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# IMPERFECT PRODUCTION SUPPLY CHAIN MODEL CONSIDERING PRICE-SENSITIVE DEMAND AND QUANTITY DISCOUNTS UNDER FREE DISTRIBUTION APPROACH

Hemapriya Selvaraj* and Uthayakumar Ramasamy

**Abstract.** During production process, we may experience with some imperfect things disregarding every single precautionary measures. The imperfect things are each of two dismissed promptly at the season of production or reworked and sold as great ones or customers are given plenty discount to keep up the generosity of the organization. This article considers about this practical circumstances and includes price-sensitive demand. As production propels, we have defective items as a part of result. The customer's demand is pretended to be price-sensitive dependent to increment the quantity of offers, and the vendor offers a quantity discount to persuade the buyer to purchase more amounts. Here, the lead time demand follows a free distribution. Therefore, the integrated model is used to find the optimizing values for the total number of shipments, order quantity, safety factor and retail price. An efficient iterative algorithm is designed to obtain the optimal solution of the model numerically and sensitivity analysis table formulate to show the impact of different parameter.

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## 1. Introduction

In this globalised economy, business professionals are mostly interested to establish integration or coordination among the participating entities in order to enhance their supply chains performance. Over the past few epochs of research on Economic Production Quantity (EPQ) models, the heaps of disputes have appeared. The traditional EPQ model is often considered some unrealistic and idealistic assumptions. Thus, the development of the manufacturing inventory models needs a certain amount of relaxation from these types of assumptions to represent the actual realistic scenario to the manufacturing industries. The foremost unrealistic assumption in using the EPQ models is that a machine can work always perfectly but in reality, a production process may not always be perfect due to facility defect, lack of facility maintenance, damage in transit, transportation delay, etc., but also may face the situation of sudden machine breakdown/failure at any random point in the duration production run. These imperfect items may affect the customer service level and company's profit margin and goodwill or reputation. It is, therefore, worth studying the effect of non-conforming or defective items on inventory decisions. To resolve the issue, the buyer may perform a 100% screening process to identify

*Keywords.* Integrated model, defective rate, quantity discount, markup rate, price-sensitive demand.

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and it can be reworked at that instant moment with an extra cost. Sarkar and Giri [10] has been devoted to the study of stochastic supply chain model with imperfect production and controllable defective rate. Dey and Giri [3] proposed an integrated inventory model with imperfect production process to study the effects of reducing defective rate. Annadurai and Uthayakumar [2] developed a periodic review inventory model under controllable lead time and lost sales reduction.

Quantity discount is a common practice in retail sales and provides economic advantages for both the buyer and vendor. The vendor will be able to benefit from sales of larger quantities by reducing the unit order and setup costs. In many cases quantity discounts can provide the buyer lower per-unit purchase cost, lower ordering costs, and decreased likelihood of shortages. The traditional quantity discount models are solely focused from the buyer's point of view to determine the optimal strategy. Giri and Sarker [5] developed a coordinating a multi-echelon supply chain under production disruption and price-sensitive stochastic demand. Mandal and Giri [9] developed a two-warehouse integrated inventory model with imperfect production process under stock-dependent demand and quantity discount offer. Jazinaninejad *et al.* [6] developed a Coordinated decision-making on manufacturer's EPQ based and buyer's period review inventory policies with stochastic price-sensitive demand [13] developed a EPQ model for returned/reworked inventories during imperfect production process under price-sensitive stock-dependent demand. Lin and Ho [8] proposed an integrated inventory model with quantity discount and price-sensitive demand. Agrawal and Yadav [1] proposed a price and profit structuring for single manufacturer multi-buyer integrated inventory supply chain under price-sensitive demand condition [14] developed a Pricing strategy for deteriorating items using quantity discount when demand is price-sensitive. Sarkar *et al.* [12] proposed an impact of safety factors and setup time reduction in a two-echelon supply chain management. Sarkar *et al.* [11] investigated the effects for variable production rate on quality of products in a single-vendor multi-buyer supply chain management.

Different researcher developed different types of model under the consideration of imperfect production system with safety stock, price-sensitive demand, but no one developed any model for single-vendor single-buyer for defective products involving price-sensitive demand under free distribution. In addition, we consider that the defective follows probability distribution function such as uniform distribution. Therefore, this research paper intends to fill this remarkable gap in the inventory literature. There is a big research gap in this direction, which is fulfilled by this research.

## 2. Problem description

In this chapter, the retailer offers different price discounts to his customers. Moreover, the demand for a product is considered as price-sensitive. The pricing strategy discussed here is one in which the vendor offers a quantity discount to the buyer. Then the buyer will adjust his retail price based on the purchasing cost, which will influence the customer demand as a result. The main objective of this study is to determine the order quantity, retail price and the number of shipments from vendor to the buyer in one production run to maximize the total profit. Finally, we solve numerical examples to substantiate the theoretical results of the underlying model and extend the numerical example by performing a sensitivity analysis of the model parameters and discuss managerial insights.

## 3. Model description

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# 4. Notations and assumptions

## 4.1. Notations

We need the following notations and assumptions, to enhance the mathematical model of this proposed model. The following terminology is used:

**Parameters**

| | |
|---|---|
| $A$ | Buyer's ordering cost per item per year. |
| $h_{b1}$ | The holding cost rate for defective items per item per year. |
| $h_{b2}$ | The holding cost rate for non-defective items per item per year. |
| $h_v$ | The vendor's holding cost per item per year. |
| $y$ | Defective rate amid the lot-size $Q$. |
| $S$ | The vendor's setup cost per item per year. |
| $P$ | Production rate per unit time. |
| $x$ | Screening rate. |
| $t_T$ | Transportation time. |
| $\pi$ | Stock out cost per unit of shortage. |
| $p_i$ | Buyer's purchase cost per unit depends on $Q$, $i = 0, 1, 2, \ldots, k$. |
| $c$ | Unit production cost. |
| $k_2$ | Backup factor of batch $2, 3, \ldots, m$. |
| $t_s$ | Setup and transportation time. |
| $t_T$ | Transportation time. |
| $F$ | The transportation cost for buyer per unit. |
| $s$ | Buyer's screening cost. |
| $N$ | Number of defective items during a production cycle. |
| $R$ | Rework cost per unit item. |
| $\eta(P)$ | Slipped by time that the procedure goes "out-of-control". |
| $t$ | Actual production run time. |
| $w$ | A unit part weight. |
| $d$ | Transportation distance. |
| $\gamma$ | Discount aspect for LTL shipments, $0 \leq \gamma < 1$. |
| $F_x$ | The freight rate for every mile for full truckload (FTL). |
| $F_y$ | The freight rate for every mile for partial load. |
| $w_x$ | Full truckload (FTL) shipping weight. |
| $w_y$ | Actual shipping weight. |

**Variables**

| | |
|---|---|
| $Q$ | Size of the shipment from the vendor to the buyer. |
| $k_1$ | Backup factor of batch 1. |
| $r_i$ | Unit retailing price for buyer is, $r_i = (1 + \Omega)p_i$, with markup rate $\Omega > 0$, $i = 0, 1, \ldots, k$. |
| $n$ | Number of lots in which the item are delivered from the vendor. |

## 4.2. Assumptions

The following hypothesis are need to be considered in this model are:

(1) This is a single-vendor and single-buyer integrated supply chain model.
(2) Replenishments are made when the on hand inventory reaches the reorder point $r$ (the inventory is reviewed continuously).
(3) The buyer orders a lot size of $nQ$ units and the vendor produces the items. In $n$ equal sized shipments these items are delivered to the buyer.
(4) The demand rate is lesser than the vendor's production rate of non-defective items ($i.e.$, $P(1 - y) > D$).

(5) The demand rate $D(r_i) = \alpha r_i^{-\delta}$ is the function of retail price where $\alpha > 0$ is a scaling factor, and $\delta > 1$ is the index of price elasticity.
(6) After the production system, the elapsed time goes "out-of-control" is an exponentially distributed random variable and the mean of the exponential distribution is a decreasing function of the production rate.
(7) The unit purchase cost for buyer is defined by:

$$p = \begin{cases} p_0 & \text{for} \quad q_0 \leq Q < q_1, \\ p_1 & \text{for} \quad q_1 \leq Q < q_2, \\ \vdots & \\ p_k & \text{for } q_k \leq Q < q_{k+1}. \end{cases}$$

where $p_0 > p_1 > \ldots p_k > 0$ and $0 = q_0 < q_1 < q_2 < \ldots < q_k < q_{k+1} = \infty$ is the arrangement of price-break quantities.
(8) The safety backup supply during the first batch is,

$$s_1 = k_1 \sigma \sqrt{L(P,Q)} = k_1 \sqrt{t_s + \frac{Q}{P}}$$

and the safety backup supply during the second batch is,

$$s_i = k_2 \sigma \sqrt{L(t_T)} = k_2 \sigma \sqrt{t_T}$$

which gives the relation between safety backup supplies as similar as in chapter 7 is,

$$k_2 = k_1 \sqrt{\frac{t_s + \frac{Q}{P}}{t_T}}$$

for batches $2, 3, \ldots, m$.

## 5. Mathematical model

This section derives the total cost involved in integrating the lot size policies between a vendor and a buyer. The mathematical approach is similar to Dey and Giri [3]. The buyer orders size $nQ$ of non-defective item to the vendor. In order to reduce the production cost, the vendor produces these $nQ$ item at one set-up and transfer of $Q$ items at each regular interval $\frac{Q(1-y)}{D}$. Therefore the length of complete production cycle is $\frac{nQ(1-y)}{D}$. The screening process is implemented by the buyer to separate defective and non-defective items with the finite screening rate $x$ per unit time.

### 5.1. Buyer's perspective

Since $A$ be the ordering cost per order, so $\frac{AD}{nQ(1-y)}$ will be the ordering cost per unit time. The expected on hand inventory per unit time is $\frac{Q}{2} + \text{SS}$, where $\text{SS} = k\sigma\sqrt{L}$. But the order quantity $Q$ have $y$ percentage of defectiveness. Therefore the buyer separates the inventory as perfect items and defective items. Therefore, the expected holding cost is,

$$\frac{nQ(1-y)}{D}\left[k\sigma\sqrt{L} + \frac{Q(1-y)}{2} + \frac{DQy}{2x(1-y)}\right].$$

Consequently, the average inventory level of defective items after the screening time in a cycle is,

$$nQ^2 y\left[\frac{1-y}{D} - \frac{1}{2x}\right].$$

Furthermore, the shortage cost for one order cycle is,

$$\frac{D\pi}{Q}E(X_1-r_1)^+ + \frac{D\pi(n-1)}{Q}E(X_2-r_2)^+.$$

The total freight cost per year given by,

$$F_y = F_x + \alpha F_x\left(\frac{w_x-w_y}{w_y}\right), \quad 0<\alpha\le 1.$$

where $\alpha$ indicated as a discount factor for LTL shipments and the anticipated total freight cost every year which is the characteristic of shipping weight and distance with adapted inverse yields is expressed as,

$$F(D,q_1,w,d) = \frac{D}{Q(1-y)}\gamma F_x w_x d + Ddw(1-\gamma)F_x.$$

So as to build up the relationship between the process quality and the production rate, we pretended that $f(P)$ as an increasing function of $P$ and it denotes the number of failure of production process with an increased production rate. Appropriately, $1/f(P)$ denotes the mean time to failure and it becomes a decreasing function of the production rate $P$ [7]. Therefore from the above analysis, it shows that when the production rate is increased, the mean time to failure decreases.

During the production cycle, the number of defective units is,

$$D = \begin{cases} 0 & \text{if } \eta \ge t \\ \alpha^* P(t-\eta(P)) & \text{if } \eta \le t \end{cases}.$$

Therefore during the production cycle, the expected number of defective units is stated as,

$$E(D) = \alpha^* P\left[\frac{Q}{P} + \frac{1}{f(P)}e^{-(\frac{Qf(P)}{P})} - \frac{1}{f(P)}\right].$$

For the small value of $f(P)$, using the Maclaurin series we obtain, $E(D) = \eta f(P)\frac{Q^2}{2P}$ and thus the expected rework cost is given by, $R\frac{D}{Q}E(D) = RD\eta f(P)\frac{Q}{2P}$.

Subsequently, the unit purchase price of the buyer is $p = p_i$, the expected total profit for the buyer per unit is

$$\begin{aligned}
\mathrm{ETP}_b(Q,r_i,n) &= \text{sales revenue} - \text{ordering cost} - \text{transportation cost} - \text{holding cost} \\
&\quad - \text{backorder cost} - \text{screening cost} - \text{purchase cost} - \text{freight cost} \\
&\quad - \text{crashing cost} - \text{rework cost} \\
&= r_i D - \frac{D(A+nF)}{nQ(1-y)} - h_{b1}\left\{Qy - \frac{DQy}{2x(1-y)}\right\} - h_{b2}\left\{r - DL\right. \\
&\quad \left. + \frac{Q(1-y)}{2} + \frac{DQy}{2x(1-y)}\right\} - \frac{\pi D E(X_1-r_1)^+}{Q(1-y)} - \frac{D\pi(n-1)E(X_2-r_2)^+}{Q(1-y)} \\
&\quad - \frac{sD}{(1-y)} - p_i D - \frac{D}{Q(1-y)}\gamma F_x W_x d - Ddw(1-\gamma)F_x - \frac{DR(L)}{nQ(1-y)} \\
&\quad - RD\eta f(P)\frac{Q}{2P}.
\end{aligned} \tag{5.1}$$

## 5.2. Vendor's perspective

Since $S$ is the setup cost per setup, therefore $\frac{SD}{nQ(1-y)}$ will be the setup cost per unit time. the average level inventory holding cost for the vendor is similar to Dey and Giri [3] which is given by, In this manner, the average inventory holding cost for the vendor can be,

$$h_v \frac{Q}{2}\left[n\left(1 - \frac{DP}{1-y}\right) - 1 + \frac{2DP}{1-y}\right].$$

Finally $R(L)$ is the lead time crashing cost, so that $\frac{DR(L)}{Q(1-y)}$ will be the total crashing cost for the entire production cycle. Subsequently, the vendor's unit selling price is $p = p_i$, the expected total profit per unit time for the vendor is,

$$\begin{aligned}\text{ETP}_v(Q,n) &= \text{sales revenue} - \text{setup cost} - \text{holding cost} - \text{inspection cost}\\ &= D(p_j - c) - \frac{SD}{nQ(1-y)} - h_v\frac{Q}{2}\left[n\left(1 - \frac{DP}{1-y}\right) - 1 + \frac{2DP}{1-y}\right] - ID. \end{aligned} \tag{5.2}$$

Conjunct with the relevant costs mentioned above, the expected annual total profit per unit time of the integrated system is expressed as follows:

$$\begin{aligned}\text{JETP}(Q,r_i,n) &= \text{ETP}_b(Q,r_i,n) + \text{ETP}_v(Q,n)\\ &= r_jD - \frac{D(A+nF)}{nQ(1-y)} - h_{b1}\left\{Qy - \frac{DQy}{2x(1-y)}\right\} - h_{b2}\left\{r - DL + \frac{Q(1-y)}{2}\right.\\ &\quad \left. + \frac{DQy}{2x(1-y)}\right\} - \frac{\pi DE(X_1 - r_1)^+}{Q(1-y)} - \frac{D\pi(n-1)E(X_2-r_2)^+}{Q(1-y)} - \frac{sD}{(1-y)}\\ &\quad - \frac{D}{Q(1-y)}\gamma F_x W_x d - Ddw(1-\gamma)F_x - \frac{DR(L)}{nQ(1-y)} - RD\eta f(P)\frac{Q}{2P}\\ &\quad - Dv - \frac{SD}{nQ(1-y)} - h_v\frac{Q}{2}\left[n\left(1 - \frac{DP}{1-y}\right) - 1 + \frac{2DP}{1-y}\right] - ID. \end{aligned} \tag{5.3}$$

As referenced before, the retail prices for the buyer is $r_i = (1+\Omega)p_i$, and we take the markup rate $\Omega$ as a decision variable rather than the $r_i$, therefore the joint expected total profit per unit time is given by,

$$\begin{aligned}\text{JETP}(Q,r_i,n) &= (1+\Omega)p_j\alpha[(1+\Omega)p_j]^{-\delta} - \frac{\alpha(A+nF)}{nQ(1-y)}[(1+\Omega)p_i]^{-\delta} - h_{b1}\left[Qy - \frac{DQy}{2x(1-y)}\right]\\ &\quad - h_{b2}\left[R - DL + \frac{Q(1-y)}{2} + \frac{Qy\alpha}{2x(1-y)}[(1+\Omega)p_i]^{-\delta}\right] - \frac{\pi\alpha E(X_1-r_1)^+}{Q(1-y)}\\ &\quad \times[(1+\Omega)p_i]^{-\delta} - \frac{\pi\alpha(n-1)E(X_2-r_2)^+}{Q(1-y)}[(1+\Omega)p_i]^{-\delta} - \frac{s\alpha[(1+\Omega)p_i]^{-\delta}}{1-y}\\ &\quad \times\frac{\alpha[(1+\Omega)p_i]^{-\delta}}{Q(1-y)}\gamma F_x W_x d - \alpha[(1+\Omega)p_i]^{-\delta}dw(1-\gamma)F_x - \frac{\alpha[(1+\Omega)p_i]^{-\delta}R(L)}{Q(1-y)}\\ &\quad - \alpha c[(1+\Omega)p_i]^{-\delta} - \frac{S\alpha[(1+\Omega)p_i]^{-\delta}}{nQ(1-y)} - h_v\frac{Q}{2}\left[n\left(1 - \frac{\alpha p[(1+\Omega)P_i]^{-\delta}}{1-y}\right)\right.\\ &\quad \left. - 1 + \frac{2\alpha P[(1+\Omega)p_i]^{-\delta}}{1-y}\right] - \frac{R\eta[(1+\Omega)p_i]^{-\delta}f(P)Q}{2P} - I\alpha[(1+\Omega)p_i]^{-\delta}. \end{aligned} \tag{5.4}$$

Since the probability distribution of $X$ is unknown, we cannot find the exact values of the expected shortage quantity $E(X_1 - r_1)^+$ and $E(X_2 - r_2)^+$. Hence we use the minimax distribution free approach to solve this problem. The problem is

$$\min_{Q,k,\Omega,n} \quad \max_{\Phi\in\mathcal{R}} \ \text{JETC}(Q,k,\Omega,n).$$

To this end, we need the following proposition that was asserted by Gallego and Moon [4].

$$\begin{aligned} E(x_1 - R_1)^+ &\le \frac{1}{2}\left\{\sqrt{\sigma^2 L(P,q_1) + (r - DL(P,q_1))} - (r - DL(P,q_1))\right\} \\ &= \frac{\sigma\sqrt{t_s + \frac{q_1}{P}}}{2}\left\{\sqrt{1+k_1^2} - k_1\right\} \end{aligned} \tag{5.5}$$

$$\begin{aligned} E(x_2 - R_2)^+ &\le \frac{1}{2}\left\{\sqrt{\sigma^2 L(t_T) + (r - DL(t_T))} - (r - DL(t_T))\right\} \\ &= \frac{\sigma\sqrt{t_T}}{2}\left\{\sqrt{1 + k_1^2\frac{t_s + \frac{Q}{P}}{t_T}} - k_1\sqrt{\frac{t_s + \frac{Q}{P}}{t_T}}\right\}. \end{aligned} \tag{5.6}$$

Using equation (5.4) and inequality (5.5) and (5.6), we get

$$\begin{aligned} \mathrm{JETP}(Q,k,\Omega,n) = {} & (1+\Omega)p_i\alpha[(1+\Omega)p_i]^{-\delta} - \frac{\alpha(A+nF)}{nQ(1-y)}[(1+\Omega)p_i]^{-\delta} - h_{b1}\left[Qy - \frac{DQy}{2x(1-y)}\right] \\ & - h_{b2}\left[k\sigma\sqrt{t_s + \frac{Q}{P}} + \frac{Q(1-y)}{2} + \frac{Qy\alpha}{2x(1-y)}[(1+\Omega)p_i]^{-\delta}\right] - \frac{\pi\alpha\sigma\sqrt{t_s + \frac{Q}{P}}}{2Q(1-y)} \\ & \times \left\{\sqrt{1+k_1^2} - k_1\right\}[(1+\Omega)p_i]^{-\delta} - \frac{\pi\alpha(n-1)\sigma\sqrt{t_T}}{2Q(1-y1)}\left\{\sqrt{1 + k_1^2\frac{t_s + \frac{Q}{P}}{t_T}} - k_1\sqrt{\frac{t_s + \frac{Q}{P}}{t_T}}\right\} \\ & \times [(1+\Omega)p_j]^{-\delta} - \frac{s\alpha[(1+\Omega)p_j]^{-\delta}}{1-y}\frac{\alpha[(1+\Omega)p_i]^{-\delta}}{Q(1-y)}\gamma F_x W_x d - \alpha[(1+\Omega)p_i]^{-\delta} \\ & \times dw(1-\gamma)F_x - \frac{\alpha[(1+\Omega)p_i]^{-\delta}R(L)}{Q(1-y)} - \frac{S\alpha[(1+\Omega)p_i]^{-\delta}}{nQ(1-y)} - h_v\frac{Q}{2} \\ & \times \left[n\left(1 - \frac{\alpha p[(1+\Omega)P_i]^{-\delta}}{1-y}\right) - 1 + \frac{2\alpha P[(1+\Omega)p_i]^{-\delta}}{1-y}\right] - \frac{R\eta[(1+\Omega)p_i]^{-\delta}f(P)Q}{2P} \\ & - \alpha c[(1+\Omega)p_i]^{-\delta} - I\alpha[(1+\Omega)p_i]^{-\delta}. \end{aligned} \tag{5.7}$$

## 6. Solution methodology

The necessary condition for the total profit per unit time in equation (5.7) to be maximum are $\frac{\partial\mathrm{JETP}}{\partial Q} = 0$, $\frac{\partial\mathrm{JETP}}{\partial k} = 0$ and $\frac{\partial\mathrm{JETP}}{\partial\Omega} = 0$.

$$\begin{aligned} \frac{\partial\mathrm{JETP}}{\partial Q} = {} & \frac{1}{Q^2}\left\{\frac{(A+nF)B}{n(1-y)} - \frac{\pi B\sigma\{\sqrt{1+k_1^2} - k_1\}}{2(1-y)}\left[\frac{Q}{2P\sqrt{t_s + \frac{Q}{P}}} - \sqrt{t_s + \frac{Q}{P}}\right] + \frac{B\gamma F_x W_x d}{(1-y)}\right. \\ & \left. + \frac{BR(L)}{(1-y)} + \frac{SB}{n(1-y)} - \frac{\pi(n-1)\sigma\sqrt{t_T}B}{2(1-y)}\left[k_1\sqrt{\frac{t_s + \frac{Q}{P}}{t_T}} - \sqrt{1 + k_1^2\left(\frac{t_s + \frac{Q}{P}}{t_T}\right)}\right]\right\} \\ & - \frac{\pi(n-1)\sigma\sqrt{t_T}B}{2Q(1-y)}\left[\frac{1}{2\sqrt{1 + k_1^2(\frac{t_s + \frac{Q}{P}}{t_T})}} - \frac{k_1}{2P\sqrt{t_T(t_s + \frac{Q}{P})}}\right] - h_{b1}\left[1 - \frac{B}{2x(1-y)}\right] \end{aligned}$$

$$
\begin{aligned}
& - h_{b2}\left[\frac{k\sigma}{2P\sqrt{t_s+\frac{Q}{P}}}+\frac{(1-y)}{2}+\frac{yB}{2x(1-y)}\right]-\frac{h_v}{2}\left[n\left(1-\frac{BP}{(1-y)}\right)-1+\frac{2BP}{(1-y)}\right]\\
& -\frac{R\eta Bf(P)}{2P}\\
= & \frac{H_1}{Q^2}-\frac{H_2}{Q}-H_3 \qquad (6.1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \text{JETP}}{\partial k} = & -\sigma\sqrt{t_s+\frac{Q}{P}}\left\{h_{b2}+\frac{\pi B}{2Q(1-y)}\left[\left\{\frac{k_1}{\sqrt{1+k_1^2}}-1\right\}+(n-1)\right.\right.\\
& \left.\left.\times\left\{k_1\sqrt{\frac{t_s+\frac{Q}{P}}{t_T+k_1^2(t_s+\frac{Q}{P})}}-1\right\}\right]\right\} \qquad (6.2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \text{JETP}}{\partial \Omega} = & \,\alpha\big[(1+\Omega)p_i-c-I\big]\big[(1+\Omega)p_i\big]^{-\delta}+\left[-\frac{(A+nF)}{nQ(1-y)}+\frac{h_{b1}Qy}{2x(1-y)}-\frac{h_{b2}Qy}{2x(1-y)}\right.\\
& -\frac{\pi\sigma\sqrt{t_s+\frac{Q}{P}}}{2Q(1-y)}\left\{\sqrt{1+k_1^2}-k_1\right\}-\frac{\pi(n-1)\sigma\sqrt{t_T}}{2Q(1-y)}\left\{\sqrt{1+k_1^2\left(\frac{t_s+\frac{Q}{P}}{t_T}\right)}\right.\\
& \left.-k_1\sqrt{\frac{t_s+\frac{Q}{P}}{t_T}}\right\}-\frac{s}{(1-y)}-\frac{\gamma F_x W_x d}{Q(1-y)}-dw(1-\gamma)F_x-\frac{R(L)}{Q(1-y)}\\
& \left.-\frac{S}{nQ(1-y)}+\frac{h_vQPn}{2(1-y)}-\frac{h_vQP}{(1-y)}-\frac{R\eta f(P)Q}{2P}\right]\alpha[(1+\Omega)p_i]^{-\delta}. \qquad (6.3)
\end{aligned}
$$

Taking the first order derivative of equations (6.1) to (6.3) with respect to $Q^*$, $k^*$ and $\Omega^*$ which gives,

$$
Q^* = \frac{H_1}{H_2+H_3Q} \qquad (6.4)
$$

$$
k^* = \frac{\pi n\alpha[(1+\Omega)p_i]^{-\delta}-2h_{b2}Q(1-y)}{\pi[(1+\Omega)p_i]^{-\delta}\left[\frac{1}{\sqrt{1+k_1^2}}+(n-1)\sqrt{\frac{t_s+\frac{Q}{P}}{t_T+k_1^2\left(t_s+\frac{Q}{P}\right)}}\right]} \qquad (6.5)
$$

$$
\begin{aligned}
\Omega^* = & \frac{\delta}{(\delta p_i-p_i)}\left[\frac{(A+nF)}{nQ(1-y)}+\frac{(h_{b2}-h_{b1})Qy}{2x(1-y)}+\frac{\pi\sigma\sqrt{t_s+\frac{Q}{P}}}{(2Q(1-y))}\{\sqrt{1+k_1^2}-k_1\}-c-I\right.\\
& +\frac{\pi(n-1)\sigma\sqrt{t_T}}{2Q(1-y)}\left\{\sqrt{1+k_1^2\left(\frac{t_s+\frac{Q}{P}}{t_T}\right)}-k_1\sqrt{\frac{t_s+\frac{Q}{P}}{t_T}}\right\}+\frac{s}{(1-y)}+\frac{\gamma F_x W_x d}{Q(1-y)}\\
& \left.+dw(1-\gamma)F_x+\frac{R(L)}{Q(1-y)}+\frac{S}{nQ(1-y)}+\frac{h_vQP}{2(1-y)}(2-n)-\frac{R\eta f(P)Q}{2P}\right]-1. \qquad (6.6)
\end{aligned}
$$

In order to examine the effect of $n$ on JETP$(Q,k,\Omega,n)$, we take the first and second partial derivatives of equation (5.7) with respect to $n$. That is,

$$
\frac{\partial \text{JETP}}{\partial n} = \frac{\alpha A}{n^2Q}[(1+\Omega)p_i]^{-\delta}+\frac{S\alpha[(1+\Omega)p_i]^{-\delta}}{n^2Q(1-y)}-h_v\frac{Q}{2}\left[1-\frac{\alpha P[(1+\Omega)p_i]^{-\delta}}{(1-y)}\right]
$$

$$- \frac{\pi\alpha\sigma\sqrt{t_T}[(1+\Omega)p_i]^{-\delta}}{2Q(1-y)} \left\{ \sqrt{1+k_1^2\left(\frac{t_s+\frac{Q}{P}}{t_T}\right)} - k_1\sqrt{\frac{t_s+\frac{Q}{P}}{t_T}} \right\}$$

$$\frac{\partial^2 \text{JETP}}{\partial n^2} = -\frac{2\alpha A}{n^3 Q(1-y)}[(1+\Omega)p_i]^{-\delta} - \frac{2S\alpha[(1+\Omega)p_i]^{-\delta}}{n^3 Q(1-y)} < 0. \tag{6.7}$$

Therefore for fixed integers $(Q, k, \Omega, n)$, the function of $\text{JETP}(Q, k, \Omega, n)$ is concave in $n$.

### 6.1. Solution algorithm

On the basis of the above discussions, the following solution algorithm can be used to determine an optimal solution.

### 6.2. Algorithm

**Step 1.** Let $n = 1$.
**Step 2.** For each $p_i$, $i = 0, 1, 2, \ldots m$, perform steps 2a–2c.
**Step 2a.** Set $\Omega_{i1} = 0$ and $k_{i1} = 0$. Substituting $\Omega_{i1}$ and $k_{i1}$ into equation (6.4) to evaluate $Q_{i1}$.
**Step 2b.** Utilize $Q_{i1}$ and $p_i$ to evaluate the values of $\Omega_{i2}$ and $k_{i2}$. Repeat steps 2a and 2b until no changes occur in the values of $Q_i$, $\Omega_i$ and $k_i$.
**Step 3.** For $i = 0, 1, 2, \ldots, m-1$.
**Step 3a.** If $q_i \le Q_i < q_{i+1}$, then $Q_i$ is an optimal solution. Set $Q_i^{(n)} = Q_i$, $\Omega_i^{(n)} = \Omega_i$ and $k_i^{(n)} = k_i$, then substitute the values of $Q_i^{(n)}, \Omega_i^{(n)}$ and $k_i^{(n)}$ into equation (5.7) to evaluate $\text{JETP}(Q_i^{(n)}, \Omega_i^{(n)}, k_i^{(n)})$.
**Step 3b.** If $Q_i > q_{i+1}$ then $Q_i$ is not a feasible solution. Thus $\text{JETP}(Q_i^{(n)}, \Omega_i^{(n)}, k_i^{(n)}) = 0$.
**Step 3c.** If $Q_i < q_i$, set $Q_i^{(n)} = q_i$, using $Q_i^{(n)}$ and $p_i$ to evaluate the values of $\Omega_i^{(n)}$ and $k_i^{(n)}$ from equations (6.5) and (6.6). Then substituting the values of $Q_i^{(n)}$, $\Omega_i^{(n)}$ and $k_i^{(n)}$ to determine the $\text{JETP}(Q_i^{(n)}, \Omega_i^{(n)}, k_i^{(n)})$.
**Step 4.** For $i = m$, perform steps 4a and 4b.
**Step 4a.** If $Q_m \ge q_m$, then $Q_m$ is an optimal solution. Set $Q_m^{(n)} = Q_m$, $\Omega_m^{(n)} = \Omega_m$ and $k_m^{(n)} = k_m$. Substituting the values of $Q_m^{(n)}, \Omega_m^{(n)}$ and $k_m^{(n)}$ in equation (5.7) to evaluate $\text{JETP}(Q_m^{(n)}, \Omega_m^{(n)}, k_m^{(n)})$.
**Step 4b.** If $Q_m < q_k$, set $Q_m^{(n)} = q_m$, and using the values of $Q_m^{(n)}$ and $p_i$ to evaluate the values of $\Omega_m^{(n)}$ and $k_m^{(n)}$ from equations (6.5) and (6.6). Then substituting the values of $Q_m^{(n)}$, $\Omega_m^{(n)}$ and the $\text{JETP}(Q_m^{(n)}, \Omega_m^{(n)}, k_m^{(n)})$.
**Step 5.** Find $\text{JETP}(Q^{(n)}, \Omega^{(n)}, k^{(n)}) = \text{Max}_{i=0,1,\ldots,m}\ \text{JETP}(Q_m^{(n)}, \Omega_m^{(n)}, k_m^{(n)})$.
**Step 6.** Set $n = n+1$, repeat steps 2–5 to get $\text{JETP}(Q^{(n)}, \Omega^{(n)}, k^{(n)})$.
**Step 7.** If $\text{JETP}(Q^{(n)}, \Omega^{(n)}, k^{(n)}) \le \text{JETP}(Q^{(n-1)}, \Omega^{(n-1)}, k^{(n-1)})$, then go to step 8 otherwise go to step 6.
**Step 8.** Set $\text{JETP}(Q^{(n)}, \Omega^{(n)}, k^{(n)}) = \text{JETP}(Q^{(n-1)}, \Omega^{(n-1)}, k^{(n-1)})$ and $\text{JETP}(Q^{(n)}, \Omega^{(n)}, k^{(n)})$ is the expected total profit then $(Q^*, \Omega^*, k^*)$ is the set of optimal solutions.

## 7. Numerical example

To illustrate the solution procedure and the results, let us apply the proposed algorithm to solve the following numerical examples. These examples are based on the following parameters in appropriate units. $P = 3200$ units/year, $A = \$50$ per order, $S = \$400$ per order, $h_v = \$4$ units/year, $h_{b1} = \$6$ units/year, $h_{b2} = \$10$ units/year, $s = \$0.25$ per unit, $x = 2152$ units/time, $w = \$20$ per unit, $y = 0.22$, $\pi = \$100$ per unit, $\alpha = 100\,000$, $c = \$7$ per unit, $I = \$12$ per unit, $\delta = 1.6$, $t_s = 0.09$, $t_T = 1.9$, $d = 400$ miles, $\gamma = 0.11246$,

TABLE 1. Expected total profit for an optimal solution.

| $n$ | $R$ | $Q$ | $\Omega$ | $k$ | JETP |
|---|---|---|---|---|---|
| 2 | 0 | 14 492 | 265.9007 | 1.418 | 82 673 |
| 2 | 5.6 | 14 581 | 257.6999 | 1.377 | 83 190 |
| 2 | 22.4 | 14 846 | 233.5950 | 1.253 | 84 734 |
| 2 | 57.4 | 15 399 | 184.7330 | 1.003 | 87 987 |

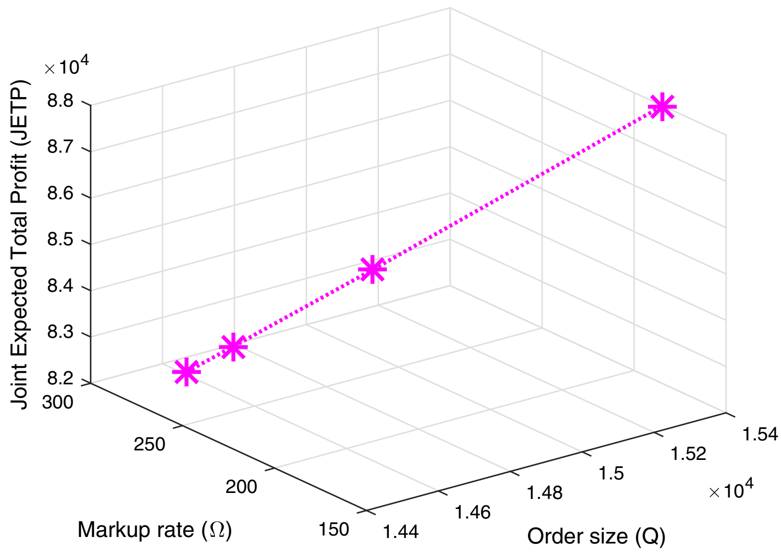


FIGURE 1. Graphical representation of Table 1.

$F_x = 0.000101343$, $w_x = 9999\,lbs$, $a_0 = 0.015$, $b_0 = 0.025$ and the unit purchase cost is defined as follows:

$$p = \begin{cases} 20 & \text{for } 0 \le Q < 10\,000, \\ 17 & \text{for } 10\,000 \le Q < 12\,000, \\ 16 & \text{for } 12\,000 \le Q < 14\,000, \\ 15 & \text{for } 14\,000 \le Q < 16\,000. \end{cases}$$

Here, the number of failures during production process with an increased production rate $f(P)$ follows a uniform distribution with probability density function is,

$$f(P) = \begin{cases} \frac{1}{b_0 - a_0}, & a_0 \le P \le b_0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus we have,

$$E(P) = \frac{a_0 + b_0}{2} = 0.02.$$

Now, applying the algorithm for this model and the outcomes of the solution procedure are outlined in Table 1. From the Table 1, we observe that the optimal values are $Q^* = 15399$, $\Omega^* = 184.7330$ and $k^* = 1.003$. Therefore the purchase cost per unit is $p^* = \$15$, the retailing price is $r_i^* = \$2785.995$ and joint expected total profit is $\text{JETP}(Q^*, \Omega^*, k^*, n) = \$87987$. The graphical representation for the Table 1 is illustrated in Figure 1.

TABLE 2. Effects of changes in parameters.

| Parameters | Changes in % | JETP |
|---|---|---|
| $P$ | +20% | 97 224 |
| | +10% | 92 556 |
| | −10% | 83 133 |
| | −20% | 77 833 |
| $S$ | +20% | 88 477 |
| | +10% | 87 887 |
| | −10% | 86 094 |
| | −20% | 84 260 |
| $\pi$ | +20% | 87 211 |
| | +10% | 87 501 |
| | −10% | 90 656 |
| | −20% | 97 614 |
| $R$ | +20% | 90 170 |
| | +10% | 89 059 |
| | −10% | 86 946 |
| | −20% | 85 940 |
| $F$ | +20% | 88 647 |
| | +10% | 88 320 |
| | −10% | 87 661 |
| | −20% | 87 336 |
| $A$ | +20% | 88 456 |
| | +10% | 88 224 |
| | −10% | 87 756 |
| | −20% | 87 519 |
| $d$ | +20% | 89 280 |
| | +10% | 88 629 |
| | −10% | 87 348 |
| | −20% | 86 711 |

## 8. Sensitivity analysis

In this section, we extend some managerial implications based on the sensitivity analysis of various key parameters of the numerical example. We investigates the effects of changes in the value of the parameters on optimal values $Q$, $\Omega$, $k$ and JETP. The sensitivity analysis is performed by changing each parameter values, taking one parameter at a time and the remaining values of the parameters are unchanged with the following data $P$, $s$, $\pi$, $R$, $F$, $A$ and $d$. The results are exhibited in Table 2 and the corresponding figures are depicted in Figures 2–8, respectively. The following inferences can be made from the results in Table 2.

(1) In Table 2, a small changes in the production rate, there is a great effect in the total profit of the integrated system.
(2) Table 2 shows that the joint expected total profit decreases as the parameter $\pi$ increases. This result is expected because higher backorder cost may amplify the total profit.
(3) In Table 2, the joint expected total profit is highly sensitive to changes in parameters $R$ and $d$, moderately sensitive to changes in parameters $S$, $F$ and $A$, respectively.

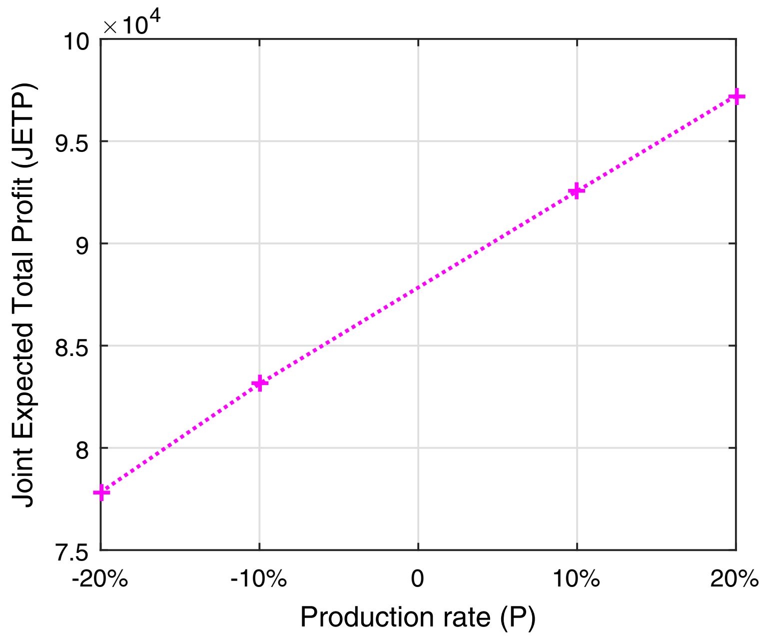


FIGURE 2. Impact of changes in $P$ on total profit.

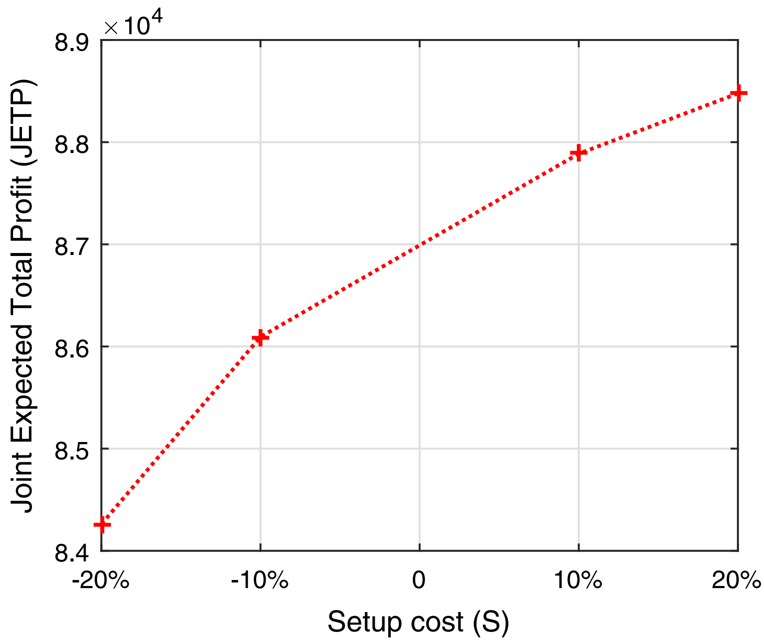


FIGURE 3. Impact of changes in $S$ on total profit.

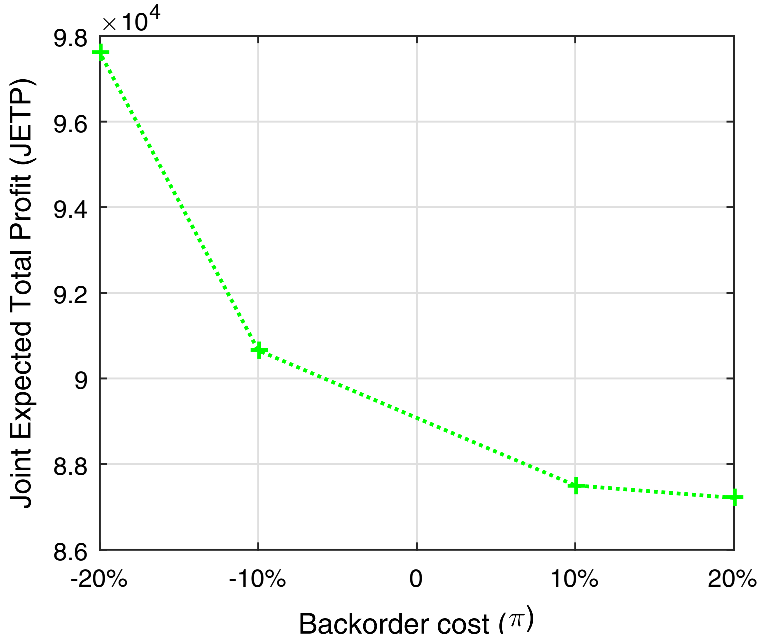


FIGURE 4. Impact of changes in $\pi$ on total profit.

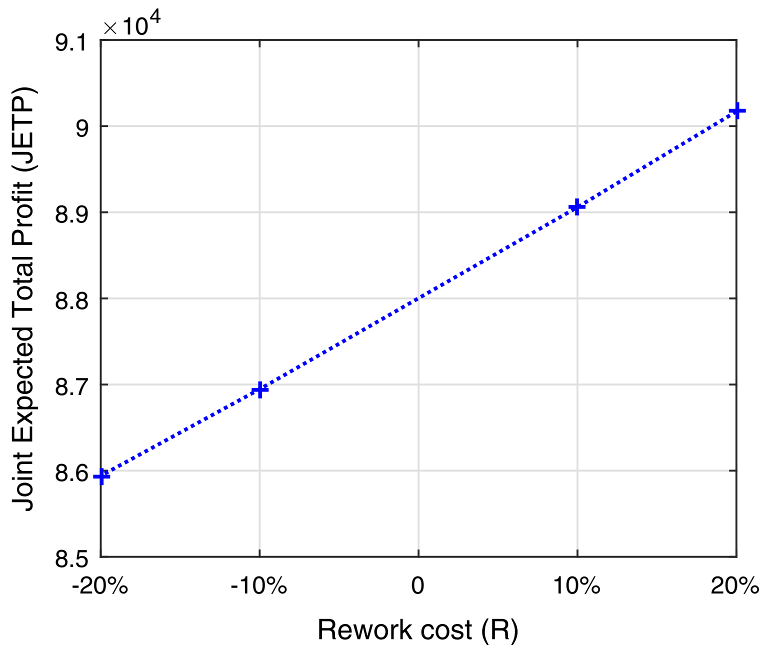


FIGURE 5. Impact of changes in $R$ on total profit.

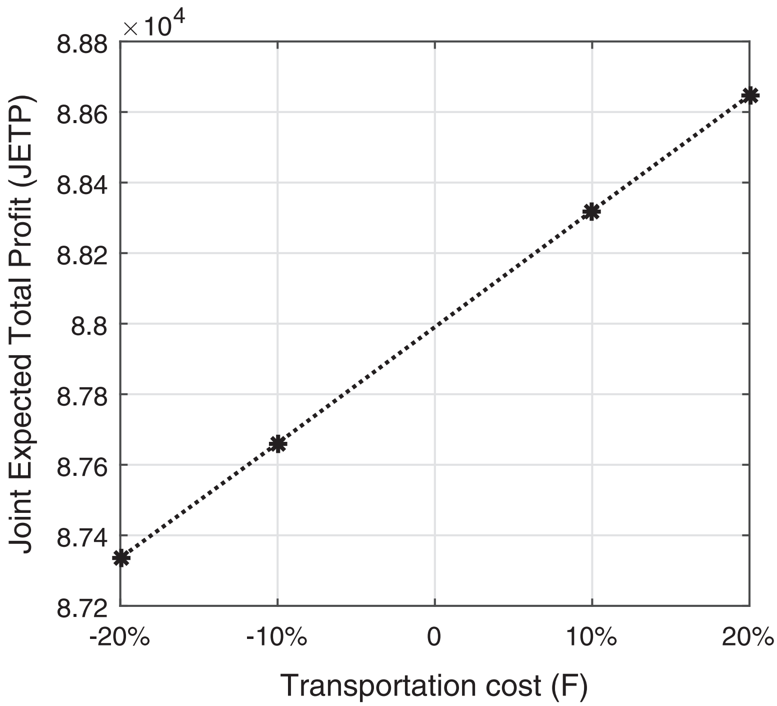


FIGURE 6. Impact of changes in $F$ on total profit.

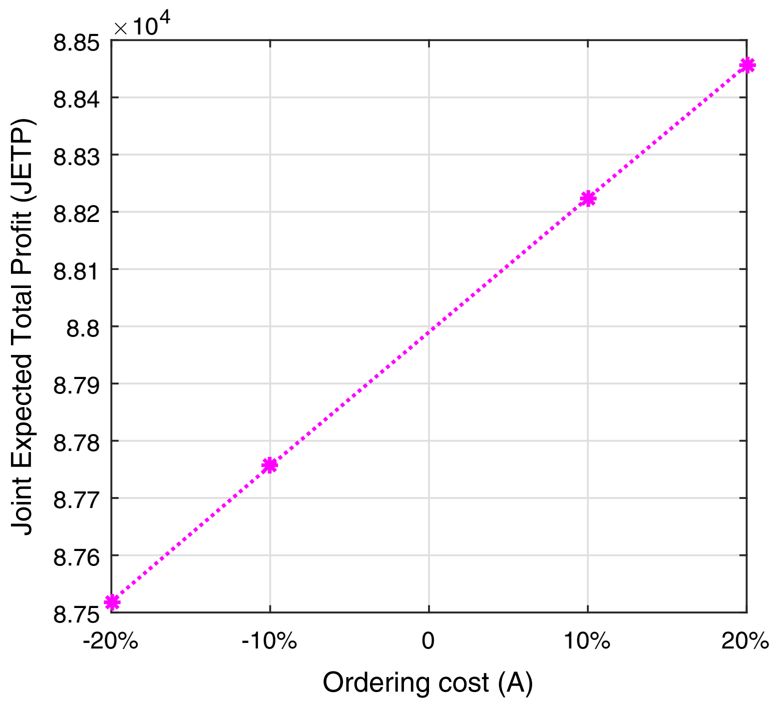


FIGURE 7. Impact of changes in $A$ on total profit.

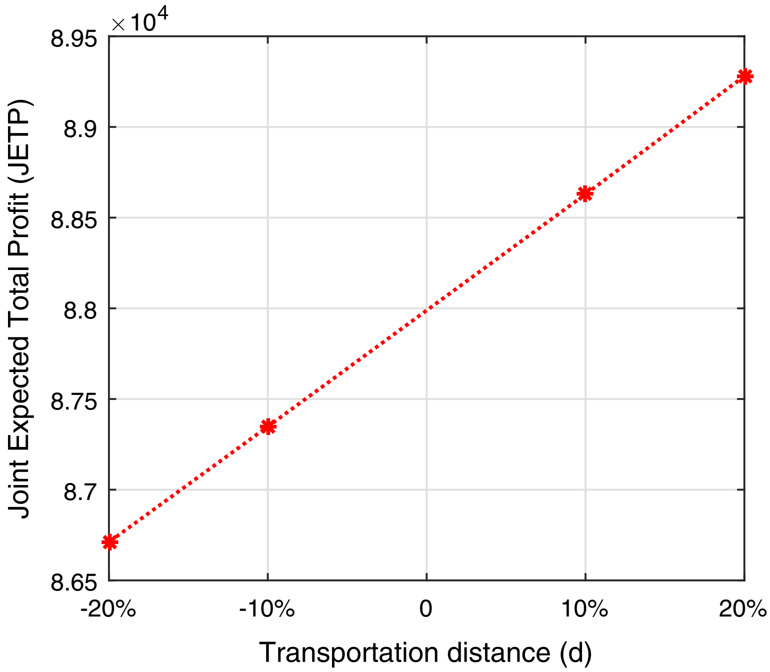


FIGURE 8. Impact of changes in $d$ on total profit.

## 9. Conclusion

In this chapter, we present an imperfect production supply chain model considering that the market demand is sensitive to the retail price and the vendor offers a quantity discount to the buyer. Here all defective/imperfect quality items are reworked after the regular production process. An efficient algorithm is proposed to obtain the optimal solutions under different conditions so as to maximize the total profit. The numerical example is provided to illustrate the proposed algorithm and the solution procedure. From Figures 2–7, we conclude that the small changes in the production rate and setup cost, there is a great effect on the total profit and the total profit increases moderately when rework cost, transportation cost and ordering cost increases. From Figure 4, the total profit decreases when price discount increases and from Figure 8, the total profit increases when transportation distance increases. The sensitivity analysis is performed to various key parameters to study the impact on optimal solutions and the managerial implications are also discussed.

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