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MINIMUM STATUS OF TREES WITH GIVEN PARAMETERS

Zhene Peng and Bo Zhou*

Abstract. The status of a vertex v in a connected graph G is defined as the sum of the distances from v to all other vertices in G. The minimum status of G is the minimum of status of all vertices of G. We give the smallest and largest values for the minimum status of a tree with fixed parameters such as the diameter, the number of pendant vertices, the number of odd vertices, and the number of vertices of degree two, and characterize the unique extremal trees.

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1. Introduction

We consider simple and undirected graphs. Let G be a connected graph of order n with vertex set $V(G)$. The distance between vertices u and v in G, denoted by $d_G(u,v)$, is the length of a shortest path connecting u and v in G. The status of a vertex u in G is defined as [9,15]

$$s_G(u) = \sum_{v \in V(G)} d_G(u,v).$$

The minimum status of G, denoted by $s(G)$, is defined as [9,24,27]

$$s(G) = \min\{s_G(u) : \ u \in V(G)\}.$$

In the literature, the status of a vertex is also known as the distance [12,26], the transmission [21,25,29,30], or the total distance [6] of a vertex, and if $n \geq 2$, the normalized status $\frac{1}{n-1}s_G(u)$ of vertex u in G is called the average distance from vertex u to all other vertices of G, see [2]. The status or normalized status of a vertex may be used to measure its closeness centrality in the network [13,14]. The minimum status of a graph G is also studied through the proximity of G, defined as $\frac{1}{n-1}s(G)$, see, *e.g.*, [1–5]. Recall that the mean vertex derivation of G, defined as $\frac{1}{n}s(G)$, was also studied in [31].

Aouchiche and Hansen [2] gave sharp lower and upper bounds for the minimum status of a graph as a function of its order, and characterized the extremal graphs, and they also gave a sharp lower bound for the minimum status of a graph with fixed order and diameter. A trivial lower bound for the minimum status of a connected graph of order n is $n-1$, which is achieved if and only if there is a vertex of degree $n-1$. Let G be a connected

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School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P.R. China.
*Corresponding author: zhoubo@scnu.edu.cn

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graph of order n. They showed that $s(G) \leq \lfloor \frac{n^2}{4} \rfloor$ with equality if and only if G is either the cycle C_n or the path P_n. If G is a tree, then this result has also been given in [31]. Aouchiche and Hansen [1] gave Nordhaus-Gaddum-type inequalities for the minimum status of a graph, *i.e.*, lower and upper bounds for $s(G)+s(\overline{G})$ and $s(G)s(\overline{G})$ in terms of the order of G with both G and $\overline{G}$ being connected, where $\overline{G}$ is the complement of G. Lin *et al.* [24] established sharp lower and upper bounds for the minimum status of a graph with maximum degree, and they characterized extremal graphs for the lower bound and gave a necessary condition for graphs attaining the upper bound. Rissner and Burkard [27] gave alternate and simple proofs for the results in [24], and showed that the minimum status and the radius achieve their minimum (maximum, respectively) values at the same type of trees when order and maximum degree are given. Liang *et al.* [22] gave sharp lower and upper bounds for the minimum status of a tree with fixed matching number (domination, respectively), and characterize the extremal trees. Related work may be found in [2, 3, 10, 28].

A vertex in a graph is a pendant vertex if its degree is one, and a vertex in a graph is an odd (even, respectively) vertex if its degree is odd (even, respectively). By Handshaking Theorem, a graph possesses an even number of odd vertices. A vertex is an internal vertex if its degree is at least two. A tree whose internal vertices all have degree at least three is called a series-reduced tree or a homeomorphically irreducible tree [7, 16, 17]. That is, a series-reduced tree has no vertices of degree two.

Following the above work, in this paper, we give sharp lower and upper bounds for the minimum status of trees using parameters, including the diameter, the number of pendant vertices, the number of odd vertices, and the number of vertices of degree two, and we also characterize all extremal cases. In some cases, the bounds may be extended to connected graphs.

2. Preliminaries

For a proper subset U of vertices of a graph G, $G-U$ denotes the subgraph of G obtained by deleting the vertices from U (and the incident edges), and in particular, we write $G-v$ for $G-\{v\}$ if $U=\{v\}$. For a subset E_1 of edges of G, $G-E_1$ denotes the subgraph obtained from G by deleting all the edges in E_1, and in particular, we write $G-uv$ for $G-\{uv\}$ if $E_1=\{uv\}$. For a subset E_2 of unordered vertex pairs of distinct vertices of G, if each element of E_2 is not an edge of G, then $G+E_2$ denotes the graph obtained from G by adding all elements of E_2 as edges, and in particular, we write $G+uv$ for $G+\{uv\}$ if $E_2=\{uv\}$.

For a vertex v in a graph G, $N_G(v)$ denotes the set of vertices adjacent to v (*i.e.*, neighbors of v) in G, and the degree of v, denoted by $\delta_G(v)$, is equal to $|N_G(v)|$. For $k \geq 1$, we say a path $P = v_0 \dots v_k$ in a graph G is a pendant path (of length k) at v_0 if $\delta_G(v_0) > 2$, $\delta_G(v_k)=1$, and if $k \geq 2$, $\delta_G(v_1) = \cdots = \delta_G(v_{k-1}) = 2$. A pendant path of length one is also known as a pendant edge.

If P is a pendant path of length ℓ at vertex v in a graph G, then G is said to be obtained from $G-(V(P)\setminus\{v\})$ by attaching a pendant path of length ℓ at v. Particularly, if $\ell=1$, say $P=vw$, then we also say that G is obtained from $G-w$ by attaching a pendant edge at v (or a pendant vertex to v).

A caterpillar is a tree with a path such that each vertex of the tree either lies on this path or is adjacent to a vertex in this path. Note that S_n and P_n are caterpillars.

A quasi-pendant vertex is a vertex that is adjacent to a pendant vertex.

The median of a connected graph G is the set of vertices of G with minimum status. The median of a tree consists of either one vertex or two adjacent vertices [12, 18, 31].

Lemma 2.1 ([19, 20, 31])**.** *Let T be a nontrivial tree of order n. Then a vertex x is in the median of T if and only if $|V(C)| \leq \frac{n}{2}$ for every component C of $T-x$.*

It follows from Lemma 2.1 that any pendant vertex in a tree with at least 3 vertices can not be in the median of this tree.

For a graph G with an cut edge uv, let $G_{uv} = G - \{vw : w \in N_G(v)\setminus\{u\}\} + \{uw : w \in N_G(v)\setminus\{u\}\}$.

Lemma 2.2 ([22])**.** *Let G be a connected graph and uv be a cut edge of G. If uv is not a pendant edge of G, then $s(G) > s(G_{uv})$.*

Lemma 2.3 ([22])**.** *Let T be a tree with $u \in V(T)$ and $N_T(u) = \{u_1, \dots, u_k\}$, where $k \geq 3$. For $1 \leq i \leq k$, let T_i be the component of $T - u$ containing u_i. For $w \in V(T_2)$, let $T' = T - \{uu_i : 3 \leq i \leq t\} + \{wu_i : 3 \leq i \leq t\}$, where $3 \leq t \leq k$. If $|V(T_1)| \geq |V(T_2)|$, then $s(T') > s(T)$.*

For integers n, r and t with $r \geq t \geq 0$ and $r + t + 2 \leq n$, we denote by $D_n(r, t)$ the caterpillar formed by attaching r pendant edges to one terminal vertex and t pendant edges to the other terminal vertex of a path P_{n-r-t}. If $r + t = n - 1$, then $D_n(r, t)$ is just the star S_n. Note that $D_n(1, 1) \cong P_n \cong D_n(1, 0)$.

Lemma 2.4 ([22])**.** *If $r + t + 2 \leq n$ and $r \geq t \geq 2$, then $s(D_n(r, t)) > s(D_n(r + 1, t - 1))$.*

Lemma 2.5. *If $2 \leq a \leq n - 1$, then $s(D_n(\lceil \frac{a}{2} \rceil, \lfloor \frac{a}{2} \rfloor)) = \lfloor \frac{n^2 - a^2 + 2a}{4} \rfloor$.*

Proof. It is evident for $a = 2, n - 1$. Suppose that $3 \leq a \leq n - 2$.

Note that the diameter of $D_n(\lceil \frac{a}{2} \rceil, \lfloor \frac{a}{2} \rfloor)$ is $n - a + 1$. By Lemma 2.1, the vertex of distance $\lfloor \frac{n-a+1}{2} \rfloor$ from pendant vertex adjacent to a vertex of degree $\lceil \frac{a}{2} \rceil + 1$ is in the median of $D_n(\lceil \frac{a}{2} \rceil, \lfloor \frac{a}{2} \rfloor)$. Thus, if $n - a$ is odd, then

$$\begin{aligned} s\left(D_n\left(\left\lceil \frac{a}{2} \right\rceil, \left\lfloor \frac{a}{2} \right\rfloor\right)\right) &= 2 \sum_{j=1}^{\frac{n-a+1}{2}-1} j + \left\lceil \frac{a}{2} \right\rceil \cdot \frac{n-a+1}{2} + \left\lfloor \frac{a}{2} \right\rfloor \cdot \frac{n-a+1}{2} \\ &= \frac{n^2 - a^2 + 2a - 1}{4}, \end{aligned}$$

and if $n - a$ is even, then

$$\begin{aligned} s\left(D_n\left(\left\lceil \frac{a}{2} \right\rceil, \left\lfloor \frac{a}{2} \right\rfloor\right)\right) &= \sum_{j=1}^{\frac{n-a}{2}-1} j + \sum_{j=1}^{\frac{n-a}{2}} j + \left\lceil \frac{a}{2} \right\rceil \cdot \frac{n-a}{2} + \left\lfloor \frac{a}{2} \right\rfloor \cdot \left(\frac{n-a}{2} + 1\right) \\ &= \left(\frac{n-a}{2}\right)^2 + \frac{a(n-a)}{2} + \left\lceil \frac{a}{2} \right\rceil. \end{aligned}$$

Thus

$$\begin{aligned} s\left(D_n\left(\left\lceil \frac{a}{2} \right\rceil, \left\lfloor \frac{a}{2} \right\rfloor\right)\right) &= \begin{cases} \frac{n^2-a^2+2a-1}{4} & \text{if } n-a \text{ is odd} \\ \frac{n^2-a^2+2a}{4} & \text{if } n-a \text{ is even and } a \text{ is even} \\ \frac{n^2-a^2+2a-2}{4} & \text{if } n-a \text{ is even and } a \text{ is odd} \end{cases} \\ &= \left\lfloor \frac{n^2 - a^2 + 2a}{4} \right\rfloor, \end{aligned}$$

as desired. □

For a nontrivial tree T with $u \in V(T)$ and integers $p, q \geq 1$, let $T_{u;p,q}$ be the tree obtained by attaching two pendant paths, one of length p and one of length q, at u.

Lemma 2.6. *Let T be a nontrivial tree with $u \in V(T)$. If $p \geq q + 2$ and $q \geq 1$, then $s(T_{u;p,q}) > s(T_{u;p-1,q+1})$.*

Proof. Let $H = T_{u;p,q}$. Let $P = v_0 \dots v_p$ and $Q = w_0 \dots w_q$ be the two pendant paths at u ($= v_0 = w_0$) in H. Let $H' = H - v_{p-1}v_p + w_q v_p$. Let x be a vertex in the median of H. As $p > q$, it follows from Lemma 2.1 that $x \notin V(Q) \setminus \{u\}$.

Case 1. $x \in V(T)$.

By Lemma 2.1, x is also in the median of H'. As we go from H to H', the distance between x and v_p is decreased by $p-(q+1)$ (which is at least 1) and the distance between x and any other vertex remains unchanged. Therefore

$$s(H') - s(H) = s_{H'}(x) - s_H(x) \leq -1 < 0,$$

implying that $s(H) > s(H')$.

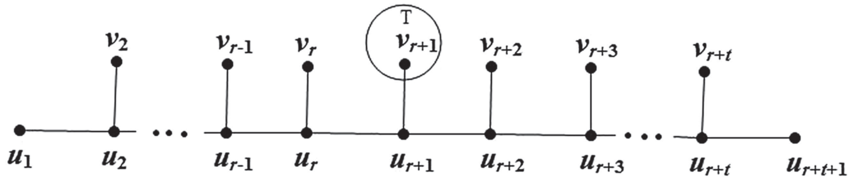

FIGURE 1. The tree $T(r,t)$.

Case 2. $x \in V(P) \setminus \{u\}$.

Assume that $x = v_i$, and if two vertices in $V(P) \setminus \{u\}$ are in the median of H, we choose v_i with smaller index i. As $\delta_H(x) = 2$, there are exactly two components in $H - x$. Let n be the order of H. By Lemma 2.1, we have $d_H(x, v_p) = \lfloor \frac{n}{2} \rfloor$, *i.e.*, $p - i = \lfloor \frac{n}{2} \rfloor$. By Lemma 2.1, v_{i-1} is in the median of H'. Note that $d_{H'}(v_{i-1}, v_p) - d_H(v_i, v_p) = i + q - \lfloor \frac{n}{2} \rfloor$, $d_{H'}(v_{i-1}, z) - d_H(v_i, z) = 1$ for $z \in \{v_{i+1}, \ldots, v_{p-1}\}$, and $d_{H'}(v_{i-1}, z) - d_H(v_i, z) = -1$ for $z \in V(H) \setminus \{v_{i-1}, \ldots, v_p\}$. Thus, we have

$$\begin{aligned} s(H') - s(H) &= s_{H'}(v_{i-1}) - s_H(v_i) \\ &= \sum_{z \in V(H')} d_{H'}(v_{i-1}, z) - \sum_{z \in V(H)} d_H(v_i, z) \\ &= \left(i + q - \left\lfloor \frac{n}{2} \right\rfloor\right) + 1 \cdot (p - 1 - i) - 1 \cdot (n - p - 2 + i) \\ &= p + q - n + 1 < 0, \end{aligned}$$

implying that $s(H) > s(H')$. □

Let r and t be positive integers. For a nontrivial tree T and a caterpillar C of degrees one or three with a diametral path $u_1 \ldots u_{r+t+1}$, where v_i is the unique neighbor of u_i outside path $u_1 \ldots u_{r+t+1}$ for each $2 \le i \le r + t$, let $T(r,t)$ be the tree of order $2(r + t) - 1 + |V(T)|$ consisting of T and C such that they share a unique common vertex v_{r+1}, see Figure 1. Note that $T(r,t) - (V(T) \setminus \{v_{r+1}\})$ is a caterpillar of degrees one or three with diametral path $u_1 \ldots u_{r+t+1}$, where v_i is the unique neighbor of u_i outside path $u_1 \ldots u_{r+t+1}$ for each $2 \le i \le r + t$.

Lemma 2.7. *Let T be a nontrivial tree. If $r \ge t \ge 2$, then*

$$s(T(r + 1, t - 1)) > s(T(r,t)).$$

Proof. Let $H = T(r,t)$. Let T_1 and T_2 be the components of $G - u_{r+1}$ containing u_1 and u_{r+t+1}, respectively. Let

$$H' = H - \{u_{r+1}v_{r+1}, u_{r+2}v_{r+2}\} + \{u_{r+1}v_{r+2}, u_{r+2}v_{r+1}\}.$$

Then $H' \cong T(r + 1, t - 1)$. As $|V(T_2)| = 2t - 1 \le 2r - 1 = |V(T_1)|$, we have $x \notin V(T_2)$ by Lemma 2.1.

Case 1. $x \in V(T)$.

By Lemma 2.1, x is also in the median of H'. Note that $d_{H'}(x, z) - d_H(x, z) = 1$ for $z \in V(T_1) \cup \{u_{r+1}\}$, $d_{H'}(x, z) - d_H(x, z) = -1$ for $z \in V(T_2) \setminus \{v_{r+2}\}$, and the distance between x and a vertex of $V(T) \cup \{v_{r+2}\}$ remains unchanged. Thus

$$\begin{aligned} s(H') - s(H) &= s_{H'}(x) - s_H(x) \\ &= 1 \cdot (|V(T_1)| + 1) - 1 \cdot (|V(T_2)| - 1) \\ &= |V(T_1)| - |V(T_2)| + 2 > 0, \end{aligned}$$

implying that $s(H') > s(H)$.

Case 2. $x \in V(T_1)$, or $x = u_{r+1}$ and $|V(T_1)| = \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - 1$.

By Lemma 2.1, x is also in the median of H'. As we go from H to H', the distance between x and a vertex of $V(T)$ is increased by 1, the distance between x and v_{r+2} is decreased by 1, and the distance between x and any other vertex remains unchanged. Thus $s(H') - s(H) = |V(T)| - 1 > 0$, *i.e.*, $s(H') > s(H)$.

Case 3. $x = u_{r+1}$ and $|V(T_1)| \le \lfloor \frac{n}{2} \rfloor - 2$.

By Lemma 2.1, u_{r+2} is a vertex in the median of H'. Note that $d_{H'}(u_{r+2}, z) - d_H(u_{r+1}, z) = 1$ for $z \in V(T_1)$, $d_{H'}(u_{r+2}, z) - d_H(u_{r+1}, z) = -1$ for $z \in V(T_2) \setminus \{u_{r+2}, v_{r+2}\}$, and $d_{H'}(u_{r+2}, z) = d_H(u_{r+1}, z)$ for $z \in V(T) \cup \{v_{r+2}\}$. Thus

$$s(H') - s(H) = s_{H'}(u_{r+2}) - s_H(u_{r+1}) = |V(T_1)| - (|V(T_2)| - 2) > 0,$$

implying that $s(H') > s(H)$. □

Lemma 2.8. *Let T be a caterpillar with the largest minimum status in the class of caterpillars of order n with $t \ge 1$ vertices of degree two and maximum degree three, or in the class of caterpillars of order n with $t \ge 1$ vertices of degree two and one or two vertices of maximum degree four, where each vertex of degree four has three pendant neighbors. Let U be the set of vertices of degree two in T. Then U induces a path in T.*

Proof. Suppose that U does not induce a path in T. Let $P = u_0 \dots u_d$ be a diametral path of T. Then there are three vertices u_i, u_j, u_ℓ with $1 \le i < j < \ell \le d - 1$ with $\delta_T(u_j) = 3$ and $u_i, u_\ell \in U$. Let T_1 and T_2 be the nontrivial components of $T - u_j$ containing u_i and u_ℓ, respectively. Let v_j be the pendant neighbor of u_j. Assume that $|V(T_1)| \ge |V(T_2)|$. Let $T' = T - u_j v_j + u_\ell v_j$. By Lemma 2.3, $s(T') > s(T)$, a contradiction. □

For $a, b \ge 0$ with $2(a + b + 1) \le n$, let $C_n(a, b)$ be the tree obtained from the path P_{n-a-b} with consecutive vertices $u_1, \dots, u_{n-a-b}$ by attaching a pendant vertex v_i to u_i for each i with $2 \le i \le a + 1$ and each i with $n - a - 2b \le i \le n - a - b - 1$. In particular $C_n(0, 0)$ is just the path P_n.

Lemma 2.9. *If $b \ge 0$, $a \ge b + 2$, and $2(a + b) + 2 < n$, then*

$$s(C_n(a - 1, b + 1)) > s(C_n(a, b)).$$

Proof. Let $T = C_n(a, b)$. Let $T' = T - u_{a+1} v_{a+1} + u_{n-a-2b-1} v_{a+1}$. Then $T' \cong C_n(a - 1, b + 1)$.

Case 1. $2a \ge \lceil \frac{n}{2} \rceil$.

Note that $T - u_{\lceil \frac{n-1}{4} \rceil + 1}$ contains exactly two nontrivial components of orders $2\lceil \frac{n-1}{4} \rceil - 1$ and $n - 1 - 2\lceil \frac{n-1}{4} \rceil$, respectively. By Lemma 2.1, $u_{\lceil \frac{n-1}{4} \rceil + 1}$ is in the median of T and T'. As we go from T to T', the distance between $u_{\lceil \frac{n-1}{4} \rceil + 1}$ and v_{a+1} is increased by $n - a - 2b - 1 - \left(\lceil \frac{n-1}{4} \rceil + 1\right) - \left(a + 1 - (\lceil \frac{n-1}{4} \rceil + 1)\right)$ (which is equal to $n - 2(a + b) - 2$, larger than 0), and the distance between $u_{\lceil \frac{n+1}{4} \rceil + 1}$ and any other vertex remains unchanged. Thus $s(T') > s(T)$, as desired.

Case 2. $2a < \lceil \frac{n}{2} \rceil$.

Let $x = u_{\lceil \frac{n}{2} \rceil - a}$ and $x' = u_{\lceil \frac{n}{2} \rceil - a + 1}$. By Lemma 2.1, x is in the median of T and x' is in the median of T'. Let T_1 be the component of $T - x$ containing u_1, and T_2 the component of $T - x'$ containing u_{n-a-b}. Note that

$$d_{T'}(x', z) - d_T(x, z) = \begin{cases} 1 & \text{if } z \in V(T_1) \setminus \{v_{a+1}\} \\ -1 & \text{if } z \in V(T_2) \end{cases}$$

and

$$\begin{aligned} d_{T'}(x', v_{a+1}) - d_T(x, v_{a+1}) &= n - a - 2b - 1 - \left(\left\lceil \frac{n}{2} \right\rceil - a + 1\right) - \left(\left\lceil \frac{n}{2} \right\rceil - a - (a + 1)\right) \\ &= n - 2\left\lceil \frac{n}{2} \right\rceil + 2(a - b) - 1 \ge 2. \end{aligned}$$

If $2a+1 = \lceil \frac{n}{2} \rceil$, then $x = u_{a+1}$, $v_{a+1} \notin V(T_1)$, and thus

$$\begin{aligned} s(T') - s(T) &= s_{T'}(x') - s_T(x) \\ &= |V(T_1)| - |V(T_2)| + d_{T'}(x', v_{a+1}) - d_T(x, v_{a+1}) \\ &\geq \left(\left\lceil \frac{n}{2} \right\rceil - 2 \right) - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + 2 > 0, \end{aligned}$$

and otherwise, we have

$$\begin{aligned} s(T') - s(T) &= s_{T'}(x') - s_T(x) \\ &= |V(T_1)| - 1 - |V(T_2)| + d_{T'}(x', v_{a+1}) - d_T(x, v_{a+1}) \\ &\geq \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) - 1 - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + 2 > 0. \end{aligned}$$

It follows that $s(T') > s(T)$, as desired. ◻

3. Minimum status and diameter

In this section, we give sharp lower and upper bounds for the minimum status of a tree with fixed diameter, and characterize the unique trees achieving these bounds.

Let $F_{n,d}$ be the caterpillar obtained by attaching $n-d-1$ pendant vertices to $v_{\lceil \frac{d}{2} \rceil}$ of the path $P_{d+1} = v_0 \ldots v_d$, where $2 \leq d \leq n-1$. Define $F_{n,n-1} = P_n$. Particularly, $F_{n,2} = S_n$ and $F_{n,3} = D_n(n-3, 1)$. The lower bound in the following theorem has been obtained in [2] for a connected graph of order n and diameter d. However, we include a proof here for completeness.

Theorem 3.1. *Suppose that T is a tree of order $n \geq 4$ with diameter d, where $2 \leq d \leq n-1$. Then*

$$s(T) \geq n - 1 - d + \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$$

with equality if and only if $T \cong F_{n,d}$.

Proof. Let x be a vertex in the median of T and P a diametral path of T. Then

$$\begin{aligned} s(T) &= \sum_{v \in V(P)} d_T(x, v) + \sum_{v \in V(T) \setminus V(P)} d_T(x, v) \\ &\geq s(P) + \sum_{v \in V(T) \setminus V(P)} 1 \\ &= n - 1 - d + \left\lfloor \frac{(d+1)^2}{4} \right\rfloor \end{aligned}$$

with equality if and only if x is in the median of P and all vertices outside P are adjacent to x, *i.e.*, $T \cong F_{n,d}$. ◻

Corollary 3.2. *Suppose that T is a tree of order n, and $T \ncong S_n$. Then $s(T) \geq n$ with equality if and only if $T \cong D_n(n-3, 1)$. Moreover, if $T \ncong D_n(n-3, 1)$, then $s(T) \geq n+1$ with equality if and only if $T \cong F_{n,4}, D_n(n-4, 2)$.*

Proof. Let d be the diameter of T. Let $f(d) = n - 1 - d + \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$ for $2 \leq d \leq n-1$. It is easy to see that $f(d+1) > f(d)$ for $2 \leq d \leq n-2$. As $T \ncong S_n$, we have $d \geq 3$. If $d \geq 4$, then by Theorem 3.1, we have $s(T) \geq f(d) \geq f(4) = n+1$ with equalities if and only if $T \cong F_{n,4}$. If $d = 3$, then $T \cong D_n(r, n-2-r)$ for some r with $\frac{n-2}{2} \leq r \leq n-3$, and in this case, $s(T) = 2n - 3 - r$, which is n if $r = n-3$, $n+1$ if $r = n-4$ and at least $n+2$ if $r \leq n-5$. ◻

Theorem 3.3. *Suppose that T is a tree of order $n \geq 4$ with diameter d, where $2 \leq d \leq n-1$. Then*

$$s(T) \leq \left\lfloor \frac{d(2n-d)+1}{4} \right\rfloor$$

with equality if and only if $T \cong D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor)$.

Proof. By Lemma 2.5, we have $s(D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor)) = \lfloor \frac{d(2n-d)+1}{4} \rfloor$.

Let T be a tree of order n with diameter d such that its minimum status is as large as possible. By the value of $s(D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor))$, we only need to show that $T \cong D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor)$.

It is trivial if $d = 2, n-1$.

Suppose that $3 \leq d \leq n-2$. Let α be the number of quasi-pendant vertices in T. Then there are at least two pendant vertices with different neighbors, *i.e.*, $\alpha \geq 2$.

We claim that $\alpha = 2$. Otherwise, $\alpha \geq 3$. Let $P = v_0 \dots v_d$ be a diametral path of T. As v_0 and v_d are pendant vertices, v_1 and v_{d-1} are quasi-pendant vertices in T. Since $\alpha \geq 3$, $\delta_T(v_i) \geq 3$ for some i with $2 \leq i \leq d-2$. For $z \in N_T(v_i)$, let T_z be the component of $T - v_i$ containing z. Let w be a neighbor of v_i outside P such that the order of T_w is maximum among the components of $T - v_i$ except $T_{v_{i-1}}$ and $T_{v_{i+1}}$. Let $\rho = \max\{d_T(v_i, x) : x \in V(T_w)\}$. As $d = d_T(v_0, v_d) = d_T(v_i, v_0) + d_T(v_i, v_d)$, we have $\rho \leq d_T(v_i, v_0) = i$ and $\rho \leq d_T(v_i, v_d) = d-i$. Assume that $|V(T_{v_{i-1}})| \geq |V(T_{v_{i+1}})|$. Suppose that $\rho < d-i$. Let $T' = T - v_iw + v_{i+1}w$. Then P is still a diametral path of T', *i.e.*, T' is a tree of order n with diameter d. By Lemma 2.3, $s(T') > s(T)$, a contradiction. Thus $\rho = d-i$.

Let x be a vertex in the median of T.

Suppose that $|V(T_{v_{i-1}})| \geq \frac{n}{2}$. Let T' be the tree obtained from $T - v_{iw}$ by deleting all edges in T_w and adding all edges in $\{v_{d-1}z : z \in V(T_w)\}$. By Lemma 2.1, x may be chosen in $V(T_{v_{i-1}})$, and x is also in the median of T'. As we go from T to T', the distance between x and any vertex different from w is increased or remains unchanged. Thus $s(T') - s(T) = s_{T'}(x) - s_T(x) \geq d_{T'}(x, w) - d_T(x, w) = d-i-1 > 0$, implying that $s(T') > s(T)$, a contradiction. It follows that $|V(T_{v_{i-1}})| < \frac{n}{2}$. Similarly, if $|V(T_w)| \geq \frac{n}{2}$, then, as above, we may form a tree T' from $T - v_iv_{i+1}$ by deleting all edges in $T_{v_{i+1}}$ and adding all edges in $\{v_1z : z \in V(T_{v_{i+1}})\}$, and $x \in V(T_w)$ is in the median of T and T', such that $s(T') - s(T) \geq d_{T'}(x, v_{i+1}) - d_T(x, v_{i+1}) = i-1 > 0$, a contradiction. So $|V(T_w)| < \frac{n}{2}$. By Lemma 2.1, $x = v_i$.

Let $a = |V(T_{v_{i-1}})|$ and $c = |V(T_w)|$. Let $V_0 = V(T_w)$ if $a + c \leq \lfloor \frac{n}{2} \rfloor$ and let V_0 be a subset of $V(T_w)$ consisting of $\lfloor \frac{n}{2} \rfloor - a$ vertices otherwise. Let T'' be the tree obtained from $T - v_{iw}$ by deleting all the edges in T_w and adding all edges in $\{v_1z : z \in V_0\} \cup \{v_{d-1}z : z \in V(T_w) \setminus V_0\}$. Then v_i is also in the median of T'' by Lemma 2.1. This is true for $a + c > \lfloor \frac{n}{2} \rfloor$ because from $a + c + |V(T_{v_{i+1}})| \leq n-1$ we have $|V(T_{v_{i+1}})| + a + c - \lfloor \frac{n}{2} \rfloor \leq \frac{n}{2}$.

As we go from T to T'', the distance between v_i and any vertex different from w is increased or remains unchanged. Thus $s(T'') - s(T) = s_{T''}(v_i) - s_T(v_i) \geq d_{T''}(v_i, w) - d_T(v_i, w) = \min\{i, d-i\} - 1 = d-i-1 > 0$, implying that $s(T'') > s(T)$, a contradiction. Now we have proved that $\alpha = 2$. That is, $T = D_n(p, q)$ for some p, q with $p + q = n - d + 1$. By Lemma 2.4, we have $T \cong D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor)$. ☐

For $3 \leq d \leq n-1$, let $F = D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor)$ and $h(n, d) = s(F)$. Let xy be the edge of F with $\delta_F(x) \geq 2$ and $\delta_F(y) = 1 + \lfloor \frac{n-d+1}{2} \rfloor$. Then $F_{xy} \cong D_n(\lceil \frac{n-(d-1)+1}{2} \rceil, \lfloor \frac{n-(d-1)+1}{2} \rfloor)$. By Lemma 2.2, we have $h(n, d) > h(n, d-1)$.

Let $P_{n,i}$ be the tree obtained from the path $P_{n-1} = v_1 \dots v_{n-1}$ by attaching a pendant edge at vertex v_i, where $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Note that $P_{n,2} \cong D_n(2, 1)$.

Corollary 3.4. *Suppose that T is a tree of order n, and $T \ncong P_n$. Then $s(T) \leq \lfloor \frac{n^2-3}{4} \rfloor$ with equality if and only if $T \cong D_n(2, 1)$. Moreover, if $T \ncong D_n(2, 1)$, then $s(T) \leq \lfloor \frac{n^2-8}{4} \rfloor$ with equality if and only if $T \cong D_n(2, 2)$, $P_{n,3}$.*

Proof. Let d be the diameter of T. Then $d \leq n-2$. If $d \leq n-3$, then by Theorem 3.3, $s(T) \leq h(n, d) \leq h(n, n-3) = \lfloor \frac{n^2-8}{4} \rfloor$ with equalities if and only if $T \cong D_n(2, 2)$. If $d = n-2$, then $T \cong P_{n,i}$ with $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$, and $s(T)$ is equal to $\lfloor \frac{n^2-3}{4} \rfloor$ if $i = 2$, $\lfloor \frac{n^2-8}{4} \rfloor$ if $i = 3$, and is at most $\lfloor \frac{n^2-8}{4} \rfloor - 1$ if $i \geq 4$. ☐

Note that, if u and v are two nonadjacent vertices in a connected graph G, then $s(G) \geq s(G + uv)$.

Suppose that G is a unicyclic graph of order $n \geq 4$ with diameter d, where $2 \leq d \leq n-2$. By Theorem 3.3,

$$s(G) \leq \left\lfloor \frac{d(2n-d)+1}{4} \right\rfloor$$

with equality if and only if G is isomorphic to a graph with diameter d obtained from $D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor)$ by adding an edge between two pendant vertices.

The eccentricity $e_G(u)$ of vertex u in a connected graph G is its distance to a farthest vertex. If u is a vertex such that $d_G(u,v) = e_G(v)$, then u is called an eccentric vertex of v. Recall that the radius of G is defined to be the minimum eccentricities of all vertices of G. Let $diam(G)$ and $r(G)$ be the diameter and radius of G, respectively. A vertex v is central if $e_G(v) = r(G)$. Buckley and Lewinter [8] characterized graphs that have diameter-preserving spanning trees. They showed that a connected graph G has a diameter-preserving spanning tree if either

(1) $diam(G) = 2r(G)$, or
(2) $diam(G) = 2r(G)$ and G contains a pair of adjacent central vertices x and y that have no common eccentric vertex.

Let G be a connected graph with diameter-preserving spanning trees, and let d be the diameter of G, where $2 \leq d \leq n-1$. Then $s(G) \leq s(T)$ for a spanning tree T of diameter d of G. Thus, by Theorem 3.3, we have $s(G) \leq s(T) \leq \lfloor \frac{d(2n-d)+1}{4} \rfloor$ with equality if $G \cong D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor)$.

4. Minimum status and number of pendant vertices

In this section, we give sharp lower and upper bounds for the minimum status of a tree with fixed number of pendant vertices, and characterize the unique trees achieving these bounds.

For $2 \leq p \leq n-1$, let $\mathbb{P}(n,p)$ be the set of trees of order n with p pendant vertices. Particularly, $\mathbb{P}(n,2) = \{P_n\}$ and $\mathbb{P}(n,n-1) = \{S_n\}$.

For $3 \leq p \leq n-1$, let $S_{n,p}$ be the tree with p pendant paths of almost equal lengths (*i.e.*, $n-1-p\lfloor \frac{n-1}{p} \rfloor$ pendant paths of length $\lfloor \frac{n-1}{p} \rfloor + 1$ and $p + p\lfloor \frac{n-1}{p} \rfloor - (n-1)$ pendant paths of length $\lfloor \frac{n-1}{p} \rfloor$) at a common vertex. Particularly, $S_{n,n-1} = S_n$. Let $S_{n,2} = P_n$.

Theorem 4.1. *Suppose that $T \in \mathbb{P}(n,p)$, where $2 \leq p \leq n-1$. Then*

$$s(T) \geq n - 1 - \frac{p\lfloor \frac{n-1}{p} \rfloor^2 - (2n-p-2)\lfloor \frac{n-1}{p} \rfloor}{2}$$

with equality if and only if $T \cong S_{n,p}$.

Proof. It is trivial for $p = 2, n-1$. Suppose that $3 \leq p \leq n-2$.

Let x be the vertex in $S_{n,p}$ of degree p. By Lemma 2.1, x is in the median of $S_{n,p}$. By direct calculation, we have

$$\begin{aligned} s(S_{n,p}) &= p \sum_{j=1}^{\lfloor \frac{n-1}{p} \rfloor} j + \left(n-1-p\left\lfloor \frac{n-1}{p} \right\rfloor\right)\left(\left\lfloor \frac{n-1}{p} \right\rfloor + 1\right) \\ &= p \cdot \frac{(1 + \lfloor \frac{n-1}{p} \rfloor)\lfloor \frac{n-1}{p} \rfloor}{2} + (n-1-p)\left\lfloor \frac{n-1}{p} \right\rfloor - p\left\lfloor \frac{n-1}{p} \right\rfloor^2 + n - 1 \\ &= n-1-\frac{p\lfloor \frac{n-1}{p} \rfloor^2 - (2n-p-2)\lfloor \frac{n-1}{p} \rfloor}{2}. \end{aligned}$$

Let T be a tree in $\mathbb{P}(n,p)$ such that its minimum status is as small as possible. By the value of $s(S_{n,p})$, it suffices to show that $T \cong S_{n,p}$.

Suppose that there are at least two vertices with degree at least three in T. Then we may choose two vertices, say w_1 and w_2, in T with degree at least three, such that $d_T(w_1,w_2)$ is as small as possible. Let $P = v_0 \dots v_\ell$ be the unique path connecting w_1 and w_2, where $v_0 = w_1$ and $v_\ell = w_2$. If the length of P is at least two, then any internal vertex of P has degree two in T. Let T_{w_1} be the component of $T - v_1$ containing w_1 and T_{w_2} the component of $T - v_{\ell-1}$ containing w_2.

Let $N_T(w_1) \setminus \{v_1\} = \{u_1,\dots,u_s\}$. For $i = 1,\dots,s$, let T_{u_i} be the component of $T - w_1$ containing u_i. Assume that $|V(T_{u_1})| \le \cdots \le |V(T_{u_s})|$. Denote $a = \sum_{i=2}^{s} |V(T_{u_i})|$.

Assume that $|V(T_{w_1})| \le |V(T_{w_2})|$. Let $T' = T - \{w_1u_i : i = 2,\dots,s\} + \{w_2u_i : i = 2,\dots,s\}$. Note that the pendant vertices of T' are just pendant vertices of T, *i.e.*, $T' \in \mathbb{P}(n,p)$. Let x be a vertex in the median of T. Then $x \in V(T_{w_2}) \cup \left\{v_{\lfloor \frac{\ell}{2} \rfloor},\dots,v_{\ell-1}\right\}$, which follows from Lemma 2.1 and the fact that $|V(T_{w_2})| + \lceil \frac{\ell}{2} \rceil \ge \frac{n}{2}$ if $\ell \ge 2$, and follows from Lemma 2.1 if $\ell = 1$.

Suppose that $x \in V(T_{w_2})$. By Lemma 2.1, x is also in the median of T'. As we go from T to T', the distance between x and a vertex of $\cup_{i=2}^{s} V(T_{u_i})$ is decreased by $d_T(w_1,w_2)$, and the distance between x and any other vertex remains unchanged. Thus

$$s(T') - s(T) = s_{T'}(x) - s_T(x) = -d_T(w_1,w_2) \cdot a < 0,$$

implying that $s(T') < s(T)$, a contradiction. It follows that $x = v_i$ for some i with $\lfloor \frac{\ell}{2} \rfloor \le i \le \ell - 1$.

Note that $s \ge 2$ and $|V(T_{w_1})| = 1 + \sum_{j=1}^{s} |V(T_{u_j})| \ge |V(T_{u_1})| + s$. Thus, we have $|V(T_{u_1})| \le |V(T_{w_2})| - 2$, implying that $|V(T_{u_1})| - |V(T_{w_2})| + 1 < 0$.

Suppose that $a \le \ell - i - 1$, *i.e.*, $i + a \le \ell - 1$. Then $\ell \ge 2$. By Lemma 2.1, v_{i+a} is in the median of T'. Note that

$$d_{T'}(v_{i+a},z) - d_T(v_i,z) = \begin{cases} a & \text{if } z \in \{v_0,\dots,v_{i-1}\} \cup V(T_{u_1}) \\ -a & \text{if } z \in V(T_{w_2}) \cup \{v_{i+a+1},\dots,v_{\ell-1}\} \\ \ell - 2i - a & \text{if } z \in V(T_{u_2}) \cup \dots \cup V(T_{u_s}) \end{cases}$$

and

$$\sum_{j=i+1}^{i+a-1} d_{T'}(v_{i+a},v_j) = \sum_{j=i+1}^{i+a-1} d_T(v_i,v_j).$$

Thus

$$\begin{aligned} s(T') - s(T) &= s_{T'}(v_{i+a}) - s_T(v_i) \\ &= a(i + |V(T_{u_1})|) - a(|V(T_{w_2})| + \ell - i - a - 1) + (\ell - 2i - a)a \\ &= a(i + |V(T_{u_1})| - |V(T_{w_2})| - \ell + i + a + 1 + \ell - 2i - a) \\ &= a(|V(T_{u_1})| - |V(T_{w_2})| + 1) < 0, \end{aligned}$$

implying that $s(T') < s(T)$, a contradiction. It follows that $a \ge \ell - i$, where $\ell \ge 1$. Then w_2 is in the median of T' by Lemma 2.1. Note that

$$d_{T'}(w_2,z) - d_T(v_i,z) = \begin{cases} \ell - i & \text{if } z \in \{v_0,\dots,v_{i-1}\} \cup V(T_{u_1}) \\ -(\ell - i) & \text{if } z \in V(T_{w_2}) \setminus \{w_2\} \\ -i & \text{if } z \in V(T_{u_2}) \cup \dots \cup V(T_{u_s}) \end{cases}$$

and

$$\sum_{j=i+1}^{\ell-1} d_{T'}(w_2,v_j) = \sum_{j=i+1}^{\ell-1} d_T(v_i,v_j).$$

Thus

$$\begin{aligned} s(T') - s(T) &= s_{T'}(w_2) - s_T(v_i) \\ &= (\ell - i)(i + |V(T_{u_1})|) - (\ell - i)(|V(T_{w_2})| - 1) - ai \\ &\le (\ell - i)(i + |V(T_{u_1})| - |V(T_{w_2})| + 1 - i) \\ &= (\ell - i)(|V(T_{u_1})| - |V(T_{w_2})| + 1) < 0, \end{aligned}$$

implying that $s(T') < s(T)$, also a contradiction. Therefore, there is exactly one vertex with degree at least three in T. As $T \in \mathbb{P}(n,p)$, T consists of p pendant paths at a common vertex. Now by Lemma 2.6, we have $T \cong S_{n,p}$. □

Corollary 4.2. *If $2 \le p \le n-2$, then $s(S_{n,p}) > s(S_{n,p+1})$.*

Proof. Let x be the vertex of degree p in $S_{n,p}$ and xy be an edge of a longest pendant path in $S_{n,p}$, where if $p = 2$, x is any vertex of degree two and xy is in the longer sub-path with a terminal vertex x. By Lemma 2.2, $s(S_{n,p}) > s\left((S_{n,p})_{xy}\right)$. Note that $(S_{n,p})_{xy} \in \mathbb{P}(n, p+1)$. By Theorem 4.1, $s\left((S_{n,p})_{xy}\right) \ge s(S_{n,p+1})$. It follows that $s(S_{n,p}) > s(S_{n,p+1})$. □

Theorem 4.3. *Suppose that $T \in \mathbb{P}(n,p)$, where $2 \le p \le n-1$. Then*

$$s(T) \le \left\lfloor \frac{n^2 - p^2 + 2p}{4} \right\rfloor$$

with equality if and only if $T \cong D_n(\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor)$.

Proof. By Lemma 2.5, we have $s(D_n(\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor)) = \lfloor \frac{n^2-p^2+2p}{4} \rfloor$.

Let T be a tree in $\mathbb{P}(n,p)$ such that its minimum status is as large as possible. From the value of the status of $D_n(\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor)$, it suffices to show that $T \cong D_n(\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor)$. It is trivial for $p = 2, n-1$. Suppose that $3 \le p \le n-2$.

Let α be the number of quasi-pendant vertices in T. Since the diameter of T is at least three, we have $\alpha \ge 2$. Suppose that $\alpha \ge 3$. Then there are at least three components, say T_v, T_w and T_z, in $T - u$ for some vertex u, and at least two of them are not nontrivial, where $v, w, z \in N_T(u)$. Assume that $|V(T_v)| \ge |V(T_w)| \ge |V(T_z)|$. Then T_v and T_w are nontrivial. Let u_1 be a quasi-pendant vertex of T_w. Then $T' = T - uz + u_1 z$ is a tree in $\mathbb{P}(n,p)$. By Lemma 2.3, $s(T') > s(T)$, a contradiction. Therefore, we have $\alpha = 2$. That is, $T \cong D_n(\ell_1, \ell_2)$, where $\ell_1 \ge \ell_2 \ge 1$ and $\ell_1 + \ell_2 = p$. By Lemma 2.4, we have $T \cong D_n(\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor)$. □

We note that Theorem 4.3 follows also from Theorem 3.3. Suppose that $T \in \mathbb{P}(n,p)$, where $2 \le p \le n-1$. Let d be the diameter of T. As a diametral path contains exactly two pendant vertices, we have $d \le n - p + 1$. By Theorem 3.3, $s(T) \le h(n,d) \le h(n, n-p+1) = \lfloor \frac{n^2-p^2+2p}{4} \rfloor$ with equalities if and only if $T \cong D_n(\lceil \frac{n-d+1}{2} \rceil, \lfloor \frac{n-d+1}{2} \rfloor)$ with $d = n-p+1$, *i.e.*, $T \cong D_n(\lceil \frac{p}{2} \rceil, \lfloor \frac{p}{2} \rfloor)$.

5. Minimum status and number of odd vertices

In this section, we give sharp lower and upper bounds for the minimum status of a tree with fixed number of odd vertices, and characterize the unique trees achieving these bounds.

For integers n and k with $1 \le k \le \lfloor \frac{n}{2} \rfloor$, let $\mathbb{O}(n,k)$ be the set of trees of order n with $2k$ odd vertices [23].

Theorem 5.1. *Suppose that $T \in \mathbb{O}(n,k)$, where $1 \le k \le \lfloor \frac{n}{2} \rfloor$. Then*

$$s(T) \ge n - 1 - k \left\lfloor \frac{n-1}{2k} \right\rfloor^2 + (n-k-1) \left\lfloor \frac{n-1}{2k} \right\rfloor$$

with equality if and only if $T \cong S_{n,2k}$, where if $k = \frac{n}{2}$, then $S_{n,n} = S_n$.

Proof. The case $k = 1$ is trivial as in this case, $T \cong P_n \cong S_{n,2}$. The case $k = \lfloor \frac{n}{2} \rfloor$ is also trivial as $S_n \cong S_{n,2\lfloor \frac{n}{2} \rfloor}$ is the unique tree of order n with smallest minimum status $n - 1$.

Suppose that $2 \le k < \lfloor \frac{n}{2} \rfloor$, *i.e.*, $4 \le 2k \le n - 2$. Let p be the number of pendant vertices of T. Then $p \le 2k$. By Theorem 4.1 and Corollary 4.2, we have

$$s(T) \ge s(S_{n,p}) \ge s(S_{n,2k}) = n - 1 - k \left\lfloor \frac{n-1}{2k} \right\rfloor^2 + (n - k - 1) \left\lfloor \frac{n-1}{2k} \right\rfloor$$

with equalities if and only if $T \cong S_{n,p}$ and $p = 2k$, *i.e.*, $T \cong S_{n,2k}$. □

Theorem 5.2. *Suppose that $T \in \mathbb{O}(n, k)$, where $1 \le k \le \lfloor \frac{n}{2} \rfloor$. Then*

$$s(T) \le \begin{cases} \left\lfloor \frac{(n+1-k)^2}{4} \right\rfloor + \left\lfloor \frac{2nk-3k^2+6k-2n}{4} \right\rfloor & \text{if } k \text{ is odd} \\ \left\lfloor \frac{(n+1-k)^2}{4} \right\rfloor + \left\lfloor \frac{2nk-3k^2+6k-2n}{4} \right\rfloor - 1 & \text{if } k \text{ is even} \end{cases}$$

with equality if and only if $T \cong C_n(\lceil \frac{k-1}{2} \rceil, \lfloor \frac{k-1}{2} \rfloor)$.

Proof. Let $a = \lceil \frac{k-1}{2} \rceil$, $b = \lfloor \frac{k-1}{2} \rfloor$ and $c = \lceil \frac{n+1-k}{2} \rceil$. Then $a + b = k - 1$ and $a = b, b + 1$. Let $H = C_n(a, b)$, whose vertices are labelled as before. Let $x = v_c$. Then x is in the median of H by Lemma 2.1. Let $U = \{u_i : i = 1, \dots, n+1-k\}$ and $W = \{v_i : i = 2, \dots, a+1, n+1-k-b, \dots, n-k\}$. By direct calculation, we have

$$\sum_{u \in U} d(x, u) = s(P_{n+1-k}) = \left\lfloor \frac{(n+1-k)^2}{4} \right\rfloor$$

and

$$\begin{aligned} \sum_{u \in W} d(x, u) &= \sum_{i=1}^{a} (c - 1 - a + i) + \sum_{i=1}^{b} (n + 1 - k - b - c + i) \\ &= a(c - 1 - a) + \frac{a(a+1)}{2} + (n + 1 - k - b - c)b + \frac{b(b+1)}{2} \\ &= \begin{cases} a(n - k - 2a) + a(a + 1) & \text{if } a = b \\ (a - 1)(n + 1 - k - 2a) + c - 1 - a - a^2 & \text{if } a = b + 1 \end{cases} \\ &= \begin{cases} \frac{(k-1)(2n-3k+3)}{4} & \text{if } k \text{ is odd,} \\ \frac{(k-2)(2n+2-3k)}{4} + \lceil \frac{n+1-k}{2} \rceil - 1 & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} s(H) &= \sum_{u \in U} d(x, u) + \sum_{u \in W} d(x, u) \\ &= \begin{cases} \left\lfloor \frac{(n+1-k)^2}{4} \right\rfloor + \left\lfloor \frac{2nk-3k^2+6k-2n}{4} \right\rfloor & \text{if } k \text{ is odd,} \\ \left\lfloor \frac{(n+1-k)^2}{4} \right\rfloor + \left\lfloor \frac{2nk-3k^2+6k-2n}{4} \right\rfloor - 1 & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

Let T be a tree in $\mathbb{O}(n, k)$ such that its minimum status is as large as possible. By the value of $s(C_n(\lceil \frac{k-1}{2} \rceil, \lfloor \frac{k-1}{2} \rfloor))$, it suffices to show that $T \cong C_n(\lceil \frac{k-1}{2} \rceil, \lfloor \frac{k-1}{2} \rfloor)$.

If $k = 1$, then it is obvious that $T \cong P_n \cong C_n(0, 0)$.

Suppose that $k \ge 2$. We claim that the maximum odd degree of T is 3. Otherwise, $\delta_T(u) = 2t + 1$ from some $u \in V(T)$ and $t \ge 2$. Let $N_T(u) = \{u_1, \dots, u_{2t+1}\}$. Let T_i be the component of $T - u$ containing u_i, where $1 \le i \le 2t+1$. Assume that $|V(T_1)| \ge |V(T_2)|$. Let z be a pendant vertex of T in $V(T_2)$, and let $T' = T - \{uu_i : 3 \le i \le 2t\} + \{zu_i : 3 \le i \le 2t\}$. Note that the degrees of u and z are still odd in T'. Then $T' \in \mathbb{O}(n,k)$. By Lemma 2.3, $s(T') > s(T)$, a contradiction. Thus, the maximum odd degree of T is 3, as claimed.

If $k = \frac{n}{2}$, then by Lemma 2.7, $T \cong C_n\left(\lceil \frac{k-1}{2} \rceil, \lfloor \frac{k-1}{2} \rfloor\right)$.

Suppose that $k < \frac{n}{2}$. Then there is at least one even vertex in T. We claim that all even vertices have degree two. Otherwise, $\delta_T(w) = 2t$ for some $w \in V(T)$, where $t \ge 2$. Let $N_T(w) = \{w_1, \ldots, w_{2t}\}$. Let T_i be the component of $T - w$ containing w_i, where $1 \le i \le 2t$. Assume that $|V(T_1)| \ge |V(T_2)|$. Let z be a pendant vertex of T in $V(T_2)$, and let $T'' = T - ww_3 + zw_3$. Note that, in T'', the degree of w is odd and the degree of z is even. Then $T'' \in \mathbb{O}(n,k)$. By Lemma 2.3, $s(T'') > s(T)$, a contradiction. Thus, all even vertices of T have degree two.

Now, we claim T is a caterpillar. Otherwise, as the maximum degree of T is three, there is a vertex u of degree three in T such that u still has degree three in the tree obtained from T by deleting all pendant vertices. Let $N_T(u) = \{u_1, u_2, u_3\}$. Then $\delta_T(u_i) \ge 2$ for $i = 1, 2, 3$. Let T_i be the component of $T - u$ containing u_i, where $i = 1, 2, 3$. Let U be the set of vertices of degree two in T. Suppose that $U \not\subseteq V(T_i)$ for any $i = 1, 2, 3$, say $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$ for $v_1, v_2 \in U$. Assume that $|V(T_1)| \ge |V(T_2)|$. Let $T^* = T - uu_3 + v_2u_3$. As $\delta_{T^*}(u) = 2$ and $\delta_{T^*}(v_2) = 3$, we have $T^* \in \mathbb{O}(n,k)$. By Lemma 2.3, $s(T^*) > s(T)$, a contradiction. Therefore, $U \subseteq V(T_i)$ for some $i = 1, 2, 3$, say $U \subseteq V(T_1)$. Suppose that $V(T_2)$ ($V(T_3)$, respectively) contains r (t, respectively) vertices of degree three of T. Then $T \cong T_1(r+1, t+1) \in \mathbb{O}(n,k)$, where $r, t \ge 1$. Assume that $r \ge t$. Note that $T_1(r+2, t) \in \mathbb{O}(n,k)$. By Lemma 2.7, $s(T_1(r+2,t)) > s(T)$, a contradiction. Thus, T is a caterpillar, as claimed. By Lemma 2.8, the set of all $n - 2k$ vertices of degree two induces a path, and thus $T \cong C_n(a,b)$ for some a and b with $a + b = k - 1$. By Lemma 2.9, we have $T \cong C_n(\lceil \frac{k-1}{2} \rceil, \lfloor \frac{k-1}{2} \rfloor)$. □

A vertex in a graph is called a branching vertex if its degree is at least three. Let T be a tree of order n with k branching vertices. Let p be the number of pendant vertices in T. Then $p + 2(n-p-k) + 3k \le 2(n-1)$, *i.e.*, $p \ge k+2$. This implies that $2k + 2 \le k + p \le n$, and thus, $k \le \frac{n}{2} - 1$.

Corollary 5.3. *Suppose that T is a tree of order n with k branching vertices, and $0 \le k \le \frac{n}{2} - 1$. Then $s(T) \le s(C_n(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor))$ with equality if and only if $T \cong C_n(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor)$.*

Proof. If $k = 0$, then the result follows from the known fact that P_n is the unique tree of order n whose minimum status is maximum [2].

Suppose that $k \ge 1$. Let T be a tree of order n with k branching vertices such that its minimum status is as large as possible.

Let Δ be the maximum degree of T. Suppose that $\Delta \ge 4$. Let $u \in V(T)$ and $N_T(u) = \{u_1, \ldots, u_\Delta\}$. Let T_i be the component of $T - u$ containing u_i, where $1 \le i \le \Delta$. Assume that $|V(T_1)| \ge |V(T_2)|$. Let w be a pendant vertex of T in $V(T_2)$, and let $T' = T - uu_3 + wu_3$. Then T' is a tree of order n with k branching vertices. By Lemma 2.3, $s(T') > s(T)$, a contradiction. Hence $\Delta = 3$. Let p be the number of pendant vertices in T. As $p + 2(n-k-p) + 3k = 2(n-1)$, we have $p = k+2$, and thus T is a tree of order n with $2k+2$ odd vertices. By Theorem 5.2, we have $T \cong C_n(\lceil \frac{k}{2} \rceil, \lfloor \frac{k}{2} \rfloor)$. □

6. Minimum status and number of vertices of degree two

For integers n and t with $0 \le t \le n-2$, let $\mathbb{H}(n,t)$ be the set of trees of order n with t vertices of degree two. Note that $\mathbb{H}(n, n-2) = \{P_n\}$, $\mathbb{H}(n, n-3) = \emptyset$, and $\mathbb{H}(n, 0)$ is the class of series-reduced trees of order n. So we only consider trees in $\mathbb{H}(n,t)$ with $0 \le t \le n-4$.

Theorem 6.1. *Suppose that $T \in \mathbb{H}(n,t)$, where $0 \le t \le n-4$. Then*

$$s(T) \ge n - 1 - \frac{(n-t-1)\lfloor \frac{n-1}{n-t-1} \rfloor^2 - (n+t-1)\lfloor \frac{n-1}{n-t-1} \rfloor}{2}$$

with equality if and only if $T \cong S_{n,n-t-1}$.

Proof. The case $t = 0$ is trivial as in this case S_n ($\cong S_{n,n-1}$) is the unique tree of order n with smallest minimum status $n-1$.

Suppose that $t \geq 1$. Let p be the number of pendant vertices of T. As there is a vertex with degree at least 3, we have $p \leq n-t-1$. By Theorem 4.1 and Corollary 4.2, we have

$$s(T) \geq s(S_{n,p}) \geq s(S_{n,n-t-1}) = n-1-\frac{(n-t-1)\lfloor\frac{n-1}{n-t-1}\rfloor^2-(n+t-1)\lfloor\frac{n-1}{n-t-1}\rfloor}{2}$$

with equalities if and only if $T \cong S_{n,p}$ and $p = n-t-1$, *i.e.*, $T \cong S_{n,n-t-1}$. □

Lemma 6.2. *Let T be a tree in $\mathbb{H}(n,t)$ such that $s(T)$ is maximum, where $0 \leq t \leq n-4$. Then the maximum degree of T is at most 4, T is caterpillar, and there are at most two vertices of degree four in T.*

Proof. Let Δ be the maximum degree of T. Suppose that $\Delta \geq 5$. Let $u \in V(T)$ with $\delta_T(u) = \Delta(T)$ and let $N_T(u) = \{u_1, \ldots, u_\Delta\}$. For $1 \leq i \leq \Delta$, let T_i be the component of $T-u$ containing u_i. Suppose without loss of generality $|V(T_1)| \geq |V(T_2)|$. Let $T' = T - \{uu_i : 4 \leq i \leq \Delta\} + \{wu_i : 4 \leq i \leq \Delta\}$, where w is a pendant vertex of T in $V(T_2)$. As $\delta_{T'}(u) = 3$, $\delta_{T'}(w) = \Delta - 2 \geq 3$ and $\delta_{T'}(v) = \delta_T(v)$ for $v \in V(T) \setminus \{u, w\}$, we have $T' \in \mathbb{H}(n,t)$. By Lemma 2.3, $s(T') > s(T)$, a contradiction. Therefore $\Delta \leq 4$.

Suppose that T is not a caterpillar. Then for some vertex u of T, $T-u$ has three nontrivial components. Note that $\delta_T(u) = 3, 4$ as $\Delta \leq 4$.

Case 1. $\delta_T(u) = 3$.

Let $N_T(u) = \{u_1, u_2, u_3\}$. Let T_i be the component of $T-u$ containing u_i for $i = 1, 2, 3$. Assume that $|V(T_1)| \geq |V(T_2)| \geq |V(T_3)|$. Suppose first that T has a vertex w of degree two in $V(T_2)$. Let $T' = T - uu_3 + wu_3$. As $\delta_{T'}(u) = 2$, $\delta_{T'}(w) = 3$ and $\delta_{T'}(z) = \delta_T(z)$ for $z \in V(T) \setminus \{u, w\}$, we have $T' \in \mathbb{H}(n,t)$. By Lemma 2.3, $s(T') > s(T)$, a contradiction. Therefore, T has no vertex of degree two in $V(T_2)$. Let w_1 be a pendant vertex of T in $V(T_2)$ such that $d_T(u, w_1) = \max\{d_T(u, s) : s \in V(T_2)\}$. Let $T' = T - \{uu_3, w_1w_2\} + \{uw_1, w_2u_3\}$, where w_2 is the neighbor of w_1. Note that $T' \in \mathbb{H}(n,t)$.

Let x be a vertex in the median of T. As $|V(T_1)| \geq |V(T_2)| \geq |V(T_3)|$, we have $x \in V(T_1) \cup \{u\}$ by Lemma 2.1.

Case 1.1. $x \in V(T_1)$.

By Lemma 2.1, x is also in the median of T'. As we go from T to T', the distance between x and a vertex of $V(T_3)$ is increased by $d_T(u, w_2)$, the distance between x and w_1 is decreased by $d_T(u, w_2)$, and the distance between x and any other vertex remains unchanged. Thus

$$\begin{aligned} s(T') - s(T) &= s_{T'}(x) - s_T(x) \\ &= d_T(u, w_2)|V(T_3)| - d_T(u, w_2) \\ &= d_T(u, w_2)(|V(T_3)| - 1) > 0, \end{aligned}$$

implying that $s(T') > s(T)$, a contradiction.

Case 1.2. $x = u$

Let x' be a vertex in the median of T'. By Lemma 2.1, x' lies on the path connecting u and w_2 in T'. Note that $d_{T'}(x', s) - d_T(x, s) = d_T(u, x')$ for $s \in V(T_1)$, $d_{T'}(x', s) - d_T(x, s) \geq -d_T(u, x')$ for $s \in V(T_2) \setminus \{w_1\}$, $d_{T'}(x', s) - d_T(x, s) = d_T(w_2, x')$ for $s \in V(T_3)$, and $d_{T'}(x', w_1) - d_T(x, w_1) = -d_T(w_2, x')$. Thus

$$\begin{aligned} s(T') - s(T) &= s_{T'}(x') - s_T(x) \\ &\geq d_T(u, x')(|V(T_1)| - |V(T_2)| + 1) + d_T(x', w_2)(|V(T_3)| - 1) \\ &\geq d_T(u, x') + d_T(x', w_2) > 0, \end{aligned}$$

a contradiction.

Case 2. $\delta_T(u) = 4$.

Let $N_T(u) = \{u_1, u_2, u_3, u_4\}$. Let T_i be the component of $T - u$ containing u_i for $i = 1, \dots, 4$. Assume that T_1, T_2 and T_3 are nontrivial, and $|V(T_1)| \geq |V(T_2)|$. Suppose that T has a vertex, say w, of degree two in $V(T_2)$. Let $T' = T - \{uu_3, uu_4\} + \{wu_3, wu_4\}$. As $\delta_{T'}(u) = 2$, $\delta_{T'}(w) = 4$ and $\delta_{T'}(v) = \delta_T(v)$ for any $v \in V(T) \setminus \{u, w\}$, we have $T' \in \mathbb{H}(n, t)$. By Lemma 2.3, $s(T') > s(T)$, a contradiction. Thus, T has no vertex of degree two in $V(T_2)$. Then there is a vertex $z \in V(T_2)$ such that $\delta_T(z) = 3, 4$. Let $T' = T - uu_3 + zu_3$. Then $T' \in \mathbb{H}(n, t)$. By Lemma 2.3, $s(T') > s(T)$, a contradiction.

By combining Cases 1 and 2, we conclude that T is a caterpillar.

Suppose that there are three vertices, say v_1, v_2, and v_3 in T with degree four. As T is a caterpillar, v_1, v_2 and v_3 lie on a diametral path of T, and so one of them, say v_3, lies on the path P connecting the other two vertices v_1 and v_2. Let T_i be the component of $T - v_3$ containing v_i for $i = 1, 2$. Assume that $|V(T_1)| \geq |V(T_2)|$. Let z be a neighbor of v_3 outside the path P in T. Let $T'' = T - v_3z + v_2z$. Then $T'' \in \mathbb{H}(n, t)$. By Lemma 2.3, $s(T'') > s(T)$, a contradiction. Therefore, there are at most two vertices of degree four in T. $\square$

Lemma 6.3. *Let $T \in \mathbb{H}(n, t)$ such that $s(T)$ is maximum, where $0 \leq t \leq n-4$. Let k_i be the number of vertices of degree i in T, where $1 \leq i \leq 4$. Then*

$$\begin{cases} k_1 = \frac{n-t+3}{2} \\ k_3 = \frac{n-t-5}{2} \\ k_4 = 1 \end{cases} \quad \text{if } n - t \text{ is odd,}$$

and

$$\begin{cases} k_1 = \frac{n-t+2}{2} \\ k_3 = \frac{n-t-2}{2} \\ k_4 = 0 \end{cases} \quad \text{or} \quad \begin{cases} k_1 = \frac{n-t+4}{2} \\ k_3 = \frac{n-t-8}{2} \\ k_4 = 2 \end{cases} \quad \text{if } n - t \text{ is even.}$$

Proof. Trivially, $k_2 = t$. By Lemma 6.1, the maximum degree of T is at most 4 and $k_4 \leq 2$. Now the result follows from the facts that $k_1 + k_2 + k_3 + k_4 = n$ and $k_1 + 2k_2 + 3k_3 + 4k_4 = 2(n-1)$. $\square$

Lemma 6.4. *Let $T \in \mathbb{H}(n, t)$ such that $s(T)$ is maximum, where $0 \leq t \leq n - 4$. If $u \in V(T)$ with $\delta_T(u) = 4$, then $T - u$ has at most one nontrivial component.*

Proof. From Lemma 6.1, T is a caterpillar with at most two vertices of degree four. Let $N_T(u) = \{u_1, u_2, u_3, u_4\}$. Suppose that $T - u$ has two nontrivial components, say T_1 and T_2, with $u_i \in V(T_i)$ for $i = 1, 2$. Assume that $|V(T_1)| \geq |V(T_2)|$. If T has a vertex w of degree two in $V(T_2)$, then by setting $T' = T - \{uu_3, uu_4\} + \{wu_3, wu_4\}$, we have $T' \in \mathbb{H}(n, t)$, and by Lemma 2.3, $s(T') > s(T)$, which is a contradiction. Thus T has no vertex of degree two in $V(T_2)$. Then $\delta_T(w) = 3, 4$ for some $w \in V(T_2)$. Let $T' = T - uu_3 + wu_3$. Then $T' \in \mathbb{H}(n, t)$. By Lemma 2.3, $s(T') > s(T)$, a contradiction. $\square$

For nonnegative integers a, b and positive integer n with $2(a + b) + 5 \leq n$, let $R_n(a, b)$ be the tree of order n obtained from a path $u_1 \dots u_{n-a-b}$ by attaching pendant vertex v_1 to u_2, and then attaching pendant vertex v_i to u_i for each $2 \leq i \leq a + 2$ and each i with $n - a - 2b - 2 \leq i \leq n - a - b - 3$. The structure of $R_n(a, b)$ is shown in Figure 2.

Lemma 6.5. *For $a \geq \max\{b, 1\}$ and $2(a + b) + 5 < n$, we have*

$$s(R_n(a - 1, b + 1)) > s(R_n(a, b)).$$

Proof. Let $R_n = R_n(a, b)$. Let $R'_n = R_n - u_{a+2}v_{a+2} + u_{n-a-2b-3}v_{a+2}$. Then $R'_n \cong R_n(a - 1, b + 1)$.

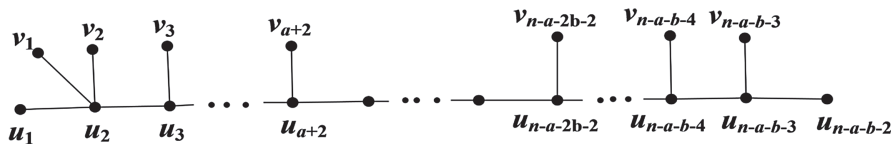

FIGURE 2. The tree $R_n(a, b)$.

Case 1. $2a + 3 \geq \lceil \frac{n}{2} \rceil$.

Let $x = u_{\lceil \frac{n}{4} \rceil}$. By Lemma 2.1, x is in the medians of R_n and R'_n. As we go from R_n to R'_n, the distance between x and v_{a+2} is increased by $n - a - 2b - 3 - \lceil \frac{n}{4} \rceil - (a + 2 - \lceil \frac{n}{4} \rceil)$ (which equals $n - 2a - 2b - 5 > 0$), and the distance between x and any other vertex remains unchanged. Thus $s(R'_n) > s(R_n)$.

Case 2. $2a + 3 < \lceil \frac{n}{2} \rceil$.

Let $x = u_{\lceil \frac{n}{2} \rceil - a - 2}$ and $x' = u_{\lceil \frac{n}{2} \rceil - a - 1}$. By Lemma 2.1, x is in the median of R_n and x' is in the median of R'_n. Let T_1 be the component of $R_n - x$ containing u_1, and T_2 the component of $R_n - x'$ containing $u_{n-a-b-2}$. Note that $d_{R'_n}(x', z) - d_{R_n}(x, z) = 1$ for $z \in V(T_1) \setminus \{v_{a+2}\}$, $d_{R'_n}(x', z) - d_{R_n}(x, z) = -1$ for $z \in V(T_2)$, and

$$\begin{aligned} d_{R'_n}(x', v_{a+2}) - d_{R_n}(x, v_{a+2}) &= n - a - 2b - 3 - \left(\left\lceil \frac{n}{2} \right\rceil - a - 1 \right) - \left(\left\lceil \frac{n}{2} \right\rceil - a - 2 - (a + 2) \right) \\ &= n - 2 \left\lceil \frac{n}{2} \right\rceil + 2(a - b) + 2. \end{aligned}$$

If x is of degree two in R_n, then

$$\begin{aligned} s(R'_n) - s(R_n) &= s_{R'_n}(x') - s_{R_n}(x) \\ &= |V(T_1)| - 1 - |V(T_2)| + d_{R'_n}(x', v_{a+2}) - d_{R_n}(x, v_{a+2}) \\ &= \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) - 1 - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + n - 2 \left\lceil \frac{n}{2} \right\rceil + 2(a - b) + 2 \\ &= 2(a - b) + 1 > 0. \end{aligned}$$

If x is of degree three in R_n, then $x = u_{a+2}$, $v_{a+2} \notin V(T_1)$, and thus

$$\begin{aligned} s(R'_n) - s(R_n) &= s_{R'_n}(x') - s_{R_n}(x) \\ &= |V(T_1)| - |V(T_2)| + d_{R'_n}(x', v_{a+2}) - d_{R_n}(x, v_{a+2}) \\ &= \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) - \left(\left\lfloor \frac{n}{2} \right\rfloor - 2 \right) + n - 2 \left\lceil \frac{n}{2} \right\rceil + 2(a - b) + 2 \\ &= 2(a - b) + 1 > 0. \end{aligned}$$

It follows that $s(R'_n) > s(R_n)$. □

Lemma 6.6. *For $b \geq a + 3$ and $2(a + b) + 5 < n$, we have*

$$s(R_n(a + 1, b - 1)) > s(R_n(a, b)).$$

Proof. Let $R_n = R_n(a, b)$. Let $R'_n = R_n - u_{n-a-2b-2}v_{n-a-2b-2} + u_{a+3}v_{n-a-2b-2}$. Then $R'_n \cong R_n(a + 1, b - 1)$.

Case 1. $2b \geq \lceil \frac{n}{2} \rceil$.

Let $x = u_{n-a-b-2-\lceil \frac{n-1}{4} \rceil}$. By Lemma 2.1, x is in the medians of R_n and R'_n. As we go from R_n to R'_n, the distance between x and $v_{n-a-2b-2}$ is increased by $n-a-b-2-\lceil \frac{n-1}{4} \rceil-(a+3)-[n-a-b-2-\lceil \frac{n-1}{4} \rceil-(n-a-2b-2)]$ (which equal to $n - 2a - 2b - 5 > 0$), and the distance between x and any other vertex remains unchanged. Thus $s(R'_n) > s(R_n)$.

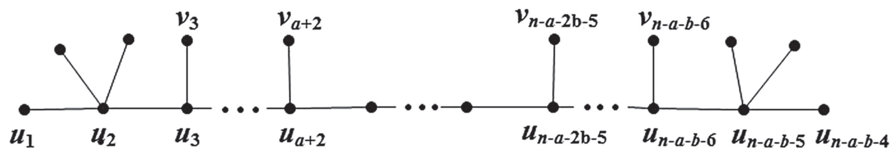

FIGURE 3. The tree $H_n(a,b)$.

Case 2. $2b < \lceil \frac{n}{2} \rceil$.

Let $x = u_{\lceil \frac{n}{2} \rceil - a - 2}$ and $x' = u_{\lceil \frac{n}{2} \rceil - a - 3}$. By Lemma 2.1, x is in the median of R_n, and x' is in the median of R'_n. Let T_1 and T_2 be the components of $R_n - x$ containing $u_{n-a-b-2}$ and $R_n - x'$ containing u_1, respectively. As we go from R_n to R'_n, we have $d_{R'_n}(x', z) - d_{R_n}(x, z) = 1$ for $z \in V(T_1) \setminus \{v_{n-a-2b-2}\}$, $d_{R'_n}(x', z) - d_{R_n}(x, z) = -1$ for $z \in V(T_2)$, and

$$\begin{aligned} d_{R'_n}(x', v_{n-a-2b-2}) - d_{R_n}(x, v_{n-a-2b-2}) &= \left\lceil \frac{n}{2} \right\rceil - a - 3 - (a+3) \\ &\quad - \left(n - a - 2b - 2 - \left(\left\lceil \frac{n}{2} \right\rceil - a - 2\right)\right) \\ &= 2\left\lceil \frac{n}{2} \right\rceil - n + 2(b-a) - 6. \end{aligned}$$

If x is of degree two in R_n, then

$$\begin{aligned} s(R'_n) - s(R_n) &= s_{R'_n}(x') - s_{R_n}(x) \\ &= |V(T_1)| - 1 - |V(T_2)| + d_{R'_n}(x', v_{n-a-2b-2}) - d_{R_n}(x, v_{n-a-2b-2}) \\ &= \left\lfloor \frac{n}{2} \right\rfloor - 1 - \left(\left\lceil \frac{n}{2} \right\rceil - 2\right) + 2\left\lceil \frac{n}{2} \right\rceil - n + 2(b-a) - 6 \\ &= 2(b-a) - 5 > 0. \end{aligned}$$

If x is of degree three in R_n, then $x = u_{n-a-2b-2}$, $v_{n-a-2b-2} \notin V(T_1)$, and thus

$$\begin{aligned} s(R'_n) - s(R_n) &= s_{R'_n}(x') - s_{R_n}(x) \\ &= |V(T_1)| - |V(T_2)| + d_{R'_n}(x', v_{n-a-2b-2}) - d_{R_n}(x, v_{n-a-2b-2}) \\ &= \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) - \left(\left\lceil \frac{n}{2} \right\rceil - 2\right) + 2\left\lceil \frac{n}{2} \right\rceil - n + 2(b-a) - 6 \\ &= 2(b-a) - 5 > 0. \end{aligned}$$

It follows that $s(R'_n) > s(R_n)$. □

For nonnegative integers a, b and positive integer n with $2(a+b)+8 \le n$, let $H_n(a,b)$ be the tree of order n obtained from the path $u_1 \dots u_{n-a-b-4}$ by attaching a pendant vertex to u_2 and $u_{n-a-b-5}$ respectively, and then attaching pendant vertex v_i to u_i for each i with $2 \le i \le a+2$ and each i with $n-a-2b-5 \le i \le n-a-b-5$, which is shown in Figure 3.

Lemma 6.7. *Suppose that $2(a+b)+8 < n$. If $a - b \ge 2$, then*

$$s(H_n(a-1, b+1)) > s(H_n(a,b)).$$

Proof. Let $T = H_n(a,b)$. Let $T' = T - u_{a+2}v_{a+2} + u_{n-a-2b-6}v_{a+2}$. Then $T' \cong H_n(a-1, b+1)$.

Case 1. $2a + 3 \ge \lceil \frac{n}{2} \rceil$.

Note that $x = u_{\lceil \frac{n}{4} \rceil}$ is in the median of T and T' by Lemma 2.1. As we go from T to T', the distance between x and v_{a+2} is increased by $n - a - 2b - 6 - \lceil \frac{n}{4} \rceil - (a + 2 - \lceil \frac{n}{4} \rceil)$ (which is equal to $n - 2(a + b) - 8 > 0$), and the distance between x and any other vertex remains unchanged. Thus $s(T') > s(T)$.

Case 2. $2a + 3 < \lceil \frac{n}{2} \rceil$.

Let $x = u_{\lceil \frac{n}{2} \rceil - a - 2}$ and $x' = u_{\lceil \frac{n}{2} \rceil - a - 1}$. Then by Lemma 2.1, x is in the median of T, and x' is in the median of T'. By considering whether v_{a+2} is in $V(T_1)$ as in the proof Lemma 2.9, we have $s(T') > s(T)$. ☐

For simplicity, let $R_{n,t} = R_n\left(\lceil \frac{n-t-5}{4} \rceil - 1, \lfloor \frac{n-t-5}{4} \rfloor + 1\right)$ for $n - t \geq 5$ and $C_{n,t} = C_n\left(\lceil \frac{n-t}{4} - \frac{1}{2} \rceil, \lfloor \frac{n-t}{4} - \frac{1}{2} \rfloor\right)$ for $n - t \geq 4$.

Theorem 6.8. *Suppose that $T \in \mathbb{H}(n, t)$, where $0 \leq t \leq n - 4$.*

(i) *If $n - t = 5$, then*

$$s(T) \leq \left\lfloor \frac{n^2 - 12}{4} \right\rfloor$$

with equality if and only if $T \cong R_n(0, 0)$. If $n - t$ is odd and at least 7, then

$$s(T) \leq \left\lfloor \frac{(n + t + 1)^2}{16} \right\rfloor + \left\lfloor \frac{n^2 + 2nt + 6n - 3t^2 - 10t}{16} \right\rfloor - 1$$

with equality if and only if $T \cong R_{n,t}$.

(ii) *If $n - t$ is even, then*

$$s(T) \leq \begin{cases} \left\lfloor \frac{(n+t+2)^2}{16} \right\rfloor + \left\lfloor \frac{n^2+2nt+4n-3t^2-12t}{16} \right\rfloor & \text{if } n - t \equiv 2 \pmod 4 \\ \left\lfloor \frac{(n+t+2)^2}{16} \right\rfloor + \left\lfloor \frac{n^2+2nt+4n-3t^2-12t}{16} \right\rfloor - 1 & \text{if } n - t \equiv 0 \pmod 4 \end{cases}$$

with equality if and only if $T \cong C_{n,t}$.

Proof. Let T be a tree in $\mathbb{H}(n, t)$ such that $s(T)$ is maximum.

Case 1. $n - t$ is odd.

By Lemmas 6.2–6.4, T is a caterpillar with exactly one vertex of maximum degree four, and the vertex of degree four has at least three pendant neighbors. If $n-t = 5$, then $T \cong R_n(0, 0)$. Suppose that $n-t \geq 7$. Then by Lemma 2.8, $T \cong R_n(a, b)$ for some a and b. Since there are $\frac{n-t-5}{2}$ vertices of degree three, we have $a+b = \frac{n-t-5}{2}$. By Lemmas 6.5 and 6.6, we have $T \cong R_n\left(\lceil \frac{a+b}{2} \rceil - 1, \lfloor \frac{a+b}{2} \rfloor + 1\right) = R_n\left(\lceil \frac{n-t-5}{4} \rceil - 1, \lfloor \frac{n-t-5}{4} \rfloor + 1\right) = R_{n,t}$.

By Lemma 2.1, $u_{\lfloor \frac{n-2}{2} \rfloor}$ is in the median of $R_n(0, 0)$. Thus, if n is even, then $s(R_n(0, 0)) = \sum_{j=1}^{\frac{n-4}{2}} j + n - 4 + \sum_{j=1}^{\frac{n-2}{2}} j = \frac{n^2-12}{4}$, and if n is odd, then $s(R_n(0, 0)) = \sum_{j=1}^{\frac{n-5}{2}} j + n - 5 + \sum_{j=1}^{\frac{n-1}{2}} j = \frac{n^2-13}{4}$. Thus $s(R_n(0, 0)) = \left\lfloor \frac{n^2-12}{4} \right\rfloor$. Suppose that $n - t \geq 7$. Let $a = \lceil \frac{n-t-5}{4} \rceil - 1$ and $b = \lfloor \frac{n-t-5}{4} \rfloor + 1$. Then $2a + 2b + t + 5 = n$ and $b = a + 1, a + 2$. Let $c = \lceil \frac{n-a-b-2}{2} \rceil$. Then u_c is in the median of $R_{n,t}$ by Lemma 2.1. Let $U = \{u_i : i = 1, \ldots, n - a - b - 2\}$ and $W = \{v_i : i = 1, \ldots, a + 2, n - a - 2b - 2, \ldots, n - a - b - 3\}$. By direct calculation, we have

$$\sum_{u \in U} d(u_c, u) = s(P_{n-a-b-2}) = \left\lfloor \frac{(n - a - b - 2)^2}{4} \right\rfloor = \left\lfloor \frac{(n + t + 1)^2}{16} \right\rfloor$$

and

$$\begin{aligned}\sum_{u\in W} d(u_c,u) &= \sum_{i=1}^{a+2}(c-a-2+i)-1+\sum_{i=1}^{b}(n-a-2b-2-c+i)\\ &= (a+2)(c-a-2)+\frac{(a+2)(a+3)}{2}-1\\ &\quad +(n-a-2b-2-c)b+\frac{b(b+1)}{2}\\ &= \begin{cases} na+n-3a^2-7a+c-5 & \text{if } b=a+1,\\ na+2n-3a^2-11a-11 & \text{if } b=a+2.\end{cases}\end{aligned}$$

If $b=a+1$, then $4a+7=n-t$, $n-t\equiv 3 \pmod 4$, and

$$c=\begin{cases}\frac{n+t+1}{4} & \text{if } n \text{ is odd},\\ \frac{n+t+3}{4} & \text{if } n \text{ is even}.\end{cases}$$

If $b=a+2$, then $4a+9=n-t$ and thus $n-t\equiv 1 \pmod 4$. So we have

$$\begin{aligned}\sum_{u\in W} d(u_c,u) &= \begin{cases}\frac{n^2-3t^2+2nt+6n-10t-27}{16} & \text{if } n-t\equiv 3 \pmod 4 \text{ and } n \text{ is odd}\\ \frac{n^2-3t^2+2nt+6n-10t-19}{16} & \text{if } n-t\equiv 3 \pmod 4 \text{ and } n \text{ is even}\\ \frac{n^2-3t^2+2nt+6n-10t-23}{16} & \text{if } n-t\equiv 1 \pmod 4\end{cases}\\ &= \left\lfloor \frac{n^2+2nt+6n-3t^2-10t}{16}\right\rfloor -1.\end{aligned}$$

It follows that

$$\begin{aligned}s(R_{n,t}) &= \sum_{u\in U} d(u_c,u)+\sum_{u\in W} d(u_c,u)\\ &= \left\lfloor \frac{(n+t+1)^2}{16}\right\rfloor + \left\lfloor \frac{n^2+2nt+6n-3t^2-10t}{16}\right\rfloor -1.\end{aligned}$$

Case 2. $n-t$ is even.

By Lemmas 6.2–6.4, T is a caterpillar with maximum degree three or four, and if the maximum degree is four, then there are exactly two such vertices, and each has exactly three pendant neighbors.

If the maximum degree of T is three, then by Lemma 2.8, $T\cong C_n(a,b)$ for some a and b with $a+b=\frac{n-t-2}{2}$. By Lemma 2.9, we have $T\cong C_n\left(\left\lceil \frac{n-t}{4}-\frac{1}{2}\right\rceil,\left\lfloor \frac{n-t}{4}-\frac{1}{2}\right\rfloor\right)=C_{n,t}$.

Suppose next that the maximum degree of T is four. By Lemma 2.8, we have $T\cong H_n(a,b)$ for some a and b with $a+b=\frac{n-t-8}{2}$. By Lemma 6.7, we have $T\cong H_n\left(\left\lceil \frac{n-t-8}{4}\right\rceil,\left\lfloor \frac{n-t-8}{4}\right\rfloor\right)$.

In the following we show that $T\cong C_{n,t}$. Set $H=H_n\left(\left\lfloor \frac{n-t-8}{4}\right\rfloor,\left\lceil \frac{n-t-8}{4}\right\rceil\right)$. We need to show that $s(C_{n,t})>s(H)$.

If $n=8$, then $t=0$, and by direct calculation, we have $s(C_{n,t})=11>10=s(H)$. Suppose that $n>8$. Note that $H\cong H_n\left(\left\lceil \frac{n-t-8}{4}\right\rceil,\left\lfloor \frac{n-t-8}{4}\right\rfloor\right)$ and the diameter of H is $\frac{n+t}{2}-1$. Let $P=u_1\ldots u_{\frac{n+t}{2}}$ be the diametral path of H. Let v_1 be a pendant vertex adjacent to u_2 and v_2 a pendant vertex adjacent to $u_{\frac{n+t}{2}-1}$ in H_n. Let $H'=H-u_2v_1-u_{\frac{n+t}{2}-1}v_2+u_1v_1+u_1v_2$. Then $H'\cong C_n\left(\left\lceil \frac{n-t}{4}-\frac{1}{2}\right\rceil,\left\lfloor \frac{n-t}{4}-\frac{1}{2}\right\rfloor\right)=C_{n,t}$. Let $x=u_{\left\lceil \frac{n+t+1}{4}\right\rceil}$ and $x'=u_{\left\lceil \frac{n+t+1}{4}\right\rceil-1}$. By Lemma 2.1, x is in the median of H, and x' is in the median of H'. Let T_1 be the component of $H-x'$ containing u_1 and T_2 the component of $H-x$ containing $u_{\frac{n+t}{2}}$. Note that $d_{H'}(x',w)-d_H(x,w)=1$

if $\delta_H(x) = 3$ and $w \in V(T_2) \cup \{z\} \setminus \{v_2\}$ with z being the pendant vertex adjacent to x, or if $\delta_H(x) = 2$ and $w \in V(T_2) \setminus \{v_2\}$,

$$d_{H'}(x', w) - d_H(x, w) = \begin{cases} -1 & \text{if } w \in V(T_1) \setminus \{v_1\} \\ 0 & \text{if } w = v_1 \end{cases}$$

and

$$\begin{aligned} d_{H'}(x', v_2) - d_H(x, v_2) &= \left\lceil \frac{n+t+1}{4} \right\rceil - 1 - 1 - \left(\frac{n+t}{2} - 1 - \left\lceil \frac{n+t+1}{4} \right\rceil \right) \\ &= 2\left\lceil \frac{n+t+1}{4} \right\rceil - \frac{n+t}{2} - 1. \end{aligned}$$

If $\delta_H(x) = 3$, then

$$\begin{aligned} s(H') - s(H) &= s_{H'}(x') - s_H(x) \\ &= |V(T_2)| - (|V(T_1)| - 1) + d_{H'}(x', v_2) - d_H(x, v_2), \end{aligned}$$

and if $\delta_H(x) = 2$, then

$$\begin{aligned} s(H') - s(H) &= s_{H'}(x') - s_H(x) \\ &= (|V(T_2)| - 1) - (|V(T_1)| - 1) + d_{H'}(x', v_2) - d_H(x, v_2). \end{aligned}$$

Thus, in either case, we have

$$\begin{aligned} s(H') - s(H) = &\left(n - \left\lceil \frac{n+t+1}{4} \right\rceil - 2 - \left\lfloor \frac{n-t-8}{4} \right\rfloor - 1 \right) \\ &- \left(\left\lceil \frac{n+t+1}{4} \right\rceil + 2 + \left\lfloor \frac{n-t-8}{4} \right\rfloor - 2 - 1 \right) \\ &+ 2\left\lceil \frac{n+t+1}{4} \right\rceil - \frac{n+t}{2} - 1 \\ = &\; n - 3 - 2\left\lfloor \frac{n-t-8}{4} \right\rfloor - \frac{n+t}{2} > 0, \end{aligned}$$

implying that $s(H') > s(H)$, *i.e.*, $s(C_{n,t}) > s(H)$, as desired.

Note that $C_{n,t}$ has exactly $n - t$ odd vertices. By Theorem 5.2 with $2k = n - t$, we have

$$s(C_{n,t}) = \begin{cases} \left\lfloor \frac{(n+t+2)^2}{16} \right\rfloor + \left\lfloor \frac{n^2+2nt+4n-3t^2-12t}{16} \right\rfloor & \text{if } n - t \equiv 2 \pmod 4, \\ \left\lfloor \frac{(n+t+2)^2}{16} \right\rfloor + \left\lfloor \frac{n^2+2nt+4n-3t^2-12t}{16} \right\rfloor - 1 & \text{if } n - t \equiv 0 \pmod 4. \end{cases}$$

This completes the proof. □

As an immediate consequence of Theorem 6.8, we have the following result on the minimum status of series-reduced trees.

Corollary 6.9. *Among all series-reduced trees on $n \geq 5$ vertices, $R_{n,0}$ for odd n and $C_{n,0}$ for even n are the unique ones with largest minimum status.*

7. Concluding remarks

In this paper, we determine the smallest and largest values for the minimum status of trees with given parameters such as the diameter, the number of pendant vertices, the number of odd vertices, and the number of vertices of degree two, and we characterize those trees which realize the minima and maxima, respectively. Recall that $n-1 \le s(T) \le \lfloor \frac{n^2}{4} \rfloor$ for a tree T of order n. The following theorem shows that every integer from $n-1$ to $\lfloor \frac{n^2}{4} \rfloor$ is the minimum status of some tree of order n.

Theorem 7.1. *For a fixed positive integer n and any integer k with $n-1 \le k \le \lfloor \frac{n^2}{4} \rfloor$, there is a tree of order n such that $s(T) = k$.*

Proof. It is trivial for $n \le 3$. Suppose that $n \ge 4$. Recall that $s(P_n) = \lfloor \frac{n^2}{4} \rfloor$. Let x be a vertex in the median of P_n. Consider a nontrivial component T_1 of $P_n - x$. The set $V(T_1) \cup \{x\}$ induces a path $x_0 \dots x_p$ with $x_0 = x$.

Let $T^{p+1,0} = P_n$. For $\ell = 2, \dots, p$ and $i = 0, \dots, \ell-2$, let

$$T^{\ell,\ell-2} = T^{\ell+1,0} - x_{\ell-1}x_\ell + x_{\ell-2}x_\ell$$

and

$$T^{\ell,i} = T^{\ell,i+1} - x_{i+1}x_\ell + x_i x_\ell \text{ for } i = 0, \dots, \ell-3.$$

The steps of the transformation from $T^{\ell+1,0}$ into $T^{\ell,0}$ is displayed as follows: $T^{\ell+1,0} \to T^{\ell,\ell-2} \to \cdots \to T^{\ell,0}$. Thus the steps of the transformation from P_n to $T^{2,0}$ is displayed as follows: $P_n \to T^{p,p-2} \to \cdots \to T^{p,1} \to T^{p,0} \to T^{p-1,p-3} \to T^{p-1,0} \to \cdots \to T^{3,0} \to T^{2,0}$. By Lemma 2.1, x is in the median of $T^{\ell,i}$ for all i and ℓ with $0 \le i \le \ell-2$ and $2 \le \ell \le p$, and thus the minimum status is decreased by 1 at each step of the above transformation. If $n \ge 5$, then we repeat above process for the unique pendant path with length at least 2 in the tree $T^{2,0}$, and as above, at each step, the minimum status is decreased by 1. Finally, we obtain the star with minimum status $n-1$. □

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