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NORDHAUS–GADDUM TYPE RESULTS FOR CONNECTED AND TOTAL DOMINATION

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Abstract. A dominating set of $G = (V, E)$ is a subset S of V such that every vertex in $V - S$ has at least one neighbor in S. A connected dominating set of G is a dominating set whose induced subgraph is connected. The minimum cardinality of a connected dominating set is the connected domination number $\gamma_c(G)$. Let $\delta^*(G) = \min\{\delta(G), \delta(\overline{G})\}$, where $\overline{G}$ is the complement of G and $\delta(G)$ is the minimum vertex degree. In this paper, we improve upon existing results by providing new Nordhaus–Gaddum type results for connected domination. In particular, we show that if G and $\overline{G}$ are both connected and $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 3$, then $\gamma_c(G) + \gamma_c(\overline{G}) \leq 4 + (\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G)-2} + \frac{1}{\gamma_c(\overline{G})-2}\right)$ and $\gamma_c(G)\gamma_c(\overline{G}) \leq 2(\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G)-2} + \frac{1}{\gamma_c(\overline{G})-2} + \frac{1}{2}\right) + 4$. Moreover, we establish accordingly results for total domination.

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1. Introduction

In extremal graph theory, many problems seek the extreme values of graph parameters on families of graphs. *Nordhaus–Gaddum type* results study the extreme values of the sum (or product) of a parameter on a graph and its complement, following the classic paper of Nordhaus and Gaddum [15] solving these problems for the chromatic number on n-vertex graphs.

For domination problems, multiple edges and loops are irrelevant, so we forbid them. We use $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* $N[v]$ is the set $N(v) \cup \{v\}$. The *open neighborhood* $N(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N(v)$, and the *closed neighborhood* $N[S]$ of S is the set $N(S) \cup S$. The *degree* of a vertex $v \in V$ is $d_G(v) = |N(v)|$. The *minimum* and *maximum vertex degrees* in G are denoted $\delta(G)$ and

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$\Delta(G)$, respectively. We denote the complement of G by $\overline{G}$, and we let $\delta^*(G) = \min\{\delta(G), \delta(\overline{G})\}$. It is worth mentioning that if G is a graph of order n, then $\delta^*(G) \le \frac{n-1}{2}$. Given graphs G and H, the *cartesian product* $G\Box H$ is the graph with vertex set $V(G) \times V(H)$ and edge set defined by making (u,v) and (u',v') adjacent if and only if either (1) $u = u'$ and $vv' \in E(H)$ or (2) $v = v'$ and $uu' \in E(G)$.

A subset S of vertices of G is a *dominating set* if $N[S] = V$. A *connected dominating set* (respectively, *total dominating set*) of G is a dominating set whose induced subgraph is connected (respectively, without isolated vertices). The minimum cardinality of a connected dominating set (respectively, a total dominating set) is the *connected domination number* $\gamma_c(G)$ (respectively, *total domination number* $\gamma_t(G)$). A connected dominating set will be abbreviated cd-set, while a total dominating set by td-set. A cd-set of minimum cardinality is called a γ_c-set. Likewise, a γ_t-set is defined similarly. Since any cd-set of order at least two is also a td-set, $\gamma_t(G) \le \gamma_c(G)$ for every nontrivial connected graph G with $\Delta(G) < |V(G)| - 1$. Moreover, it is worth noting that $\text{diam}(G) \ge 3$ if and only if $\gamma_c(\overline{G}) \le 2$.

Inequalities of Nordhaus–Gaddum type have been proved for many graph invariants including various domination parameters. The excellent survey by Aouchiche and Hansen [1] provides a large collection of Nordhaus–Gaddum relations up to the year 2013. Furthermore, by imposing constraints on graphs and their complements, many of these results can be improved. For the connected and total domination numbers that are the focus of our study, the following bounds have been proved.

Theorem 1.1. *If G and $\overline{G}$ are nontrivial connected graphs of order n, then*

(i) ([12]) $\gamma_c(G) + \gamma_c(\overline{G}) \le \delta^*(G) + 4 - (\gamma_c(G) - 3)(\gamma_c(\overline{G}) - 3)$*; sharp for* $\delta^*(G) \ge 2$.
(ii) ([4]) $(\gamma_c(G) - 2)(\gamma_c(\overline{G}) - 2) \le \delta^*(G) + 2$.
(iii) ([12]) $\gamma_c(G) + \gamma_c(\overline{G}) \le \frac{3n}{4}$ *when* $\delta^*(G) \ge 3$ *and* $n \ge 14$*; sharp when* 4 *divides* n.
(iv) ([12]) $\gamma_c(G) + \gamma_c(\overline{G}) \le \delta^*(G) + 2$ *when* $\gamma_c(G), \gamma_c(\overline{G}) \ge 4$*, with equality possible if and only if* $\delta^*(G) = 6$.
(v) ([10]) $\gamma_t(G) + \gamma_t(\overline{G}) \le n + 2$.

Throughout this paper, G is a connected graph of order n whose complement $\overline{G}$ is also connected. Note that this yields $n \ge 4$. For such graphs G, we establish the following sharp upper bound for $\gamma_c(G) + \gamma_c(\overline{G})$ which improves the bound of item (i) in Theorem 1.1.

$$\gamma_c(G) + \gamma_c(\overline{G}) \le 4 + (\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2}\right).$$

This bound is our main result and most of results of Theorem 1.1 and others follow from a closer examination of its proof. In the last two sections, we will also provide upper bounds on the sums $\gamma_t(G) + \gamma_t(\overline{G})$ and $\gamma_t(G) + \text{sd}_{\gamma_t}(\overline{G})$ where $\text{sd}_{\gamma_t}(\overline{G})$ is the minimum number of edges that must be subdivided in order to increase the total domination number.

Before closing this section, we recall a result of [8] and that every connected graph G contains a spanning tree with at least $\Delta(G)$ leaves.

Theorem 1.2 ([8]). *If G is a connected n-vertex graph, then* $\gamma_c(G) \le n - \Delta(G)$.

2. Bounds on $\gamma_c(G) + \gamma_c(\overline{G})$

In this section we present sharp upper bounds on the sum $\gamma_c(G) + \gamma_c(\overline{G})$.

Theorem 2.1. *If G and $\overline{G}$ are connected graphs with* $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \ge 3$*, then*

$$\gamma_c(G) + \gamma_c(\overline{G}) \le 4 + (\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2}\right).$$

This bound is sharp for every value of $\delta^*(G) \ge 2$.

Proof. We first observe that since $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 3$, we have $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. Let x be a vertex in G of degree $\delta(G)$, and let $X = V(G) - N[x]$. We deduce from $\gamma_c(G) \geq 3$ that $X \neq \emptyset$. Also, since $\mathrm{diam}(G) = 2$, $N(x)$ dominates X.

In the sequel, we consecutively select disjoint sets $S_0, \ldots, S_{k-1}$ in $N(x)$ that almost dominate X, and disjoint sets $X_0, \ldots, X_{k-1}$ in X which are not dominated by $S_0, \ldots, S_{k-1}$, respectively. Let $T_0 = N(x)$, and let S_0 be a largest subset of $N(x)$ that does not dominate X. Let $X_0 = X - N(S_0)$ and $T_1 = T_0 - S_0$. By the choice of S_0, for any vertex $y \in T_1$, the set $S_0 \cup \{y\}$ dominates X and thus y dominates X_0. Note that T_1 may possibly dominate X. Now, if T_1 does not dominate X, then we stop and if T_1 dominates X, then let S_1 be a largest subset of T_1 that does not dominate X. Let $X_1 = X - N(S_1)$ and $T_2 = T_1 - S_1$. We continue constructing sets $T_0, \ldots, T_k$ with $T_0 \supset \ldots \supset T_k$ (where $k \geq 1$), sets $S_0, \ldots, S_{k-1}$ and $X_0, \ldots, X_{k-1}$ such that:

(a) For each $i < k$, T_i dominates X.
(b) For each $i < k$, S_i is a largest subset of T_i that does not dominate X, and $T_{i+1} = T_i - S_i$.
(c) For each $i < k$, $X_i = X - N(S_i)$.
(d) T_k does not dominate X.

Since T_i dominates X but S_i does not (for any $i < k$), all of $T_0, \ldots, T_k$ are nonempty. Moreover, by construction, $S_i \cup \{y_i\}$ dominates X whenever $y_i \in T_{i+1}$. Thus $S_i \cup \{x, y_i\}$ is a cd-set of G, and hence

$$|S_i| \geq \gamma_c(G) - 2 \tag{2.1}$$

for each $i \in \{0, 1, \ldots, k-1\}$. For each $i \in \{0, \ldots, k-1\}$, let x_i be a vertex of X_i, and let x_k be a vertex of X that is not dominated by T_k. Since $N(x) = \left(\bigcup_{i=0}^{k-1} S_i\right) \cup T_k$, the set $\{x, x_0, \ldots, x_k\}$ is a cd-set of $\overline{G}$ and thus

$$k \geq \gamma_c(\overline{G}) - 2. \tag{2.2}$$

Since $\delta(G) = |T_k| + \sum_{i=0}^{k-1} |S_i|$ and $|T_k| \geq 1$, inequality (2.1) implies

$$\delta(G) \geq 1 + k(\gamma_c(G) - 2). \tag{2.3}$$

Hence

$$\gamma_c(G) \leq \frac{\delta(G) - 1}{k} + 2 \tag{2.4}$$

and

$$k \leq \frac{\delta(G) - 1}{\gamma_c(G) - 2}. \tag{2.5}$$

By (2.2), (2.4) and (2.5), we have

$$\begin{aligned} \gamma_c(G) + \gamma_c(\overline{G}) &\leq \left(\frac{\delta(G) - 1}{\gamma_c(\overline{G}) - 2} + 2\right) + \left(\frac{\delta(G) - 1}{\gamma_c(G) - 2} + 2\right) \\ &= 4 + (\delta(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2}\right). \end{aligned} \tag{2.6}$$

By symmetry, we also have $\gamma_c(G) + \gamma_c(\overline{G}) \leq 4 + (\delta(\overline{G}) - 1)\left(\frac{1}{\gamma_c(G)-2} + \frac{1}{\gamma_c(\overline{G})-2}\right)$, and the desired inequality is proved.

To prove the sharpness, for each integer $\ell \geq 3$, we will provide a connected graph G_ℓ of order $\ell^2 + \ell + 1$ such that $\delta(G_\ell) = \ell$, $\gamma_c(G_\ell) = \ell + 1$, $\delta(\overline{G}_\ell) = \ell^2 - \ell + 1$, $\gamma_c(\overline{G}_\ell) = 3$, and $\gamma_c(G_\ell) + \gamma_c(\overline{G}_\ell) = \ell + 4$, hereby achieving the bound. The graph G_ℓ is constructed as follows. Let $H_1 = H_2 = K_\ell$, with $V(H_2) = \{v_1, \ldots, v_\ell\}$, and consider the cartesian product $H_1 \Box H_2$. Then add a star of order $\ell + 1$ with center y and leaves $x_1, \ldots, x_\ell$, where for each $i \in \{1, \ldots, \ell\}$ we join x_i to all vertices of the ith copy of H_1 in $H_1 \Box H_2$, that is to all vertices

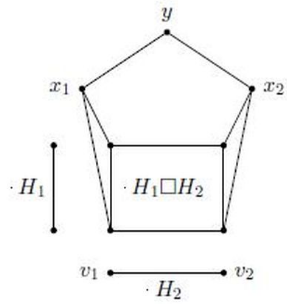

FIGURE 1. The graph G_2, plus H_1 and H_2.

of $H_1\square H_2$ with second coordinate v_i. See Figure 1 for an example of G_3 (along with H_1 and H_2). Note that $\mathrm{diam}(G_\ell) = \mathrm{diam}(\overline{G}_\ell) = 2$ and $\delta^*(G_\ell) = \ell$.

It remains to show that $\gamma_c(\overline{G}_\ell) = 3$ and $\gamma_c(G_\ell) = \ell + 1$. Since $\mathrm{diam}(G_\ell) = 2$, we have $\gamma_c(\overline{G}_\ell) \geq 3$. On the other hand, if u and w are neighbors of x_1 and x_2 in G_ℓ other than y, then $\{y, u, w\}$ is a connected dominating set of $\overline{G}_\ell$ and so $\gamma_c(\overline{G}_\ell) = 3$. Now to see that $\gamma_c(G_\ell) = \ell+1$, we first note that $\{y, x_1, \ldots, x_\ell\}$ is a cd-set of G_ℓ, and thus $\gamma_c(G_\ell) \leq \ell+1$. To get the lower bound, let S be a cd-set of G_ℓ, and let $T_i = N[x_i] - \{y\}$. If S does not intersect T_i, which includes x_i and vertices of a copy of H_1, then dominating T_i requires that S contains y and a vertex from each copy of H_2. This requires $\ell + 1$ vertices. Thus $|S| \geq \ell + 1$ unless S intersects each of the ℓ disjoint sets $T_1, \ldots, T_\ell$ exactly once. But then dominating y without reaching size $\ell+1$ requires that S contains some x_i, and the latter (x_i) has no neighbor in S, which is again not connected. Therefore $\gamma_c(G) = \ell + 1$. □

Corollary 2.2. *If G and $\overline{G}$ are connected n-vertex graphs, then $\gamma_c(G) + \gamma_c(\overline{G}) \leq n + 1$.*

Proof. If $\min\{\gamma_c(G), \gamma_c(\overline{G})\} = 2$, then the result follows from Theorem 1.2, and if $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 3$, then the result follows from Theorem 2.1. □

Theorem 2.1 will be useful to establish the next upper bound for the product of $\gamma_c(G) - 2$ and $\gamma_c(\overline{G}) - 2$ that was first shown in [4]. However, we will provide in addition a characterization of extremal graphs attaining this upper bound.

Corollary 2.3. *If G and $\overline{G}$ are connected graphs, then*

$$(\gamma_c(G) - 2)(\gamma_c(\overline{G}) - 2) \leq \delta^*(G) - 1.$$

Proof. If $\gamma_c(G) = 2$ or $\gamma_c(\overline{G}) = 2$, then the result is immediate. Hence we assume that $\gamma_c(G), \gamma_c(\overline{G}) \geq 3$. By Theorem 2.1, $\gamma_c(G) + \gamma_c(\overline{G}) - 4 \leq (\delta^*(G) - 1)\left(\frac{\gamma_c(G)+\gamma_c(\overline{G})-4}{(\gamma_c(G)-2)(\gamma_c(\overline{G})-2)}\right)$ and the result follows from the fact that $\gamma_c(G) + \gamma_c(\overline{G}) - 4 > 0$. □

Let $\mathcal{F}$ be the family of graphs G such that $\delta^*(G) = 1$, or $\gamma_c(G) = \delta(G) + 1$ and $\gamma_c(\overline{G}) = 3$, or $\gamma_c(G) = 3$ and $\gamma_c(\overline{G}) = \delta(\overline{G}) + 1$.

Theorem 2.4. *Let G and $\overline{G}$ be connected graphs with $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 2$. Then $(\gamma_c(G) - 2)(\gamma_c(\overline{G}) - 2) = \delta^*(G) - 1$ if and only if $G \in \mathcal{F}$.*

Proof. If $\delta^*(G) = 1$, then clearly $\gamma_c(G) = 2$ or $\gamma_c(\overline{G}) = 2$ and thus $(\gamma_c(G) - 2)(\gamma_c(\overline{G}) - 2) = \delta^*(G) - 1$. Hence assume that $\delta^* \geq 2$, and let $\gamma_c(G) = \delta(G)+1$ and $\gamma_c(\overline{G}) = 3$ (the case $\gamma_c(G) = 3$ and $\gamma_c(\overline{G}) = \delta(\overline{G})+1$ is similar). By Corollary 2.3 we have $\delta(G) - 1 = (\gamma_c(G) - 2)(\gamma_c(\overline{G}) - 2) \leq \delta^*(G) - 1$ and hence $(\gamma_c(G) - 2)(\gamma_c(\overline{G}) - 2) = \delta^*(G) - 1$.

Conversely, assume that $(\gamma_c(G) - 2)(\gamma_c(\overline{G}) - 2) = \delta^*(G) - 1$. If $\delta^*(G) = 1$, then obviously $G \in \mathcal{F}$. Hence let $\delta^*(G) \geq 2$. In the sequel, we will use the same notations as in the proof of Theorem 2.1. Clearly, since $\delta^*(G) \geq 2$, we have $\gamma_c(G) \geq 3$ and $\gamma_c(\overline{G}) \geq 3$. Then all inequalities (2.1)–(2.6) occurring in the proof of Theorem 2.1 become equalities, in particular

$$|S_i| = \gamma_c(G) - 2 \tag{2.7}$$

for each $i \in \{0, 1, \ldots, k-1\}$,

$$k = \gamma_c(\overline{G}) - 2 \tag{2.8}$$

and

$$\delta(G) = 1 + \sum_{i=0}^{k-1} |S_i| = 1 + k(\gamma_c(G) - 2). \tag{2.9}$$

Thus

$$\gamma_c(G) = \frac{\delta(G) - 1}{k} + 2 \tag{2.10}$$

and

$$k = \frac{\delta(G) - 1}{\gamma_c(G) - 2}. \tag{2.11}$$

We consider two cases.

Case 1. $\gamma_c(G) = 3$.

Then by (2.7) and (2.9) we have $|S_i| = 1$ for each $i \in \{0, 1, \ldots, k-1\}$, $|T_k| = 1$ and $\delta(G) = k + 1$. Let $S_i = \{z_i\}$ for $i \in \{0, \ldots, k-1\}$ and $T_k = \{z_k\}$. Let G_1 be the subgraph of G induced by $\{z_0, \ldots, z_k\}$. Assume first that G_1 has an isolated vertex, say z_0. Since $\gamma_c(G) = 3$, there exists a vertex $y \in V(G) - N[x]$ that is not dominated by z_0. It follows that $\{x, y, z_0\}$ is a cd-set of $\overline{G}$ and thus $\gamma_c(\overline{G}) = 3$. We conclude from (2.8) and (2.9) that $k = 1$ and $\gamma_c(G) = \delta(G) + 1 = \delta(G^*) + 1$ yielding $G \in \mathcal{F}$. Assume now that G_1 has no isolated vertex. Without loss of generality, let $z_0z_1 \in E(G)$. If each z_j $(j \geq 2)$ has a neighbor in $\{z_0, z_1\}$, then $\{z_0, z_1\}$ is a cd-set of G, a contradiction. Hence, we may assume, without loss of generality, that z_2 has no neighbor in $\{z_0, z_1\}$. Then $\{x, z_2, x_2, \ldots, x_k\}$, where $x_i \in X_i$ and x_k is a vertex of X not dominated by T_k, is a cd-set of $\overline{G}$ of cardinality $k + 1$, contradicting the fact that $\gamma_c(\overline{G}) = k + 2$.

Case 2. $\gamma_c(G) \geq 4$.

By (2.7) and (2.9), we have $|S_0| = |S_1| = \cdots = |S_{k-1}| = \gamma_c(G) - 2$, $|T_k| = 1$ and $\delta(G) = k(\gamma_c(G) - 2) + 1$. It follows that (i) any subset of $N(x)$ of size $\gamma_c(G) - 1$ dominates $X = V(G) - N[x]$, and (ii) for any subset W of $N(x)$ of size $\gamma_c(G) - 2$, there exists a subset W' of $X = V(G) - N[x]$ that is not dominated by W and any vertex of $N(x) - W$ is adjacent to all W'. Let G_2 be the subgraph induced by $N(x)$. We distinguish the following situations.

Subcase 2.1. $\mathrm{diam}(G_2) \geq 3$.

Let $z_1, z_2 \in V(G_2)$ be two vertices at distance at least three in G_2. Since $\gamma_c(G) \geq 4$, there is a vertex $z \in X - (N(z_1) \cup N(z_2))$. Then $\{z_1, z_2, z, x\}$ is a cd-set of $\overline{G}$ and so $\gamma_c(\overline{G}) \leq 4$. It follows from (2.8) that $k \leq 2$. If $k = 1$, then we have $\delta = |S_0| + 1 = \gamma_c(G) - 1$ and $\gamma_c(\overline{G}) = 3$. Hence $\gamma_c(G) = \delta(G) + 1$ and $\gamma_c(\overline{G}) = 3$ and thus $G \in \mathcal{F}$. Now, let $k = 2$. Then we have $\delta = |S_0| + |S_1| + 1 = 2(\gamma_c(G) - 2) + 1 = 2\gamma_c(G) - 3$ and $\gamma_c(\overline{G}) = 4$. If $d_{G_2}(y) \leq \gamma_c(G) - 3$ for some $y \in V(G_2)$, then $|N_{G_2}[y]| \leq \gamma_c(G) - 2$ and for $y' \in X$ not dominated by $N_{G_2}[y]$ in G, $\{x, y, y'\}$ is a cd-set of $\overline{G}$ which is a contradiction. Thus $d_{G_2}(y) \geq \gamma_c(G) - 2$ for each $y \in V(G_2)$. But then $\delta(G) \geq |N_{G_2}(z_1)| + |N_{G_2}(z_2)| + 2 \geq 2\gamma_c(G) - 2$, a contradiction.

Subcase 2.2. $\operatorname{diam}(G_2) = 2$.

Let $y \in V(G_2)$ be an arbitrary vertex and $Y = V(G_2) - N[y]$. Using an argument similar to that described in the proof of Theorem 2.1, we can construct sets $T'_0, \ldots, T'_s$ with $T'_0 \supset \ldots \supset T'_s$ (where $s \geq 1$), sets $S'_0, \ldots S'_{s-1}$ and sets $X'_0, \ldots X'_{s-1}$ such that:

(a) For each $i < s$, T'_i dominates Y.
(b) For each $i < s$, S'_i is a largest subset of T'_i that does not dominate Y, and $T'_{i+1} = T'_i - S'_i$.
(c) For each $i < s$, $X'_i = Y - N(S'_i)$.
(d) T'_s does not dominate Y.

First let $d_{G_2}(y) \leq \gamma_c(G) - 3$. Thus $|N_{G_2}[y]| \leq \gamma_c(G) - 2$. Let y' be a vertex of X not dominated by $N_{G_2}[y]$. Then $\{x, y, y'\}$ is a cd-set of $\overline{G}$, implying that $\gamma_c(\overline{G}) = 3$. By (2.8), we have $k = 1$, and by (2.9) we get $\gamma_c(G) = \delta(G) + 1$. Therefore $G \in \mathcal{F}$. Now, we can assume that $d_{G_2}(y) \geq \gamma_c(G) - 2$ for each $y \in N_G(x)$. If $|S'_i| \leq \gamma_c(G) - t$ for some i, with $t \geq 3$, then $S'_i \cup \{x, y'\}$ whenever $y' \in X - N(S'_i)$ (see (ii) of Case 2), is a cd-set of G which leads to a contradiction. Hence $|S'_i| \geq \gamma_c(G) - 2$ for each $i \in \{0, \ldots, s-1\}$. Let $|T'_s| + \sum_{i=0}^{s} |X'_i| = m(\gamma_c(G) - 2) + j$, where $m \geq 0$ and $0 \leq j \leq \gamma_c(G) - 3$. Note that either $m \neq 0$ or $j \neq 0$. Using (2.9) we obtain

$$\begin{aligned} 1 + k(\gamma_c(G) + 2) &= \delta(G) \\ &= 1 + \sum_{i=0}^{s-1} |S'_i| + |T'_s| + \sum_{i=0}^{s} |X'_i| \\ &\geq s(\gamma_c(G) - 2) + 1 + |T'_s| + \sum_{i=0}^{s} |X'_i| \\ &= s(\gamma_c(G) - 2) + 1 + m(\gamma_c(G) - 2) + j, \end{aligned}$$

and by (2.8) we have

$$\gamma_c(\overline{G}) - 2 \geq s + m + \frac{j}{\gamma_c(G) + 2}. \tag{2.12}$$

If $m = 0$ or $m = 1$ and $j = 0$, then for any vertex $y' \in X$ not dominated by $T'_s \cup (\cup_{i=0}^{s} X'_i)$, the set $\{x, y', z_1, \ldots, z_{s-1}\}$ whenever $z_i \in X'_i$ for each i, is a cd-set of $\overline{G}$ and (2.12) leads to $s \geq s + m + \frac{j}{\gamma_c(G)+2}$, which is a contradiction. Hence $m \geq 2$ or $m = 1$ and $j \geq 1$. By construction of the sets and fact (i), we have $|T'_s| \leq \gamma_c(G) - 2$. For any vertex $y' \in X$ not dominated by T'_s, the set $\{x, y', y, z_1, \ldots, z_{s-1}\}$ whenever $z_i \in X'_i$ for each i, is a cd-set of $\overline{G}$ and (2.12) leads to $s + 1 \geq s + m + \frac{j}{\gamma_c(G)+2}$ which is a contradiction. □

The next result shows that the bound in Theorem 1.1(i) is a consequence of Theorem 2.1 when $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 3$.

Corollary 2.5 ([12])**.** *If both G and $\overline{G}$ are connected graphs with $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 3$, then*

$$\gamma_c(G) + \gamma_c(\overline{G}) \leq 4 + \delta^*(G) - (\gamma_c(G) - 3)(\gamma_c(\overline{G}) - 3).$$

Proof. If $\min\{\gamma_c(G), \gamma_c(\overline{G})\} = 3$, then by Corollary 2.3, $\max\{\gamma_c(G), \gamma_c(\overline{G})\} \leq 1 + \delta^*(G)$. Therefore $\gamma_c(G) + \gamma_c(\overline{G}) \leq 4 + \delta^*(G) = 4 + \delta^*(G) - (\gamma_c(G) - 3)(\gamma_c(\overline{G}) - 3)$. Hence we can assume that $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 4$. Let $x = \gamma_c(G) - 2$ and $y = \gamma_c(\overline{G}) - 2$. Thus by Theorem 2.1, $\gamma_c(G) + \gamma_c(\overline{G}) \leq 4 + (\delta^*(G) - 1)(\frac{1}{x} + \frac{1}{y})$. Now to complete the proof, it is enough to show that

$$\begin{aligned} (\delta^*(G) - 1)\left(\frac{1}{x} + \frac{1}{y}\right) &\leq \delta^*(G) - (x-1)(y-1) \\ &= (\delta^*(G) - 1) - xy + xy\left(\frac{1}{x} + \frac{1}{y}\right), \end{aligned}$$

or equivalently

$$(\delta^*(G) - 1 - xy)\left(\frac{1}{x} + \frac{1}{y}\right) \leq \delta^*(G) - 1 - xy.$$

This last inequality is always true because of $\frac{1}{x} + \frac{1}{y} \leq 1$ and $\delta^*(G) - 1 - xy \geq 0$ (by Cor. 2.3). □

The following result also shows that the upper bound in Theorem 1.1(iv) can be easily obtained by using Theorem 2.4 and Corollaries 2.3 and 2.5.

Corollary 2.6 ([12]). *If both G and $\overline{G}$ are connected graphs with $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 4$, then $\gamma_c(G) + \gamma_c(\overline{G}) \leq \delta^*(G) + 2$.*

Proof. If $\gamma_c(G) > 4$ or $\gamma_c(\overline{G}) > 4$, then the result immediately follows from Corollary 2.5. Hence we assume that $\gamma_c(G) = \gamma_c(\overline{G}) = 4$. By Corollary 2.3, we have $\delta^*(G) \geq 5$. But with these data, we deduce from Theorem 2.4 that G does not belong to $\mathcal{F}$, and thus $\delta^*(G) \geq 6$. Clearly, the result is valid in this case. □

Now, we turn our attention to the product of $\gamma_c(G)$ and $\gamma_c(\overline{G})$ for which we provide the next upper bound.

Theorem 2.7. *If G and $\overline{G}$ are connected n-vertex graphs with $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 3$, then*

$$\gamma_c(G)\gamma_c(\overline{G}) \leq 2(\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2} + \frac{1}{2}\right) + 4.$$

Furthermore, this bound is sharp for the cycle C_5.

Proof. Expanding and collecting terms in the inequality of Corollary 2.3 yields $\gamma_c(G)\gamma_c(\overline{G}) \leq 2(\gamma_c(G) + \gamma_c(\overline{G})) + \delta^*(G) - 5$. Moreover, Theorem 2.1 implies that

$$\begin{aligned}
\gamma_c(G)\gamma_c(\overline{G}) &\leq 2(\gamma_c(G) + \gamma_c(\overline{G})) + \delta^*(G) - 5 \\
&\leq 2\left(4 + (\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2}\right)\right) + \delta^*(G) - 5 \\
&= 2(\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2}\right) + \delta^*(G) + 3 \\
&= 2(\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2} + \frac{1}{2}\right) + 4.
\end{aligned}$$

□

Corollary 2.8. *Let G and $\overline{G}$ be connected n-vertex graphs.*

(1) ([4]) *If $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 4$, then $\gamma_c(G)\gamma_c(\overline{G}) \leq \frac{3}{2}(n-1)$.*

(2) *If $\min\{\gamma_c(G), \gamma_c(\overline{G})\} = 3$, then $\gamma_c(G)\gamma_c(\overline{G}) \leq \frac{3}{2}(n+1)$. This bound is sharp for the cycle C_5.*

(3) ([12]) *If $\min\{\gamma_c(G), \gamma_c(\overline{G})\} = 2$, then $\gamma_c(G)\gamma_c(\overline{G}) \leq 2n - 4$. This bound is sharp for the path P_4.*

Proof. (1) Let $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 4$. Corollary 2.6 yields $\delta^*(G) \geq 6$. Now, if $\delta^*(G) < \frac{n-1}{2}$, then Theorem 2.7 implies that

$$\gamma_c(G)\gamma_c(\overline{G}) \leq 2(\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2} + \frac{1}{2}\right) + 4 \leq \frac{3}{2}(n-4) + 4 < \frac{3}{2}(n-1).$$

Hence assume that $\delta^*(G) = \frac{n-1}{2}$, and thus $n \geq 11$. If $\gamma_c(G) = \gamma_c(\overline{G}) = 4$, then the result is immediate and if $\max\{\gamma_c(G), \gamma_c(\overline{G})\} \geq 5$, then Theorem 2.7 yields $\gamma_c(G)\gamma_c(\overline{G}) \leq \frac{4}{3}(n-3) + 4 \leq \frac{3}{2}(n-1)$.

(2) Let $\min\{\gamma_c(G), \gamma_c(\overline{G})\} = 3$. Corollary 2.6 yields $\max\{\gamma_c(G), \gamma_c(\overline{G})\} \leq \frac{n+1}{2}$ and so $\gamma_c(G)\gamma_c(\overline{G}) = \min\{\gamma_c(G), \gamma_c(\overline{G})\} \cdot \max\{\gamma_c(G), \gamma_c(\overline{G})\} \leq \frac{3}{2}(n+1)$.

(3) Let $\min\{\gamma_c(G), \gamma_c(\overline{G})\} = 2$. Then the result follows from Theorem 1.2.

□

3. Bounds on the sum and product of $\gamma_t(G)$ and $\gamma_t(\overline{G})$

In this section, we give some upper bounds on the sum $\gamma_t(G) + \gamma_t(\overline{G})$ and the product $\gamma_t(G)\gamma_t(\overline{G})$. Most of these results are immediate consequences of Theorems 2.1, 2.4, 2.7 and Corollary 2.3. Recall that $\gamma_t(G) \le \gamma_c(G)$ for every connected graph G with $\Delta(G) < n(G) - 1$.

Theorem 3.1. *If G and $\overline{G}$ are connected graphs with $\min\{\gamma_t(G), \gamma_t(\overline{G})\} \ge 3$, then*

$$\gamma_t(G) + \gamma_t(\overline{G}) \le 4 + (\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2}\right).$$

This bound is sharp for the graph G_ℓ constructed in Theorem 2.1.

Corollary 3.2. *If G and $\overline{G}$ are non-trivial connected n-vertex graphs, then*

$$\gamma_t(G) + \gamma_t(\overline{G}) \le n + 1.$$

Theorem 3.3. *If G and $\overline{G}$ are connected n-vertex graphs with $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \ge 3$, then*

$$\gamma_t(G)\gamma_t(\overline{G}) \le 2(\delta^*(G) - 1)\left(\frac{1}{\gamma_c(G) - 2} + \frac{1}{\gamma_c(\overline{G}) - 2} + \frac{1}{2}\right) + 4.$$

This bound is sharp for the cycle C_5.

Theorem 3.4. *If G and $\overline{G}$ are connected graphs with $\min\{\gamma_t(G), \gamma_t(\overline{G})\} \ge 2$, then*

$$(\gamma_t(G) - 2)(\gamma_t(\overline{G}) - 2) \le \delta^*(G) - 1.$$

The equality holds if and only if $\delta^*(G) = 1$ or $\gamma_t(G) = \gamma_c(G) = \delta(G) + 1$ and $\gamma_t(\overline{G}) = \gamma_c(\overline{G}) = 3$ or $\gamma_t(G) = \gamma_c(G) = 3$ and $\gamma_t(\overline{G}) = \gamma_c(\overline{G}) = \delta(\overline{G}) + 1$.

For the proof of the next result, it is necessary to recall the following two results.

Theorem 3.5 ([16])**.** *If G is a connected graph of order n with $\delta(G) \ge 4$, then $\gamma_t(G) \le \frac{3n}{7}$.*

Theorem 3.6 ([14])**.** *If G is a n-vertex graph with $\delta(G) \ge 2$ such that $d(u) + d(v) \ge 5$ for every two adjacent vertices u and v of G, then $\gamma_t(G) \le n/2$.*

Theorem 3.7. *If G and $\overline{G}$ are connected graphs of order $n \ge 14$ with $\delta(G) \ge 2$ such that $5 \le d_G(u) + d_G(v) \le n - 3$ for every two adjacent vertices u and v of G, then $\gamma_t(G) + \gamma_t(\overline{G}) \le n/2 + 2$.*

Proof. If $\gamma_t(\overline{G}) = 2$, then the result is immediate from Theorem 3.6. Hence, we assume that $\gamma_t(\overline{G}) \ge 3$. Since $d_G(u) + d_G(v) \le n - 3$ for every two adjacent vertices u and v of G, we have $\gamma_t(G) \ge 3$. Moreover, since $\min\{\gamma_t(G), \gamma_t(\overline{G})\} \ge 3$, we have $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. Assume first that $\delta(G) = 2$, and let u be a vertex of degree 2, with $N_G(u) = \{u_1, u_2\}$. Since $\mathrm{diam}(G) = 2$, $\{u, u_1, u_2\}$ is a td-set of G and so $\gamma_t(G) = 3$. On the other hand, it follows from $d_G(u) + d_G(u_i) \le n - 3$ that there is a vertex $u'_i \in V(G) - \{u, u_1, u_2\}$ such that $u_i u'_i \notin E(G)$ for $i = 1, 2$. Then $\{u, u'_1, u'_2\}$ is a td-set of $\overline{G}$, implying that $\gamma_t(\overline{G}) = 3$. Hence $\gamma_t(G) + \gamma_t(\overline{G}) = 6 \le n/2 + 2$ because $n \ge 10$.

Now let $\delta(G) \ge 3$. Since $\gamma_t(\overline{G}) \ge 3$ and since $d_G(x) + d_G(y) \le n - 3$ for every two adjacent vertices x and y of G, we have $\delta(\overline{G}) \ge 3$. If $\min\{\gamma_t(G), \gamma_t(\overline{G})\} \ge 4$, then by Theorem 1.1(iv), we have

$$\begin{aligned}\gamma_t(G) + \gamma_t(\overline{G}) &\le \delta^*(G) + 2\\ &\le \frac{n-1}{2} + 2\\ &< \frac{n}{2} + 2.\end{aligned}$$

Hence, we may assume, without loss of generality, that $\gamma_t(G) = 3$. If $\gamma_t(\overline{G}) = 4$, then the result is immediate since $n \geq 10$. Thus let $\gamma_t(\overline{G}) \geq 5$. Since $\operatorname{diam}(\overline{G}) = 2$, we have $\delta(\overline{G}) \geq 4$. By Theorem 3.5, it follows that

$$\gamma_t(G) + \gamma_t(\overline{G}) \leq 3 + \frac{3n}{7} \leq \frac{n}{2} + 2,$$

and the proof is complete. □

The next result that was first proven in [10], follows from a closer examination of the proof of Theorem 3.7.

Corollary 3.8. *If G and $\overline{G}$ are both connected with $n(G) \geq 14$ and $\delta^*(G) \geq 3$, then $\gamma_t(G) + \gamma_t(\overline{G}) \leq n/2 + 2$.*

Before seeing the sharpness of the bound in Corollary 3.8, we have to note that for every graph G of order n and minimum degree at least 3, $\gamma_t(G) \leq n/2$ (see [2]), and this bound is attained for the following family $\mathcal{G}$ of cubic graphs constructed in [5]. For $k \geq 1$, let G_k be the graph constructed as follows. Consider two copies of the path P_{2k} with respective vertex sequences $a_1b_1a_2b_2 \dots a_kb_k$ and $c_1d_1c_2d_2 \dots c_kd_k$. Let $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$, $C = \{c_1, c_2, \dots, c_k\}$, and $D = \{d_1, d_2, \dots, d_k\}$. For each $i \in \{1, 2, \dots, k\}$, join a_i to d_i and b_i to c_i. To complete the construction of the graph G_k, join a_1 to c_1 and b_k to d_k. Let $\mathcal{G} = \{G_k \mid k \geq 1\}$. Moreover, for $k \geq 2$, one can easily see that for every $G_k \in \mathcal{G}$, we have $\delta(\overline{G_k}) = n - 4 \geq 3$ and $\gamma_t(\overline{G_k}) = 2$. Since Corollary 14 is stated for graphs of order $n \geq 14$ and $\delta^*(G) \geq 3$, the upper bound of Corollary 14 is sharp for any graph $G_k \in \mathcal{G}$ with $k \geq 4$.

4. Bounds on $\gamma_t(G) + sd_{\gamma_t}(\overline{G})$

In this section we present upper bounds on the sum of the total domination number of a graph G and the total domination subdivision number of the complement of G. Recall that the *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(G)$ of a graph G is the minimum number of edges that must be subdivided in order to increase the total domination number. Let us first recall some well-known results.

Theorem 4.1 ([3]). *If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq \frac{2n}{3}$.*

Let G_{10} be the graph obtained from the 10-cycle $C_{10} = (v_1v_2 \dots v_{10})$ by adding the edge v_1v_6 and H_{10} be the graph obtained from the 10-cycle $C_{10} = (v_1v_2 \dots v_{10})$ by adding the edges v_1v_6 and v_5v_{10}.

Theorem 4.2 ([9]). *If $G \notin \{C_3, C_5, C_6, C_{10}, G_{10}, H_{10}\}$ is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_t(G) \leq \frac{4n}{7}$.*

Theorem 4.3 ([2]). *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) \leq \frac{n}{2}$.*

Theorem 4.4 ([7]). *If G is a graph of order $n \geq 3$ and $\gamma_t(G) = 2$ or 3, then $1 \leq \mathrm{sd}_{\gamma_t}(G) \leq 3$.*

The join of two graphs G and H, $G \vee H$, is a graph formed from disjoint copies of G and H by connecting every vertex of G to every vertex of H.

Theorem 4.5 ([11]). *Let G be a connected graph of order n. The following statements are equivalent.*

(1) $\gamma_t(G) = 2$ *and* $\mathrm{sd}_{\gamma_t}(G) = 3$.
(2) *G is isomorphic to $\overline{K_m} \vee K_{n-m}$ for some $1 \leq m \leq n - 3$.*

Theorem 4.6 ([6]). *If G is a connected graph of order $n \geq 3$ different from K_4, then $\mathrm{sd}_{\gamma_t}(G) \leq \lfloor \frac{2n}{3} \rfloor$ with equality if and only if G is isomorphic to $P_3, K_3, K_{1,3}, K_{1,3} + e, K_4 - e, K_5 - e$ or K_5.*

Theorem 4.7 ([13]). *If G is a connected graph of order $n \geq 3$ different from K_4, then $\mathrm{sd}_{\gamma_t}(G) \leq \frac{n+1}{2}$.*

Theorem 4.8 ([13]). *If G is a connected graph of order $n \geq 3$, then $\mathrm{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$.*

Theorem 4.9. *If G and $\overline{G}$ are connected graphs of order $n \geq 6$, then*

$$\gamma_t(G) + \mathrm{sd}_{\gamma_t}(\overline{G}) \leq \frac{2n}{3} + 2.$$

Proof. If $\gamma_t(G) = 2$, then the result is immediate by Theorem 4.6. If $\gamma_t(G) = 3$, then we deduce from Theorem 4.6 and the fact $n \geq 6$ that $\mathrm{sd}_{\gamma_t}(G) \leq \lfloor \frac{2n}{3} \rfloor - 1$ and so $\gamma_t(G) + \mathrm{sd}_{\gamma_t}(\overline{G}) \leq \frac{2n}{3} + 2$. Hence we can assume that $\gamma_t(G) \geq 4$. If $\gamma_t(\overline{G}) = 2$, then we deduce from Theorems 4.4, 4.5 and the connectedness of G that $\mathrm{sd}_{\gamma_t}(\overline{G}) \leq 2$ and thus the result follows from Theorem 4.1. Hence we assume that $\gamma_t(\overline{G}) \geq 3$. It is clear that $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. Observe that if $\delta(G) = 1$, then $\gamma_t(\overline{G}) = 2$, a contradiction. Also, if $\delta(G) = 2$, then any vertex of degree two and its neighbors form a td-set of G, contradicting the fact that $\gamma_t(G) \geq 4$. Thus let $\delta(G) \geq 3$. If $\gamma_t(\overline{G}) = 3$, then $\mathrm{sd}_{\gamma_t}(\overline{G}) \leq 3$ and the result follows from Theorem 4.3 and the fact that $n \geq 6$. Hence assume that $\gamma_t(\overline{G}) \geq 4$. By Theorem 3.1 that $\gamma_t(G) + \gamma_t(\overline{G}) \leq 4 + (\delta^*(G) - 1) \leq 3 + \frac{n-1}{2} < 2 + \frac{2n}{3}$ and the proof is complete. □

Theorem 4.10. *If G and $\overline{G}$ are connected graphs of order $n \geq 11$ with $\delta^*(G) \geq 2$, then*

$$\gamma_t(G) + \mathrm{sd}_{\gamma_t}(\overline{G}) \leq \frac{4n}{7} + 3.$$

Proof. If $\gamma_t(\overline{G}) \leq 3$, then the result follows by Theorems 4.4 and 4.2, and if $\gamma_t(G) \leq 3$, then the result follows from Theorem 4.7. Hence assume that $\gamma_t(G) \geq 4$ and $\gamma_t(\overline{G}) \geq 4$. By Theorems 3.1 and 4.8 we conclude that

$$\begin{aligned}\gamma_t(G) + \mathrm{sd}_{\gamma_t}(\overline{G}) &\leq \gamma_t(G) + \gamma_t(\overline{G}) + 1 \leq \delta^*(G) + 4\\ &\leq \frac{n-1}{2} + 4 < \frac{4n}{7} + 3.\end{aligned}$$

□

We conclude this section with the following open problem.

Problem 4.11. Is it true that for every connected graph G of order $n \geq 3$, $\gamma_t(G) + \mathrm{sd}_{\gamma_t}(G) \leq \frac{2n}{3} + 3$?

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