RAIRO-Oper. Res. 55 (2021) S2221–S2240
https://doi.org/10.1051/ro/2020081

**RAIRO Operations Research**
www.rairo-ro.org

# OPTIMALITY AND DUALITY FOR NONSMOOTH SEMI-INFINITE MATHEMATICAL PROGRAM WITH EQUILIBRIUM CONSTRAINTS INVOLVING GENERALIZED INVEXITY OF ORDER $\sigma > 0$

Bhuwan Chandra Joshi*

**Abstract.** In this paper, we derive sufficient condition for global optimality for a nonsmooth semi-infinite mathematical program with equilibrium constraints involving generalized invexity of order $\sigma > 0$ assumptions. We formulate the Wolfe and Mond–Weir type dual models for the problem using convexificators. We establish weak, strong and strict converse duality theorems to relate the semi-infinite mathematical program with equilibrium constraints and the dual models in the framework of convexificators.

**Mathematics Subject Classification.** 90C30, 90C46, 49J52.

Received October 21, 2019. Accepted July 24, 2020.

## 1. Introduction

In nonsmooth optimization the notion of convexificator has proved that it provides nice calculus rules for establishing necessary optimality conditions. The notion of compact and convex convexificator is introduced by Demyanov [7], which was later developed by Demyanov and Jeyakumar (see [8]). The notion of convexificator is a generalization of the notions of upper convex and lower concave approximations (see [9]). Jeyakumar and Luc [19, 20] introduced the notions of nonconvex closed convexificator and approximate Jacobian, respectively. Making use of these notions, different Lagrange multiplier rules for efficiency were established (see, *e.g.*, [10, 11, 19, 21, 24, 25, 47]).

The notion of convexificator is a generalization of known subdifferentials such as Michel-Penot [29], Clarke [3], and Mordukhovich [32]. Later Luc [25] derived a Fritz John type necessary condition for a weak minimum of constrained multiobjective programming without set constraint *via* approximate Jacobians. Dutta and Chandra [10] established some necessary optimality conditions of Fritz John type for weak minima in terms of upper semi-regular and upper convexifactors for multiobjective optimization problems with inequality constraints.

In the recent years, numerous generalizations of convex functions have been derived which proved to be useful for extending optimality conditions and some classical duality results, previously restricted to only convex programs, to larger classes of optimization problems. One of them which is mostly used is invexity introduced by Hanson [16]. Invex functions are extremly significant in optimization theory mainly due to the properties regarding their global optima. In 1985, Cravan *et al.* [5] defined non smooth invex functions. Israel and Mond [1]

*Keywords.* Nonsmooth analysis, convexificators, duality, constraint qualification.

Department of Mathematics, Graphic Era Deemed to be University, Dehradun, India.
*Corresponding author: bhuwanjoshi007@gmail.com

 

discussed optimality in his paper "What is Invexity?" In last few decades, duality and optimality condition in invex optimization theory have been discussed by several authors (see [22, 23, 43]) and references therin.

The class of mathematical programs with equilibrium constraints is an extension of the class of bilevel programming problems [4,6,27,46]. Mathematical programming problems with equilibrium constraints belong to a difficult class of nonlinear optimization problems. Since the feasible region of these problems are not necessarily convex. In MPEC problems many constraint qualifications like Mangasarian-Fromovitz constraint qualification, Abadie constraint qualification, Slater constraint qualification do not hold (see [51]). Flegel *et al.* [12] considered optimization problems with a disjunctive structure of the feasible set and obtained optimality conditions for disjunctive programs with application to MPEC using Guignard-type constraint qualifications. Movahedian and Nobakhtian [33] introduced nonsmooth strong stationarity, M-stationarity and generalized Abadie and Guignard-type constraint qualifications for nonsmooth MPEC. Movahedian and Nobakhtian [34] introduced a nonsmooth type M-stationary condition based on the Michel-Penot subdifferential and established the Kuhn–Tucker, Fritz-John-type, M-stationary necessary conditions for the nonsmooth MPEC. Further, Movahedian and Nobakhtian [35] established necessary optimality conditions for Lipschitz MPEC on Asplund space and sufficient optimality conditions for nonsmooth MPEC in Banach space. (MPEC) plays a very important role in many fields such as economic equilibrium, engineering design, capacity enhancement in traffic, dynamic pricing in telecommunication networks and multilevel games [2, 17, 40]. For recent results in MPEC, we refer to the articles [15, 31, 36–38, 42] and references therin.

Semi-infinite programming problem (SIP) involves finitely many decision variables and infinitely many constraints. SIP has been widely applied in many fields, such as engineering design problem [39], transportation problem [26], disjunctive programming [44], robot trajectory design problem [28], robust optimization and design centering problem [45], air pollution control problem [49], optimal power flow problems in power systems with transient stability constraints [48], lapidary cutting problems [50]. For more literature, we refer to the survey articles [18, 26, 39], to the monograph [13] as well as to the books [14, 41] which contain several tutorial articles on theory, numerics and many possible applications of semi-infinite programming to the modeling and solution of real-life problems.

In practice, it is natural that an (MPEC) problem may arise where infinite many restrictions are present rather than finite many restrictions in finitely many variables. This motivated us to the mathematical formulation of a semi-infinite mathematical program with equilibrium constraints (SIMPEC) for generalized convex functions of order $\sigma > 0$ assumptions. The organization of this paper is as follows: in Section 2, we give some preliminary, definitions and results. In Section 3, we show that M-stationary condition is sufficient for global optimality under some SIMPEC generalized strong invexity of order $\sigma > 0$ assumptions. In Section 4, we formulate Wolfe and Mond–Weir type dual models for the SIMPEC and establish weak and strong duality theorems relating to the MPEC and the two dual models under strong invexity of order $\sigma > 0$ and generalized strong invexity of order $\sigma > 0$ assumptions. In Section 5, we conclude the results of this paper.

## 2. Preliminaries

Throughout this paper, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and $C$ is a non-empty subset of $\mathbb{R}^n$. The convex hull of $C$ and convex cone generated by $C$ are denoted by $co\ C$ and $cone\ C$ respectively.

The negative polar cone is defined by $C^- = \{u \in \mathbb{R}^n | \forall x \in C, \langle x, u \rangle \leq 0\}$.

Let $C$ be a nonempty subset of $\mathbb{R}^n$ and $x \in clC$ (closure of $C$), then the contingent cone $T(x, C)$ to $C$ at $x$ is defined by

$$T(x, C) = \{u \in \mathbb{R}^n | \exists t_n \downarrow 0, \exists u_n \to u \text{ such that } x + t_n u_n \in C\}.$$

We consider SIMPEC in the following form:

$$\begin{aligned} \text{SIMPEC} \qquad \min \quad & F(u) \\ \text{subject to}: \quad & g(u,t) \leq 0, \forall t \in T, \quad h(u) = 0, \\ & \theta(u) \geq 0, \ \psi(u) \geq 0, \ \langle \theta(u), \psi(u) \rangle = 0, \end{aligned}$$

where the set $T$ is an infinite compact subset of $\mathbb{R}^n, F : \mathbb{R}^n \to \mathbb{R}, \ g : \mathbb{R}^n \times T \to \mathbb{R}^k, \ h : \mathbb{R}^n \to \mathbb{R}^p, \ \theta : \mathbb{R}^n \to \mathbb{R}^l$ and $\psi : \mathbb{R}^n \to \mathbb{R}^l$ are given functions. If we take $h(u) := 0, \ \theta(u) := 0, \ \psi(u) := 0$, then, the optimization problem with equilibrium constraint coincides with the standard nonlinear programming problem.

The following index sets will be used throughout the paper:

$$\begin{aligned} T_g &:= T_g(\tilde{u}) := \{t \in T : g(\tilde{u}, t) = 0\}, \\ \delta &:= \delta(\tilde{u}) := \{i = 1, 2, \ldots, l : \theta_i(\tilde{u}) = 0, \psi_i(\tilde{u}) > 0\}, \\ \omega &:= \omega(\tilde{u}) := \{i = 1, 2, \ldots, l : \theta_i(\tilde{u}) = 0, \psi_i(\tilde{u}) = 0\}, \\ \kappa &:= \kappa(\tilde{u}) := \{i = 1, 2, \ldots, l : \theta_i(\tilde{u}) > 0, \psi_i(\tilde{u}) = 0\}, \end{aligned}$$

where $\tilde{u} \in X$ is a feasible vector for the problem SIMPEC and the set $\omega$ denotes the degenerate set.

**Definition 2.1.** Let $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, $u \in \mathbb{R}^n$, and let $F(u)$ be finite. Then, the *lower and upper Dini directional derivatives* of $F$ at $u$ in the direction $y$ are defined, respectively, by

$$F_d^-(u, y) := \liminf_{t \to 0^+} \frac{F(u + ty) - F(u)}{t},$$

and

$$F_d^+(u, y) := \limsup_{t \to 0^+} \frac{F(u + ty) - F(u)}{t}.$$

**Definition 2.2** (see [20])**.** A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have upper convexificators, $\partial^* F(u)$ at $u \in \mathbb{R}^n$ if $\partial^* F(u) \subseteq \mathbb{R}^n$ is a closed set and for each $y \in \mathbb{R}^n$,

$$F_d^-(u, y) \leq \sup_{\xi \in \partial^* F(u)} \langle \xi, y \rangle.$$

**Definition 2.3** (see [20])**.** A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have lower convexificators, $\partial_* F(u)$ at $u \in \mathbb{R}^n$ if $\partial_* F(u) \subseteq \mathbb{R}^n$ is a closed set and for each $y \in \mathbb{R}^n$,

$$F_d^+(u, y) \geq \inf_{\xi \in \partial_* F(u)} \langle \xi, y \rangle.$$

The function $F$ is said to have a convexificator $\partial^* F(u) \subseteq \mathbb{R}^n$ at $u \in \mathbb{R}^n$, iff $\partial^* F(u)$ is both upper and lower convexificators of $F$ at $u$.

**Definition 2.4** (see [10])**.** A function $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have upper semi-regular convexificators, $\partial^* F(u)$ at $u \in \mathbb{R}^n$ if $\partial^* F(u) \subseteq \mathbb{R}^n$ is a closed set and for each $y \in \mathbb{R}^n$

$$F_d^+(u, y) \leq \sup_{\xi \in \partial^* F(u)} \langle \xi, y \rangle. \tag{2.1}$$

Based on the definitions of an invex function [30] and generalized invex functions [43], we are introducing the definition of strongly invex function of order $\sigma > 0$ and generalized strongly invex functions of order $\sigma > 0$ in terms of convexificators.

**Definition 2.5.** Let $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real valued function, which admit convexificator at $\tilde{u} \in \mathbb{R}^n$ and $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a kernel function then, $f$ is said to be

(i) strongly $\partial^*$-invex of order $\sigma > 0$ at $\tilde{u}$ with respect to $\eta$ if for every $u \in \mathbb{R}^n, u \neq \tilde{u}$, there exist $\mu > 0$ such that

$$F(u) \geq F(\tilde{u}) + \langle \xi, \eta(u,\tilde{u}) \rangle + \mu \|\eta(u,\tilde{u})\|^\sigma, \quad \forall\, \xi \in \partial^* F(\tilde{u}).$$

(ii) strongly $\partial^*$-pseudoinvex of order $\sigma > 0$ at $\tilde{u}$ with respect to $\eta$ if for every $u \in \mathbb{R}^n, u \neq \tilde{u}$, there exist $\mu > 0$ such that

$$\exists\, \xi \in \partial^* F(\tilde{u}),\ \langle \xi, \eta(u,\tilde{u}) \rangle + \mu \|\eta(u,\tilde{u})\|^\sigma \geq 0 \Rightarrow F(u) \geq F(\tilde{u}).$$

(iii) strongly $\partial^*$-quasiinvex of order $\sigma > 0$ at $\tilde{u}$ with respect to $\eta$ if for every $u \in \mathbb{R}^n, u \neq \tilde{u}$, there exist $\mu > 0$ such that

$$F(u) \leq F(\tilde{u}) \Rightarrow \langle \xi, \eta(u,\tilde{u}) \rangle + \mu \|\eta(u,\tilde{u})\|^\sigma \leq 0, \quad \forall\, \xi \in \partial^* F(\tilde{u}).$$

We provide following examples in support of the definition of $\partial^*$-invex function and generalized $\partial^*$-invex functions of order $\sigma > 0$, respectively.

**Example 2.6.** Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by

$$F(u) = \begin{cases} 2 + u^2; & u \geq 0, \\ 2 - \frac{u}{2}; & u < 0 \end{cases}$$

if we take point $\tilde{u} = 0$, then the function becomes strongly $\partial^*$-invex of order $\sigma$ at $\tilde{u} = 0$ with respect to the kernel function, $\eta(u,\tilde{u}) = u^2$ and $\partial^* F(0) = \{-\frac{1}{2}, 0\}$.

**Example 2.7.** Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by

$$F(u) = \begin{cases} 1 + \frac{u}{4}; & u \geq 0, \\ 1 + u^2; & u < 0 \end{cases}$$

if we take point $\tilde{u} = 0$, then the function becomes strongly $\partial^*$-pseudoinvex function of order $\sigma$ at $\tilde{u} = 0$ with respect to the kernel function, $\eta(u,\tilde{u}) = u + \tilde{u}$ and $\partial^* F(0) = \{\frac{1}{4}, 0\}$.

**Example 2.8.** Consider the function $F : [-1,1] \to \mathbb{R}$ is given by $F(u) = -u$, if we take point $\tilde{u} = -1$, on the set $[-1,1]$, then the function becomes strongly $\partial^*$-quasiinvex function of order $\sigma$ at $\tilde{u} = -1$ with respect to the kernel function, $\eta(u,\tilde{u}) = u^2 + \tilde{u}^2$, and $\partial^* F(-1) = \{-1\}$.

Pandey and Mishra [38] presented the following notations for SIMPEC:

$$\begin{aligned}
g &= \bigcup_{i=1}^{m} co\partial^* g(\tilde{u}, t_i), \quad h = \bigcup_{i=1}^{p} co\partial^* h_i(\tilde{u}) \cup co\partial^*(-h_i)(\tilde{u}), \\
\Phi_\delta &= \bigcup_{i\in\delta} co\partial^* \Phi_i(\tilde{u}) \cup co\partial^*(-\Phi_i)(\tilde{u}), \quad \Phi_\omega = \bigcup_{i\in\omega} co\partial^* \Phi_i(\tilde{u}), \\
\Psi_\kappa &= \bigcup_{i\in\kappa} co\partial^* \Psi_i(\tilde{u}) \cup co\partial^*(-\Psi_i)(\tilde{u}), \quad \Psi_\omega = \bigcup_{i\in\omega} co\partial^* \Psi_i(\tilde{u}), \\
(\Phi\Psi)_\omega &= \bigcup_{i\in\omega} co\partial^*(-\Phi_i)(\tilde{u}) \cup co\partial^*(-\Psi_i)(\tilde{u}), \\
\Gamma(\tilde{u}) &:= g^- \cap h^- \cap \Phi_\delta^- \cap \Psi_\kappa^- \cap (\Phi\Psi)_\omega^-,
\end{aligned}$$

where, $t_1, t_2, \ldots, t_m \in T_g(\tilde{u}),\ m \leq n+1$, and $\tilde{u}$ is a feasible point of the problem SIMPEC.

The following definitions are taken from Pandey and Mishra [38] for SIMPEC.

**Definition 2.9.** Let $\tilde{u}$ be a feasible point of SIMPEC, and assume that all functions have convexificators considered above at $\tilde{u}$. We say that the generalized standard Abadie constraint qualification (GS Abadie CQ) holds at $\tilde{u}$ if at least one of the dual sets used in the definition of $\Gamma(\tilde{u})$ is nonzero and

$$\Gamma(\tilde{u}) \subset T(C, \tilde{u}).$$

The following definitions of a generalized alternatively stationary point and a generalized strong stationary point are taken from Joshi *et.al.* [23].

**Definition 2.10.** A feasible point $\tilde{u}$ of SIMPEC is called a generalized alternatively stationary (GA-stationary) point if there are vectors $\tau = (\tau^g, \tau^h, \tau^\theta, \tau^\psi) \in \mathbb{R}^{k+p+2l}, \gamma = (\gamma^h, \gamma^\theta, \gamma^\psi) \in \mathbb{R}^{p+2l}$ and $t_1, t_2, \dots, t_m \in T_g(\tilde{u})$, $m \leq n+1$ satisfying the following conditions:

$$\begin{aligned} 0 \in co\partial^* F(\tilde{u}) &+ \sum_{i=1}^{m} \tau_i^g co\partial^* g(\tilde{u}, t_i) + \sum_{m=1}^{p} \left[\tau_m^h co\partial^* h_m(\tilde{u}) + \gamma_m^h co\partial^*(-h_m)(\tilde{u})\right] \\ &+ \sum_{i=1}^{l} \left[\tau_i^\theta co\partial^*(-\theta_i)(\tilde{u}) + \tau_i^\psi co\partial^*(-\psi_i)(\tilde{u})\right] \\ &+ \sum_{i=1}^{l} \left[\gamma_i^\theta co\partial^*(\theta_i)(\tilde{u}) + \gamma_i^\psi co\partial^*(\psi_i)(\tilde{u})\right], \end{aligned} \tag{2.2}$$

$$\tau_i^g \geq 0 (i = 1, 2, \dots, m),\ \tau_m^h,\ \gamma_m^h \geq 0,\ m = 1, 2, \dots, p, \tag{2.3}$$

$$\tau_i^\theta, \tau_i^\psi, \gamma_i^\theta, \gamma_i^\psi \geq 0,\ i = 1, 2, \dots, l, \tag{2.4}$$

$$\tau_\kappa^\theta = \tau_\delta^\psi = \gamma_\kappa^\theta = \gamma_\delta^\psi = 0, \tag{2.5}$$

$$\forall\ i \in \omega, \gamma_i^\theta = 0 \text{ or } \gamma_i^\Psi = 0. \tag{2.6}$$

**Definition 2.11.** A feasible point $\tilde{u}$ of SIMPEC is called a generalized strong stationary (GS-stationary) point if there are vectors $\tau = (\tau^g, \tau^h, \tau^\theta, \tau^\psi) \in \mathbb{R}^{k+p+2l}, \gamma = (\gamma^h, \gamma^\theta, \gamma^\psi) \in \mathbb{R}^{p+2l}$ and $t_1, t_2, \dots, t_m \in T_g(\tilde{u}), m \leq n+1$ satisfying (2.2) to (2.5) together with the following condition

$$\forall\ i \in \omega,\ \gamma_i^\theta = 0,\ \gamma_i^\Psi = 0.$$

The following result shows that GS-stationarity is a necessary optimality condition for SIMPEC.

**Theorem 2.12** ([38])**.** *Let $\tilde{u}$ be a local optimal solution of SIMPEC. Suppose that $F$ is locally Lipschitz function at $\tilde{u}$, which admits a bounded upper semi-regular convexificator $\partial^* F(\tilde{u})$. Assume also that GS-ACQ holds at $\tilde{u}$ and that the cone*

$$\delta = cone\ co\ g + cone\ co\ h + cone\ co\ \Phi_\delta + cone\ co\ \Psi_\kappa + cone\ co\ (\Phi\Psi)_\omega,$$

*is closed, then $\tilde{u}$ is a GS-stationary point.*

**Corollary 2.13** ([38])**.** *Let $\tilde{u}$ be a local optimal solution of SIMPEC. Suppose that $F$ is locally Lipschitz near $\tilde{u}$. Assume also that $F$ and effective constraint functions admit bounded upper semi-regular convexificators at $\tilde{u}$. If GS-ACQ holds at $\tilde{u}$, then $\tilde{u}$ is a GS-stationary point.*

In the next section, we prove global sufficient optimality conditions and we show that under certain SIMPEC generalized strong invexity of order $\sigma$ assumptions, generalized alternatively (GA)-stationarity turns into a global sufficient optimality condition.

## 3. Optimality condition

We consider the following index sets:

$$\omega_\gamma^\theta := \{i \in \omega : \gamma_i^\psi = 0, \gamma_i^\theta > 0\},$$
$$\omega_\gamma^\psi := \{i \in \omega : \gamma_i^\psi > 0, \gamma_i^\theta = 0\},$$
$$\delta_\gamma^+ := \{i \in \delta : \gamma_i^\theta > 0\},$$
$$\kappa_\gamma^+ := \{i \in \kappa : \gamma_i^\psi > 0\}.$$

**Theorem 3.1.** *Let $\tilde{u}$ be a feasible GA-stationary point of SIMPEC, assume that $F$ is strongly $\partial^*$-pseudoinvex function of order $\sigma$ at $\tilde{u}$ with respect to the kernel $\eta$ and $g(\cdot,t)(t \in T_g), \pm h_m(m = 1,2,\ldots,p)$, $-\theta_i(i \in \delta \cup \omega), -\psi_i(i \in \omega \cup \kappa)$ are strongly $\partial^*$-quasiinvex functions of order $\sigma$ at $\tilde{u}$ with respect to the common kernel $\eta$. If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then $\tilde{u}$ is a global optimal solution of SIMPEC.*

*Proof.* Let $u$ be any arbitrary feasible point of SIMPEC. Then, for any $t_i \in T_g(\tilde{u})$, we have

$$g(u,t_i) \leq g(\tilde{u},t_i), \ \forall\ t_i \in T_g(\tilde{u}).$$

By strong $\partial^*$-quasiinvexity of $g_i$ of order $\sigma$ at $\tilde{u}$, there exist $\mu_i^g > 0$, such that

$$\langle \xi_i^g, \eta(u,\tilde{u})\rangle + \mu_i^g \|\eta(u,\tilde{u})\|^\sigma \leq 0, \ \forall\ \xi_i^g \in \partial^* g(\tilde{u},t_i), \ \forall\ t_i \in T_g(\tilde{u}). \tag{3.1}$$

Similarly, we have

$$\Big\langle \zeta_m, \eta(u,\tilde{u})\Big\rangle + \mu_m \|\eta(u,\tilde{u})\|^\sigma \leq 0, \quad \forall\ \zeta_m \in \partial^* h_m(\tilde{u}), \quad \forall\ m = \{1,2,\ldots,p\}, \tag{3.2}$$

$$\Big\langle \nu_m, \eta(u,\tilde{u})\Big\rangle + \mu_m \|\eta(u,\tilde{u})\|^\sigma \leq 0, \quad \forall\ \nu_m \in \partial^*(-h_m)(\tilde{u}), \quad \forall\ m = \{1,2,\ldots,p\}, \tag{3.3}$$

$$\Big\langle \xi_i^\theta, \eta(u,\tilde{u})\Big\rangle + \mu_i^\theta \|\eta(u,\tilde{u})\|^\sigma \leq 0, \quad \forall\ \xi_i^\theta \in \partial^*(-\theta_i)(\tilde{u}), \quad \forall\ i \in \delta \cup \omega, \tag{3.4}$$

$$\Big\langle \xi_i^\Psi, \eta(u,\tilde{u})\Big\rangle + \mu_i^\Psi \|\eta(u,\tilde{u})\|^\sigma \leq 0, \quad \forall\ \xi_i^\Psi \in \partial^*(-\Psi_i)(\tilde{u}), \quad \forall\ i \in \omega \cup \kappa. \tag{3.5}$$

If $\omega_\gamma^\theta \cup \omega_\gamma^\Psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, multiplying (3.1)–(3.5) by $\tau_i^g \geq 0$ $(i = 1,2,\ldots,m)$, $\tau_m^h > 0$ $(m = 1,2,\ldots,p)$, $\gamma_m^h > 0$ $(m = 1,2,\ldots,p)$, $\tau_i^\theta > 0$ $(i \in \delta \cup \omega)$, $\tau_i^\Psi > 0$ $(i \in \omega \cup \kappa)$, respectively and adding, we obtain

$$\Bigg\langle \left(\sum_{i=1}^m \tau_i^g \xi_i^g + \sum_{m=1}^p \left[\tau_m^h \zeta_m + \gamma_m^h \nu_m\right] + \sum_{i=1}^l \tau_i^\theta \xi_i^\theta + \sum_{i=1}^l \tau_i^\Psi \xi_i^\Psi\right), \eta(u,\tilde{u}) \Bigg\rangle + \sum_{i \in I_g} \tau_i^g \mu_i^g \|\eta(u,\tilde{u})\|^\sigma$$
$$+ \sum_{m=1}^p \tau_m^h \mu_m \|\eta(u,\tilde{u})\|^\sigma + \sum_{m=1}^p \gamma_m^h \mu_m \|\eta(u,\tilde{u})\|^\sigma + \sum_{i=1}^l \tau_i^\theta \mu_i^\theta \|\eta(u,\tilde{u})\|^\sigma + \sum_{i=1}^l \tau_i^\Psi \mu_i^\Psi \|\eta(u,\tilde{u})\|^\sigma \leq 0,$$

where $\xi_i^g \in co\partial^* g(\tilde{u},t_i)$, $\zeta_m \in co\partial^* h_m(\tilde{u})$, $\nu_m \in co\partial^*(-h_m)(\tilde{u})$, $\xi_i^\theta \in co\partial^*(-\theta_i)(\tilde{u})$, and $\xi_i^\Psi \in co\partial^*(-\Psi_i)(\tilde{u})$. Since all $\mu_i > 0$ that is the above inequality can also be written as

$$\Bigg\langle \left(\sum_{i=1}^m \tau_i^g \xi_i^g + \sum_{m=1}^p \left[\tau_m^h \zeta_m + \gamma_m^h \nu_m\right] + \sum_{i=1}^l \tau_i^\theta \xi_i^\theta + \sum_{i=1}^l \tau_i^\Psi \xi_i^\Psi\right), \eta(u,\tilde{u}) \Bigg\rangle \leq 0.$$

Thus by GA-stationarity of $\tilde{u}$, we can select $\xi \in co\partial^* F(\tilde{u})$, so that,

$$\langle \xi, \eta(u,\tilde{u})\rangle + \mu\|\eta(u,\tilde{u})\|^\sigma \geq 0.$$

By strong $\partial^*$-pseudoinvexity of order $\sigma$ of $F$ at $\tilde{u}$ with respect to the common kernel $\eta$, we get

$$F(u) \geq F(\tilde{u})$$

for all feasible points $u$. Hence $\tilde{u}$ is a global optimal solution of SIMPEC. □

The example given below illustrates Theorem 3.1.

**Example 3.2.** Consider the following SIMPEC problem

SIMPEC

$$\min\ F(u) := \begin{cases} u; & u \geq 0, \\ 1-e^u; & u < 0, \end{cases}$$

subject to:

$$g(u,t) := -u^2 - t \leq 0, \forall\ t\ \in [0,1],$$

$$\theta(u) := \begin{cases} u; & u \geq 0, \\ u^2; & u < 0, \end{cases}$$

$$\Psi(u) := \begin{cases} u^2|\cos\frac{\pi}{u}|; & u \neq 0, \\ 0; & u = 0, \end{cases}$$

$\langle \theta(u), \Psi(u)\rangle := 0.$

Here $F$ is $\partial^*$-pseudoinvex of order $\sigma > 0$ at $\tilde{u} = 0$ with respect to the kernel, $\eta(u,\tilde{u}) = \cos u \sin\tilde{u}$. Further, $g, -\theta$ and $-\Psi$ are $\partial^*$-quasiinvex of order $\sigma > 0$ at $\tilde{u} = 0$ with respect to the common kernel $\eta(u,\tilde{u}) = \cos u \sin\tilde{u}$. The feasible point for the given SIMPEC is $\tilde{u} = 0$. We have $co\partial^* F(0) = [-1,1]$, $co\partial^* g(0,t_1) = 0$, $t_1 = 0$, $co\partial^*(-\theta)(0) = [-1,0]$ and $co\partial^*(-\Psi)(0) = [-\pi,\pi]$. One can easily verify that, there exist $\tau^g = 1, \tau^\theta = 1$ and $\tau^\Psi = 1$ such that $\tilde{u} = 0$ is a GA- stationary point and $\tilde{u} = 0$ is a global optimal solution for the given primal problem SIMPEC. Hence the assumptions of the Theorem 3.1, are satisfied.

**Remark 3.3.** Based on the Definition 2.5, the definitions of strongly invex function of order $\sigma$ and generalized strongly invex function of order $\sigma$ can also be given in terms of upper semi-regular convexificators.

## 4. Duality

In this section, we formulate and study a Wolfe type dual problem for the problem SIMPEC using the asumption of strong $\partial^*$-invexity of order $\sigma$. We also formulate Mond–Weir type dual problem and study the problem SIMPEC using strong $\partial^*$-invexity of order $\sigma$ and generalized strong $\partial^*$-invexity of order $\sigma$ assumptions.

The formulation of Wolfe type dual problem for the problem SIMPEC is as follows:

$$\text{WD} \qquad \max_{v,\tau} \left\{ F(v) + \sum_{i=1}^{m} \tau_i^g g(v,t_i) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v)\right] \right\}$$

subject to :

$$0 \in co\partial^* F(v) + \sum_{i=1}^{m} \tau_i^g co\partial^* g(v, t_i) + \sum_{m=1}^{p} \left[\tau_m^h co\partial^* h_m(v) + \gamma_m^h co\partial^*(-h_m)(v)\right]$$

$$+ \sum_{i=1}^{l} \left[\tau_i^\theta co\partial^*(-\theta_i)(v) + \tau_i^\psi co\partial^*(-\psi_i)(v)\right],$$

$$\tau_i^g \geq 0 (i = 1, 2, \ldots, m),\ \tau_m^h,\ \gamma_m^h \geq 0,\ m = 1, 2, \ldots, p,$$

$$\tau_i^\theta, \tau_i^\psi, \gamma_i^\theta, \gamma_i^\psi \geq 0,\ i = 1, 2, \ldots, l,$$

$$\tau_\kappa^\theta = \tau_\delta^\psi = \gamma_\kappa^\theta = \gamma_\delta^\psi = 0,\ \forall\ i \in \omega,\ \gamma_i^\theta = 0,\ \gamma_i^\psi = 0, \tag{4.1}$$

where $\rho_m^h = \tau_m^h - \gamma_m^h, \tau = (\tau^g, \tau^h, \tau^\theta, \tau^\psi) \in \mathbb{R}^{k+p+2l}$, $\gamma = (\gamma^h, \gamma^\theta, \gamma^\psi) \in \mathbb{R}^{p+2l}$, and $t_1, t_2, \ldots, t_m \in T_g(\tilde{u})$, $m \leq n+1$.

**Theorem 4.1** (Weak duality)**.** *Let $\tilde{u}$ be feasible for the problem SIMPEC, $(v, \tau)$ be feasible for the dual WD and the index sets $I_g, \delta, \omega, \kappa$ are defined accordingly. Suppose that $F$, $g(\cdot, t)$ $(t \in T_g)$, $\pm h_m$ $(m = 1, 2, \ldots, p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are strongly $\partial^*$-invex functions of order $\sigma$ at $v$, with respect to the common kernel $\eta$. If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then for any $u$ feasible for the problem $SIMPEC$, we have*

$$F(u) \geq F(v) + \sum_{i=1}^{m} \tau_i^g g(v, t_i) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v)\right].$$

*Proof.* Let us suppose that $u$ be any feasible point for the problem SIMPEC. Then, we have

$$g(u, t) \leq 0,\ \forall\ t \in T \text{ and } h_m(u) = 0, \quad \forall\ m = \{1, 2, \ldots, p\}.$$

Since $F$ is strongly invex function of order $\sigma$ at $v$, with respect to the kernel $\eta$, then there exist $\mu$ such that

$$F(u) - F(v) \geq \langle \xi, \eta(u, v) \rangle + \mu \|\eta(u, v)\|^\sigma, \quad \forall\ \xi \in \partial^* F(v). \tag{4.2}$$

Similarly, we have

$$g(u, t_i) - g(v, t_i) \geq \langle \xi_i^g, \eta(u, v) \rangle + \mu_i^g \|\eta(u, v)\|^\sigma, \qquad \forall\ \xi_i^g \in \partial^* g(v, t_i), \forall\ t_i \in T_g(v), \tag{4.3}$$

for all $m = \{1, 2, \ldots, p\}$, we have

$$h_m(u) - h_m(v) \geq \langle \zeta_m, \eta(u, v) \rangle + \mu_m \|\eta(u, v)\|^\sigma, \quad \forall\ \zeta_m \in \partial^* h_m(v), \tag{4.4}$$

$$-h_m(u) + h_m(v) \geq \langle \nu_m, \eta(u, v) \rangle + \mu_m \|\eta(u, v)\|^\sigma, \quad \forall\ \nu_m \in \partial^*(-h_m)(v), \tag{4.5}$$

in the same manner, we have

$$-\theta_i(u) + \theta_i(v) \geq \langle \xi_i^\theta, \eta(u, v) \rangle + \mu_i^\theta \|\eta(u, v)\|^\sigma, \quad \forall\ \xi_i^\theta \in \partial^*(-\theta_i)(v), \quad \forall\ i \in \delta \cup \omega, \tag{4.6}$$

$$-\psi_i(u) + \psi_i(v) \geq \langle \xi_i^\psi, \eta(u, v) \rangle + \mu_i^\psi \|\eta(u, v)\|^\sigma, \quad \forall \xi_i^\psi \in \partial^*(-\psi_i)(v), \quad \forall\ i \in \omega \cup \kappa. \tag{4.7}$$

If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then multiplying (4.3) to (4.7) by $\tau_i^g \geq 0$ $(i = 1, 2, \ldots, m)$, $\tau_m^h > 0$ $(m = 1, 2, \ldots, p)$, $\gamma_m^h > 0$ $(m = 1, 2, \ldots, p)$, $\tau_i^\theta > 0$ $(i \in \delta \cup \omega)$, $\tau_i^\psi > 0$ $(i \in \omega \cup \kappa)$, respectively and adding (4.2) to (4.7), we obtain

$$
\begin{aligned}
&F(u) - F(v) + \sum_{i=1}^{m} \tau_i^g g(u,t_i) - \sum_{i=1}^{m} \tau_i^g g(v,t_i) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\
&\quad + \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^\theta \theta_i(u) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) - \sum_{i=1}^{l} \tau_i^\psi \psi_i(u) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \\
&\quad \geq \left\langle \xi + \sum_{i=1}^{m} \tau_i^g \xi_i^g + \sum_{m=1}^{p} \left[\tau_m^h \zeta_m + \gamma_m^h \nu_m\right] + \sum_{i=1}^{l} \left[\tau_i^\theta \xi_i^\theta + \tau_i^\psi \xi_i^\psi\right], \eta(u,v) \right\rangle \\
&\qquad + \mu \|\eta(u,v)\|^\sigma + \sum_{i \in I_g} \tau_i^g \mu_i^g \|\eta(u,v)\|^\sigma + \sum_{m=1}^{p} \tau_m^h \mu_m \|\eta(u,v)\|^\sigma \\
&\qquad + \sum_{m=1}^{p} \gamma_m^h \mu_m \|\eta(u,v)\|^\sigma + \sum_{i=1}^{l} \tau_i^\theta \mu_i^\theta \|\eta(u,v)\|^\sigma + \sum_{i=1}^{l} \tau_i^\Psi \mu_i^\Psi \|\eta(u,v)\|^\sigma,
\end{aligned}
$$

where $\tilde{\xi} \in co\partial^* F(v),\ \tilde{\xi}_i^g \in co\partial^* g(v,t_i)(t_i \in T_g),\ \tilde{\zeta}_m \in co\partial^* h_m(v),\ \tilde{\nu}_m \in co\partial^*(-h_m)(v),\ \tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(v),$ and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(v)$. From (2.2), we have

$$
\tilde{\xi} + \sum_{i=1}^{m} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] + \sum_{i=1}^{l} \left[\tau_i^\theta \tilde{\xi}_i^\theta + \tau_i^\psi \tilde{\xi}_i^\psi\right] = 0.
$$

Therefore,

$$
\begin{aligned}
&F(u) - F(v) + \sum_{i=1}^{m} \tau_i^g g(u,t_i) - \sum_{i=1}^{m} \tau_i^g g_i(v,t_i) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\
&\quad + \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^\theta \theta_i(u) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) - \sum_{i=1}^{l} \tau_i^\psi \psi_i(u) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \geq 0.
\end{aligned}
$$

Now, using feasibility condition of SIMPEC, *i.e.*, $g(u,t_i) \leq 0,\ h_m(u) = 0,\ \theta_i(u) \geq 0,\ \psi_i(u) \geq 0$, it follows that

$$
F(u) - F(v) - \sum_{i=1}^{m} \tau_i^g g(v,t_i) - \sum_{m=1}^{p} \tau_m^h h_m(v) + \sum_{m=1}^{p} \gamma_m^h h_m(v) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \geq 0.
$$

Hence,

$$
F(u) \geq F(v) + \sum_{i=1}^{m} \tau_i^g g_i(v,t_i) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v)\right],
$$

where, $\rho_m^h = \tau_m^h - \gamma_m^h$, and the proof is completed. □

**Theorem 4.2** (Strong duality)**.** *Let $\tilde{u}$ be a local optimal solution of the problem SIMPEC and assume that $F$ is locally Lipschitz near $\tilde{u}$. Suppose that $F,\ g(\cdot,t)\ (t \in T),\ \pm h_m\ (m = 1,2,\ldots,p), -\theta_i\ (i \in \delta \cup \omega), -\psi_i\ (i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are strongly $\partial^*$-invex functions of order $\sigma$ at $\tilde{u}$ with respect to the common kernel $\eta$. If GS-ACQ holds at $\tilde{u}$, then, $\exists\ \tilde{\tau} = (\tilde{\tau}^g, \tilde{\tau}^h, \tilde{\tau}^\theta, \tilde{\tau}^\psi) \in \mathbb{R}^{k+p+2l}$, such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of the dual WD and the corresponding objective values of SIMPEC and WD are equal.*

*Proof.* Since $\tilde{u}$ is a local optimal solution of the problem SIMPEC and the GS-ACQ is satisfied at $\tilde{u}$, now, using Corollary 2.13, *i.e.*, $\exists\ \tilde{\tau} = (\tilde{\tau}^g, \tilde{\tau}^h, \tilde{\tau}^\theta, \tilde{\tau}^\psi) \in \mathbb{R}^{k+p+2l}$ and $\tilde{\gamma} \in (\tilde{\gamma}^h, \tilde{\gamma}^\theta, \tilde{\gamma}^\psi) \in \mathbb{R}^{p+2l}$, such that the GS-stationarity conditions for the problem SIMPEC are satisfied, it follows that $\exists\ \tilde{\xi} \in co\partial^* F(\tilde{u})$, $\tilde{\xi}_i^g \in co\partial^* g(\tilde{u}, t_i)$, $\tilde{\zeta}_m \in co\partial^* h_m(\tilde{u})$, $\tilde{\nu}_m \in co\partial^*(-h_m)(\tilde{u})$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(\tilde{u})$ and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(\tilde{u})$, such that

$$\begin{aligned}
&\tilde{\xi} + \sum_{i=1}^{m} \tilde{\tau}_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tilde{\tau}_m^h \tilde{\zeta}_m + \tilde{\gamma}_m^h \tilde{\nu}_m\right] + \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \tilde{\xi}_i^\theta + \tilde{\tau}_i^\psi \tilde{\xi}_i^\psi\right] = 0,\\
&\tilde{\tau}_i^g \geq 0 (i = 1, 2, \ldots, m),\ \tilde{\tau}_m^h,\ \tilde{\gamma}_m^h \geq 0,\quad m = 1, 2, \ldots, p,\\
&\tilde{\tau}_i^\theta, \tilde{\tau}_i^\psi, \tilde{\gamma}_i^\theta, \tilde{\gamma}_i^\psi \geq 0,\ i = 1, 2, \ldots, l,\\
&\tilde{\tau}_\kappa^\theta = \tilde{\tau}_\delta^\psi = \tilde{\gamma}_\kappa^\theta = \tilde{\gamma}_\delta^\psi = 0, \forall\ i \in \omega,\ \tilde{\gamma}_i^\theta = 0,\ \tilde{\gamma}_i^\psi = 0.
\end{aligned}$$

Therefore $(\tilde{u}, \tilde{\tau})$ is feasible for the dual WD. Now, using Theorem 4.1, we obtain

$$F(\tilde{u}) \geq F(v) + \sum_{i=1}^{m} \tau_i^g g(v, t_i) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v)\right], \tag{4.8}$$

where, $\rho_m^h = \tau_m^h - \gamma_m^h$, for any feasible solution $(v, \tau)$ for the dual WD. Using the feasibility condition of SIMPEC and its dual WD, *i.e.*, for $i = 1, 2, \ldots, m$, $g_i(\tilde{u}) = 0$, $h_m(\tilde{u}) = 0$, $(m = 1, 2, \ldots, p)$, $\theta_i(\tilde{u}) = 0$, $\forall i \in \delta \cup \omega$, and $\psi_i(\tilde{u}) = 0, \forall i \in \omega \cup \kappa$, we get

$$F(\tilde{u}) = F(\tilde{u}) + \sum_{i=1}^{m} \tilde{\tau}_i^g g_i(\tilde{u}) + \sum_{m=1}^{p} \tilde{\rho}_m^h h_m(\tilde{u}) - \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \theta_i(\tilde{u}) + \tilde{\tau}_i^\psi \psi_i(\tilde{u})\right], \tag{4.9}$$

where, $\tilde{\rho}_m^h = \tilde{\tau}_m^h - \tilde{\gamma}_m^h$. Using (4.8) and (4.9), we obtain

$$\begin{aligned}
F(\tilde{u}) &+ \sum_{i=1}^{m} \tilde{\tau}_i^g g(\tilde{u}, t_i) + \sum_{m=1}^{p} \tilde{\rho}_m^h h_m(\tilde{u}) - \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \theta_i(\tilde{u}) + \tilde{\tau}_i^\psi \psi_i(\tilde{u})\right]\\
&\geq F(v) + \sum_{i=1}^{m} \tau_i^g g(v, t_i) + \sum_{m=1}^{p} \rho_m^h h_m(v) - \sum_{i=1}^{l} \left[\tau_i^\theta \theta_i(v) + \tau_i^\psi \psi_i(v)\right].
\end{aligned}$$

Hence, $(\tilde{u}, \tilde{\tau})$ is an optimal solution for the dual WD and the corresponding objective values of SIMPEC and WD are equal. □

**Theorem 4.3** (Strict converse duality)**.** *Let $\tilde{u}$ be an optimal solution for (SIMPEC) and $\bar{u}$ be an optimal solution for (WD). If the assumptions of the strong duality theorem are satisfied and $F$ be strictly strong invex of order $\sigma$ at $\tilde{u}$ then $\bar{u} = \tilde{u}$.*

*Proof.* We proceed by contradiction. Assume $\bar{u} \neq \tilde{u}$. By the strong duality theorem there exist $\bar{\tau} = (\bar{\tau}^g, \bar{\tau}^h, \bar{\tau}^\theta, \bar{\tau}^\psi) \in \mathbb{R}^{k+p+2l}$, $\bar{\gamma} \in (\bar{\gamma}^h, \bar{\gamma}^\theta, \bar{\gamma}^\psi) \in \mathbb{R}^{p+2l}$, and indices $t_1, t_2, \ldots, t_m \in T_g(\bar{u}), m \leq n+1$, such that $(\bar{u}, \bar{\tau})$ be an optimal solution for (WD) and

$$\begin{aligned}
f(\bar{u}) &= F(\bar{u}) + \sum_{i=1}^{m} \bar{\tau}_i^g g(\bar{u}, t_i) + \sum_{m=1}^{p} \bar{\rho}_m^h h_m(\bar{u}) - \sum_{i=1}^{l} \left[\bar{\tau}_i^\theta \theta_i(\bar{u}) + \bar{\tau}_i^\psi \psi_i(\bar{u})\right]\\
&= F(\tilde{u}) + \sum_{i=1}^{m} \tilde{\tau}_i^g g(\tilde{u}, t_i) + \sum_{m=1}^{p} \tilde{\rho}_m^h h_m(\tilde{u}) - \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \theta_i(\tilde{u}) + \tilde{\tau}_i^\psi \psi_i(\tilde{u})\right].
\end{aligned} \tag{4.10}$$

Now, strict strong invexity of $F$ at $\tilde{u}$ gives

$$F(\bar{u}) - F(\tilde{u}) > \langle \xi, \eta(\bar{u}, \tilde{u})\rangle + \mu \|\eta(\bar{u}, \tilde{u})\|^\sigma, \quad \forall\ \xi \in \partial^* F(\tilde{u}). \tag{4.11}$$

Similarly, we have

$$g(\bar{u},t_i) - g(\tilde{u},t_i) \geq \langle \xi_i^g, \eta(\bar{u},\tilde{u})\rangle + \mu_i^g \|\eta(\bar{u},\tilde{u})\|^{\sigma}, \quad \forall\, \xi_i^g \in \partial^* g(\tilde{u},t_i), \quad \forall\, t_i \in T_g(\tilde{u}), \tag{4.12}$$

for all $m = \{1,2,\ldots,p\}$, we have

$$h_m(\bar{u}) - h_m(\tilde{u}) \geq \langle \zeta_m, \eta(\bar{u},\tilde{u})\rangle + \mu_m \|\eta(\bar{u},\tilde{u})\|^{\sigma}, \quad \forall\, \zeta_m \in \partial^* h_m(\tilde{u}), \tag{4.13}$$

$$-h_m(\bar{u}) + h_m(\tilde{u}) \geq \langle \nu_m, \eta(\bar{u},\tilde{u})\rangle + \mu_m \|\eta(\bar{u},\tilde{u})\|^{\sigma}, \quad \forall\, \nu_m \in \partial^* (-h_m)(\tilde{u}), \tag{4.14}$$

in the same manner, we have

$$-\theta_i(\bar{u}) + \theta_i(\tilde{u}) \geq \langle \xi_i^{\theta}, \eta(\bar{u},\tilde{u})\rangle + \mu_i^{\theta} \|\eta(\bar{u},\tilde{u})\|^{\sigma}, \quad \forall\, \xi_i^{\theta} \in \partial^* (-\theta_i)(\tilde{u}), \quad \forall\, i \in \delta \cup \omega, \tag{4.15}$$

$$-\psi_i(\bar{u}) + \psi_i(\tilde{u}) \geq \langle \xi_i^{\psi}, \eta(\bar{u},\tilde{u})\rangle + \mu_i^{\psi} \|\eta(\bar{u},\tilde{u})\|^{\sigma}, \quad \forall\, \xi_i^{\psi} \in \partial^* (-\psi_i)(\tilde{u}), \quad \forall\, i \in \omega \cup \kappa. \tag{4.16}$$

If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then multiplying (4.12) to (4.16) by $\tilde{\tau}_i^g \geq 0$ $(i = 1,2,\ldots,m)$, $\tilde{\tau}_m^h > 0$ $(m = 1,2,\ldots,p)$, $\tilde{\gamma}_m^h > 0$ $(m = 1,2,\ldots,p)$, $\tilde{\tau}_i^\theta > 0$ $(i \in \delta \cup \omega)$, $\tilde{\tau}_i^\psi > 0$ $(i \in \omega \cup \kappa)$, respectively and adding (4.11) to (4.16), we obtain

$$\begin{aligned}
&F(\bar{u}) - F(\tilde{u}) + \sum_{i=1}^{m} \tilde{\tau}_i^g g(\bar{u},t_i) - \sum_{i=1}^{m} \tilde{\tau}_i^g g(\tilde{u},t_i) + \sum_{m=1}^{p} \tilde{\tau}_m^h h_m(\bar{u}) - \sum_{m=1}^{p} \tilde{\tau}_m^h h_m(\tilde{u}) - \sum_{m=1}^{p} \tilde{\gamma}_m^h h_m(\bar{u}) \\
&\quad + \sum_{m=1}^{p} \tilde{\gamma}_m^h h_m(\tilde{u}) - \sum_{i=1}^{l} \tilde{\tau}_i^\theta \theta_i(\bar{u}) + \sum_{i=1}^{l} \tilde{\tau}_i^\theta \theta_i(\tilde{u}) - \sum_{i=1}^{l} \tilde{\tau}_i^\psi \psi_i(\bar{u}) + \sum_{i=1}^{l} \tilde{\tau}_i^\psi \psi_i(\tilde{u}) \\
&\quad > \left\langle \xi + \sum_{i=1}^{m} \tilde{\tau}_i^g \xi_i^g + \sum_{m=1}^{p} \left[\tilde{\tau}_m^h \zeta_m + \tilde{\gamma}_m^h \nu_m\right] + \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \xi_i^\theta + \tilde{\tau}_i^\psi \xi_i^\psi\right], \eta(\bar{u},\tilde{u}) \right\rangle \\
&\quad + \mu \|\eta(\bar{u},\tilde{u})\|^{\sigma} + \sum_{i=1}^{m} \tilde{\tau}_i^g \mu_i^g \|\bar{u} - \tilde{u}\|^{\sigma} + \sum_{m=1}^{p} \tilde{\tau}_m^h \mu_m \|\eta(\bar{u},\tilde{u})\|^{\sigma} \\
&\quad + \sum_{m=1}^{p} \tilde{\gamma}_m^h \mu_m \|\eta(\bar{u},\tilde{u})\|^{\sigma} + \sum_{i=1}^{l} \tilde{\tau}_i^\theta \mu_i^\theta \|\eta(\bar{u},\tilde{u})\|^{\sigma} + \sum_{i=1}^{l} \tilde{\tau}_i^\Psi \mu_i^\Psi \|\eta(\bar{u},\tilde{u})\|^{\sigma}.
\end{aligned}$$

From (4.1), we have $\tilde{\xi} \in co\partial^* F(\tilde{u})$, $\tilde{\xi}_i^g \in co\partial^* g(\tilde{u},t_i)(t_i \in T_g)$, $\tilde{\zeta}_m \in co\partial^* h_m(\tilde{u})$, $\tilde{\nu}_m \in co\partial^*(-h_m)(\tilde{u})$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(\tilde{u})$, and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(\tilde{u})$, such that

$$\tilde{\xi} + \sum_{i=1}^{m} \tilde{\tau}_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tilde{\tau}_m^h \tilde{\zeta}_m + \tilde{\gamma}_m^h \tilde{\nu}_m\right] + \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \tilde{\xi}_i^\theta + \tilde{\tau}_i^\psi \tilde{\xi}_i^\psi\right] = 0.$$

That is one has

$$\begin{aligned}
&F(\bar{u}) - F(\tilde{u}) + \sum_{i=1}^{m} \tilde{\tau}_i^g g(\bar{u},t_i) - \sum_{i=1}^{m} \tilde{\tau}_i^g g(\tilde{u},t_i) + \sum_{m=1}^{p} \tilde{\tau}_m^h h_m(\bar{u}) - \sum_{m=1}^{p} \tilde{\tau}_m^h h_m(\tilde{u}) - \sum_{m=1}^{p} \tilde{\gamma}_m^h h_m(\bar{u}) \\
&\quad + \sum_{m=1}^{p} \tilde{\gamma}_m^h h_m(\tilde{u}) - \sum_{i=1}^{l} \tilde{\tau}_i^\theta \theta_i(\bar{u}) + \sum_{i=1}^{l} \tilde{\tau}_i^\theta \theta_i(\tilde{u}) - \sum_{i=1}^{l} \tilde{\tau}_i^\psi \psi_i(\bar{u}) + \sum_{i=1}^{l} \tilde{\tau}_i^\psi \psi_i(\tilde{u}) > 0.
\end{aligned}$$

Now, as $\bar{u}$ is an optimal solution for (SIMPEC), we get

$$F(\bar{u}) - F(\tilde{u}) + \sum_{i=1}^{m} \tilde{\tau}_i^g g_i(\tilde{u}) + \sum_{m=1}^{p} \tilde{\rho}_m^h h_m(\tilde{u}) - \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \theta_i(\tilde{u}) + \tilde{\tau}_i^\psi \psi_i(\tilde{u})\right] > 0,$$

*i.e.*,

$$F(\bar{u}) > F(\tilde{u}) + \sum_{i=1}^{m} \tilde{\tau}_i^g g_i(\tilde{u}) + \sum_{m=1}^{p} \tilde{\rho}_m^h h_m(\tilde{u}) - \sum_{i=1}^{l} \left[\tilde{\tau}_i^\theta \theta_i(\tilde{u}) + \tilde{\tau}_i^\psi \psi_i(\tilde{u})\right].$$

Which contradicts (4.10). Therefore $\bar{u} = \tilde{u}$ □

**Example 4.4.** Consider the following SIMPEC in $\mathbb{R}^2$;
SIMPEC

$$\min\ f(u_1, u_2) := |u_1| + u_2^2$$

subject to:

$$\begin{aligned} g(u,t) &:= |u_1| - t \leq 0, t \in [0,1], \\ \theta(u_1,u_2) &:= |u_1| + u_2 \geq 0, \\ \psi(u_1,u_2) &:= -u_2 \geq 0, \\ \theta(u_1,u_2)\psi(u_1,u_2) &:= u_2(|u_1| + u_2) = 0. \end{aligned}$$

Now, we formulate Wolfe-type dual problem WD of SIMPEC

$$\max_{v,\tau} |v_1| + v_2^2 + \tau^g |v_1| - \left[\tau^\theta(|v_1| + v_2) + \tau^\psi(-v_2)\right]$$

$$\text{subject to: } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} \xi \\ 2v_2 \end{pmatrix} + \tau^g \begin{pmatrix} \xi^g \\ 0 \end{pmatrix} - \tau^\theta \begin{pmatrix} \xi^\theta \\ 1 \end{pmatrix} - \tau^\psi \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

where $\xi, \xi^g,$ and $\xi^\theta \in [-1,1]$.

$$\begin{aligned} \tau^g\,(|v_1| - t) &\geq 0,\ t \in T_g(\tilde{u}) = \{0\}, \\ \tau^\theta\,(|v_1| + v_2) &\leq 0, \\ \tau^\psi\,(-v_2) &\leq 0. \end{aligned}$$

Here one can see that, if $\omega$ is nonempty then either

$$\tau^\theta > 0, \tau^\psi > 0 \text{ or } \tau^\theta\tau^\psi = 0.$$

If we take $\bar{u} = (0,0)$, as a feasible point then the index sets $\delta(0,0)$ and $\kappa(0,0)$ are empty sets, but $\omega(0,0)$ is non-empty. Also, solving a constraint equation in the feasible region of WD we get a number of choices for $\tau^\theta$ *and* $\tau^\psi$. Since $\omega$ is non-empty so we can consider $\omega, \omega_\theta^+, \omega_\psi^+$ to decide the feasible region of WD. Now it is clear that the assumptions of Theorem 4.1 are satisfied. Hence we can say that Theorem 4.1, holds between SIMPEC and WD.

Since $\tilde{u} = (0,0)$ is an optimal solution for the SIMPEC also one can see that the GS-ACQ is satisfied at $\tilde{u}$. If we take

$$\xi = \frac{1}{2}, \tau^g = 1, \xi^g = \frac{1}{2}, \tau^\theta = 1, \xi^\theta = 1, \text{ and } \tau^\psi = 1$$

then $\tilde{u} = (0,0)$ is a generalized strong stationary point. Further, we have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + 1\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of WD and the optimal values corresponding to the objective function of the primal and its dual are equal. Hence the assumptions of the Theorem 4.2, are satisfied.

Now, we formulate the Mond–Weir type dual problem (MWD) for the problem SIMPEC and establish duality theorems using convexificators.

$$\text{MWD} \qquad \max_{v,\tau}\{F(v)\}$$

subject to:

$$\begin{aligned}
0 \in co\partial^* F(v) &+ \sum_{i=1}^{m} \tau_i^g co\partial^* g(v,t_i) + \sum_{m=1}^{p} \left[\tau_m^h co\partial^* h_m(v) + \gamma_m^h co\partial^*(-h_m)(v)\right] \\
&+ \sum_{i=1}^{l} \left[\tau_i^\theta co\partial^*(-\theta_i)(v) + \tau_i^\psi co\partial^*(-\psi_i)(v)\right], \\
g(v,t_i) &\geq 0 \ (t_i \in T_g), \quad h_m(v) = 0 \ (m = 1,2,\ldots,p), \\
\theta_i(v) &\leq 0 \ (i \in \delta \cup \omega), \quad \psi_i(v) \leq 0 \ (i \in \omega \cup \kappa), \\
\tau_i^g &\geq 0 \ (i = 1,2,\ldots,m), \ \tau_m^h, \gamma_m^h \geq 0, \ m = 1,2,\ldots,p, \\
\tau_i^\theta, \tau_i^\psi, \gamma_i^\theta, \gamma_i^\psi &\geq 0, \ i = 1,2,\ldots,l, \\
\tau_\kappa^\theta = \tau_\delta^\psi &= \gamma_\kappa^\theta = \gamma_\delta^\psi = 0, \forall\, i \in \omega, \gamma_i^\theta = 0, \gamma_i^\psi = 0,
\end{aligned} \tag{4.17}$$

where, $\tau = (\tau^g, \tau^h, \tau^\theta, \tau^\psi) \in \mathbb{R}^{k+p+2l}$, $\gamma = (\gamma^h, \gamma^\theta, \gamma^\psi) \in \mathbb{R}^{p+2l}$ and, $t_1, t_2, \ldots, t_m \in T_g(\tilde{u}), m \leq n+1$.

**Theorem 4.5** (Weak duality). *Let $\tilde{u}$ be feasible for the problem SIMPEC, $(v,\tau)$ be feasible for the dual MWD and the index sets $I_g, \delta, \omega, \kappa$ be defined accordingly. Suppose that $F, g(\cdot,t)$ $(t \in T), \pm h_m$ $(m = 1,2,\ldots,p), -\theta_i$ $(i \in \delta \cup \omega), -\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are strongly $\partial^*$-invex functions of order $\sigma$ at $v$, with respect to the common kernel $\eta$. If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then for any $u$ feasible for the problem SIMPEC, we have*

$$F(u) \geq F(v).$$

*Proof.* Since $f$ is strongly $\partial^*$-invex function of order $\sigma$ at $v$, with respect to the kernel $\eta$, then for some $\mu > 0$, we have

$$F(u) - F(v) \geq \langle \xi, \eta(u,v) \rangle + \mu \|\eta(u,v)\|^\sigma, \quad \forall\, \xi \in \partial^* F(v). \tag{4.18}$$

Similarly, we have

$$g(u,t_i) - g(v,t_i) \geq \langle \xi_i^g, \eta(u,v) \rangle + \mu_i^g \|\eta(u,v)\|^\sigma, \quad \forall\, \xi_i^g \in \partial^* g(v,t_i), \forall\, t_i \in T_g, \tag{4.19}$$

for all $m = \{1,2,\ldots,p\}$, we have

$$h_m(u) - h_m(v) \geq \langle \zeta_m, \eta(u,v) \rangle + \mu_m \|\eta(u,v)\|^\sigma, \ \forall\, \zeta_m \in \partial^* h_m(v), \tag{4.20}$$

$$-h_m(u) + h_m(v) \geq \langle \nu_m, \eta(u,v) \rangle + \mu_m \|\eta(u,v)\|^\sigma, \ \forall\, \nu_m \in \partial^*(-h_m)(v), \tag{4.21}$$

in the same manner, we have

$$-\theta_i(u) + \theta_i(v) \geq \langle \xi_i^\theta, \eta(u,v) \rangle + \mu_i^\theta \|\eta(u,v)\|^\sigma, \ \forall\, \xi_i^\theta \in \partial^*(-\theta_i)(v), \quad \forall\, i \in \delta \cup \omega, \tag{4.22}$$

$$-\psi_i(u) + \psi_i(v) \geq \langle \xi_i^\psi, \eta(u,v) \rangle + \mu_i^\psi \|\eta(u,v)\|^\sigma, \quad \forall\, \xi_i^\psi \in \partial^*(-\psi_i)(v), \quad \forall\, i \in \omega \cup \kappa. \tag{4.23}$$

If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, multiplying (4.19) to (4.23) by $\tau_i^g \geq 0$ $(i = 1, 2, \ldots, m)$, $\tau_m^h > 0(m = 1, 2, \ldots, p)$, $\gamma_m^h > 0$ $(m = 1, 2, \ldots, p)$, $\tau_i^\theta > 0$ $(i \in \delta \cup \omega)$, $\tau_i^\psi > 0$ $(i \in \omega \cup \kappa)$, respectively and adding (4.18) to (4.23), we obtain

$$\begin{aligned}
&F(u) - F(v) + \sum_{i=1}^{m} \tau_i^g g(u, t_i) - \sum_{i=1}^{m} \tau_i^g g(v, t_i) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\
&\quad + \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^\theta \theta_i(u) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) - \sum_{i=1}^{l} \tau_i^\psi \psi_i(u) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \\
&\quad \geq \left\langle \xi + \sum_{i=1}^{m} \tau_i^g \xi_i^g + \sum_{m=1}^{p} \left[\tau_m^h \zeta_m + \gamma_m^h \nu_m\right] + \sum_{i=1}^{l} \left[\tau_i^\theta \xi_i^\theta + \tau_i^\psi \xi_i^\psi\right], \eta(u, v) \right\rangle \\
&\quad + \mu \|\eta(u, v)\|^\sigma + \sum_{i \in I_g} \tau_i^g \mu_i^g \|\eta(u, v)\|^\sigma + \sum_{m=1}^{p} \tau_m^h \mu_m \|\eta(u, v)\|^\sigma \\
&\quad + \sum_{m=1}^{p} \gamma_m^h \mu_m \|\eta(u, v)\|^\sigma + \sum_{i=1}^{l} \tau_i^\theta \mu_i^\theta \|\eta(u, v)\|^\sigma + \sum_{i=1}^{l} \tau_i^\Psi \mu_i^\Psi \|\eta(u, v)\|^\sigma,
\end{aligned}$$

where $\tilde{\xi} \in co\partial^* F(v)$, $\tilde{\xi}_i^g \in co\partial^* g(v, t_i)(t \in T), \tilde{\zeta}_m \in co\partial^* h_m(v)$, $\tilde{\nu}_m \in co\partial^*(-h_m)(v)$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(v)$, and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(v)$. From (4.17), we have

$$\tilde{\xi} + \sum_{i=1}^{m} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] + \sum_{i=1}^{l} \left[\tau_i^\theta \tilde{\xi}_i^\theta + \tau_i^\psi \tilde{\xi}_i^\psi\right] = 0.$$

Therefore,

$$\begin{aligned}
&F(u) - F(v) + \sum_{i=1}^{m} \tau_i^g g(u, , t_i) - \sum_{i=1}^{m} \tau_i^g g(v, t_i) + \sum_{m=1}^{p} \tau_m^h h_m(u) - \sum_{m=1}^{p} \tau_m^h h_m(v) - \sum_{m=1}^{p} \gamma_m^h h_m(u) \\
&\quad + \sum_{m=1}^{p} \gamma_m^h h_m(v) - \sum_{i=1}^{l} \tau_i^\theta \theta_i(u) + \sum_{i=1}^{l} \tau_i^\theta \theta_i(v) - \sum_{i=1}^{l} \tau_i^\psi \psi_i(u) + \sum_{i=1}^{l} \tau_i^\psi \psi_i(v) \geq 0.
\end{aligned}$$

Now using the feasibility of $u$ and $v$ for SIMPEC and MWD, it follows that

$$F(u) \geq F(v).$$

Hence, the proof is completed. □

**Theorem 4.6** (Strong duality)**.** *Let $\tilde{u}$ be a local optimal solution of the problem SIMPEC and let $F$ be locally Lipschitz near $\tilde{u}$. Suppose that $F$, $g_i$ $(i = 1, 2, \ldots, m)$, $\pm h_m$ $(m = 1, 2, \ldots, p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are strongly $\partial^*$-invex functions of order $\sigma$ at $\tilde{u}$ with respect to the common kernel $\eta$. If GS-ACQ holds at $\tilde{u}$, then there exists $\tilde{\tau}$, such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of the dual MWD and the corresponding objective values of SIMPEC and MWD are equal.*

*Proof.* The proof can be done similar to the proof of Theorem 4.2 by invoking Theorem 4.5. □

**Theorem 4.7** (Strict converse duality)**.** *Let $\tilde{u}$ be an optimal solution for (SIMPEC) and $\bar{u}$ be an optimal solution for (MWD). If the assumptions of the strong duality theorem are satisfied and $F$ be strictly strong $\partial^*$-invex of order $\sigma$ at $\tilde{u}$ then $\bar{u} = \tilde{u}$.*

*Proof.* We proceed by contradiction. Assume $\bar{u} \neq \tilde{u}$. By the strong duality theorem there exist $\bar{\tau} = (\bar{\tau}^g, \bar{\tau}^h, \bar{\tau}^\theta, \bar{\tau}^\psi) \in \mathbb{R}^{k+p+2l}$, $\bar{\gamma} \in (\bar{\gamma}^h, \bar{\gamma}^\theta, \bar{\gamma}^\psi) \in \mathbb{R}^{p+2l}$, and indices $t_1, t_2, \ldots, t_m \in T_g(\bar{u}), m \leq n+1$, such that $(\bar{u}, \bar{\tau})$ be an optimal solution for (MWD) and

$$f(\bar{u}) = F(\tilde{u}). \tag{4.24}$$

Now strictly strong $\partial^*$-invexity of $F$ at $\tilde{u}$ gives

$$F(\bar{u}) - F(\tilde{u}) > \langle \xi, \eta(\bar{u}, \tilde{u}) \rangle + \mu \|\eta(\bar{u}, \tilde{u})\|^\sigma, \quad \forall \xi \in \partial^* F(\tilde{u}). \tag{4.25}$$

Similarly, we have

$$g(\bar{u}, t_i) - g(\tilde{u}, t_i) \geq \langle \xi_i^g, \eta(\bar{u}, \tilde{u}) \rangle + \mu_i^g \|\eta(\bar{u}, \tilde{u})\|^\sigma, \qquad \forall\ \xi_i^g \in \partial^* g(\tilde{u}, t_i), \quad \forall\ t_i \in T_g(\tilde{u}), \tag{4.26}$$

for all $m = \{1, 2, \ldots, p\}$, we have

$$h_m(\bar{u}) - h_m(\tilde{u}) \geq \langle \zeta_m, \eta(\bar{u}, \tilde{u}) \rangle + \mu_m \|\eta(\bar{u}, \tilde{u})\|^\sigma, \quad \forall\ \zeta_m \in \partial^* h_m(\tilde{u}), \tag{4.27}$$

$$-h_m(\bar{u}) + h_m(\tilde{u}) \geq \langle \nu_m, \eta(\bar{u}, \tilde{u}) \rangle + \mu_m \|\eta(\bar{u}, \tilde{u})\|^\sigma, \quad \forall\ \nu_m \in \partial^*(-h_m)(\tilde{u}), \tag{4.28}$$

in the same manner, we have

$$-\theta_i(\bar{u}) + \theta_i(\tilde{u}) \geq \langle \xi_i^\theta, \eta(\bar{u}, \tilde{u}) \rangle + \mu_i^\theta \|\eta(\bar{u}, \tilde{u})\|^\sigma, \quad \forall\ \xi_i^\theta \in \partial^*(-\theta_i)(\tilde{u}), \quad \forall\ i \in \delta \cup \omega, \tag{4.29}$$

$$-\psi_i(\bar{u}) + \psi_i(\tilde{u}) \geq \langle \xi_i^\psi, \eta(\bar{u}, \tilde{u}) \rangle + \mu_i^\psi \|\eta(\bar{u}, \tilde{u})\|^\sigma, \quad \forall\ \xi_i^\psi \in \partial^*(-\psi_i)(\tilde{u}), \quad \forall\ i \in \omega \cup \kappa. \tag{4.30}$$

If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then multiplying (4.26) to (4.30) by $\tilde{\tau}_i^g \geq 0$ $(i = 1, 2, \ldots, m)$, $\tilde{\tau}_m^h > 0$ $(m = 1, 2, \ldots, p)$, $\tilde{\gamma}_m^h > 0$ $(m = 1, 2, \ldots, p)$, $\tilde{\tau}_i^\theta > 0$ $(i \in \delta \cup \omega)$, $\tilde{\tau}_i^\psi > 0$ $(i \in \omega \cup \kappa)$, respectively and adding (4.25) to (4.30), we obtain

$$\begin{aligned}
F(\bar{u}) - F(\tilde{u}) &+ \sum_{i=1}^m \tilde{\tau}_i^g g(\bar{u}, t_i) - \sum_{i=1}^m \tilde{\tau}_i^g g(\tilde{u}, t_i) + \sum_{m=1}^p \tilde{\tau}_m^h h_m(\bar{u}) - \sum_{m=1}^p \tilde{\tau}_m^h h_m(\tilde{u}) - \sum_{m=1}^p \tilde{\gamma}_m^h h_m(\bar{u}) \\
&+ \sum_{m=1}^p \tilde{\gamma}_m^h h_m(\tilde{u}) - \sum_{i=1}^l \tilde{\tau}_i^\theta \theta_i(\bar{u}) + \sum_{i=1}^l \tilde{\tau}_i^\theta \theta_i(\tilde{u}) - \sum_{i=1}^l \tilde{\tau}_i^\psi \psi_i(\bar{u}) + \sum_{i=1}^l \tilde{\tau}_i^\psi \psi_i(\tilde{u}) \\
&> \left\langle \xi + \sum_{i=1}^m \tilde{\tau}_i^g \xi_i^g + \sum_{m=1}^p \left[\tilde{\tau}_m^h \zeta_m + \tilde{\gamma}_m^h \nu_m\right] + \sum_{i=1}^l \left[\tilde{\tau}_i^\theta \xi_i^\theta + \tilde{\tau}_i^\psi \xi_i^\psi\right], \eta(\bar{u}, \tilde{u}) \right\rangle \\
&+ \mu \|\eta(\bar{u}, \tilde{u})\|^\sigma + \sum_{i=1}^m \tilde{\tau}_i^g \mu_i^g \|\eta(\bar{u}, \tilde{u})\|^\sigma + \sum_{m=1}^p \tilde{\tau}_m^h \mu_m \|\eta(\bar{u}, \tilde{u})\|^\sigma \\
&+ \sum_{m=1}^p \tilde{\gamma}_m^h \mu_m \|\eta(\bar{u}, \tilde{u})\|^\sigma + \sum_{i=1}^l \tilde{\tau}_i^\theta \mu_i^\theta \|\eta(\bar{u}, \tilde{u})\|^\sigma + \sum_{i=1}^l \tilde{\tau}_i^\Psi \mu_i^\Psi \|\eta(\bar{u}, \tilde{u})\|^\sigma.
\end{aligned}$$

From (4.17), we have $\tilde{\xi} \in co\partial^* F(\tilde{u})$, $\tilde{\xi}_i^g \in co\partial^* g(\tilde{u}, t_i)(t_i \in T_g)$, $\tilde{\zeta}_m \in co\partial^* h_m(\tilde{u})$, $\tilde{\nu}_m \in co\partial^*(-h_m)(\tilde{u})$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(\tilde{u})$, and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(\tilde{u})$, such that

$$\tilde{\xi} + \sum_{i=1}^m \tilde{\tau}_i^g \tilde{\xi}_i^g + \sum_{m=1}^p \left[\tilde{\tau}_m^h \tilde{\zeta}_m + \tilde{\gamma}_m^h \tilde{\nu}_m\right] + \sum_{i=1}^l \left[\tilde{\tau}_i^\theta \tilde{\xi}_i^\theta + \tilde{\tau}_i^\psi \tilde{\xi}_i^\psi\right] = 0.$$

That is one has

$$\begin{aligned}
F(\bar{u}) - F(\tilde{u}) &+ \sum_{i=1}^m \tilde{\tau}_i^g g(\bar{u}, t_i) - \sum_{i=1}^m \tilde{\tau}_i^g g(\tilde{u}, t_i) + \sum_{m=1}^p \tilde{\tau}_m^h h_m(\bar{u}) - \sum_{m=1}^p \tilde{\tau}_m^h h_m(\tilde{u}) - \sum_{m=1}^p \tilde{\gamma}_m^h h_m(\bar{u}) \\
&+ \sum_{m=1}^p \tilde{\gamma}_m^h h_m(\tilde{u}) - \sum_{i=1}^l \tilde{\tau}_i^\theta \theta_i(\bar{u}) + \sum_{i=1}^l \tilde{\tau}_i^\theta \theta_i(\tilde{u}) - \sum_{i=1}^l \tilde{\tau}_i^\psi \psi_i(\bar{u}) + \sum_{i=1}^l \tilde{\tau}_i^\psi \psi_i(\tilde{u}) > 0.
\end{aligned}$$

Now, using the feasibilty of $\tilde{u}$ and $\bar{u}$ for (SIMPEC) and (MWD), respectively, we get

$$F(\bar{u}) - F(\tilde{u}) > 0.$$

*i.e.*,

$$F(\bar{u}) > F(\tilde{u}).$$

Which contradicts (4.24). Therefore $\bar{u} = \tilde{u}$. □

**Example 4.8.** Consider the following SIMPEC in $\mathbb{R}^2$;
SIMPEC

$$\min |u_1| + u_2$$

subject to:

$$\begin{aligned} g(u,t) &:= |u_1| - t \leq 0, \forall\ t \in [0,1], \\ \theta(u_1,u_2) &:= |u_1| + u_2 \geq 0, \\ \psi(u_1,u_2) &:= u_2 - |u_1| \geq 0, \\ (|u_1| + u_2)(u_2 - |u_1|) &:= 0. \end{aligned}$$

Now, we formulate Mond–Weir type dual problem (MWD) for SIMPEC

$$\max_{v,\tau} |v_1| + v_2$$
$$\text{subject to: } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} \xi \\ 1 \end{pmatrix} + \tau^g \begin{pmatrix} \xi^g \\ 0 \end{pmatrix} - \tau^\theta \begin{pmatrix} \xi^\theta \\ 1 \end{pmatrix} - \tau^\psi \begin{pmatrix} \xi^\psi \\ 1 \end{pmatrix},$$

where $\xi, \xi^g, \xi^\theta \in [-1,1]$ and $\xi^\psi \in \{-1,1\}$.

$$\tau^g(|v_1| - t) \geq 0,\ t \in T_g(\tilde{u}) = \{0\}, \tag{4.31}$$
$$\tau^\theta(|v_1| + v_2) \leq 0, \tag{4.32}$$
$$\tau^\psi(v_2 - |v_1|) \leq 0. \tag{4.33}$$

If $\omega$ is nonempty then either

$$\tau^\theta > 0, \tau^\psi > 0 \text{ or } \tau^\theta\tau^\psi = 0.$$

If we take the point $\tilde{u} = (0,0)$ from the feasible region, then the index sets $\delta(0,0)$ and $\kappa(0,0)$ are empty sets, but $\omega(0,0)$ is non-empty. Also, solving constraints equation in the feasible region of MWD we get a number of choices for $\tau^\theta$ and $\tau^\psi$. Since $\omega$ is non-empty so we can consider $\omega, \omega^+_\theta, \omega^+_\psi$ to decide the feasible region of MWD. Now it is clear that the assumptions of Theorem 4.5 are satisfied. Hence we can say that Theorem 4.5, holds between SIMPEC and MWD.

Since $\tilde{u} = (0,0)$ is the optimal solution of SIMPEC and GS-ACQ is satisfied at $\tilde{u}$. Now if we take

$$\xi = \frac{1}{2}, \xi^g = -\frac{1}{2}, \xi^\theta = 1, \xi^\psi = -1, \tau^g = 1, \tau^\theta = \frac{1}{2}, \text{ and } \tau^\psi = \frac{1}{2},$$

then $\tilde{u} = (0,0)$ is the generalized strong stationary point and

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + 1 \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Now it is clear that, we have $\tilde{\tau}$ such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of MWD and the optimal values corresponding to the objective function of the primal and its dual are equal. Hence, the assumptions of the Theorem 4.6 are satisfied.

In order to justify Theorem 4.7, let we start with a contradiction *i.e.*, let $\tilde{u} \neq \tilde{v}$ (*i.e.*, the value of the objective function of the primal SIMPEC is not equal to the value of the objective function of its dual MWD).

We have shown in our example, that there exist $\tilde{v} = (0, 0)$ and

$$\xi = \frac{1}{2}, \xi^g = -\frac{1}{2}, \xi^\theta = 1, \xi^\psi = -1, \tau^g = 1, \tau^\theta = \frac{1}{2}, \text{ and } \tau^\psi = \frac{1}{2},$$

such that $\tilde{v}$ is an optimal solution corresponding to the dual MWD. One can also see that for $u \neq \tilde{u}$ function $f$ is strictly strong invex. It is also clear that corresponding to the objective function $f$ given in the primal SIMPEC problem $\tilde{u} = (0, 0)$ is an optimal solution. Which is a contradiction. Hence Theorem 4.7, is satisfied.

Next, we establish weak duality and strong duality theorems for SIMPEC and its Mond–Weir type dual problem (MWD) under the assumptions of generalized strong $\partial^*$-invexity of order $\sigma$ assumptions.

**Theorem 4.9** (Weak duality)**.** *Let $\tilde{u}$ be feasible for the problem SIMPEC, $(v, \tau)$ be feasible for the dual MWD and the index sets $I_g, \delta, \omega, \kappa$ are defined accordingly. Suppose that $F$ is strongly $\partial^*$-pseudoinvex of order $\sigma$ at $v$, with respect to the kernel $\eta$ and $g(\cdot, t)$ $(t \in T_g)$, $\pm h_m$ $(m = 1, 2, \ldots, p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are strongly $\partial^*$-quasiinvex functions order $\sigma$ at $v$, with respect to the common kernel $\eta$. If $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, then for any $u$ feasible for the problem SIMPEC, there exist $\mu > 0$, such that*

$$F(u) \geq F(v).$$

*Proof.* Assume that, for some feasible point $u$, such that $F(u) < F(v)$, then by strong $\partial^*$-pseudoinvexity of $F$ of order $\sigma$ at $v$, with respect to the kernel $\eta$, we get

$$\langle \xi, \eta(u, v) \rangle + \mu \|\eta(u, v)\|^\sigma < 0, \forall\, \xi \in \partial^* F(v). \tag{4.34}$$

From (4.17), $\exists$ $\tilde{\xi} \in co\partial^* F(v)$, $\tilde{\xi}_i^g \in co\partial^* g(v, t_i)$, $\tilde{\zeta}_m \in co\partial^* h_m(v)$, $\tilde{\nu}_m \in co\partial^*(-h_m)(v)$, $\tilde{\xi}_i^\theta \in co\partial^*(-\theta_i)(v)$, and $\tilde{\xi}_i^\psi \in co\partial^*(-\psi_i)(v)$, such that

$$-\sum_{i=1}^{m} \tau_i^g \tilde{\xi}_i^g - \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] - \sum_{\delta \cup \omega} \tau_i^\theta \tilde{\xi}_i^\theta - \sum_{\omega \cup \kappa} \tau_i^\psi \tilde{\xi}_i^\psi \in \partial^* F(v). \tag{4.35}$$

From (4.34), we have

$$\left\langle \left(\sum_{i=1}^{m} \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^{p} \left[\tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m\right] + \sum_{\delta \cup \omega} \tau_i^\theta \tilde{\xi}_i^\theta + \sum_{\omega \cup \kappa} \tau_i^\psi \tilde{\xi}_i^\psi\right), \eta(u, v) \right\rangle > 0. \tag{4.36}$$

For each $t_i \in T_g$, $g(u, t_i) \leq 0 \leq g(v, t_i)$. Hence, by strong $\partial^*$-quasiinvexity of order $\sigma$ of $g(\cdot, t)$, we obtain

$$\left\langle \xi_i^g, \eta(u, v) \right\rangle + \mu_i^g \|\eta(u, v)\|^\sigma \leq 0, \quad \forall \xi_i^g \in \partial^* g(v, t_i), \quad \forall\, t_i \in T_g. \tag{4.37}$$

Similarly, we have

$$\left\langle \zeta_m, \eta(u, v) \right\rangle + \mu_m \|\eta(u, v)\|^\sigma \leq 0, \quad \forall \zeta_m \in \partial^* h_m(v), \quad \forall\, m = \{1, 2, \ldots, p\}, \tag{4.38}$$

for any feasible point $v$ of the dual MWD, and for every $m, -h_m(v) = -h_m(u) = 0$. On the other hand, $-\theta_i(u) \leq -\theta_i(v), \forall i \in \delta \cup \omega$, and $-\psi_i(u) \leq -\psi_i(v), \forall i \in \omega \cup \kappa$. By $\partial^*$-quasiinvexity, we obtain

$$\left\langle \nu_m, \eta(u, v) \right\rangle + \mu_m \|\eta(u, v)\|^\sigma \leq 0, \quad \forall\, \nu_m \in \partial^*(-h_m)(v), \quad \forall\, m = \{1, 2, \ldots, p\}, \tag{4.39}$$

$$\left\langle \xi_i^\theta, \eta(u, v) \right\rangle + \mu_i^\theta \|\eta(u, v)\|^\sigma \leq 0, \quad \forall\, \xi_i^\theta \in \partial^*(-\theta_i)(v), \forall\, i \in \delta \cup \omega, \tag{4.40}$$

$$\left\langle \xi_i^\psi, \eta(u, v) \right\rangle + \mu_i^\psi \|\eta(u, v)\|^\sigma \leq 0, \quad \forall\, \xi_i^\psi \in \partial^*(-\psi_i)(v), \forall\, i \in \omega \cup \kappa. \tag{4.41}$$

From equations (4.37) to (4.41), we have

$$\left\langle \tilde{\xi}_i^g, \eta(u,v) \right\rangle \leq 0 \ (i = 1, 2, \ldots, m), \ \left\langle \tilde{\zeta}_m, \eta(u,v) \right\rangle \leq 0, \ \left\langle \tilde{\nu}_m, \eta(u,v) \right\rangle \leq 0 \ (m = \{1, 2, \ldots, p\}),$$
$$\left\langle \tilde{\xi}_i^\theta, \eta(u,v) \right\rangle \leq 0, \forall \ i \in \delta \cup \omega, \ \ \left\langle \tilde{\xi}_i^\psi, \eta(u,v) \right\rangle \leq 0, \forall \ i \in \omega \cup \kappa.$$

Since $\omega_\gamma^\theta \cup \omega_\gamma^\psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \phi$, we have

$$\left\langle \sum_{i=1}^m \tau_i^g \tilde{\xi}_i^g, \eta(u,v) \right\rangle \leq 0, \ \ \left\langle \sum_{m=1}^p \left[ \tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\gamma}_m \right], \eta(u,v) \right\rangle \leq 0,$$
$$\left\langle \sum_{\delta \cup \omega} \tau_i^\theta \tilde{\xi}_i^\theta, \eta(u,v) \right\rangle \leq 0, \ \ \left\langle \sum_{\omega \cup \kappa} \tau_i^\psi \tilde{\xi}_i^\psi, \eta(u,v) \right\rangle \leq 0.$$

Therefore,

$$\left\langle \left( \sum_{i=1}^m \tau_i^g \tilde{\xi}_i^g + \sum_{m=1}^p \left[ \tau_m^h \tilde{\zeta}_m + \gamma_m^h \tilde{\nu}_m \right] + \sum_{\delta \cup \omega} \tau_i^\theta \tilde{\xi}_i^\theta + \sum_{\omega \cup \kappa} \tau_i^\psi \tilde{\xi}_i^\psi \right), \eta(u,v) \right\rangle \leq 0,$$

which contradicts (4.36). Therefore $F(u) \geq F(v)$. Hence the proof is completed. □

**Theorem 4.10** (Strong duality). *Let $\tilde{u}$ be a local optimal solution of the problem SIMPEC and let $F$ be locally Lipschitz near $\tilde{u}$. Suppose that $F$ is strongly $\partial^*$-pseudoinvex of order $\sigma$ at $\tilde{u}$, with respect to the kernel $\eta$, $g(\cdot,t)$ $(t \in T)$, $\pm h_m$ $(m = 1, 2, \ldots, p)$, $-\theta_i$ $(i \in \delta \cup \omega)$, $-\psi_i$ $(i \in \omega \cup \kappa)$ admit bounded upper semi-regular convexificators and are strongly $\partial^*$-quasiinvex functions of order $\sigma$ at $\tilde{u}$ with respect to the common kernel $\eta$. If GS-ACQ holds at $\tilde{u}$, then there exists $\tilde{\tau}$, such that $(\tilde{u}, \tilde{\tau})$ is an optimal solution of the dual MWD and the respective objective values are equal.*

*Proof.* The proof can be done similar to the proof of Theorem 4.2, by invoking Theorem 4.9. □

**Example 4.11.** Consider the following SIMPEC problem

$$\min f(u) = |u|$$

subject to:

$$\begin{aligned} g(u,t) &:= |u| - t \leq 0, \quad \forall \ t \in [0,1], \\ \theta(u) &:= u \geq 0, \\ \psi(u) &:= |u| \geq 0, \\ \theta(u)\psi(u) &:= 0. \end{aligned}$$

Now, we formulate Mond–Weir type dual problem MWD for SIMPEC,

$$\max_{v,\tau} f(v) = |v| \tag{4.42}$$

subject to:

$$0 \in \xi + \tau^g \xi^g - [\tau^\theta \xi^\theta + \tau^\psi \xi^\psi],$$

where $\xi, \xi^g, \xi^\psi \in [-1,1]$ and $\xi^\theta = 1$,

$$\tau^g(|u| - t) \geq 0,\ t \in T_g(\tilde{u}) = \{0\},$$
$$\tau^\theta u \leq 0,$$
$$\tau^\psi |u| \leq 0.$$

The optimal solution for SIMPEC is $u = 0$. For $u = 0$ index sets $\delta$ and $\kappa$ are empty sets but $\beta$ is nonempty *i.e.*, $\beta = 1$. Further, we have $\xi = \frac{1}{2}, \xi^g = \frac{1}{2},\ \xi^\theta = 1, \xi^\psi = \frac{1}{2}, \tau^g = 1, \tau^\theta = \frac{1}{2}$ and $\tau^\psi = 1$, such that the value of the objective function of the primal is equal to the value of the objective function of its dual. Hence strong duality theorem is satisfied.

## 5. Conclusions

We have studied semi-infinite mathematical program with equilibrium constraints (SIMPEC) and derived the sufficient conditions for global optimality for SIMPEC using generalized strong invexity of order $\sigma$ assumptions. We have formulated the Wolfe type and Mond–Weir type dual models for the problem SIMPEC in the framework of convexificators. We have established weak, strong and strict converse duality theorems relating to the problem SIMPEC and two dual models using strong $\partial^*$-invexity of order $\sigma$ and generalized strong $\partial^*$-invexity of order $\sigma$ assumptions. We have Provided examples in order to understand the concepts of global optimality conditions and the dual models for the SIMPEC.

*Acknowledgements.* The author would like to thank both of the reviewers for their helpful and effective comments which help us to improve this paper scientifically.

## References

[1] A. Ben-Israel and B. Mond, What is Invexity? *J. Aust. Math. Soc. Ser. B.* **28** (1986) 1–9.
[2] W. Britz, M. Ferris and A. Kuhn, Modeling water allocating institutions based on multiple optimization problems with equilibrium constraints. *Environ. Model. Softw.* **46** (2013) 196–207.
[3] F.H. Clarke, Optimization and Nonsmooth Analysis. John Wiley & Sons, New York, NY (1983).
[4] B. Colson, P. Marcotte and G. Savard, An overview of bilevel optimization. *Ann. Oper. Res.* **153** (2007) 235-256.
[5] B.D. Craven and B.M. Glover, Invex Functions and Duality. *J. Aust. Math. Soc. Ser. A.* **39** (1985) 1–20.
[6] S. Dempe and A.B. Zemkoho, Bilevel road pricing: theoretical analysis and optimality conditions. *Ann. Oper. Res.* **196** (2012) 223–240.
[7] V.F. Demyanov, Convexification and concavification of a positively homogeneous function by the same family of linear functions. Universia di Pisa, Report 3, 208, 802 (1994).
[8] V.F. Demyanov and V. Jeyakumar, Hunting for a smaller convex subdifferential. *J. Glob. Optim.* **10** (1997) 305-326.
[9] V.F. Demyanov and A.M. Rubinov, Constructive Nonsmooth Analysis. Verlag Peter Lang, Frankfurt (1995).
[10] J. Dutta and S. Chandra, Convexifactors, generalized convexity and vector optimization. *Optimization* **53** (2004) 77–94.
[11] J. Dutta and S. Chandra, Convexifactors, generalized convexity and optimality conditions. *J. Optim. Theory Appl.* **113** (2002) 41–65.
[12] M.L. Flegel, C. Kanzow and J.V. Outrata, Optimality conditions for disjunctive programs with application to mathematical programs with equilibrium constraints. *Set-Valued Anal.* **15** (2006) 139–162.
[13] M.A. Goberna and M.A. López, Linear Semi-Infinite Optimization. John Wiley & Sons, Chichester (1998).
[14] M.A. Goberna and M.A. López, editors. Semi-Infinite Programming: Recent Advances. Kluwer, Dordrecht (2001).
[15] S.M. Guu, Y. Singh and S.K. Mishra, On strong KKT type sufficient optimality conditions for multiobjective semi-infinite programming problems with vanishing constraints. *J. Inequal. Appl.* **2017** (2017) 1–9.
[16] M.A. Hanson, On sufficiency of the Kuhn–Tucker conditions. *J. Math. Anal. Appl.* **80** (1981) 545–550.
[17] P.T. Harker and J.S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. *Math. Program.* **48** (1990) 161–220.
[18] R. Hettich and K.O. Kortanek, Semi-infinite programming: theory, methods, and applications. *SIAM Rev.* **35** (1993) 380–429.
[19] V. Jeyakumar and D.T. Luc, Approximate Jacobian matrices for nonsmooth continuous maps and $C^1$-optimization. *SIAM J. Control Optim.* **36** (1998) 1815–1832.
[20] V. Jeyakumar and D.T. Luc, Nonsmooth calculus, minimality, and monotonicity of convexificators. *J. Optim. Theory Appl.* **101** (1999) 599–621.

[21] V. Jeyakumar, D.T. Luc and S. Schaible, Characterizations of generalized monotone nonsmooth continuous maps using approximate Jacobians. *J. Convex Anal.* **5** (1998) 119–132.

[22] B.C. Joshi, Higher order duality in multiobjective fractional programming problem with generalized convexity. *Yugosl. J. Oper. Res.* **27** (2017) 249–264.

[23] B.C. Joshi, S.K. Mishra and P. Kumar, On Semi-infinite Mathematical Programming Problems with Equilibrium Constraints Using Generalized Convexity. *J. Oper. Res. Soc. Ch.* (2019). DOI: 10.1007/s40305-019-00263-y.

[24] B. Kohli, Optimality conditions for optimistic bilevel programming problem using convexifactors. *J. Optim. Theory Appl.* **152** (2012) 632–651.

[25] D.T. Luc, A multiplier rule for multiobjective programming problems with continuous data. *SIAM J. Optim.* **13** (2002) 168–178.

[26] M.A. López and G. Still, Semi-infinite programming. *Eur. J. Oper. Res.* **180** (2007) 491–518.

[27] Z.Q. Luo, J.S. Pang and D. Ralph, Mathematical Programs with Equilibrium Constraints. Cambridge University Press, Cambridge (1996).

[28] S.P. Marin, Optimal parameterization of curves for robot trajectory design. *IEEE T Automat. Contr.* **33** (1988) 209–214.

[29] P. Michel and J.-P. Penot, Calcul sous-différentiel pour des fonctions lipschitziennes et nonlipschitziennes. *C. R. Math. Acad. Sci.* **12** (1984) 269–272.

[30] S.K. Mishra and G. Giorgi, Invexity and Optimization, Springer-Verlag, Berlin-Heidelberg (2008).

[31] S.K. Mishra, V. Singh and V. Laha, On duality for mathematical programs with vanishing constraints. *Ann. Oper. Res.* **243** (2016) 249–272.

[32] B.S. Mordukhovich and Y. Shao, On nonconvex subdifferential calculus in Banach spaces. *J. Convex Anal.* **2** (1995) 211–228.

[33] N. Movahedian and S. Nobakhtian, Constraint qualifications for nonsmooth mathematical programs with equilibrium constraints. *Set-Valued Anal.* **17** (2009) 65–95.

[34] N. Movahedian and S. Nobakhtian, Nondifferentiable multiplier rules for optimization problems with equilibrium constraints. *J. Convex Anal.* **16** (2009) 187–210.

[35] N. Movahedian and S. Nobakhtian, Necessary and sufficient conditions for nonsmooth mathematical programs with equilibrium constraints. *Nonlinear Anal.* **72** (2010) 2694–2705.

[36] Y. Pandey and S.K. Mishra, Duality for nonsmooth optimization problems with equilibrium constraints, using convexificators. *J. Optim. Theory Appl.* **171** (2016) 694–707.

[37] Y. Pandey and S.K. Mishra, On strong KKT type sufficient optimality conditions for nonsmooth multiobjective semi-infinite mathematical programming problems with equilibrium constraints. *Oper. Res. Lett.* **44** (2016) 148–151.

[38] Y. Pandey and S.K. Mishra, Optimality conditions and duality for semi-infinite mathematical programming problems with equilibrium constraints, using convexificators. *Ann. Oper. Res.* **269** (2018) 549–564.

[39] E. Polak, On the mathematical foundations of nondifferentiable optimization in engineering design. *SIAM Rev.* **29** (1987) 21–89.

[40] A.U. Raghunathan and L.T. Biegler, Mathematical programs with equilibrium constraints in process engineering. *Comput. Chem. Eng.* **27** (2003) 1381–1392.

[41] R. Reemtsen and J.J. Ruckmann, editors. Semi-Infinite Programming. Kluwer, Dordrecht (1998).

[42] Y. Singh, Y. Pandey and S.K. Mishra, Saddle point optimality criteria for mathematical programming problems with equilibrium constraints. *Oper. Res. Lett.* **45** (2017) 254–258.

[43] M. Soleimani-damaneh, Characterizations and applications of generalized invexity and monotonicity in Asplund spaces. *Top.* **20** (2012) 592–613.

[44] O. Stein, Bi-level strategies in semi-infinite programming. In: Vol. 71 of *Nonconve Optimization and its Applications*. Kluwer Academic Publishers, Boston (2003).

[45] O. Stein, How to solve a semi-infinite optimization problem. *Eur. J. Oper. Res.* **223** (2012) 312–320.

[46] S. Suh and T.J. Kim, Solving nonlinear bilevel programming models of the equilibrium network design problem: a comparative review. *Ann. Oper. Res.* **34** (1992) 203–218.

[47] S.K. Suneja and B. Kohli, Optimality and duality results for bilevel programming problem using convexifactors. *J. Optim. Theory Appl.* **150** (2011) 1–19.

[48] X. Tong, C. Ling and L. Qi, A semi-infinite programming algorithm for solving optimal power flow with transient stability constraints. *J Comput. Appl. Math.* **217** (2008) 432–447.

[49] A.I.F. Vaz and E.C. Ferreira, Air pollution control with semi-infinite programming. *Appl. Math. Model.* **33** (2009) 1957–1969.

[50] A. Winterfeld, Application of general semi-infinite programming to lapidary cutting problems. *Eur. J. Oper. Res.* **191** (2008) 838–854.

[51] J.J. Ye, Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints. *J. Math. Anal. Appl.* **307** (2005) 350–369.