RAIRO-Oper. Res. 55 (2021) S1195–S1206
https://doi.org/10.1051/ro/2020050

RAIRO Operations Research
www.rairo-ro.org

OPTIMALITY AND DUALITY IN NONSMOOTH VECTOR OPTIMIZATION WITH NON-CONVEX FEASIBLE SET

Sunila Sharma[1] and Priyanka Yadav[2,*]

Abstract. For a convex programming problem, the Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient for optimality under suitable constraint qualification. Recently, Suneja *et al.* [*Am. J. Oper. Res.* **6** (2013) 536–541] proved KKT optimality conditions for a differentiable vector optimization problem over cones in which they replaced the cone-convexity of constraint function by convexity of feasible set and assumed the objective function to be cone-pseudoconvex. In this paper, we have considered a nonsmooth vector optimization problem over cones and proved KKT type sufficient optimality conditions by replacing convexity of feasible set with the weaker condition considered by Ho [*Optim. Lett.* **11** (2017) 41–46] and assuming the objective function to be generalized nonsmooth cone-pseudoconvex. Also, a Mond–Weir type dual is formulated and various duality results are established in the modified setting.

Mathematics Subject Classification. 90C29, 90C46, 90C30, 90C26.

Received August 18, 2019. Accepted May 6, 2020.

1. Introduction

Most of the real world problems require simultaneous optimization of two or more objectives. Such problems are classified as Multiobjective optimization problems. Several authors have studied these problems under convexity and generalized convexity assumptions. Mishra [17] and Mishra *et al.* [20] have obtained first-order optimality conditions and duality results for differentiable multiobjective optimization problems under generalized type I univexity assumptions. Similar results have also been proved by Mishra *et al.* [18, 19, 21, 22] for nonsmooth multiobjective optimization problems wherein the functions involved are either locally Lipschitz or directionally differentiable, using V-invex and $(F, \alpha, \eta, \rho, d)$-type I functions along with their generalizations.

When ordering in the multiobjective optimization problems is defined by general cones rather than the positive orthant, these problems are referred to as Vector optimization problems. Such problems are encountered in many upcoming fields like game theory, variational inequalities, aircraft and automobile design, mechanical engineering and many more [3, 24–26]. The general ordering cones in vector optimization problem help to adjust the set of efficient solutions. By choosing general cones of different sizes, we can modify the size of the set of

Keywords. Vector optimization, cones, generalized nonsmooth cone-pseudoconvex, KKT type optimality conditions, duality.

[1] Department of Mathematics, Miranda House, University of Delhi, New Delhi, Delhi 110007, India.
[2] Department of Mathematics, Atma Ram Sanatan Dharma College, University of Delhi, New Delhi, Delhi 110021, India.
*Corresponding author: 88.priyanka@gmail.com

© EDP Sciences, ROADEF, SMAI 2021

efficient solutions. Since differentiability is a restrictive condition in nature, it is important to develop theory for nonsmooth optimization problems.

Given differentiable functions $f, g_j : \mathbb{R}^n \longrightarrow \mathbb{R}, j = 1, 2, \ldots, m$, consider the minimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{subject to} & x \in S, \end{array} \tag{CP}$$

where $S = \{x \in \mathbb{R}^n : g_j(x) \leq 0, j = 1, 2, \ldots, m\}$.

If f and g_j's are convex, then (CP) reduces to the convex programming problem that has been studied by various researchers [2, 13]. Constantin [6] has proved second-order necessary conditions for a nonsmooth nonconvex nonlinear programming problem in terms of second-order directional derivatives of the functions involved, using some new constraint qualification introduced by the author in the same paper. Ivanov [15] obtained higher-order necessary and sufficient optimality conditions for the same problem in terms of n-th order upper and lower Dini directional derivatives. The sufficient optimality conditions are proved by assuming the objective function to be pseudoinvex and constraint function to be prequasiinvex with respect to same η. Recently, Constantin [7–9] has also studied nonconvex nonsmooth multiobjective optimization problems with both equality and inequality constraints and set constraints, its particular case problems like the ones with equality and inequality constraints only and ones with inequality constraint (or equality constraint) and set constraints only. The functions involved are either locally Lipschitz or Gâteaux differentiable and optimality conditions have been obtained under some regularity conditions and constrain qualifications introduced by the author.

A crucial feature of convex programming is that, the KKT optimality conditions are necessary and sufficient under Slater's constraint qualification[1]. However, Lasserre [16] replaced the convexity of g_j's with convexity of the feasible set S and obtained KKT necessary and sufficient optimality conditions for minimization problem (CP) when f is a convex function and Slater's constraint qualification holds along with following nondegeneracy assumption on g_j's:

Definition 1.1. For every $j = 1, 2, \ldots, m, \nabla g_j(x) \neq 0$, whenever $x \in S$ and $g_j(x) = 0$.

Giorgi [12] generalized Lassere's results using suitable generalized convex functions. Dutta and Lalitha [11] studied nondifferentiable minimization problem (CP) wherein f is nondifferentiable convex function and g_j's are locally Lipschitz. They proved that, under Slater's constraint qualification and nonsmooth degeneracy condition on g_j's, KKT type optimality conditions are necessary and sufficient when f is a convex function, S is a convex set and g_j's are regular in the sense of Clarke.

Suneja *et al.* [28] extended Lasserre's work to the vector optimization case. They considered the following vector optimization problem over cones.

$$\begin{array}{ll} K\text{-Minimize} & f(x) \\ \text{subject to} & -g(x) \in Q, \end{array} \tag{VP}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p, g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are differentiable vector valued functions, K and Q are closed, convex and pointed cones with non-empty interiors in $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively. $X = \{x \in \mathbb{R}^n : -g(x) \in Q\}$ denotes the set of all feasible solutions of (VP).

In the absence of cone-convexity condition on g, Suneja *et al.* [28] proved KKT necessary and sufficient optimality conditions for (VP) under the assumption that f is cone-convex (cone-pseudoconvex) function, X is a convex set and Slater-type cone constraint qualification holds. Unlike Lasserre [16] they did not impose any nondegeneracy assumption on g_j's. However, they introduced the following non-degeneracy condition with respect to cones for (VP) and showed that Slater-type cone-constraint qualification implies non-degeneracy condition with respect to cones.

[1]Slater's constraint qualification holds for S if there exists $x_0 \in S, g_j(x_0) < 0$ for all $j = 1, 2, \ldots, m$.

Definition 1.2 ([28])**.** The problem (VP) is said to satisfy non-degeneracy condition if for all $\mu \in Q^+ \setminus \{0\}$, $\mu^T \nabla g(z) \neq 0$, whenever $z \in X$ and $\mu^T g(z) = 0$.

Recently, Ho [14] considered the minimization problem (CP) with non-convex feasible set. They replaced convexity of feasible set S by following weaker condition on S at some feasible point x:

$$\forall\, y \in S \quad \exists\, t_n \downarrow 0 \quad \text{such that} \quad x + t_n(y - x) \in S. \tag{1.1}$$

By imposing the weaker condition (1.1) on S and without convexity of f and g_j's, KKT optimality conditions were proved to be necessary and sufficient for the minimization problem (CP) when nondegeneracy assumption (see Def. 1.1) holds at a feasible point and Slater's constraint qualification holds.

In 2019, Suneja *et al.* [29] extended Ho's work to the differentiable vector optimization problem (VP) with non-convex feasible set. Without cone-convexity (cone-pseudoconvexity) of the objective as well as constraint functions, KKT optimality conditions are obtained under condition (1.1) on the feasible set X and using the non-degeneracy condition with respect to cones (see Def. 1.2). The KKT sufficient optimality conditions are proved by replacing cone-convexity (cone-pseudoconvexity) of the objective function with convexity of strict level set of the objective function. Also, Mond–Weir type duality results are studied for the problem (VP).

The present paper is motivated by the works of Dutta and Lalitha [11], Suneja *et al.* [28] and Ho [14]. In this paper, we consider the following nonsmooth vector optimization problem (VOP):

$$\begin{aligned} &K\text{-Minimize} \quad f(x) \\ &\text{subject to} \quad -g(x) \in Q, \end{aligned} \tag{VOP}$$

where $f = (f_1, f_2, \ldots, f_p)^T : \mathbb{R}^n \longrightarrow \mathbb{R}^p, g = (g_1, g_2, \ldots, g_m)^T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are locally Lipschitz functions on $\mathbb{R}^n$, K and Q are closed, convex and pointed cones with non-empty interiors in $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively. $F_0 = \{x \in \mathbb{R}^n : -g(x) \in Q\}$ denotes the set of all feasible solutions of (VOP).

The paper extends the work of Dutta and Lalitha [11]. In this paper, KKT type sufficient optimality conditions are obtained for (VOP) under the assumption that f is (strictly, strongly) nonsmooth cone-pseudoconvex and each $g_j, j = 1, 2, \ldots, m$ is regular in the sense of Clarke. The convexity of feasible set F_0 is replaced by the weaker condition (1.1). Unlike Dutta and Lalitha [11], we do not assume any nonsmooth degeneracy assumption. Further, we have associated a Mond–Weir type dual with (VOP) and proved duality theorems in the modified setting. Examples are given to substantiate the results proved.

2. Notations and definitions

Let $B \subseteq \mathbb{R}^p$ be a closed, convex and pointed ($B \cap (-B) = \{0\}$) cone with non-empty interior ($\text{int}B \neq \emptyset$). We denote $B \setminus \{0\}$ by B_0. The positive dual cone B^+ is defined as follows:

$$B^+ := \left\{ b \in \mathbb{R}^p : z^T b \geq 0, \quad \forall\, z \in B \right\}.$$

We recall the notions of locally Lipschitz function, Clarke generalized directional derivative and Clarke generalized gradient in the form of following definitions [4].

A real valued function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be locally Lipschitz at a point $x \in \mathbb{R}^n$ if there exists $k > 0$ such that $|\phi(q) - \phi(\bar{q})| \leq k\|q - \bar{q}\| \;\forall\, q, \bar{q}$ in a neighbourhood of x.

The real valued function ϕ is said to be locally Lipschitz on $\mathbb{R}^n$ if it is locally Lipschitz at each point of $\mathbb{R}^n$.

Let $f = (f_1, f_2, \ldots, f_p) : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a vector valued function where each f_i is a real valued function defined on $\mathbb{R}^n$. If each component f_i is locally Lipschitz on $\mathbb{R}^n$, then f is locally Lipschitz on $\mathbb{R}^n$ and for all $\lambda \in B^+, \lambda^T f = \sum_{i=1}^p \lambda_i f_i$ is locally Lipschitz on $\mathbb{R}^n$.

Definition 2.1. Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be locally Lipschitz at $x \in \mathbb{R}^n$. Then, the Clarke generalized directional derivative of ϕ at x in the direction $v \in \mathbb{R}^n$ is given by

$$\phi^\circ(x, v) = \limsup_{y \to x, t \downarrow 0} \frac{\phi(y + tv) - \phi(y)}{t}.$$

Definition 2.2. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x \in \mathbb{R}^n$. Then, the Clarke generalized gradient of ϕ at x is defined as

$$\partial^c \phi(x) = \{\xi \in \mathbb{R}^n \ : \ \phi^\circ(x, v) \geq \langle \xi, v \rangle , \quad \forall\, v \in \mathbb{R}^n\}.$$

It follows that $\phi^\circ(x, v) = \max\{\langle \xi, v\rangle : \xi \in \partial^c\phi(x)\}$, for any $v \in \mathbb{R}^n$.

Let $f = (f_1, f_2, \ldots, f_p) : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a vector valued function such that each f_i is locally Lipschitz at $x \in \mathbb{R}^n$. The Clarke generalized directional derivative of f at x in the direction $v \in \mathbb{R}^n$ is given by $f^\circ(x, v) = \left(f_1^\circ(x, v), f_2^\circ(x, v), \ldots, f_p^\circ(x, v)\right)$ where for each $i = 1, 2, \ldots, p$, $f_i^\circ(x, v)$ is the Clarke generalized directional derivative of f_i at x in the direction d. The Clarke generalized gradient of f at x is the set $\partial^c f(x) = \partial^c f_1(x) \times \ldots \times \partial^c f_p(x)$ where $\partial^c f_i(x)$ is the Clarke generalized gradient of f_i at x for $i = 1, 2, \ldots, p$. Any $y \in \partial^c f(x)$ is a $p \times n$ matrix of the form $y = [\xi_1, \xi_2, \ldots, \xi_p]^T$ where $\xi_i \in \partial^c f_i(x)$ for all $i = 1, 2, \ldots, p$.

In our results, we assume g_j's to be regular in the sense of Clarke [4], so we briefly state the notion of regularity.

Definition 2.3. Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real valued function. The directional derivative of ϕ at $x \in \mathbb{R}^n$ in the direction of $v \in \mathbb{R}^n$ is given by

$$\phi'(x, v) = \lim_{t \downarrow 0} \frac{\phi(x + tv) - \phi(x)}{t},$$

provided limit exists.

Definition 2.4. Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The function ϕ is said to be regular in the sense of Clarke if

(i) $\phi'(x, v)$ exists for all $x \in \mathbb{R}^n$ and for every direction $v \in \mathbb{R}^n$.
(ii) $\phi'(x, v) = \phi^\circ(x, v)$.

The solution concepts of weak minimum, minimum and strong minimum for (VOP) and Karush–Kuhn–Tucker (KKT) point of (VOP) are defined as follows:

Definition 2.5 ([5])**.** A point $u^* \in F_0$ is called a

(i) weak minimum of (VOP) if $f(u^*) - f(z) \notin \text{int}K$, $\forall\, z \in F_0$.
(ii) minimum of (VOP) if $f(u^*) - f(z) \notin K_0$, $\forall\, z \in F_0$.
(iii) strong minimum of (VOP) if $f(z) - f(u^*) \in K$, $\forall\, z \in F_0$.

Definition 2.6 ([10])**.** A point $u^* \in F_0$ is said to be a Karush–Kuhn–Tucker or a KKT point of (VOP) if

$$0 \in \partial^c \left(\lambda^T f + \mu^T g\right)(u^*) \quad \text{and} \quad \mu^T g(u^*) = 0, \quad \text{for some} \quad \lambda \in K^+ \setminus \{0\}, \quad \mu \in Q^+.$$

Suneja *et al.* [27] defined generalizations of cone-invex functions namely K-nonsmooth pseudo-invex, strongly K-nonsmooth pseudo-invex and strictly K-nonsmooth pseudo-invex functions with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We are giving below these definitions by taking $\eta(x, u) = x - u$ and call them K-nonsmooth pseudoconvex, strongly K-nonsmooth pseudoconvex and strictly K-nonsmooth pseudoconvex functions.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a locally Lipschitz function, $u \in \mathbb{R}^n$ and $K \subseteq \mathbb{R}^p$ be a closed convex cone with non empty interior.

Definition 2.7. f is said to be K-nonsmooth pseudoconvex at u, if for every $x \in \mathbb{R}^n$,

$$-f^{\circ}(u, x-u) \notin \text{int}K \Longrightarrow -(f(x)-f(u)) \notin \text{int}K.$$

Definition 2.8. f is said to be strongly K-nonsmooth pseudoconvex at u, if for every $x \in \mathbb{R}^n$,

$$-f^{\circ}(u, x-u) \notin \text{int}K \Longrightarrow f(x)-f(u) \in K.$$

Definition 2.9. f is said to be strictly K-nonsmooth pseudoconvex at u, if for every $x \in \mathbb{R}^n$,

$$-f^{\circ}(u, x-u) \notin \text{int}K \Longrightarrow -(f(x)-f(u)) \notin K_0.$$

3. Optimality conditions

Suneja *et al.* [28] obtained KKT optimality conditions for (VP) under the convexity of feasible set and cone-convexity (cone-pseudoconvexity) of the objective function along with Slater-type cone constraint qualification. We prove KKT type sufficient optimality conditions for (VOP) by replacing the convexity of feasible set F_0 with the weaker condition (1.1) and the objective function f is assumed to be (strictly, strongly) nonsmooth cone-pseudoconvex.

We begin by proving the following lemma which will be used to establish KKT type sufficient optimality conditions later.

Lemma 3.1. *Suppose that the feasible set F_0 of (VOP) satisfies condition (1.1) at $u^* \in F_0$ and each $g_j, j = 1, 2, \ldots, m$ is regular in the sense of Clarke. If $\mu \in Q^+$ satisfying $\mu^T g(u^*) = 0$, then $\mu^T g^{\circ}(u^*, x-u^*) \leq 0$ for all $x \in F_0$.*

Proof. The result trivially holds for $\mu = 0$. Let $\mu \in Q^+ \setminus \{0\}$ such that $\mu^T g(u^*) = 0$. Assume on contrary that for some $x \in F_0$

$$\mu^T g^{\circ}(u^*, x-u^*) > 0. \tag{3.1}$$

Since g_j's are regular in the sense of Clarke, we obtain

$$\sum_{j=1}^{m} \mu_j g_j'(u^*, x-u^*) > 0.$$

Using definition of directional derivative, we get

$$\lim_{t \downarrow 0} \frac{\mu^T g(u^* + t(x-u^*)) - \mu^T g(u^*)}{t} > 0.$$

This means that there exists some $\epsilon > 0$ such that $\mu^T g(u^* + t(x-u^*)) - \mu^T g(u^*) > 0$, for all $0 < t < \epsilon$. From $\mu^T g(u^*) = 0$, we get

$$\mu^T g(u^* + t(x-u^*)) > 0, \quad \text{for all} \quad 0 < t < \epsilon. \tag{3.2}$$

□

Since F_0 satisfies condition (1.1) at u^*, for $x \in F_0$ there exists a sequence $t_n \downarrow 0$ such that $u^* + t_n(x-u^*) \in F_0$. Then, $\mu^T g(u^* + t_n(x-u^*)) \leq 0$. In particular for sufficiently small t_n, we have

$$\mu^T g(u^* + t_n(x-u^*)) \leq 0.$$

This contradicts equation (3.2). Hence the result.

In the following example we show that, condition (1.1) on the feasible set can not be relaxed in the above lemma.

Example 3.2. Let $g = (g_1, g_2) : \mathbb{R} \longrightarrow \mathbb{R}^2$ be defined as

$$g_1(x) = \begin{cases} -x, & x \geq 0 \\ -2x, & x < 0 \end{cases} \quad \text{and} \quad g_2(x) = \sin x + x.$$

For $Q = \{(p, q) \in \mathbb{R}^2 : p \geq 0, p \geq -q\}$, the feasible set $F_0 = \{x \in \mathbb{R} : -g(x) \in Q\} = \{0\} \bigcup \cup_{k=0}^{\infty} [(2k+1)\pi, (2k+2)\pi]$. Let $u^* = 0$. Clearly, F_0 is a non-convex set and does not satisfy condition (1.1) at u^*. Moreover, g_1 and g_2 are regular in the sense of Clarke.

We have $Q^+ = \{(p, q) \in \mathbb{R}^2 : p \geq q \geq 0\}$ and $\mu^T g(0) = 0$ for all $\mu \in Q^+$. We calculate $g^\circ(0, x)$.

Since $\partial^c g_1(0) = [-2, -1]$, we get

$$g_1^\circ(0, x) = \begin{cases} -x, & x \geq 0 \\ -2x, & x < 0. \end{cases}$$

Also, g_2 is continuously differentiable so $\partial^c g_2(0) = \{2\}$ and $g_2^\circ(0, x) = 2x$. Hence,

$$\begin{aligned} g^\circ(0, x) &= (g_1^\circ(0, x), g_2^\circ(0, x)) \\ &= \begin{cases} (-x, 2x)^T, & x \geq 0 \\ (-2x, 2x)^T, & x < 0. \end{cases} \end{aligned}$$

Now,

$$\mu^T g^\circ(0, x) = \begin{cases} (2\mu_2 - \mu_1)\, x, & x \geq 0 \\ (\mu_2 - \mu_1)\, 2x, & x < 0. \end{cases}$$

For $(\mu_1, \mu_2) = (1, 1) \in Q^+, \mu^T g^\circ(0, x) = x \nleq 0$ for all $x \in F_0 \setminus \{0\}$. Thus, conclusion of Lemma 3.1 does not hold.

Theorem 3.3. *Let u^* be a KKT point of (VOP). Assume that f is K-nonsmooth pseudoconvex at u^* and F_0 satisfies condition (1.1) at u^*. Further, if g_j's, $j = 1, 2, \ldots, m$ are regular in the sense of Clarke and $\mathbb{R}_+^p \subseteq K$, $\mathbb{R}_+^m \subseteq Q$, then u^* is a weak minimum of (VOP).*

Proof. It is given that u^* is a KKT point of (VOP), therefore there exist $\lambda \in K^+ \setminus \{0\}, \mu \in Q^+$ such that

$$0 \in \partial^c \left(\lambda^T f + \mu^T g\right)(u^*) \quad \text{and} \quad \mu^T g(u^*) = 0. \tag{3.3}$$

From (3.3), we get

$$0 \in \left(\sum_{i=1}^{p} \lambda_i \partial^c f_i(u^*) + \sum_{j=1}^{m} \mu_j \partial^c g_j(u^*) \right).$$

Further $\mathbb{R}_+^p \subseteq K, \mathbb{R}_+^m \subseteq Q$ yields

$$\lambda^T f^\circ(u^*, h) + \mu^T g^\circ(u^*, h) \geq 0, \quad \text{for all} \quad h \in \mathbb{R}^n. \tag{3.4}$$

Assume on contrary, that u^* is not a weak minimum of (VOP). Then,

$$f(u^*) - f(\hat{u}) \in \text{int} K, \quad \text{for some} \quad \hat{u} \in F_0.$$

Since f is K-nonsmooth pseudoconvex at u^*, $-f^\circ(u^*, \hat{u} - u^*) \in \text{int} K$ which implies

$$\lambda^T f^\circ(u^*, \hat{u} - u^*) < 0. \tag{3.5}$$

Using (3.4) for $h = \hat{u} - u^*$, we get

$$\lambda^T f^\circ(u^*, \hat{u} - u^*) + \mu^T g^\circ(u^*, \hat{u} - u^*) \geq 0.$$

Using Lemma 3.1 above reduces to $\lambda^T f^\circ(u^*, \hat{u} - u^*) \geq 0$ which is a contradiction to (3.5). Hence, our assumption is wrong and u^* is a weak minimum of (VOP). ☐

Next we prove KKT type sufficient optimality conditions for a feasible point to be a minimum and strong minimum of (VOP).

Theorem 3.4. *Let u^* be a KKT point of (VOP). Assume that f is strictly K-nonsmooth pseudoconvex at u^* and F_0 satisfies condition (1.1) at u^*. Further, if g_j's, $j = 1, 2, \ldots, m$ are regular in the sense of Clarke and $\mathbb{R}^p_+ \subseteq K$, $\mathbb{R}^m_+ \subseteq Q$, then u^* is a minimum of (VOP).*

Proof. Assume on contrary, that u^* is not a minimum of (VOP). Then,

$$f(u^*) - f(\hat{u}) \in K_0, \quad \text{for some} \quad \hat{u} \in F_0.$$

Since f is strictly K-nonsmooth pseudoconvex at u^*, $-f^\circ(u^*, \hat{u} - u^*) \in \text{int}K$. This implies

$$\lambda^T f^\circ(u^*, \hat{u} - u^*) < 0.$$

This is same as equation (3.5). Proceeding on the lines of Theorem 3.3, we get a contradiction. Thus, u^* is a minimum of (VOP). □

Theorem 3.5. *Let u^* be a KKT point of (VOP). Assume that f is strongly K-nonsmooth pseudoconvex at u^* and F_0 satisfies condition (1.1) at u^*. Further, if g_j's, $j = 1, 2, \ldots, m$ are regular in the sense of Clarke and $\mathbb{R}^p_+ \subseteq K$, $\mathbb{R}^m_+ \subseteq Q$, then u^* is a strong minimum of (VOP).*

Proof. Assume on contrary, that u^* is not a strong minimum of (VOP). Then,

$$f(\hat{u}) - f(u^*) \notin K, \quad \text{for some} \quad \hat{u} \in F_0.$$

Since f is strongly K-nonsmooth pseudoconvex at u^*, $-f^\circ(u^*, \hat{u} - u^*) \in \text{int}K$. This implies

$$\lambda^T f^\circ(u^*, \hat{u} - u^*) < 0.$$

This is same as equation (3.5). Proceeding on the lines of Theorem 3.3, we get a contradiction. Thus, u^* is a strong minimum of (VOP). □

Remark 3.6. Theorem 3.3 related to nonsmooth vector optimization problem (VOP) generalizes (Dutta and Lalitha [11] Thm. 2.4) for the nonsmooth nonlinear programming problem (CP). Also, Suneja *et al.* [28] derived optimality results for the vector optimization problem (VP) involving differentiable functions whereas we have derived optimality results for the nonsmooth vector optimization problem (VOP) involving nondifferentiable functions. Suneja *et al.* [27] proved the above sufficient optimality results by assuming f to be (strictly, strongly) K-nonsmooth pseudo-invex and g to be Q-nonsmooth quasi-invex with respect to same $\eta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We have proved these results by assuming f to be (strictly, strongly) K-nonsmooth pseudoconvex and replacing the Q-nonsmooth quasiconvexity of g by condition (1.1) on the feasible set F_0.

In the following examples we have shown that if f is (strictly, strongly) K-nonsmooth pseudo-invex with respect to $\eta(x, u) = x - u$ but g is not Q-nonsmooth quasi-invex with respect to same η, still the problem (VOP) has weak minimum, minimum and strong minimum, provided condition (1.1) is satisfied.

Example 3.7. Consider the following vector optimization problem (VOP) where
$f = (f_1, f_2) : \mathbb{R} \longrightarrow \mathbb{R}^2, g = (g_1, g_2) : \mathbb{R} \longrightarrow \mathbb{R}^2$ are defined as

$$f_1(x) = \sin |x|, f_2(x) = -2\sin |x|,$$

$$g_1(x) = \begin{cases} (1-x)(x-2), & x \geq 0 \\ -2, & x < 0 \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} 0, & x \geq 0 \\ -2x, & x < 0. \end{cases}$$

$K = \{(p,q) \in \mathbb{R}^2 : p \geq 0, p \geq -q\}, Q = \{(p,q) \in \mathbb{R}^2 : q \geq 0, p \geq -q\}$. The feasible set $F_0 = \{x \in \mathbb{R} : -g(x) \in Q\} = [0,1] \bigcup [2,\infty)$. Let $u^* = 0$. Clearly, F_0 is a non-convex set and satisfies condition (1.1) at u^*. It is easy to see that g_1 and g_2 are regular in the sense of Clarke.

Here, $K^+ = \{(p,q) \in \mathbb{R}^2 : p \geq q \geq 0\}, Q^+ = \{(p,q) \in \mathbb{R}^2 ; q \geq p \geq 0\}$. For $\lambda = (1,0) \in K^+ \setminus \{0\}$, $\mu = (0,1) \in Q^+$, let $h(x) = (\lambda^T f + \mu^T g)(x)$. Then

$$h(x) = \begin{cases} \sin x, & x \geq 0 \\ -\sin x - 2x, & x < 0. \end{cases}$$

As $0 \in \partial^c h(0) = [-3,1]$ and $\mu^T g(0) = 0$, therefore $u^* = 0$ is a KKT point of (VOP). Next, we calculate $-f^\circ(u^*, x - u^*)$ at $u^* = 0$. Since $\partial^c f_1(0) = [-1,1]$ and $\partial^c f_2(0) = [-2,2]$,

$$f_1^\circ(0,x) = \mid x \mid \quad \text{and} \quad f_2^\circ(0,x) = 2 \mid x \mid$$

and we get,

$$\begin{aligned} -f^\circ(0,x) &= -\left(f_1^\circ(0,x), f_2^\circ(0,x)\right) \\ &= \begin{cases} (-x,-2x), & x \geq 0 \\ (x,2x), & x < 0. \end{cases} \end{aligned}$$

Also,

$$-(f(x) - f(0)) = \begin{cases} (-\sin x, 2\sin x), & x \geq 0 \\ (\sin x, -2\sin x), & x < 0. \end{cases}$$

f is K-nonsmooth pseudoconvex at $u^* = 0$ because, $\forall\, x \in \mathbb{R}$

$$-f^\circ(0,x) \notin \text{int}K \Longrightarrow -(f(x) - f(0)) \notin \text{int}K.$$

From Theorem 3.3, u^* is a weak minimum of (VOP).

Example 3.8. Consider the following vector optimization problem (VOP) where $f : \mathbb{R} \longrightarrow \mathbb{R}^2, g : \mathbb{R} \longrightarrow \mathbb{R}^2$ are defined as

$$f_1(x) = \begin{cases} x, & x \geqslant 0 \\ -e^x + 1, & x < 0, \end{cases} \quad f_2(x) = \begin{cases} x, & x \geqslant 0 \\ -\sin x, & x < 0, \end{cases}$$

$$g_1(x) = \begin{cases} (1-x)(x-2), & x \geqslant 0 \\ -2, & x < 0 \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} 0, & x \geq 0 \\ -2x, & x < 0. \end{cases}$$

$$K = \{(p,q) \in \mathbb{R}^2 : p \geqslant 0, p \geqslant -q\}, \quad Q = \{(p,q) \in \mathbb{R}^2 : q \geqslant 0, p \geqslant -q\}.$$

As shown in Example 3.7, g_1 and g_2 are regular in the sense of Clarke and the feasible set $F_0 = [0,1] \bigcup [2,\infty)$ satisfies condition (1.1) at $u^* = 0$.

Choose $\lambda = (1,0)^T \in K^+ \setminus \{0\}$, $\mu = (0,1)^T \in Q^+$ and set $h(x) = (\lambda^T f + \mu^T g)(x)$. Then,

$$0 \in \partial^c h(0) = [-3,1] \quad \text{and} \quad \mu^T g(0) = 0.$$

Thus, $u^* = 0$ is a KKT point of (VOP).

Since $\partial^c f_1(0) = \partial^c f_2(0) = [-1,1]$, $f_1^\circ(0,x) = f_2^\circ(0,x) = \mid x \mid$ and we get,

$$\begin{aligned} -f^\circ(0,x) &= -\left(f_1^\circ(0,x), f_2^\circ(0,x)\right) \\ &= \begin{cases} (-x,-x), & x \geq 0 \\ (x,x), & x < 0. \end{cases} \end{aligned}$$

Also,

$$-(f(x)-f(0)) = \begin{cases} (-x,-x), & x \geq 0 \\ (e^x - 1, \sin x), & x < 0. \end{cases}$$

f is strictly K-nonsmooth pseudoconvex at $u^* = 0$ because, $\forall\, x \in \mathbb{R}$

$$-f^\circ(0,x) \notin \mathrm{int}K \Longrightarrow -(f(x)-f(0)) \notin K_0.$$

From Theorem 3.4, u^* is a minimum of (VOP).

Example 3.9. Consider the following vector optimization problem (VOP) where $f : \mathbb{R} \longrightarrow \mathbb{R}^2, g : \mathbb{R} \longrightarrow \mathbb{R}^2$ are defined as

$$f_1(x) = f_2(x) = \begin{cases} x, & x \geqslant 0 \\ -e^x + 1, & x < 0 \end{cases}, \quad g_1(x) = \begin{cases} (1-x)(x-2), & x \geqslant 0 \\ -2, & x < 0 \end{cases} \quad \text{and}$$

$$g_2(x) = \begin{cases} 0, & x \geq 0 \\ -2x, & x < 0. \end{cases}$$

$$K = \left\{(p,q) \in \mathbb{R}^2 : p \geqslant 0, p \geqslant -q\right\}, \quad Q = \left\{(p,q) \in \mathbb{R}^2 : q \geqslant 0, p \geqslant -q\right\}.$$

As shown in Example 3.7, g_1 and g_2 are regular in the sense of Clarke and the feasible set $F_0 = [0,1] \bigcup [2,\infty)$ satisfies condition (1.1) at $u^* = 0$.

Choose $\lambda = (1,0)^T \in K^+ \setminus \{0\}$, $\mu = (0,1)^T \in Q^+$ and set $h(x) = \left(\lambda^T f + \mu^T g\right)(x)$. Then,

$$0 \in \partial^c h(0) = [-3,1] \quad \text{and} \quad \mu^T g(0) = 0.$$

Thus, $u^* = 0$ is a KKT point of (VOP).

Since $\partial^c f_1(0) = \partial^c f_2(0) = [-1,1]$, $f_1^\circ(0,x) = f_2^\circ(0,x) = \mid x \mid$ and we get,

$$\begin{aligned} -f^\circ(0,x) &= -\left(f_1^\circ(0,x), f_2^\circ(0,x)\right) \\ &= \begin{cases} (-x,-x), & x \geq 0 \\ (x,x), & x < 0. \end{cases} \end{aligned}$$

Also,

$$f(x) - f(0) = \begin{cases} (x,x), & x \geq 0 \\ (-e^x + 1, -e^x + 1), & x < 0. \end{cases}$$

f is strongly K-nonsmooth pseudoconvex at $u^* = 0$ because, $\forall\, x \in \mathbb{R}$

$$-f^\circ(0,x) \notin \mathrm{int}K \Longrightarrow f(x) - f(0) \in K.$$

From Theorem 3.5, u^* is a strong minimum of (VOP).

4. Duality

In this section, we associate a Mond–Weir type dual with (VOP) and establish duality results. Mond–Weir dual was proposed by Mond and Weir [23] for the nonlinear programming problem. This dual has same objective function as that of primal problem and duality theorems hold under weaker notions of convexity. Weir and Mond [30] extended duality results for Mond–Weir dual of nonlinear programming problem to the multiobjective optimization problem. Aggarwal [1] formulated a Mond–Weir type dual for the vector optimization problem over cones.

On the lines of Suneja *et al.* [27], we consider the following Mond–Weir type dual problem for (VOP).

$$K\text{-Maximize} \quad f(z) \qquad \text{(MWD)}$$
$$\text{subject to} \quad 0 \in \partial^c \left(\lambda^T f + \mu^T g\right)(z), \qquad (4.1)$$
$$\mu^T g(z) \geq 0 \qquad (4.2)$$

$\lambda \in K^+ \setminus \{0\}, \mu \in Q^+, z \in F_0$.

Let F_D be the feasible set of (MWD).

Definition 4.1. A point $\left(\bar{z}, \bar{\lambda}, \bar{\mu}\right) \in F_D$ is called a weak maximum of (MWD) if $f(z) - f(\bar{z}) \notin \text{int}K, \forall (z, \lambda, \mu) \in F_D$.

We now prove Weak and Strong Duality results for the dual (MWD).

Theorem 4.2 (Weak Duality)**.** *Let $x \in F_0$ and $(z, \lambda, \mu) \in F_D$. Assume that f is K-nonsmooth pseudoconvex at z and F_0 satisfies condition (1.1) at z. Further, if g_j's, $j = 1, 2, \ldots, m$ are regular in the sense of Clarke and $\mathbb{R}^p_+ \subseteq K$, $\mathbb{R}^m_+ \subseteq Q$,then $f(z) - f(x) \notin \text{int}K$.*

Proof. The proof follows on the lines of Theorem 3.3. □

Example 4.3. Consider the same vector optimization problem as in Example 3.7 and call it (P1). Let (D1) be the corresponding Mond–Weir type dual for (P1).

Consider $(z, \lambda, \mu) = (0, (1, 0), (0, 1))$ where $z \in F_0, \lambda \in K^+ \setminus \{0\}$ and $\mu \in Q^+$. As shown in Example 3.7, $0 \in \partial^c \left(\lambda^T f + \mu^T g\right)(0)$ and $\mu^T g(0) = 0$. Therefore, $(z, \lambda, \mu) \in F_D$. It was deduced in Example 3.7 that f is K-nonsmooth pseudoconvex at z and F_0 satisfies condition (1.1) at z. Also, g_1, g_2 are regular in the sense of Clarke and $\mathbb{R}^2_+ \subseteq K, \mathbb{R}^2_+ \subseteq Q$. Consider $x = 3\pi/2 \in F_0$. Then $f(z) - f(x) = (1, -2)^T \notin \text{int}K$. Hence, Weak Duality holds for feasible point x of (P1) and feasible point (z, λ, μ) of (D1).

To prove Strong Duality result, we utilize the following Fritz John type necessary optimality conditions for a nonsmooth vector optimization problem from the work of Craven [10].

Theorem 4.4. *Let u^* be a weak minimum of (VOP), then there exists $(0, 0) \neq (\lambda, \mu) \in K^+ \times Q^+$ such that*

$$0 \in \partial^c \left(\lambda^T f + \mu^T g\right)(u^*) \quad \text{and} \quad \mu^T g(u^*) = 0.$$

KKT type necessary optimality conditions can be obtained under the assumption of following Slater-type constraint qualification for the set $\partial^c g(\cdot)$.

Definition 4.5 ([10])**.** The problem (VOP) is said to satisfy Slater-type constraint qualification at u^* if for all $y \in \partial^c g(u^*)$, there exists $u \in \mathbb{R}^n$ such that $yu \in -\text{int}Q$.

Theorem 4.6 (Strong Duality)**.** *Let u^* be a weak minimum of (VOP). If (VOP) satisfies Slater-type constraint qualification at u^*, then there exist $\lambda^* \in K^+ \setminus \{0\}, \mu^* \in Q^+$ such that (u^*, λ^*, μ^*) is feasible for the dual problem (MWD) and the values of the objective functions of (VOP) and (MWD) are equal. Further, if $\mathbb{R}^p_+ \subseteq K$, $\mathbb{R}^m_+ \subseteq Q$ and conditions of Weak Duality Theorem 4.2 hold for all $(z, \lambda, \mu) \in F_D$, then (u^*, λ^*, μ^*) is a weak maximum of (MWD).*

Proof. Since u^* is a weak minimum of (VOP), by Theorem 4.4 and Slater-type constraint qualification there exist $\lambda^* \in K^+ \setminus \{0\}, \mu^* \in Q^+$ such that Definition 4.1 and Theorem 4.2 are satisfied. Thus, (u^*, λ^*, μ^*) is feasible for the dual problem (MWD) and objective function values of (VOP) and (MWD) are equal. Assume by contradiction that (u^*, λ^*, μ^*) is not a weak maximum of (MWD). Then, there exists $(z, \lambda, \mu) \in F_D$ such that $f(z) - f(u^*) \in \text{int}K$, which contradicts the Weak Duality. Hence, (u^*, λ^*, μ^*) is a weak maximum of (MWD). □

Remark 4.7. Suneja *et al.* [27] obtained the Strong Duality result for feasibility of the dual problem by assuming f to be K-generalized invex and g to be Q-generalized invex with respect to η whereas in the above Strong Duality result, we have not used any condition on f and g in order to obtain the dual feasible point. In the following example, f is not K-generalized invex with respect to $\eta(x,u) = x - u$, still we have obtained a dual feasible point.

Example 4.8. Consider the following vector optimization problem (VOP) where $f : \mathbb{R} \longrightarrow \mathbb{R}^2, g : \mathbb{R} \longrightarrow \mathbb{R}^2$ are defined as

$$f_1(x) = \sin \mid x \mid, f_2(x) = -2\sin \mid x \mid \quad \text{and}$$

$$g_1(x) = g_2(x) = \begin{cases} -\sin x, & x \geqslant 0 \\ -2x, & x < 0. \end{cases}$$

$$K = \left\{(p,q) \in \mathbb{R}^2 : p \geqslant 0, p \geqslant -q\right\}, \quad Q = \left\{(p,q) \in \mathbb{R}^2 : q \geqslant 0, p \geqslant -q\right\}.$$

The feasible set $F_0 = \{x \in \mathbb{R} : -g(x) \in Q\} = \bigcup_{k=0}^{\infty} [2k\pi, (2k+1)\pi]$. Let $u^* = 0$. Then,

$$-(f(x) - f(0)) = \begin{cases} (-\sin x, 2\sin x), & x \geq 0 \\ (\sin x, -2\sin x), & x < 0 \end{cases} \notin \text{int}K, \forall\, x \in F_0.$$

Thus, $u^* = 0$ is a weak minimum of (VOP).

Clearly, $\partial^c g(0) = \partial^c g_1(0) \times \partial^c g_2(0) = \left\{(\xi_1, \xi_2)^T : \xi_1, \xi_2 \in [-2, -1]\right\}$. Therefore, for all $y \in \partial^c g(0)$ there exists $u = 1 \in \mathbb{R}$ such that $yu \in -\text{int}Q$. Thus, (VOP) satisfies Slater-type constraint qualification at u^*.

Using Theorem 4.6, we can find $\lambda^* = (1,0)^T \in K^+ \setminus \{0\}$ and $\mu^* = (0,1)^T \in Q^+$ such that

$$0 \in \partial^c \left(\lambda^{*T} f + \mu^{*T} g\right)(0) \quad \text{and} \quad \mu^{*T} g(0) = 0.$$

Thus, $(u^* = 0, \lambda^* = (1,0), \mu^* = (0,1))$ is feasible for the corresponding Mond–Weir dual (MWD).

5. Conclusion

We have obtained KKT type sufficient optimality conditions and duality theorems for (VOP) by assuming f to be (strictly, strongly) nonsmooth cone-pseudoconvex function. Also, convexity of the feasible set F_0 is replaced by a weaker condition. We have not used any degeneracy assumption to prove the results. It will be interesting to derive aforesaid results without the condition of regularity.

Acknowledgements. The authors would like to express their sincere thanks to anonymous referees and the editor for valuable suggestions and comments for the paper which have considerably improved the presentation of the paper.

References

[1] S. Aggarwal, *Optimality and duality in mathematical programming involving generalized convex functions.* Ph.D. thesis, University of Delhi, Delhi (1998).

[2] M.S. Bazaraa, H.D. Sherali and C.M. Shetty, Nonlinear Programming. Wiley, New Jersey (2006).

[3] A. Chinchuluun, A. Migdalas, P.M. Pardalos and L. Pitsoulis, Pareto Optimality: Game Theory and Equilibria. Springer, New York (2008).

[4] F.H. Clarke, Optimization and Nonsmooth Analysis. John Wiley & Sons, Inc., New York (1983).

[5] L. Coladas, Z. Li and S. Wang, Optimality conditions for multiobjective and nonsmooth minimisation in abstract spaces. *Bull. Aust. Math. Soc.* **50** (1994) 205–218.

[6] E. Constantin, Second-order necessary conditions in locally Lipschitz optimization with inequality constraints. *Optim. Lett.* **9** (2015) 245–261.

[7] E. Constantin, First-order necessary conditions in locally Lipschitz multiobjective optimization. *Optimization* **67** (2018) 1447–1460.

[8] E. Constantin, Necessary conditions for weak efficiency for nonsmooth degenerate multiobjective optimization problems. *J. Global Optim.* **75** (2019) 111–129.

[9] E. Constantin, Higher-order sufficient conditions for optimization problems with Gâteaux differentiable data. *Rev. Roum. Math. Pures Appl.* **64** (2019) 25–41.

[10] B.D. Craven, Nonsmooth multiobjective programming. *Numer. Funct. Anal. Optim.* **10** (1989) 49–64.

[11] J. Dutta and C.S. Lalitha, Optimality conditions in convex optimization revisited. *Optim. Lett.* **7** (2013) 221–229.

[12] G. Giorgi, Optimality conditions under generalized convexity revisited. *Ann. Univ. Buchar. (Math. Ser.)* **4(LXII)** (2013) 479–490.

[13] J.B. Hiriart-Urruty and C. Lemarechal, Convex Analysis and Minimization Algorithms I. Springer, Berlin (1993).

[14] Q. Ho, Necessary and sufficient KKT optimality conditions in non-convex optimization. *Optim. Lett.* **11** (2017) 41–46.

[15] V.I. Ivanov, Higher-order optimality conditions for inequality-constrained problems. *Appl. Anal.* **92** (2013) 2152–2167.

[16] J.B. Lasserre, On representations of the feasible set in convex optimization. *Optim. Lett.* **4** (2010) 1–5.

[17] S.K. Mishra, On multiple-objective optimization with generalized univexity. *J. Math. Anal. App.* **224** (1998) 131–148.

[18] S.K. Mishra and R.N. Mukherjee, On generalised convex multi-objective nonsmooth programming. *J. Aust. Math. Soc. Ser. B* **38** (1996) 140–148.

[19] S.K. Mishra, S.Y. Wang and K.K. Lai, Optimality and duality in nondifferentiable and multiobjective programming under generalized d-invexity. *J. Global Optim.* **29** (2004) 425–438.

[20] S.K. Mishra, S.Y. Wang and K.K. Lai, Optimality and duality for multiple-objective optimization under generalized type I univexity. *J. Math. Anal. App.* **303** (2005) 315–326.

[21] S.K. Mishra, S.Y. Wang and K.K. Lai, Optimality and duality for V-invex non-smooth multiobjective programming problems. *Optimization* **57** (2008) 635–641.

[22] S.K. Mishra, S.Y. Wang and K.K. Lai, Optimality and duality for a nonsmooth multiobjective optimization involving generalized type I functions. *Math. Methods Oper. Res.* **67** (2008) 493–504.

[23] B. Mond and T. Weir, Generalized concavity and duality. In: Generalized Concavity in Optimization and Economics, edited by S. Schaible and W. Ziemba. Academic Press, New York (1981) 263–279.

[24] A.A. Schy and D.P. Giesy, Multicriteria optimization methods for design of aircraft control systems, edited by W. Stadler. In: Multicriteria Optimization in Engineering and in the Sciences. Plenum, New York (1981) 225–262.

[25] W. Stadler, Multicriteria optimization in mechanics: a survey. *Appl. Mech. Rev.* **37** (1984) 277–286.

[26] W. Stadler, Multicriteria Optimization in Engineering and in the Sciences. Plenum Press, New York (1988).

[27] S.K. Suneja, S. Khurana and Vani, Generalized nonsmooth invexity over cones in vector optimization. *Eur. J. Oper. Res.* **186** (2008) 28–40.

[28] S.K. Suneja, S. Sharma, M. Grover and M. Kapoor, A different approach to cone-convex optimization. *Am. J. Oper. Res.* **6** (2013) 536–541.

[29] S.K. Suneja, S. Sharma and P. Yadav, Optimality and duality for vector optimization problem with non-convex feasible set. *OPSEARCH* **57** (2020) 1–12.

[30] T. Weir and B. Mond, Generalised convexity and duality in multiple objective programming. *Bull. Aust. Math. Soc.* **39** (1989) 287–299.