline{z}_i\right) \geq 0, \quad \forall y_i \in F_i(x), \quad \forall z_i \in G_i(x), \quad \forall x \in V \cap C. \tag{3.2}$$

– Let us prove that $\overline{h}_i\overline{z}_i = \overline{h}_i\xi_i(\overline{x})$ and $\overline{y}_i = \sigma_i(\overline{x})$.
  - For $x = \overline{x}$ and $y_i = \overline{y}_i$, we get

$$\overline{h}_i\overline{z}_i \geq \overline{h}_i z_i, \quad \forall z_i \in G_i(\overline{x}).$$

    - Since $\overline{h}_i \geq 0$, we get

$$\overline{h}_i\overline{z}_i \geq \overline{h}_i\xi_i(\overline{x}). \tag{3.3}$$

    - Since $\overline{z}_i \in G_i(\overline{x})$ and since $\xi_i(\overline{x}) = \sup_{z\in G_i(\overline{x})} z$, we also have $\overline{z}_i \leq \xi_i(\overline{x})$. Thus,

$$\overline{h}_i\overline{z}_i \leq \overline{h}_i\xi_i(\overline{x}). \tag{3.4}$$

    From (3.3) and (3.4), it follows $\overline{h}_i\overline{z}_i = \overline{h}_i\xi_i\left(\overline{x}\right)$.

- For $x = \overline{x}$ and $z_i = \overline{z}_i$, we get

$$y_i \geq \overline{y}_i, \quad \forall y_i \in F_i(\overline{x}).$$

Then,

$$\sigma_i(\overline{x}) \geq \overline{y}_i. \tag{3.5}$$

Since $\overline{y}_i \in F_i(\overline{x})$ and since $\sigma_i(\overline{x}) = \inf_{y \in F_i(\overline{x})} y$, we also have

$$\sigma_i(\overline{x}) \leq \overline{y}_i. \tag{3.6}$$

From (3.5) and (3.6), it follows $\overline{y}_i = \sigma_i(\overline{x})$.

– From (3.2), since $\overline{h}_i \overline{z}_i = \overline{h}_i \xi_i(\overline{x})$ and $\overline{y}_i = \sigma_i(\overline{x})$, we have

$$y_i - \sigma_i(\overline{x}) \geq \overline{h}_i z_i - \overline{h}_i \xi_i(\overline{x}), \quad \forall y_i \in F_i(x), \quad \forall z_i \in G_i(x), \quad \forall x \in V \cap C.$$

Thus,

$$\sigma_i(x) - \sigma_i(\overline{x}) \geq \overline{h}_i \xi_i(x) - \overline{h}_i \xi_i(\overline{x}), \quad \forall x \in V \cap C.$$

Setting

$$\psi(x) := \max_{1 \leq i \leq n} \left[\sigma_i(x) - \overline{h}_i \xi_i(x) - \sigma_i(\overline{x}) + \overline{h}_i \xi_i\left(\overline{x}\right) \right],$$

we have

$$\psi(\overline{x}) = 0 \quad \text{and} \quad \psi(x) \geq 0, \quad \forall x \in V \cap C.$$

We deduce that $\overline{x}$ minimizes locally ψ over C.

- Let $d \in T(C, \overline{x})$. By definition there exist $t_n \searrow 0$ and $d_n \to d$ such that $\overline{x} + t_n d_n \in C$ for all n. For n large enough, $\overline{x} + t_n d_n \in V$. Moreover,

$$\frac{\psi\left(\overline{x} + t_n d_n\right) - \psi\left(\overline{x}\right)}{t_n} \geq 0.$$

Remarking that

$$\frac{\psi\left(\overline{x} + t_n d_n\right) - \psi\left(\overline{x}\right)}{t_n} = \frac{\psi\left(\overline{x} + t_n d_n\right) - \psi\left(\overline{x} + t_n d\right)}{t_n} + \frac{\psi\left(\overline{x} + t_n d\right) - \psi(\overline{x})}{t_n}$$

and that ψ is locally Lipschitz, we have

$$\begin{aligned} \psi_d^+\left(\overline{x}, d\right) &= \limsup_n \frac{\psi\left(\overline{x} + t_n d\right) - \psi(\overline{x})}{t_n} \\ &= \limsup_n \frac{\psi\left(\overline{x} + t_n d_n\right) - \psi(\overline{x})}{t_n} \\ &\geq 0. \end{aligned}$$

Thus,

$$\psi_d^+\left(\overline{x}, d\right) \geq 0, \quad \text{for all} \quad d \in T(C, \overline{x}).$$

- Using the upper semiregularity of $\partial^* \psi(\overline{x})$ at $\overline{x}$, we get

$$\sup_{\eta \in \partial^* \psi(\overline{x})} \langle \eta, d \rangle \geq 0, \quad \text{for all} \quad d \in T(C, \overline{x}).$$

Here, $\partial^* \psi(\overline{x})$ is a bounded (USRCF) of ψ at $\overline{x}$. From this, we can conclude from the calculus of the support functions that

$$0 \in \overline{\overline{co}\left(\partial^* \psi\left(\overline{x}\right)\right) + \left[T\left(C, \overline{x}\right)\right]^-}.$$

– Since the nonsmooth generalized Guignard constraint qualification (3.1) holds at $\overline{x}$, we get

$$0 \in cl \left[cl\ conv\left(\partial^{*}\psi\left(\overline{x}\right)\right) + \overline{cone}\left(\left(\bigcup_{j\in J_0(\overline{x})} \partial^{*}\varphi_j\left(\overline{x}\right) \right) \bigcup \left(\bigcup_{t\in T} \partial^{*}\phi_t(\overline{x}) \right) \bigcup \left(\bigcup_{t\in T} \partial^{*}\left(-\phi_t\right)\left(\overline{x}\right) \right) \right) \right].$$

Since $\partial^{*}\psi(\overline{x})$ is also a closed set, $conv\ (\partial^{*}\psi(\overline{x}))$ is a compact set (see [15], Thm. 1.4.3); consequently,

$$0 \in conv\ \left(\partial^{*}\psi(\overline{x})\right) + \overline{cone}\ \left(\left(\bigcup_{j\in J_0(\overline{x})} \partial^{*}\varphi_j(\overline{x}) \right) \bigcup \left(\bigcup_{t\in T} \partial^{*}\phi_t\left(\overline{x}\right) \right) \bigcup \left(\bigcup_{t\in T} \partial^{*}\left(-\phi_t\right)\left(\overline{x}\right) \right) \right).$$

By Lemma 2.15, we get scalars $\alpha_i \geq 0$ such that $\sum_{i\in I}\alpha_i = 1$ and

$$0 \in \sum_{i\in I}\alpha_i co\left[\partial^{*}\sigma_i(\overline{x}) + \overline{h}_i\partial^{*}\left(-\xi_i\right)\left(\overline{x}\right)\right]$$
$$+ cl\left(\sum_{j\in J_0(\overline{x})} cone\ \partial^{*}\varphi_j(\overline{x}) + \sum_{t\in T} cone\ \partial^{*}\phi_t(\overline{x}) + \sum_{t\in T} cone\ \partial^{*}\left(-\phi_t\right)\left(\overline{x}\right) \right).$$

□

Let $f_i : \mathbb{R}^p \to \mathbb{R}, g_i : \mathbb{R}^p \to \mathbb{R}, h_i : \mathbb{R}^p \to \mathbb{R}$ and $k_i : \mathbb{R}^p \to \mathbb{R}$ be given locally Lipschitz functions such that

$$l_i(x) \geq f_i(x) \geq 0, k_i(x) \geq g_i(x) > 0, \quad \forall x \in C.$$

Setting

$$F_i(x) = [f_i(x), l_i(x)] \quad \text{and} \quad G_i(x) = [g_i(x), k_i(x)],$$

we get the following corollary.

Corollary 3.5. *Let $\overline{x} \in C$. Suppose that φ_j, ϕ_t, f_i and k_i admit bounded upper semi-regular convexificator $\partial^{*}\varphi_j(\overline{x}), \partial^{*}\phi_t(\overline{x})$, $\partial^{*}f_i(\overline{x})$ and $\partial^{*}\left(-k_i\right), i \in I, j \in J, t \in T$. Suppose that $\left(\overline{x}, \overline{h}\right)$ is a weak local Pareto minimal point with respect to $\mathbb{R}^n_+$ of the problem (P) such that the nonsmooth Guignard constraint qualification* (NGCQ) *holds at $\overline{x}$. Then, we get scalars $\alpha_i \geq 0$ such that $\sum_{i\in I}\alpha_i = 1$ and*

$$0 \in \sum_{i\in I}\alpha_i co\left[\partial^{*}f_i(\overline{x}) + \overline{h}_i\partial^{*}\left(-k_i\right)\left(\overline{x}\right)\right]$$
$$+ cl\left(\sum_{j\in J_0(\overline{x})} cone\ \partial^{*}\varphi_j(\overline{x}) + \sum_{t\in T} cone\ \partial^{*}\phi_t(\overline{x}) + \sum_{t\in T} cone\ \partial^{*}\left(-\phi_t\right)\left(\overline{x}\right) \right).$$

Consider now the following fractional multiobjective optimization problem

$$(Q) : \begin{cases} \text{Min } \left(\dfrac{f_1(x)}{g_1(x)}, \ldots, \dfrac{f_n(x)}{g_n(x)}\right) \\ \text{subject to: } x \in C \end{cases}$$

where f_i, g_i, h_j and k_j are locally Lipschitz functions such that

$$f_i(x) \geq 0 \quad \text{and} \quad g_i(x) > 0, \quad \text{for all} \quad i \quad \text{and all} \quad x \in \mathbb{R}^p.$$

Setting $F_i(x) = \{f_i(x)\}$ and $G_i(x) = \{g_i(x)\}$, we have

$$\overline{h} \in H(\overline{x}) \Leftrightarrow \overline{h} = \left(\frac{f_1(\overline{x})}{g_1\left(\overline{x}\right)}, \ldots, \frac{f_n(\overline{x})}{g_n(\overline{x})}\right).$$

We have the following result.

Corollary 3.6. *Let $\overline{x} \in C$. Suppose that φ_j, ϕ_t, f_i and g_i admit bounded upper semi-regular convexificator $\partial^*\varphi_j(\overline{x}), \partial^*\phi_t(\overline{x})$, $\partial^* f_i(\overline{x})$ and $\partial^*(-g_i)$, $i \in I, j \in J, t \in T$. Suppose that $(\overline{x}, \overline{h})$ is a weak local Pareto minimal point with respect to $\mathbb{R}^n_+$ of the problem (P) such that the nonsmooth Guignard constraint qualification* (NGCQ) *holds at $\overline{x}$. Then, we get scalars $\alpha_i \geq 0$ such that $\sum_{i\in I}\alpha_i = 1$ and*

$$0 \in \sum_{i\in I}\alpha_i co\left[\partial^* f_i(\overline{x}) + \frac{f_i(\overline{x})}{g_i(\overline{x})}\partial^*\left(-g_i\right)(\overline{x})\right]$$
$$+\, cl\left(\sum_{j\in J_0(\overline{x})} cone\ \partial^*\varphi_j(\overline{x}) + \sum_{t\in T} cone\ \partial^*\phi_t(\overline{x}) + \sum_{t\in T} cone\ \partial^*\left(-\phi_t\right)(\overline{x})\right).$$

Example 3.7. Consider the following multiobjective optimization problem:

$$(P^*) : \begin{cases} \min\limits_x H(x) \\ \text{subject to: } x \in C \end{cases}$$

where

$$C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0 \quad \text{and} \quad x_1x_2 = 0\right\}$$

and

$$F_1(x_1,x_2) = \left\{y \in \mathbb{R} : (2x_1+1)^2 + x_2 + 1 \leq y \leq (2x_1+1)^2 + (x_2+1)^2\right\},$$
$$F_2(x_1,x_2) = \left\{y \in \mathbb{R} : x_2^2 + 1 \leq y \leq (x_2+1)^2\right\},$$
$$G_1(x_1,x_2) = \left\{z \in \mathbb{R} : x_1 + 1 \leq z \leq 2x_1 + 2\right\},$$
$$G_2(x_1,x_2) = \left\{y \in \mathbb{R} : x_1 + 1 \leq z \leq \left(x_2^2+1\right)(x_1+1)\right\}.$$

- According to Example 3.2, the Guignard constraint qualification holds at $\overline{x} = (0,0)$.
- In this case, the set-valued mappings H_1 and H_2 are given by

$$H_1(x_1,x_2) = \left\{h_1 \in \mathbb{R} : \frac{(2x_1+1)^2 + x_2 + 1}{2x_1+2} \leq h_1 \leq \frac{(2x_1+1)^2 + (x_2+1)^2}{x_1+1}\right\}$$

and

$$H_2(x_1,x_2) = \left\{h_2 \in \mathbb{R} : \frac{1}{x_1+1} \leq h_2 \leq \frac{(x_2+1)^2}{x_1+1}\right\}.$$

Moreover,

$$F_1(0,0) = \{2\}, F_2(0,0) = \{1\}, G_1(0,0) = [1,2], G_2(0,0) = \{1\}, H_1(0,0) = [1,2] \text{ and } H_2(0,0) = \{1\}.$$

For $\overline{y}_1 = 2, \overline{z}_1 = 2, \overline{y}_2 = 1$ and $\overline{z}_2 = 1$, we have $\overline{h}_1 = \dfrac{\overline{y}_1}{\overline{z}_1} = 1$ and $\overline{h}_2 = \dfrac{\overline{y}_2}{\overline{z}_2} = 1$.

- On the one hand, $(\overline{x}, \overline{h}) = ((0,0),(1,1))$ is a weak local Pareto minimal point with respect to $\mathbb{R}^2_+$ of the problem (P^*). Indeed, since

$$\frac{(2x_1+1)^2 + x_2 + 1}{2x_1+2} - 1 = \frac{4x_1^2 + 4x_1 + 1 + x_2 + 1 - 2x_1 - 2}{2x_1+2} = \frac{4x_1^2 + 2x_1 + x_2}{2x_1+2} \geq 0.$$

one deduces that

$$(h_1,h_2) - (1,1) \notin -int\mathbb{R}^2_+, \quad \forall h_1 \in F_1(x_1,x_2), \quad \forall h_2 \in F_2(x_1,x_2), \quad \forall (x_1,x_2) \in C.$$

Consequently,

$$h - \overline{h} \notin -intR^2_+, \quad \forall h \in H(x_1,x_2), \quad \forall (x_1,x_2) \in C.$$

– On the other hand, we have

$$\sigma_1(x_1,x_2) = (2x_1+1)^2 + x_2 + 1,\ \sigma_1(0,0) = 2,\ \sigma_2(x_1,x_2) = x_2^2+1,\ \sigma_2(0,0) = 1,$$
$$\xi_1(x_1,x_2) = 2x_1+2,\ \xi_1(0,0) = 2,\ \xi_2(x_1,x_2) = \left(x_2^2+1\right)(x_1+1),\ \xi_2(0,0) = 1,$$
$$\varphi_1(x_1,x_2) = -x_1,\ \varphi_1(x_1,x_2) = -x_2 \quad \text{and} \quad \phi_1(x_1,x_2) = x_1x_2.$$

Consequently, as upper semi-regular convexificators, one has

$$\partial^*\sigma_1(0,0) = \left\{\begin{pmatrix}4\\1\end{pmatrix}\right\}, \qquad \partial^*\sigma_2(0,0) = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\},$$
$$\partial^*(-\xi_1)(0,0) = \left\{\begin{pmatrix}-2\\0\end{pmatrix}\right\}, \quad \partial^*(-\xi_2)(0,0) = \left\{\begin{pmatrix}-1\\0\end{pmatrix}\right\}$$
$$\partial^*\varphi_1(0,0) = \left\{\begin{pmatrix}-1\\0\end{pmatrix}\right\}, \quad \partial^*\varphi_2(0,0) = \left\{\begin{pmatrix}0\\-1\end{pmatrix}\right\} \quad \text{and} \quad \partial^*\phi(0,0) = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}.$$

Since

$$0 \in \frac{3}{4}conv\left[\begin{pmatrix}4\\1\end{pmatrix} + 1\begin{pmatrix}-2\\0\end{pmatrix}\right] + \frac{1}{4}conv\left[\begin{pmatrix}0\\0\end{pmatrix} + 1\begin{pmatrix}-1\\0\end{pmatrix}\right] + cl\left(\frac{5}{4}\begin{pmatrix}-1\\0\end{pmatrix} + \frac{3}{4}\begin{pmatrix}0\\-1\end{pmatrix}\right)$$

one deduces that (3.1) is satisfied for $\alpha_1 = \frac{3}{4}$ and $\alpha_2 = \frac{1}{4}$. Remark that

$$\overline{h}_1\overline{z}_1 = 2 = \overline{h}_1\xi_1(0,0), \quad \overline{h}_2\overline{z}_2 = 1 = \overline{h}_2\xi_2(0,0), \quad \overline{y}_1 = 2 = \sigma_1(0,0) \quad \text{and} \quad \overline{y}_2 = 1 = \sigma_2(0,0).$$

Example 3.8. Consider the following multiobjective optimization problem:

$$(Q^*) : \begin{cases} \min\limits_{x} H(x) \\ \text{subject to:}\, x \in C \end{cases}$$

where

$$C = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0 \quad \text{and} \quad x_1x_2 - x_2 = 0\right\}$$

and

$$F_1(x_1,x_2) = \left\{y \in \mathbb{R} : |x_1| + |x_2| \leq y \leq 2\left(|x_1|+|x_2|\right) + x_1^2 + \frac{1}{2}\right\},$$
$$F_2(x_1,x_2) = \left\{y \in \mathbb{R} : 2x_1x_2 + \frac{1}{2} \leq y \leq {x_1}^2 + x_2^2 + 1\right\},$$
$$G_1(x_1,x_2) = \left\{z \in \mathbb{R} : x_1 + x_2 + \frac{\lambda}{2} \leq z \leq x_1 + x_2 + 2x_1^2 + \lambda\right\}, \quad \lambda > 0,$$
$$G_2(x_1,x_2) = \left\{y \in \mathbb{R} : 2\left(x_1+1\right) \leq z \leq x_1^2 + x_2^2 + 3\right\}.$$

– In this case, the set-valued mappings H_1 and H_2 are given by

$$H_1(x_1,x_2) = \left\{h_1 \in \mathbb{R} : \frac{|x_1|+|x_2|}{x_1+x_2+2x_1^2+\lambda} \leq h_1 \leq \frac{2\left(|x_1|+|x_2|\right) + x_1^2 + \frac{1}{2}}{x_1+x_2+\frac{\lambda}{2}}\right\}$$

and

$$H_2(x_1,x_2) = \left\{h_2 \in \mathbb{R} : \frac{2x_1x_2+\frac{1}{2}}{x_1^2+x_2^2+3} \leq h_2 \leq \frac{{x_1}^2+x_2^2+1}{2\left(x_1+1\right)}\right\}.$$

Moreover,

$$F_1(0,0)=\left[0,\tfrac{1}{2}\right],\quad F_2(0,0)=\left[\tfrac{1}{2},1\right],\quad G_1(0,0)=\left[\tfrac{\lambda}{2},\lambda\right],$$
$$G_2(0,0)=[2,3],\quad H_1(0,0)=\left[0,\tfrac{1}{\lambda}\right],\quad H_2(0,0)=\left[\tfrac{1}{6},\tfrac{1}{2}\right].$$

For $\overline{y}_1=0,\overline{z}_1=\lambda,\overline{y}_2=\dfrac{1}{2}$ and $\overline{z}_2=3$, we have $\overline{h}_1=\dfrac{\overline{y}_1}{\overline{z}_1}=0$ and $\overline{h}_2=\dfrac{\overline{y}_2}{\overline{z}_2}=\dfrac{1}{6}$.

– On the one hand, $(\overline{x},\overline{h})=\left((0,0),\left(0,\tfrac{1}{6}\right)\right)$ is a weak local Pareto minimal point with respect to $\mathbb{R}^2_+$ of the problem (Q^*). In addition, we have

$$\sigma_1(x_1,x_2)=|x_1|+|x_2|,\quad \sigma_1(0,0)=0,\quad \sigma_2(x_1,x_2)=2x_1x_2+\frac{1}{2},\quad \sigma_2(0,0)=\frac{1}{2},$$
$$\xi_1(x_1,x_2)=x_1+x_2+2x_1^2+\lambda,\quad \xi_1(0,0)=\lambda,\quad \xi_2(x_1,x_2)=x_1^2+x_2^2+3,\quad \xi_2(0,0)=3,$$
$$\varphi_1(x_1,x_2)=-x_1,\quad \varphi_1(x_1,x_2)=-x_2\quad\text{and}\quad \phi_1(x_1,x_2)=x_1x_2-x_2.$$

Consequently, as upper semi-regular convexificators, one has

$$\partial^*\sigma_1(0,0)=\left\{\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\},\quad \partial^*\sigma_2(0,0)=\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\},$$
$$\partial^*(-\xi_1)(0,0)=\left\{\begin{pmatrix}-1\\-1\end{pmatrix}\right\},\quad \partial^*(-\xi_2)(0,0)=\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\},\quad \partial^*\varphi_1(0,0)=\left\{\begin{pmatrix}-1\\0\end{pmatrix}\right\},$$
$$\partial^*\varphi_2(0,0)=\left\{\begin{pmatrix}0\\-1\end{pmatrix}\right\},\quad \partial^*(-\phi)(0,0)=\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}\quad\text{and}\quad \partial^*\phi(0,0)=\left\{\begin{pmatrix}0\\-1\end{pmatrix}\right\}.$$

– In the other hand, we have

$$T(C,(0,0))=\mathbb{R}^+\times\{0\},[T(C,(0,0))]^-=\mathbb{R}^-\times\mathbb{R}$$

and

$$\Lambda(0,0)=\{(-1,0),(0,-1),(0,1)\}.$$

Thus,

$$\overline{cone}\ \Lambda(0,0)=\mathbb{R}^-\times\mathbb{R}.$$

Then,

$$[T(C,(0,0))]^-=\overline{cone}\ \Lambda(0,0).$$

Hence, the Guignard constraint qualification holds at $\overline{x}=(0,0)$.

– Finally, since

$$\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\in conv\left\{\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}$$

and

$$0\in\frac{1}{3}\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}+\frac{2}{3}\begin{pmatrix}0\\0\end{pmatrix}+cl\left(\frac{1}{6}\begin{pmatrix}-1\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0\\-1\end{pmatrix}+\frac{5}{6}\begin{pmatrix}0\\-1\end{pmatrix}+1\begin{pmatrix}0\\1\end{pmatrix}\right)$$

one deduces that (3.1) is satisfied for $\alpha_1=\dfrac{1}{3}$ and $\alpha_2=\dfrac{2}{3}$. Remark that

$$\overline{h}_1\overline{z}_1=0=\overline{h}_1\xi_1(0,0),\quad \overline{h}_2\overline{z}_2=\frac{1}{2}=\overline{h}_2\xi_2(0,0),\quad \overline{y}_1=0=\sigma_1(0,0)\quad\text{and}\quad \overline{y}_2=\frac{1}{2}=\sigma_2(0,0).$$

4. Conclusion

In the past few years, the set-valued optimization theory has attracted the attention of many researchers towards this expanding branch of optimization. In this paper, a fractional vector problem involving set-valued mappings has been investigated. Using the notion of convexificator together with support functions of set-valued mappings, necessary optimality conditions have been described. For future research, it would be interesting to investigate sufficient optimality conditions and duality results.

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