RAIRO-Oper. Res. 55 (2021) S343–S350
https://doi.org/10.1051/ro/2020023

RAIRO Operations Research
www.rairo-ro.org

USING MULTIFLOW FORMULATIONS TO SOLVE THE STEINER TREE PROBLEM IN GRAPHS

Laura Bahiense[1], Arthur Besso[2], Rogerio Tostas[1] and Nelson Maculan[1,*]

Abstract. We present three different mixed integer linear models with a polynomial number of variables and constraints for the Steiner tree problem in graphs. The linear relaxations of these models are compared to show that a good (strong) linear relaxation can be a good approximation for the problem. We present computational results for the STP OR-Library (J.E. Beasley) instances of type b, c, d and e.

Mathematics Subject Classification. 90C27, 90Cxx, 90C90.

Received July 20, 2019. Accepted February 21, 2020.

1. Introduction

Using multiflow formulations, we present three mixed integer linear models with a polynomial number of variables and constraints for the Steiner tree problem in graphs. This approach has been used since the 80's by [1–6, 9, 10, 13].

We study the different models and we compare their linear relaxations to show that a good (strong) linear relaxation can be a good approximation for the problem.

Multiflow formulations with strong linear relaxations are very promising as they can be solved efficiently by good Lagrangian algorithms despite having a large (although polynomial) number of variables and constraints.

We present computational results for the STP OR-Library (J.E. Beasley) instances of type b, c, d and e, that can be found at `http://people.brunel.ac.uk/~mastjjb/jeb/orlib/steininfo.html`.

2. The Steiner problem in graphs

The Steiner Problem derives from the Fermat Problem, which consists in, given a triangle, finding the point whose sum of the distances to the vertices is minimal. This problem was generalized to any figure in any dimension is known as the Euclidean Steiner Problem.

The Steiner Problem in Graphs appeared later, and consists in, given a graph $G = (V, E)$, where V is the set of vertices and E is the set of edges (each edge is associated with a cost), and a subset $V_0 \subseteq V$ of vertices of G, finding a connected subgraph $G' = (V', E')$ of minimal cost that contains these vertices.

Keywords. Steiner problem in graphs, multiflow formulations, linear relaxations.

[1] Systems Engineering and Computer Science Program, Alberto Luiz Coimbra Institute – Graduate School and Research in Engineering, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil.

[2] IBGE Foundation – CRM/GEFET, Rio de Janeiro, Brazil.

*Corresponding author: maculan@cos.ufrj.br

The vertices of V_0 are mandatory and are called "terminal points", and the vertices of $V \setminus V_0$ are optional. The elements of $V' \setminus V_0$, *i.e.*, the optional vertices that compose the solution, are known as "Steiner's points". The graph-solution is called the Steiner's Tree, since the existence of cycles always worsens the value of the objective function.

The Steiner problem in graphs is a classical $\mathcal{NP}$-hard problem [7,8] with many applications in the network design of communication, distribution and transportation systems in general and VLSI design in particular.

3. Models

Let $G = (V, E)$ be a connected graph, where $V = \{1, 2, 3, \ldots, n\}$ is the set of vertices or nodes, and E is the set of edges. Let $G_d = (V, A)$ be a directed graph derived from G, where $A = \{(i, j), (j, i) \mid \{i, j\} \in E\}$, and each edge $u = \{i, j\} \in E$ is associated with two arcs (i, j) and $(j, i) \in A$.

We consider $y = (y_u)_{u \in E} \in \{0, 1\}^{|E|}$, $\Gamma^+(i) = \{j \mid (i, j) \in A\}$, $\Gamma^-(i) = \{j \mid (j, i) \in A\}$, $m = |E|$, $n = |V|$. We also consider $E(i)$ the set of edges $u \in E$ such that an endpoint is $i \in V$. Associated with each edge we have a weight $w_u > 0$, $u \in E$.

Let (V_0, V_1) be a partition of V, that is $V = V_0 \cup V_1$, $V_0 \cap V_1 = \phi$. Supposing $3 \leq |V_0| < n$, we want to find a connected sub-graph of G, $G_T = (V_T, E_T)$, where $V_0 \subset V_T$ such that $\sum_{u \in E_T} w_u y_u$ is minimum. It is easy to see that G_T is a sub-tree of G, defined as a Steiner tree.

A vertex $s \in V_0$ is chosen to be the source offering $|V_0 \setminus \{s\}|$ commodities for the remaining terminal vertices. Variables $z_{ij}^k \geq 0$, $(i, j) \in A$, $k \in V_0 \setminus \{s\}$ indicate the continuous flow amount of commodity k going through arc (i, j) having s as source and k as terminal.

In this paper we are looking for a special connected subgraph of G, that contains all points of S with minimum weight. This subgraph will be a minimum weight tree since $w_u > 0, u \in E$ [11].

3.1. First model – STP$_1$

The first model (STP$_1$) is a non-oriented formulation that can be found in [11].

$$(\text{STP}_1): \quad \min \sum_{\{i,j\} \in E} w_{ij} y_{ij}, \tag{3.1}$$

subject to :

$$\sum_{j \in \Gamma^+(s)} z_{sj}^k = 1, \quad k \in V_0 \setminus \{s\}, \tag{3.2}$$

$$\sum_{j \in \Gamma^+(k)} z_{kj}^k - \sum_{j \in \Gamma^-(k)} z_{jk}^k = -1, \quad k \in V_0 \setminus \{s\}, \tag{3.3}$$

$$\sum_{j \in \Gamma^+(i)} z_{ij}^k - \sum_{j \in \Gamma^-(i)} z_{ji}^k = 0, \quad i \in V \setminus \{s, k\}, \quad k \in V \setminus \{s\}, \tag{3.4}$$

$$z_{ij}^k \leq y_{ij} \text{ and } z_{ji}^k \leq y_{ij}, \quad \{i, j\} \in E, \quad k \in V_0 \setminus \{s\}, \tag{3.5}$$

$$z_{ij}^k \geq 0, \quad (i, j) \in A, \quad k \in V_0 \setminus \{s\}, \tag{3.6}$$

$$y_{ij} \in \{0, 1\}, \quad \{i, j\} \in E. \tag{3.7}$$

Objective function (3.1) minimizes the Steiner tree weight; equations (3.2)–(3.4) represent flow conservation for the terminal vertex s chosen to be the source, for the remaining terminal vertices and for the non-terminal vertices, respectively; constraints (3.5) allow a non-zero flow z_{ij}^k or z_{ji}^k through an arc $(i, j) \in A$ only if edge $\{i, j\} \in E$ is included in the solution; and, finally, constraints (3.6) and (3.7) define flow variables z_{ij}^k as continuous and arc variables y_{ij} as binary, respectively.

3.2. Second model – $\mathbf{STP_2}$

If in STP_1 we replace constraints (3.5) by

$$z_{ij}^k + z_{ji}^k \le y_{ij}, \quad \{i,j\} \in E, \quad k \in V_0 \setminus \{s\}, \tag{3.8}$$

we will have a new model STP_2.

3.3. Third model – $\mathbf{STP_3}$

Model (STP_3) represents the relationship between flows and arcs in an oriented way: in this formulation, each oriented arc has associated an utilization cost that is independent from the opposite arc, even the value of this cost being equal for both arcs ($w_{ij} = w_{ji} > 0,\ (i,j) \in A$).

In order to distinguish the oriented variables related to the arcs in (STP_3), we will change $y_{ij} \in \{0,1\},\ \{i,j\} \in E$ by $x_{ij} \in \{0,1\},\ (i,j) \in A$ in this oriented formulation.

$$(\text{STP}_3): \quad \min \sum_{(i,j)\in A} w_{ij}\ x_{ij}, \tag{3.9}$$

subject to:
Flow constraints (3.2), (3.3), (3.4).

$$z_{ij}^k \le x_{ij}, \quad (i,j) \in A, \quad k \in V_0 \setminus \{s\}, \tag{3.10}$$

$$z_{ij}^k \ge 0, \quad (i,j) \in A, \quad k \in V_0 \setminus \{s\}, \tag{3.11}$$

$$x_{ij} \in \{0,1\}, \quad (i,j) \in A. \tag{3.12}$$

We note that $(i,j) \in A$ if and only if $(j,i) \in A$. Let x_{ij}^* for $(i,j) \in A$ be an optimal solution for STP_3. It is easy to observe that $x_{ij}^* + x_{ji}^* \le 1$, which means that or $x_{ij}^* = x_{ji}^* = 0$, or $x_{ij}^* = 0$ and $x_{ji}^* = 1$, or $x_{ij}^* = 0$ and $x_{ji}^* = 1$, since $w_{ij} = w_{ji} > 0$. We obtain a minimum directed tree, rooted in s, which also contains all points of S. This means that we have only one directed path from s to each vertex different from s in the subtree of the original graph G induced by s. Therefore, if we find an optimal solution for STP_3 we are also solving the indirect Steiner tree problem.

4. Linear relaxations

If in (STP_1) and (STP_2) we replace $y_{ij} \in \{0,1\},\ \{i,j\} \in E$, by $0 \le y_{ij} \le 1$, and in (STP_3) we replace $x_{ij} \in \{0,1\},\ (i,j) \in A$, by $0 \le x_{ij} \le 1$, we will have three linear programming relaxations defined by (LSTP_1), (LSTP_2) and (LSTP_3). We call $val(\cdot)$ the optimum value of the objective function associated with the optimization problem $(\cdot)$.

We know that $val(\text{STP}_1) = val(\text{STP}_2) = val(\text{STP}_3)$, and also $val(\text{LSTP}_q) \le val(\text{STP}_q), q = 1,2,3$. We call $val(\text{LSTP}_q)$ a lower bound of $val(\text{STP}_q)$. In the next two lemmas we address the strengh of these lower bounds.

Lemma 4.1. $val(\text{LSTP}_1) \le val(\text{LSTP}_2)$.

Proof. In (LSTP_1) we have:

$$z_{ij}^k \le y_{ij} \text{ and } z_{ji}^k \le y_{ij},\ \{i,j\} \in E,\ k \in V_0 \setminus \{s\}. \tag{4.1}$$

In (LSTP_2) we consider:

$$z_{ij}^k + z_{ji}^k \le y_{ij},\ \{i,j\} \in E,\ k \in V_0 \setminus \{s\}. \tag{4.2}$$

All feasible solutions of (4.2) are feasible solutions of (4.1), but some solutions of (4.1) are not feasible solutions of (4.2). It means that the set of feasible solutions of (LSTP_2) is included in the set of feasible solutions of (LSTP_1), which implies that $val(\text{LSTP}_1) \le val(\text{LSTP}_2)$. ☐

Lemma 4.2. $val(\text{LSTP}_2) \leq val(\text{LSTP}_3)$.

Proof. We know that $w_{ij} = w_{ji} > 0$, for all $\{i,j\}$ and $\{j,i\} \in A$. We will prove that at the optimal solutions of (LSTP_2) and (LSTP_3) we have:

(i) $z_{ij}^k \geq 0$ and $z_{ji}^k = 0$, or $z_{ij}^k = 0$ and $z_{ji}^k \geq 0$;
(ii) $z_{ij}^k + z_{ji}^k \leq 1$; and
(iii) $val(\text{LSTP}_2) \leq val(\text{LSTP}_3)$.

In order to prove (i), consider by absurd an optimal solution with $z_{ij}^k > 0$ and $z_{ji}^k > 0$, which is associated with a circular flow using both arcs (i,j) and $(j,i) \in A$. Define $d = z_{ij}^k - z_{ji}^k,\ |d| < 1,\ 0 \leq z_{ij}^k \leq 1, (i,j) \in A$. Now replace $\overline{z}_{ij}^k = d$ and $\overline{z}_{ji}^k = 0$, if $d \geq 0$; and $\overline{z}_{ji}^k = -d$ and $\overline{z}_{ij}^k = 0$, if $d < 0$. These new $\overline{z}_{ji}^k \in \mathbb{Z}$ and $\overline{z}_{ij}^k + \overline{z}_{ji}^k \leq 1$, even for ($\text{LSTP}_3$). From model ($\text{STP}_2$) we have the correspondence $\overline{y}_{ij} \geq \overline{z}_{ij}^k + \overline{z}_{ji}^k$, and from model ($\text{STP}_3$) the correspondences $\overline{x}_{ij} \geq \overline{z}_{ij}^k$ and $\overline{x}_{ji} \geq \overline{z}_{ji}^k$. In addition, we also have that $z_{ij}^k + z_{ji}^k \geq \overline{z}_{ij}^k + \overline{z}_{ji}^k,\ \{i,j\} \in E$ and $z_{ij}^k \geq \overline{z}_{ij}^k,\ (i,j) \in A$. Therefore, since $w_{ij} > 0$, we have that $w_{ij}y_{ij} \geq w_{ij}\overline{y}_{ij}$, and $w_{ij}(x_{ij} + x_{ji}) \geq w_{ij}(\overline{x}_{ij} + \overline{x}_{ji})$, which is a contradiction.

The proof of (ii) is straightforward: directly from (i) we have that $z_{ij}^k + z_{ji}^k \leq y_{ij} \leq 1,\ \forall\ \{i,j\} \in E$ for (LSTP_2); and $z_{ij}^k \leq x_{ij} \leq 1,\ \forall\ (i,j) \in A$ for (LSTP_3).

Finally, in order to prove (iii), if we define $z_{ij}^a = \max_{k \in V_0 \setminus \{s\}} \{z_{ij}^k\}$ and $z_{ji}^b = \max_{k \in V_0 \setminus \{s\}} \{z_{ji}^k\}$, then:

$$\begin{cases} \forall\ \{i,j\} \in E \text{ in } (\text{LSTP}_2) :\ y_{ij} = \max\left\{z_{ij}^a,\ z_{ji}^b\right\} \text{ ; and} \\ \forall\ (i,j) \in A \text{ in } (\text{LSTP}_3) : \begin{cases} x_{ij} = z_{ij}^a \ , \\ x_{ji} = z_{ji}^b \ . \end{cases} \end{cases}$$

Therefore, we have that:

$$val(\text{LSTP}_2) = \sum_{\{i,j\} \in E} w_{ij}\ \max\left\{z_{ij}^a,\ z_{ji}^b\right\} \text{ ; and}$$
$$val(\text{LSTP}_3) = \sum_{\{i,j\} \in A;\ i<j} w_{ij}\ \left(z_{ij}^a +\ z_{ji}^b\right) .$$

Since $z_{ij}^a +\ z_{ji}^b \geq \max\left\{z_{ij}^a,\ z_{ji}^b\right\}$, we have that $val(\text{LSTP}_2) \leq val(\text{LSTP}_3)$. □

Corollary 4.3. $val(\text{LSTP}_1) \leq val(\text{LSTP}_2) \leq val(\text{LSTP}_3)$.

Figure 1 shows the strength of (STP_3) throughout an example of $val(\text{LSTP}_1) = val(\text{LSTP}_2) = 6.5 < val(\text{LSTP}_3) = val(\text{STP}) = 7$ (integer problem value). For LSTP_1 and LSTP_2 Figure 1a depicts the same fractional solution highlighted in bold: $y_{12} = y_{23} = y_{34} = 1,\ y_{45} = y_{48} = y_{58} = 0.5$, the other values of y_{ij} are zero. For LSTP_3 Figure 1b shows the integer solution highlighted in bold: $x_{12} = x_{23} = x_{38} = x_{85} = 1$, the other values of x_{ij} are zero. The yellow vertices in Figure 1 form the set V_0, where $s = 1$.

5. Computational results

The models presented in Section 3 were tested over the STP OR-Library (J.E. Beasley) for instances of type b, c, d and e, that can be found at http://people.brunel.ac.uk/~mastjjb/jeb/orlib/steininfo.html. Table 1 presents the sizes of these instances.

The computing environment consisted of a computer with Intel XeonTM Processor (3.07 GHz, 48 GB RAM), running under Ubuntu 12.04 LTS operating system. The modeling software used was AMPL version 20150721 (Linux 64 bits) together with the IBM ILOG CPLEX Optimization Studio Interactive Optimizer 12.6.0.0 as solver.

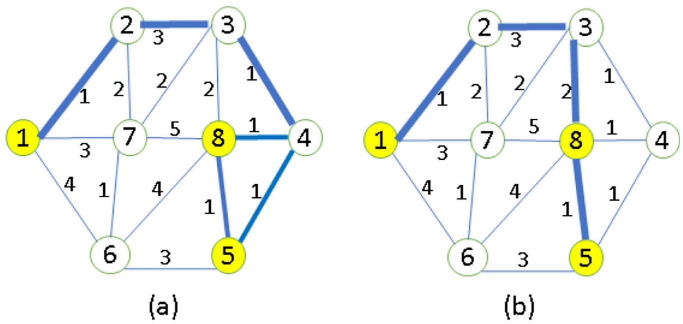

FIGURE 1. Example of the strength of (STP_3): $val(LSTP_1) = val(LSTP_2) < val(LSTP_3)$.

TABLE 1. Sizes of instances b, c, d and e from STP OR-Library.

Instance	$\|V\|$	$\|A\|$	$\|V_0\|$	Instance	$\|V\|$	$\|A\|$	$\|V_0\|$	Instance	$\|V\|$	$\|A\|$	$\|V_0\|$	Instance	$\|V\|$	$\|A\|$	$\|V_0\|$
$b1$	50	63	9	$c3$	500	625	83	$d3$	1000	1250	167	$e3$	2500	3125	417
$b2$	50	63	13	$c4$	500	625	125	$d4$	1000	1250	250	$e4$	2500	3125	625
$b3$	50	63	25	$c5$	500	625	250	$d5$	1000	1250	500	$e5$	2500	3125	1250
$b4$	50	100	9	$c6$	500	1000	5	$d6$	1000	2000	5	$e6$	2500	5000	5
$b5$	50	100	13	$c7$	500	1000	10	$d7$	1000	2000	10	$e7$	2500	5000	10
$b6$	50	100	25	$c8$	500	1000	83	$d8$	1000	2000	167	$e8$	2500	5000	417
$b7$	75	94	13	$c9$	500	1000	125	$d9$	1000	2000	250	$e9$	2500	5000	625
$b8$	75	94	19	$c10$	500	1000	250	$d10$	1000	2000	500	$e10$	2500	5000	1250
$b9$	75	94	38	$c11$	500	2500	5	$d11$	1000	5000	5	$e11$	2500	12 500	5
$b10$	75	150	13	$c12$	500	2500	10	$d12$	1000	5000	10	$e12$	2500	12 500	10
$b11$	75	150	19	$c13$	500	2500	83	$d13$	1000	5000	167	$e13$	2500	12 500	417
$b12$	75	150	38	$c14$	500	2500	125	$d14$	1000	5000	250	$e14$	2500	12 500	625
$b13$	100	125	17	$c15$	500	2500	250	$d15$	1000	5000	500	$e15$	2500	12 500	1250
$b14$	100	125	25	$c16$	500	12 500	5	$d16$	1000	25 000	5	$e16$	2500	62 500	5
$b15$	100	125	50	$c17$	500	12 500	10	$d17$	1000	25 000	10	$e17$	2500	62 500	10
$b16$	100	200	17	$c18$	500	12 500	83	$d18$	1000	25 000	167	$e18$	2500	62 500	417
$b17$	100	200	25	$c19$	500	12 500	125	$d19$	1000	25 000	250	$e19$	2500	62 500	625
$b18$	100	200	50	$c20$	500	12 500	250	$d20$	1000	25 000	500	$e20$	2500	62 500	1250
$c1$	500	625	5	$d1$	1000	1250	5	$e1$	2500	3125	5				
$c2$	500	625	10	$d2$	1000	1250	10	$e2$	2500	3125	10				

Tables 2–4 show the computational results of models (STP_3) and ($LSTP_3$) for the STP OR-Library instances of type b, c, d and e. Of the 78 instances available in the Library, 49 were successfully executed, and the others required an amount of computational memory above the capacity of the available computer.

Table 2 shows that the linear relaxation ($LSTP_3$) has performed faster than the integer model (STP_3) in 55.56% of type b instances. Besides, both models were able to find all the optimal integer solutions for these instances.

Table 3 shows that the linear relaxation ($LSTP_3$) has performed faster than the integer model (STP_3) in 63.16% of type b instances. In addition, both models were able to find all the optimal integer solutions for these instances, except for instance $c18$ where $LSTP_3$ found 11.21 instead of 113.

Table 4 shows that the linear relaxation ($LSTP_3$) has performed faster than the integer model (STP_3) for all the instances of types d and e that could be solved to optimality, except for instance $e7$ that could only be solved by the integer model.

TABLE 2. Computational results of (STP_3) and ($LSTP_3$) for STP OR-Library instances *b*.

Instance	Optimal solution	Time STP_3 (s)	Time $LSTP_3$ (s)
*b*1	82	0.03	0.01
*b*2	83	0.02	0.02
*b*3	138	0.03	0.03
*b*4	59	0.03	0.03
*b*5	61	0.03	0.03
*b*6	122	0.12	0.12
*b*7	111	0.04	0.03
*b*8	104	0.03	0.04
*b*9	220	0.09	0.09
*b*10	86	0.09	0.06
*b*11	88	0.10	0.10
*b*12	174	0.48	0.45
*b*13	165	0.45	0.44
*b*14	235	0.11	0.11
*b*15	318	0.15	0.15
*b*16	127	0.13	0.13
*b*17	131	0.20	0.20
*b*18	218	0.60	0.60

TABLE 3. Computational results of (STP_3) and ($LSTP_3$) for STP OR-Library instances *c*.

Instance	Optimal solution	Time STP_3 (s)	Time $LSTP_3$ (s)
*c*1	85	0.14	0.06
*c*2	144	0.49	0.29
*c*3	754	5.66	6.80
*c*4	1079	23.60	22.98
*c*5	1579	60.81	71.32
*c*6	55	0.69	0.18
*c*7	102	1.49	1.44
*c*8	509	149.86	205.77
*c*9	707	1004.31	2218.25
*c*10	1093	1865.51	575.52
*c*11	32	2.85	1.52
*c*12	46	15.91	7.62
*c*13	258	4856.99	8440.30
*c*14	323	4130.90	10 246.80
*c*15	556	7076.49	119 248.00
*c*16	11	12.74	5.94
*c*17	18	57.15	39.41
*c*18	113 ($LSTP_3 = 112.21$)	48 009.80	31 724.10
*c*19	146	361 601.00	110 588.00

In all but one Beasley problems for which we had enough memory to solve through the linear relaxation ($LSTP_3$) we were able to find an integer solution. Instance *c*18 was the only exception. However, as the value of its objective function was equal to 112.21 and all coefficients are integer, the bound to be considered is indeed 113.

TABLE 4. Computational results of (STP_3) and ($LSTP_3$) for STP OR-Library instances d and e.

Instance	Optimal solution	Time STP_3 (s)	Time $LSTP_3$ (s)
$d1$	106	1.37	0.29
$d2$	220	1.92	0.34
$d6$	67	6.13	1.27
$d7$	103	11.75	2.78
$d11$	29	8.13	5.76
$d12$	42	49.80	37.59
$d16$	13	30.60	22.17
$e1$	11	1.06	0.55
$e2$	214	3.64	1.77
$e6$	73	10.10	4.08
$e7$	145	84.64	Not enough memory

6. Conclusions and suggestions for future work

The hypothesis initially formulated was that linear relaxations derived from multiflow formulations would be good (strong) approximations for the Steiner problem in graphs.

The computational results presented for the STP OR-Library (J.E. Beasley) instances of type b, c, d and e corroborated the initial hypothesis, since the linear relaxation $LSTP_3$ presented, in a faster way, the same results obtained by the integer model.

In addition, it is worth mentioning that the multiflow mixed integer linear formulations presented have a polynomial number (although very large) of variables and constraints.

As future work we suggest two fronts. The first one is related to the application of VUB (variable upper bound) techniques [12] to constraints (3.10) in model STP_3, as did in [10]. We think this can lead us to solve much larger problems.

The second front consists in using more sophisticated algorithms, such as Lagrangian relaxation algorithms, for the resolution of formulation STP_3. The works [1, 2] have already shown that the Volume Algorithm and the Revised Volume Algorithm work very well for these kind of multiflow formulations. Besides, the suboptimal primal solutions generated by these algorithms can be used as a starting point for a Simplex-based method or for the development of heuristics and metaheuristics that generate better primal solutions.

Acknowledgements. This work was partially supported by the National Council for Scientific and Technological Development – CNPq, under Grants 305180/2016-9 and 302578/2014-5. The authors also thank CAPES-MEC and COPPETEC.

References

[1] L. Bahiense, N. Maculan and C. Sagastizábal, The volume algorithm revisited: relation with bundle methods. *Math. Program.* **94** (2002) 41–69.

[2] L. Bahiense, F. Barahona and O. Porto, Solving Steiner tree problems in graphs with lagrangian relaxation. *J. Comb. Optim.* **7** (2003) 259–282.

[3] J.E. Beasley, An algorithm for the Steiner problem in graphs. *Networks* **14** (1984) 147–159.

[4] A. Besso, *Problema de Steiner em grafos: Uma experiência numérica para problemas de médio porte utilizando formulações compactas de multi-fluxo*. Master's thesis. Programa de Engenharia de Sistemas e Computação, COPPE, Federal University of Rio de Janeiro, Rio de Janeiro (2015).

[5] A. Claus and N. Maculan, Une nouvelle formulation du problème de Steiner sur un graphe orienté. In: Publication 315, Centre de Recherche sur les Transports, Université de Montréal, Montréal (1983).

[6] M.X. Goemans and Y.S. Myung, A catalog of Steiner tree formulations. *Networks* **23** (1993) 19–28.

[7] F.K. Hwang, D.S. Richards and P. Winter, The Steiner tree problem. In: Vol. 53 of *Annals of Discrete Mathematics*. North-Holland, Amsterdam (1992).

[8] R.M. Karp, Reducibility among combinatorial problems, edited by R.E. Miller, J.W. Thatcher and J.D. Bohlinger. In: Complexity of Computer Computations. The IBM Research Symposia Series. Springer, New York, NY (1972) 85–103.

[9] N. Maculan, The Steiner problem in graphs. *Ann. Discrete Math.* **31** (1987) 185–211.

[10] N. Maculan, D. Arpin and S. Nguyen, Le problème de Steiner sur un graphe orienté: formulations et relaxations. *Comput. Appl. Math.* **7** (1988) 109–118.

[11] N. Maculan, G. Plateau and A. Lisser, Integer linear models with a polynomial number of variables and constraints for some classical combinatorial optimization problems. *Pesquisa Oper.* **23** (2003) 161–168.

[12] L. Schrage, Implicit representation of generalized variable upper bounds in linear programming. *Math. Program.* **14** (1978) 11–20.

[13] R.T. Wong, A dual ascent approach to Steiner tree problems on a directed graph. *Math. Program.* **28** (1984) 271–287.