RAIRO-Oper. Res. 55 (2021) S2027–S2053
https://doi.org/10.1051/ro/2020074

RAIRO Operations Research
www.rairo-ro.org

BERNOULLI VACATION MODEL FOR $M^X/G/1$ UNRELIABLE SERVER RETRIAL QUEUE WITH BERNOULLI FEEDBACK, BALKING AND OPTIONAL SERVICE

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Abstract. The study of unreliable server retrial bulk queue with multiphase optional service is analyzed by incorporating the features of balking, Bernoulli vacation and Bernoulli feedback. On the occasion when the server is occupied with the service of the customers, an arriving customer finding the long queue, can join the retrial orbit and receives its service later on by making re-attempt. The system is reinforced with multi phase optional service along with essential service and joining customer can opt any one of optional services after getting essential service. Furthermore, the essential/optional service can be aborted due to abrupt failure of the server. There is an immediate support of multi phase repair facility to take care of the failed server, but sometimes repair may be put on hold by virtue of any unexpected cause. If the service is unsatisfactory, the customer can rejoin the queue as feedback customer. Bernoulli vacation is permitted to the server following the respective busy period. For evaluating the queue size distribution and other system performance metrics, supplementary variable technique (SVT) is used. The approximate solutions for the steady state probabilities and waiting time are suggested using maximum entropy principle (MEP). We perform a comparative study of the exact waiting time obtained by the supplementary variable technique and the approximate waiting time derived by using maximum entropy principle by taking the numerical illustration. Quasi Newton method is used to find optimal cost. To verify the outcomes of the model, numerical illustrations and senstivity analysis have been accomplished.

Mathematics Subject Classification. 60K25, 68M20.

Received November 28, 2019. Accepted June 29, 2020.

1. Introduction

Retrial queueing models have applicability in a varied real world congestion circumstances emerging in telecommunication systems, call centres, distribution systems, manufacturing industry and computer operating systems, etc. The phenomenon of retrial queue may be involved in queueing scenarios in which the server occupied in providing service, can be re-tried for service after a random interval by the units/customers residing in the retrial orbit. To amplify the concept we cite the airplane landing in which pilot cannot take off the plane

Keywords. Retrial, bulk queue, balking, unreliable server, multiphase service/repair, Bernoulli vacations, feedback, supplementary variable, queue length.

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if runway is not free and has to wait for the permission of ATC to land; in such a case the pilot has to keep circling in the sky and re-attempts to land after sometime. The features of unreliable server and balking are also important and should be incorporated while developing the queueing model for the performance prediction and acquiring realistic outcomes of delay situations. The feature of vacationing server is also a salient attribute to enhance the server's capability and to maintain the efficiency of the system at optimum cost. The server can also utilize the vacation time for some supplementary tasks such as maintenance jobs, etc. In many practical queueing scenarios, the arrivals may occur in groups or batches; for instance packet switching technique is used in computer networks for transmission of data in the form of packets of variable sizes. These packets contain the required information of data and it might be possible that packet cannot be successfully transmitted to the destination. The concept of feedback queue can also be involved in packet transmission as some packets are again feedback to the queue and are retransmitted to the destination. The provision of optional services along with necessary regular service can be observed in many places including at a diagnostic lab where patients arrive for their regular blood test and to diagnose a particular disease, there is also a provision of additional tests as per requirement for the treatment.

An $M/G/1$ retrial queue inclusive of optional vacation and delayed repair is analyzed in Choudhury and Ke [7]. In the present paper we have extended their work by incorporating some more realistic attributes namely batch input, multiphase optional service, server breakdown, delayed and multiphase repair, Bernoulli feedback and server vacation provision. To cite an application of our model, we illustrate its application in banking system wherein customers arrive in batches and require multi-phases services provided at the bank's counter such as saving account updation, withdrawal/deposit of money, fixed deposits, loans and medical insurance, etc. It can be observed that during the busy hours of banks, the arriving customer either may wait in the retrial orbit for its turn or may balk from the system. Also, the employee at bank may avail regular vacation when there is no customer. The computer system which acts as server is subject to failure due to power failure, cyber attack, internet availability, technical issue or any unforeseen reason. The failed server needs urgent repair but delay in repair may occur in the process of finding out the reason of failure of the system. Furthermore, the repair is accomplished in multiphase. In bank there may be provision of customer's feedback in case if the customer finds its service unsatisfactory and can request for repeat its service.

Due to enormous applications, it is noticed that there is a significant amount of literature on retrial queues and its applications with distinct assumptions to develop queueing models [1,2,24]. Gao and Wang [13] investigated the non-Markovian retrial queue including impatient nature of the customers as a consequence of server breakdown being caused by occurrence of negative units. Yang *et al.* [38] have examined $M^X/G/1$ retrial queue with server failure under J optional vacations and obtained various performance measures using supplementary variable technique (SVT). An $M/G/1$ model discussed by Kim and Kim [18] was based on the concept that the arriving customers are classified into two categories separately join retrial orbit or infinite queue during unavailability of the server. The queue size distributions of both types of customers and waiting time distribution are evaluated for the concerned system by employing Laplace–Stieltjes transform. Rajadurai *et al.* [25] studied an $M/G/1$ retrial queueing model with multiphase service, working vacation and vacation interruptions using supplementary variable technique.

The queueing modeling with Bernoulli feedback has also been done by some researchers under various conditions to analyze the queueing characteristics [9,17,19]. An $M/G/1$ retrial queue having optional feedback along with a pair of heterogeneous essential service, was investigated by Lakshmi and Ramanath [20]. Ayyappan and Shyamala [3] have studied the bulk arrival feedback queueing model by utilizing Laplace Stieltjes transform. Lately, Chang *et al.* [4] have discussed unreliable server queue with impatient customers and Bernoulli feedback and derived steady state queue size probabilities using quasi progression algorithm. Due to real life applications of server vacation in various congestion situations such as in production system, fabricating industry and PC operating system, many queueing analysts contributed their significant researches in queueing theory and developed a variant of vacation queueing models [11, 12, 22, 29]. Yang and Ke [37] have analyzed the unreliable server non-Markovian model with server vacation under (p, N)-policy and obtained various analytical as well as numerical outcomes along with optimized cost. Bulk arrival retrial queues with Bernoulli vacation was discussed by Singh *et al.* [30] using SVT to evaluate performance characteristics by considering the optional service in addition to regular service. Choudhury and Deka [6] have considered the multiple vacation policy for unreliable server $M^X/G/1$ model using SVT to evaluate the busy period distribution, reliability functions and other performance measures.

In many queueing scenarios, the facility of optional services can be noticed apart from regular essential service. The example of such type of services can be seen in clinical laboratories, production management, automobile repair stations, etc. In the queueing literature, a few researchers including Medhi [23], Wang [35], Choudhury and Deka [5] worked on this aspect under varied assumptions. Bulk queue with multiphase services was discussed by Jain and Bhagat [15] by considering the features of retrial, unreliable server and modified vacation policy. Recently, Jain *et al.* [31] explored the bulk arrival queue with optional service by incorporating the balking behaviour of customers, unreliable server and multiphase repairs. In this work, they have derived the system state probabilities and distributions at steady state using supplementary variable technique. The server breakdown is a common phenomenon in queueing systems and broken server needs the immediate repair to resume the services without any halt. However, sometimes failed server has to wait for the repair facility due to unavailability of spare parts or due to any other technical reason. Some queueing theorists have done research works in different frameworks by considering unreliable server [8, 21]. $M^X/G/1$ queueing system with reneging having unreliable server, and multi optional services was analyzed by Jain and Bhagat [16] by employing SVT. The balking behaviour in the arriving units may also emerged due to sudden failure of system; this concept was incorporated in the investigation done by Singh *et al.* [32] for the study of queueing system with bulk input, Bernoulli feedback, breakdown and multiphase repairs. A queueing model with server breakdown and balking for $M^X/G(a,b)/1$ queue with Bernoulli vacation schedule under multiple vacation policy was investigated by Govindan and Marimuthu [14].

In many complex queueing situations, the derivation of performance measures in explicit form is a challenging task by applying analytical methods. In aforesaid circumstances the probability distribution, waiting time and other metrics can be evaluated by applying maximum entropy principle (MEP). Following the preamble research work done by Shannon [26], diversified studies on queueing models were accomplished by various researchers [28, 36] *via* maximum entropy principle. Recently, Singh *et al.* [33] developed the $M^X/G/1$ model under randomised vacation policy and used MEP to facilitate the comparison of analytical and approximate results for the waiting time.

Queueing models under individual or a few of the realistic concepts *viz.* unreliable server, retrial, balking, Bernoulli feedback, phase service and vacation are discussed by some researchers. However, there is need of research work by combining all these features together due to applications of such models in real time systems wherein these features are prevalent. In order to fill up this research gap, our proposed model is devoted to the study of $M^X/G/1$ retrial queueing model under the realistic concepts of (i) server breakdown, (ii) Bernoulli feedback, (iii) optional service, (iv) Bernoulli vacation, (v) balking, (vi) delay repair (vii) multiphase repair and (viii) service/vacation/delay/repair processes governed by i.i.d. general distributions. The remaining paper is presented as follows. In Section 2, we have outlined the assumptions and notations to develop present $M^X/G/1$ model. Governing equations and boundary conditions are framed in Section 3. The PGF's and stability condition are specified in Section 3. Section 4 is contributed for the evaluation of analytical metrics for the steady state queue size distributions. The expressions for steady state probabilities, mean system size, reliability indices and total cost are evaluated in Section 5. The formulation of MEP results is done in Section 6. Section 7 is concerned with special cases deduced by fixing specific parameters. Section 8 is devoted to numerical results. The last section provides the conclusion and future scope of the work done.

2. Model description

The batch arrival retrial queue by taking into consideration the different attributes namely balking, vacation, Bernoulli feedback and multiphase optional services, etc. The server may encounter a sudden breakdown due to any unexpected cause and instantly joins the repair station, where repair is completed in multi-phases after a random delay. To formulate the model mathematically, underlying assumptions and notations are described as follows.

2.1. Assumptions

The batch arrival queueing system with optional phase service/repair, Bernoulli feedback and Bernoulli vacation is investigated. Moreover, the discouragement factor of the units is considered while designing the model. We outline the basic assumptions which are used to formulate $M^X/G/1$ model mathematically to evaluate various system indices are as follows.

(i) *Arrival process and balking.* The units arrive at the system in bulk according to Poisson process in random batch size X with arrival rate λ; the probability mass function X is $c_j = \text{Prob.}[X = j]$ with $c, c_{(2)}$ as first and second factorial moments. Balking may take place in units while joining the system during different states of server. The joining probabilities of the units are b, b_1, b_2 and b_3 in different states of the server *viz.* busy, delayed repair, under repair and in vacation states, respectively.

(ii) *Retrial and Bernoulli feedback.* There is provision of waiting in the retrial orbit for incoming units; if units find the server unavailable for the service, they can join the retrial orbit and repeat their attempt of getting service at later stage. After getting the essential/optional service, if the unit is unsatisfied from its service, then it can rejoin the original queue as a feedback unit for receiving one more regular service and subsequently optional service with probability $\theta(0 \le \theta \le 1)$; or exit from the system with probability $(1-\theta)$.

(iii) *Service process.* The arriving units are served by the server for two types of services; the first type of service is essential for all the units joining the system and after receiving essential service, the units can demand for second type of optional services from total available l-optional services and take dth $(d = 1, 2, \ldots, l)$ phase optional service with probability r_d; or choose to depart from the system with probability $r_0 = 1 - \sum_{d=1}^{l} r_d$.

(iv) *Server vacation.* The server is eligible to avail the vacation after each busy period under Bernoulli vacation schedule *i.e.* the server can take vacation with probability p after each service completion epoch or may continue to serve next unit with probability $1-p$.

(v) *Delay in repair and repair processes.* The server is prone to failure as such considered to be unreliable and may breakdown in Poisson pattern while rendering any of the essential or optional services with rates α_d, $(0 \le d \le l)$. Due to breakdown, the server becomes unable to provide service and needs urgent repair; but sometimes repair facility is not immediately available as such the server has to wait for repair *i.e.* delay in repair occurs. In order to recover the failed system, the repair is accomplished in m-phases.

(vi) The retrial duration, essential as well as optional phase service times, delay to repair and repair times, vacation times are i.i.d. and general distributed.

Notations used for different distributions *viz.* bulk arrival, retrial time and general i.i.d distributed dth $(d = 0, 1, 2, \ldots, l)$ phase service, delay to repair and kth phase repair while failed during dth phase service are expressed as follows:

Notations

$N(t)$	Number of units in retrial orbit.
$M(x)$	CDF of retrial time.
$B_d(x)$	The CDF of dth $(d = 0, 1, 2, \ldots, l)$ phase service time, with first two moments as β_d and $\beta_d^{(2)}$; $d = 0$ denotes the essential phase whereas $d = 1, 2, \ldots, l$ represent optional phase services.
$D_d(y), G_{d,k}(y)$	The CDF of delay in repair time and repair time, respectively of server failed in dth phase of service $(0 \le d \le l, 1 \le k \le m)$.
$\gamma_d, \gamma_d^{(2)}$	First and second moments of $D_d(y), (d = 0, 1, 2, \ldots, l)$
$g_{dk}, g_{dk}^{(2)}$	First and second moments of $G_{d,k}(y)(0 \le d \le l, 1 \le k \le m)$.
$V(x)$	The vacation time CDF.
$v, v^{(2)}$	First two moments of $V(x)$.

P_0	Prob. that the server is idle and system is empty.
$A_n(x)\mathrm{d}x$	Prob. that there are n units in the system when server is idle and elapsed retrial time lies in $(x, x+\mathrm{d}x], n \in W_\infty$.
$P_n^0(x)\mathrm{d}x(P_n^d(x)\mathrm{d}x)$	Prob. that there are n units in the system when server is busy in implementing first phase essential service (dth phase optional service) and elapsed service time lies in $(x, x+\mathrm{d}x], n \in W_\infty$, $d \in W_l$.
$D_n^0(x,y)\mathrm{d}y(D_n^d(x,y)\mathrm{d}y)$	Prob. that there are n units in the system when server fails in first phase essential service (dth phase optional service), waiting for repair and elapsed delay time in repair lies in $(y, y+\mathrm{d}y]$, $n \in W_\infty, d \in W_l$.
$R_{k,n}^0(x,y)\mathrm{d}y(R_{k,n}^d(x,y)\mathrm{d}y)$	Prob. that there are n units in the system when server fails in first phase essential service (dth phase optional service), under repair in kth repair phase and elapsed repair time lies in $(y, y+\mathrm{d}y]$, $n \in W_\infty, d \in W_l, k \in W_m$.
$V_n(y)\mathrm{d}y$	Prob. that there are n units in the system when server is on vacation and elapsed vacation time lies in $(y, y+\mathrm{d}y], n \in W_\infty$.
π_j	Probability of j units being in the queue at service completion epoch.
W_J	The set containing numbers $\{1, 2, 3, \ldots, J\}$.

For solving the model, we define the following probability generating functions (PGFs):

$$P^d(x,z) = \sum_{n\geq 1} z^n P_n^d(x), A(x,z) = \sum_{n\geq 1} z^n A_n(x) (\text{also valid for } x=0)$$
$$D^d(x,y,z) = \sum_{n\geq 1} z^n D_n^d(x,y), R_k^d(x,y,z) = \sum_{n\geq 1} z^n R_{k,n}^d(x,y), V(y,z) = \sum_{n\geq 1} z^n V_n(y) (\text{also valid for } y=0).$$

We use $\widetilde{F}(s)$ as Laplace transform of any CDF $F(u)$ and also $\overline{\widetilde{F}(s)} = 1 - \widetilde{F}(s)$.

Also, $f(u)\mathrm{d}u = dF(u)(\overline{F(u)})^{-1}$ is hazard rate function of any CDF $F(u)$ (Here $f(\cdot)$ can take values $\mu_d(\cdot), K(\cdot), v(\cdot), \eta_d(\cdot), \xi_{d,k}(\cdot)$ with corresponding $F(\cdot)$ as $B_d(\cdot), M(\cdot), V(\cdot), D_d(\cdot), G_{d,k}(\cdot)$) with $(0 \leq d \leq l, 1 \leq k \leq m)$.

3. Governing equations

In order to develop the model based on different assumptions and notations as described in Section 2, the governing system state equations, boundary conditions and normalizing condition are framed using supplementary variables corresponding to elapsed time of general distributed processes.

3.1. Governing equations

$$\lambda P_0 = \bar{p}\left[\bar{\theta}\left(r_0 \int_0^\infty \mu_0(x) P_1^0(x)\mathrm{d}x + \sum_{d=1}^{l} \int_0^\infty \mu_i(x) P_1^d(x)\mathrm{d}x\right)\right] + \int_0^\infty \nu(y) V_1(y)\mathrm{d}y \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} A_n(x) + [\lambda b + K(x)] A_n(x) = 0, \quad n \in W_\infty, x > 0 \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n^d(x) + [\lambda b + \alpha_d + \mu_d(x)] P_n^d(x) = \lambda b \sum_{j=1}^{n} c_j P_{n-j}^d(x) + \int_0^\infty \xi_{d,m}(y) R_{m,n}^d(x,y)\mathrm{d}y, \quad n \in W_\infty;$$
$$d \in W_l \cup \{0\}, x, y > 0 \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}D_n^d(x,y)+[\lambda b_1+\eta_d(y)]D_n^d(x,y)=\lambda b_1\sum_{j=1}^{n}c_jD_{n-j}^d(x,y),\quad n\in W_\infty, d\in W_l\cup\{0\}, x,y>0 \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}R_{k,n}^d(x,y)+[\lambda b_2+\xi_{d,k}(y)]R_{k,n}^d(x,y)=\lambda b_2\sum_{j=1}^{n}c_jR_{k,n-j}^d(x,y),\quad n\in W_\infty, k\in W_m, d\in W_l\cup\{0\}, x,y>0 \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}V_n(y)+[\lambda b_3+v(y)]V_n(y)=\lambda b_3\sum_{j=1}^{n}c_jV_{n-j}(y),\quad n\in W_\infty, y>0. \tag{3.6}$$

3.2. Boundary conditions

$$\begin{aligned}A_n(0)=&\int_0^\infty v(y)V_{n+1}(y)\mathrm{d}y+\bar{p}\left[\bar{\theta}\left(r_0\int_0^\infty\mu_0(x)P_{n+1}^0(x)\mathrm{d}x+\sum_{d=1}^{l}\int_0^\infty\mu_d(x)P_{n+1}^d(x)\mathrm{d}x\right)\right.\\&\left.+\ \theta\left(r_0\int_0^\infty\mu_0(x)P_n^0(x)\mathrm{d}x+\sum_{d=1}^{l}\int_0^\infty\mu_d(x)P_n^d(x)\mathrm{d}x\right)\right],\quad n\in W_\infty\end{aligned} \tag{3.7}$$

$$P_n^0(0)=\lambda c_nP_0+\lambda b\sum_{j=1}^{n}c_j\int_0^\infty A_{n-j}(x)\mathrm{d}x+\int_0^\infty k(x)A_n(x)\mathrm{d}x,\quad n\in W_\infty \tag{3.8}$$

$$P_n^d(0)=r_d\int_0^\infty\mu_d(x)P_n^d(x)\mathrm{d}x,\quad n\in W_\infty, d\in W_l \tag{3.9}$$

$$D_n^d(x,0)=\alpha_dP_n^d(x),\quad n\in W_\infty, d\in W_l\cup\{0\} \tag{3.10}$$

$$R_{1,n}^d(x,0)=\int_0^\infty\eta_d(y)D_n^d(x,y)\mathrm{d}y,\quad n\in W_\infty, d\in W_l\cup\{0\} \tag{3.11}$$

$$R_{k,n}^d(x,0)=\int_0^\infty\xi_{d,k-1}(y)R_{k-1,n}^d(x,y)\mathrm{d}y,\quad n\in W_\infty, d\in W_l\cup\{0\}, k\in\{2,3,\ldots,m\} \tag{3.12}$$

$$\begin{aligned}V_n(0)=p&\left[\bar{\theta}\left(r_0\int_0^\infty\mu_0(x)P_n^0(x)\mathrm{d}x+\sum_{d=1}^{l}\int_0^\infty\mu_d(x)P_n^d(x)\mathrm{d}x\right)\right.\\&\left.+\ (1-\delta_{n,1})\ \theta\left(r_0\int_0^\infty\mu_0(x)P_{n-1}^0(x)\mathrm{d}x+\sum_{d=1}^{l}\int_0^\infty\mu_d(x)P_{n-1}^d(x)\mathrm{d}x\right)\right],\quad n\in W_\infty.\end{aligned} \tag{3.13}$$

3.3. Normalizing condition

$$\begin{aligned}P_0+\sum_{d=0}^{l}\sum_{n\geq1}&\left[\int_0^\infty P_n^d(x)\mathrm{d}x+\int_0^\infty\int_0^\infty D_n^d(x,y)\mathrm{d}x\,\mathrm{d}y+\int_0^\infty\int_0^\infty\sum_{k=1}^{m}R_{k,n}^d(x,y)\mathrm{d}x\,\mathrm{d}y\right]\\+&\sum_{n\geq1}\int_0^\infty A_n(x)\mathrm{d}x+\sum_{n\geq1}\int_0^\infty V_n(y)\mathrm{d}y=1.\end{aligned} \tag{3.14}$$

4. Queue size distribution

In this section, we establish the stability condition. The PGFs for the queue size distribution under the stability condition are also determined.

4.1. Stability condition

Lemma 4.1. *At equilibrium state, stability condition of system is expressed as*

$$\kappa_1 + \theta + c\overline{\widetilde{M}(\lambda b)} < 1. \tag{4.1}$$

Proof. The above stability condition is attained by following Takagi [34]. □

4.2. Joint and marginal distributions

Lemma 4.2. *The PGFs for the joint distributions of server states are as follows:*

$$A(x,z) = \lambda b \varepsilon_1 (\overline{M(x)}) e^{-\lambda b x} E_1(z) \tag{4.2}$$

$$P^d(x,z) = U_d(x,z), \quad d \in W_l \cup \{0\} \tag{4.3}$$

$$V(y,z) = pE_2(z)\overline{V(y)} \exp\{-\phi_5(z)y\} \tag{4.4}$$

$$D^d(x,y,z) = \alpha_d U_d(x,z)\overline{D_d(y)} \exp\{-\phi_3(z)y\}, \quad d \in W_l \cup \{0\} \tag{4.5}$$

$$R_1^d(x,y,z) = \alpha_d U_d(x,z)\widetilde{D}_0(\phi_3(z))\overline{G_{d,1}(y)} \exp\{-\phi_4(z)y\}, \quad d \in W_l \cup \{0\} \tag{4.6}$$

$$R_k^d(x,y,z) = \alpha_d U_d(x,z)\widetilde{D}_0(\phi_3(z))\overline{G_{d,1}(y)} \exp(-\phi_4(z)y)\widetilde{D}_0(\phi_3(z)) \prod_{j=1}^{k-1} \widetilde{G}_{d,j}(\phi_4(z)),$$
$$d \in W_l \cup \{0\}, k \in \{2,3,\dots m\}, \tag{4.7}$$

where

$$\overline{F(u)} = 1 - F(u), \text{ for any CDF } F(u)$$

$$P_0 = b\varepsilon_1[\varepsilon_2]^{-1}, \phi_1(z) = \lambda\overline{X(z)}, \phi_2(z) = b\phi_1(z), \phi_h(z) = b_{h-2}\phi_1(z); h = 3,4,5$$

$$U_d(x,z) = \begin{cases} \Omega(z)\overline{B_0(x)} \exp\{-\tau_0(z)x\}, & d = 0 \\ r_d\Omega(z)\widetilde{B}_0(\tau_0(z)\overline{B_d(x)} \exp\{-\tau_d(z)x\}, & d \in \{1,2,3,4,\dots,l\} \end{cases}$$

$$\Omega(z) = zb\varepsilon_1\phi_1(z)\widetilde{M}(\lambda b)[\varepsilon_2 S(z)]^{-1}, Y(z) = (\theta z + \bar{\theta})\widetilde{B}_0(\tau_0(z)) \left\{ r_0 + \sum_{d=1}^{l} r_d \widetilde{B}_d(\tau_d(z)) \right\}$$

$$E_1(z) = [z - E_3(z)X(z)][\varepsilon_2 S(z)]^{-1}, E_2(z) = \Omega(z)Y(z), E_3(z) = Y(z)\left\{\bar{p} + p\widetilde{V}(\phi_5(z))\right\}$$

$$\tau_d(z) = \phi_2(z) + \alpha_d \left(1 - \widetilde{D}_d(\phi_3(z)) \prod_{j=1}^{m} \widetilde{G}_{d,j}(\phi_4(z))\right), \quad d = 0,1,2\dots,l$$

$$\chi_1 = \overline{\widetilde{M}(\lambda b)}\left[\bar{b}c - 1\right], S(z) = Y(z)\left\{\bar{p} + p\widetilde{V}(\phi_5(z))\right\}\left[\widetilde{M}(\lambda b) + X(z)\overline{\widetilde{M}(\lambda b)}\right] - z$$

$$\psi_d = \left(b + \alpha_d\left(b_1\gamma_d + b_2\sum_{j=1}^{m} g_{dj}\right)\right), \varphi_d = \left(1 + \alpha_d\left(\gamma_d + \sum_{j=1}^{m} g_{dj}\right)\right); \quad 0 \le d \le l$$

$$\kappa_1 = \lambda c\left[\beta_0\psi_0 + \sum_{d=1}^{l} r_d\beta_d\psi_d + pvb_3\right], \kappa_2 = \lambda c\left[\beta_0(b\varphi_0 - \psi_0) + \sum_{d=1}^{l} r_d\beta_d(b\varphi_d - \psi_d) + (b - b_3)pv\right]$$

$$\varepsilon_1 = \left[\bar{\theta} - \kappa_1 - c\overline{\widetilde{M}(\lambda b)}\right], \varepsilon_2 = \left[b\bar{\theta} + \chi_1 + \bar{b}\kappa_1 + \widetilde{M}(\lambda b)\kappa_2\right].$$

Proof. Results given in equations (4.2–4.7) can be evaluated by multiplying governing equations (3.1–3.6) by permissible power of z and further solving along with boundary conditions (3.7–3.13) and normalizing condition (3.14). (For detailed proof, see Appendix A). □

Lemma 4.3. *The PGFs for the marginal distributions of server states are*

$$A(z) = \varepsilon_1\overline{\widetilde{M}(\lambda b)}E_1(z) \tag{4.8}$$

$$P^d(z) = Q_d(z), \quad 0 \le d \le l \tag{4.9}$$

$$V(z) = \left[E_2(z)p\overline{\widetilde{V}(\phi_5(z))}\right][\phi_5(z)]^{-1} \tag{4.10}$$

$$D^d(z) = \left[\alpha_d Q_d(z)\overline{\widetilde{D}_d(\phi_3(z))}\right][\phi_3(z)]^{-1}, \quad 0 \le d \le l \tag{4.11}$$

$$R_1^d(z) = \left[\alpha_d Q_d(z)\widetilde{D}_d(\phi_3(z))\overline{\widetilde{G}_{d,1}(\phi_4(z))}\right][\phi_4(z)]^{-1}, \quad 0 \le d \le l \tag{4.12}$$

$$R_k^d(z) = \left[\alpha_d Q_d(z)\overline{\widetilde{G}_{d,k}(\phi_4(z))}\widetilde{D}_d(\phi_3(z))\prod_{j=1}^{k-1}\widetilde{G}_{d,j}(\phi_4(z))\right][\phi_4(z)]^{-1}, \quad 2 \le k \le m; 0 \le d \le l \tag{4.13}$$

where

$$Q_d(z) = \begin{cases} \Omega(z)\overline{\widetilde{B}_0(\tau_0(z)}(\tau_0(z))^{-1}; & d = 0 \\ r_d\Omega(z)\widetilde{B}_0(\tau_0(z)\overline{\widetilde{B}_d(\tau_d(z)}(\tau_d(z))^{-1}; & d \in \{1,2,3,4,\ldots,l\}. \end{cases}$$

Proof. Utilizing the relation $\int_0^\infty e(1-F(u))\mathrm{d}u = [1-\widetilde{F}(s)][s]$, the results given in (4.8–4.13) are obtained by integrating equations (4.2–4.7) w.r.t. relevant variables. □

4.3. Stationary queue length distribution

Theorem 4.4. *The PGF for stationary queue length distribution at service completion epoch is*

$$\pi(z) = \left[\varepsilon_1\overline{X(z)}E_3(z)\right][cS(z)]^{-1}. \tag{4.14}$$

Proof. Using the probability π_j, we obtain equation (5.1) as follows

$$\pi_j = K_0\left[\int_0^\infty \nu(y)V_{j+1}(y)\mathrm{d}y + \bar{p}\left\{\bar{\theta}\left(r_0\int_0^\infty \mu_0(x)P_{j+1}^0(x)\mathrm{d}x + \sum_{d=1}^{l}\int_0^\infty \mu_d(x)P_{j+1}^d(x)\mathrm{d}x\right)\right.\right.$$
$$\left.\left. + \ \theta(1-\delta_{j,0})\left(r_0\int_0^\infty \mu_0(x)P_j^0(x)\mathrm{d}x + \sum_{d=1}^{l}\int_0^\infty \mu_d(x)P_j^d(x)\mathrm{d}x\right)\right\}\right]; \quad j \ge 0. \tag{4.15}$$

with normalizing constant K_0.

By using $\pi(z) = \sum_{j\ge0}\pi_j z^j$, equation (4.15) yields

$$\pi(z) = \left[K_0 b\varepsilon_1\phi_1(z)\widetilde{M}(\lambda b)E_3(z)\right][\varepsilon_2]^{-1}. \tag{4.16}$$

The normalizing condition $\pi(1) = 1$ gives

$$K_0 = \varepsilon_2[\lambda cb\widetilde{M}(\lambda b)]^{-1}. \tag{4.17}$$

Using equations (4.16) and (4.17), we acquire the result given in equation (4.14). □

Theorem 4.5. *The PGF for stationary queue length distribution at departure epoch is*

$$\omega(z) = \left[\varepsilon_1 \overline{X(z)} E_3(z)\right] \left[c(\bar{\theta} + \theta z) S(z)\right]^{-1}. \tag{4.18}$$

Proof. Utilizing the relation $\omega(z) = \pi(z)(\bar{\theta} + \theta z)^{-1}$, we obtain the above expression. □

Theorem 4.6. *At arbitrary epoch, the PGF's for the system size and orbit size, respectively are*

(i)
$$P(z) = \varepsilon_1 \left\{ \overline{\widetilde{M}(\lambda b)} E_1(z) + b\varepsilon_2^{-1} \right\} + p\overline{\widetilde{V}(\phi_5(z))} E_2(z)[\phi_5(z)]^{-1} + \sum_{d=0}^{l} Q_d(z)\{\Lambda_d(z) + (z-1)\} \tag{4.19}$$

(ii)
$$O(z) = \varepsilon_1 \left\{ \overline{\widetilde{M}(\lambda b)} E_1(z) + p\overline{\widetilde{V}(\phi_5(z))} E_2(z)[\phi_5(z)]^{-1} + b\varepsilon_2^{-1} \right\} + \sum_{d=0}^{l} Q_d(z)\Lambda_d(z), \tag{4.20}$$

with

$$\Lambda_d(z) = \left\{ 1 + \alpha_d \left(\overline{\widetilde{D}_d(\phi_3(z))} (\phi_3(z))^{-1} + \widetilde{D}_d(\phi_3(z))(\phi_4(z))^{-1} \left(1 - \prod_{j=1}^{m} \widetilde{G}_{d,j}(\phi_4(z)) \right) \right) \right\}; \quad 0 \le d \le l.$$

Proof. We have

$$P(z) = P_0 + A(z) + z \sum_{d=0}^{l} \left\{ P^d(z) + D^d(z) + \sum_{k=1}^{m} R_k^d(z) \right\} + V(z) \tag{4.21}$$

$$O(z) = P_0 + A(z) + \sum_{d=0}^{l} \left\{ P^d(z) + D^d(z) + \sum_{k=1}^{m} R_k^d(z) \right\} + V(z). \tag{4.22}$$

Utilizing the above relations (4.21) and (4.22), we get the results given by equations (4.19) and (4.20). □

5. Performance measures

In this segment, various performance metrics of the concerned queueing model are acquired as follows.

5.1. Steady state probabilities

Long run probabilities of the model are primary features to analyze the behaviour of the system. The server state probabilities of being in idle, accumulation, busy, delay in repair, under repair and vacation states are denoted by $P(I), P(N), P(B_d), P(D_d), P(R_d)$ and $P(V)$, respectively.

Theorem 5.1. *The steady state probabilities at distinct server's status are as follows:*

(i) *The server is idle and system is empty.*

$$P(I) = (\bar{\theta})^{-1}(\bar{\theta} - \rho) - \chi_3 = E(I)[E(C)]^{-1}. \tag{5.1}$$

(ii) *The server is in accumulation state.*

$$P(N) = \chi_3. \tag{5.2}$$

(iii) *The server is in busy state while rendering essential service.*

$$P(B_0) = \chi_2 \beta_0 = E(B_0)[E(C)]^{-1}. \tag{5.3}$$

(iv) *The server is in busy state while rendering* dth *phase optional service.*

$$P(B_d) = r_d\chi_2\beta_d = E(B_d)[E(C)]^{-1}, \quad d \in W_l. \tag{5.4}$$

(v) *The server is in busy state.*

$$P(B) = \chi_2\left(\beta_0 + \sum_{d=1}^{l} r_d\beta_d\right). \tag{5.5}$$

(vi) *The server is in vacation state.*

$$P(V) = p\chi_2 v = E(V)[E(C)]^{-1}. \tag{5.6}$$

(vii) *The server is waiting for repair when failed while rendering essential service.*

$$P(D_0) = \alpha_0\chi_2\gamma_0\beta_0 = E(D_0)[E(C)]^{-1}. \tag{5.7}$$

(viii) *The server is waiting for repair when failed while rendering* dth *phase optional service.*

$$P(D_d) = \alpha_d r_d\chi_2\gamma_d\beta_d = E(D_d)[E(C)]^{-1}, \quad d \in W_l. \tag{5.8}$$

(ix) *The server is waiting for repair.*

$$P(D) = \chi_2\left(\alpha_0\beta_0\gamma_0 + \sum_{d=1}^{l}\alpha_d r_d\gamma_d\beta_d\right). \tag{5.9}$$

(x) *The server is under* kth *phase repair when failed while rendering essential service.*

$$P(R_k^0) = \alpha_0\chi_2 g_{0k}\beta_0 = E(R_{0k})[E(C)]^{-1}, \quad k \in W_m. \tag{5.10}$$

(xi) *The server is under* kth *phase repair when failed while rendering* dth *phase optional service.*

$$P(R_k^d) = \alpha_d\chi_2 r_d g_{dk}\beta_d = E(R_{dk})[E(C)]^{-1}, \quad d \in W_l, k \in W_m. \tag{5.11}$$

(xii) *The server is under repair.*

$$P(R) = \sum_{k=1}^{m}\chi_2\left(\alpha_d g_{0k}\beta_0\gamma_0 + \sum_{d=1}^{l}\alpha_d r_d g_{dk}\beta_d\right), \tag{5.12}$$

where

$$\lambda_e = \left[\lambda b\bar{\theta}\widetilde{M}(\lambda b)\right]\left[b\bar{\theta} + \chi_1 + \bar{b}\kappa_1 + \widetilde{M}(\lambda b)\kappa_2\right]^{-1}, \chi_2 = \lambda_e c(\bar{\theta})^{-1}$$

$$\chi_3 = \left\{\lambda_e[\kappa_1 + \theta + c - 1]\overline{\widetilde{M}(\lambda b)}\right\}\left[\lambda b\bar{\theta}\widetilde{M}(\lambda b)\right]^{-1}, \rho = \lambda_e c\left[\beta_0\varphi_0 + \sum_{d=1}^{l} r_d\beta_d\varphi_d + pv\right]$$

$$[E(C)]^{-1} = \lambda_e c\left[(\bar{\theta})^{-1}(\bar{\theta} - \rho) - \chi_3\right].$$

Proof. Equations (5.2–5.4), (5.6–5.8), (5.10) and (5.11) are evaluated by taking limit $z = 1$ in equations (4.8–4.13). Also, equation (5.1) is obtained on utilizing the expression given by

$$P(I) = 1 - \left[\sum_{d=0}^{l}\left[P(B_d) + P(D_d) + \sum_{k=1}^{m} P(R^d)\right] + P(V) + P(N)\right].$$

Now, (5.5), (5.9) and (5.12) are obtained using following relations:

$$P(B)=\sum_{d=0}^{l}[P(B_d)],P(D)=\sum_{d=0}^{l}[P(D_d)],P(R)=\sum_{k=1}^{m}\sum_{d=0}^{l}[P(R_k^d)].$$

Note that λ_e is the effective arrival rate and is determined by using

$$\lambda_e=\lambda P_0+\lambda b\left(\sum_{d=0}^{l}P^d(1)+A(1)\right)+\lambda b_1\left(\sum_{d=0}^{l}D^d(1)\right)+\lambda b_2\sum_{k=1}^{m}\left(\sum_{d=0}^{l}R_k^d(1)\right)+\lambda b_3V(1).$$

Also, estimated cycle length is $E(C)=E(I)+E(H)$ with $E(I)=(\lambda_e c)^{-1},E(H)=\sum_{d=0}^{l}[E(B_d)+E(D_d)+E(R_d)]+E(V)$. □

5.2. Mean system size

Theorem 5.2. *The mean queue length* (L_{dep}) *at departure epoch is*

$$L_{\text{dep}}=\kappa_1+c_{(2)}(2c)^{-1}+S''(1)(2\varepsilon_1)^{-1}, \tag{5.13}$$

where

$$\begin{aligned}S''(1)=&\Bigg[2\theta\left(\kappa_1+c\overline{\widetilde{M}(\lambda b)}\right)+c_{(2)}\overline{\widetilde{M}(\lambda b)}+\sum_{d=1}^{l}r_d\Big\{2\beta_0\beta_d(\lambda c)^2\psi_0\psi_d-\beta_d\tau_d''(1)+2pvb_3\beta_dc^2\psi_d\\&+2\beta_d\lambda(c)^2\psi_d\overline{\widetilde{M}(\lambda b)}\Big\}-\beta_0\tau_0''(1)+2\beta_0\lambda c^2\psi_0\overline{\widetilde{M}(\lambda b)}+pv^{(2)}(\lambda b_3c)^2\\&+pv\lambda b_3\left\{c_{(2)}+2\lambda\beta_0c^2\psi_0+2c^2\overline{\widetilde{M}(\lambda b)}\right\}+\sum_{d=1}^{l}r_d\beta_d^{(2)}\times\{\psi_d\lambda c\}^2+\beta_0^{(2)}\{\psi_0\lambda c\}^2\Bigg]\end{aligned}$$

$$\psi_0=\left(b+\alpha_0\left(b_1\gamma_0+b_2\sum_{j=1}^{m}g_{0j}\right)\right);\psi_d=\left(b+\alpha_d\left(b_1\gamma_d+b_2\sum_{j=1}^{m}g_{dj}\right)\right);\quad 1\le d\le l$$

$$\begin{aligned}\tau_d''(1)=&-\Bigg\{\lambda bc_{(2)}+\alpha_d\Bigg(2b_1b_2(\lambda c)^2\gamma_d\sum_{j=1}^{m}(g_{dj})+2(\lambda b_2c)^2\sum_{k=2}^{m}\sum_{j=1}^{k-1}(g_{dj}g_{dk})\\&+\left[\lambda b_1c_{(2)}\gamma_d+(\lambda b_1c)^2\gamma_d^{(2)}\right]+\sum_{j=1}^{m}\left[\lambda b_2c_{(2)}g_{dj}+(\lambda b_2c)^2g_{dj}^{(2)}\right]\Bigg)\Bigg\};\quad 0\le d\le l.\end{aligned}$$

Proof. The expression (5.13) is obtained by using $L_{\text{dep}}=\left(\frac{\mathrm{d}\omega(z)}{\mathrm{d}z}\right)_{z=1}$. □

Theorem 5.3. *The mean system size* (L_q) *at arbitrary epoch is*

$$\begin{aligned}L_q=\varepsilon_1(\varepsilon_2)^{-1}&\Bigg[\overline{\widetilde{M}(\lambda b)}\{-\varepsilon_1\chi_4-(\bar{\theta}-\kappa_1-c)S''(1)\}[2\varepsilon_1^2]^{-1}+\chi_2\chi_7+b\widetilde{M}(\lambda b)\\&\times\left\{\lambda c\left(M_0+\sum_{d=1}^{l}\{r_d\lambda c\beta_0\beta_d\psi_0\varphi_d+M_d\}\right)(\varepsilon_1)^{-1}+\chi_6\times\left(\varepsilon_1(\lambda c_{(2)}+2\lambda c)+\lambda cS''(1)\right)[2\varepsilon_1^2]^{-1}\right\}\\&+bp\widetilde{M}(\lambda b)b_3^{-1}\{-\varepsilon_1\chi_5+\lambda b_3vcS''(1)\}[2\varepsilon_1^2]^{-1}\Bigg],\end{aligned}\tag{5.14}$$

where

$$\chi_4 = -\left(S''(1) + \widetilde{M}(\lambda b)\left\{c_{(2)} + 2\sum_{d=1}^{l} r_d\beta_d\lambda c^2\psi_d + 2\beta_0\lambda c^2\psi_0 + 2pv\lambda b_3c^2 + 2\theta c\right\}\right)$$

$$M_d = \lambda c(2^{-1})\left(\beta_d\alpha_d\left\{b_1\gamma_d + b_2\left(\sum_{i=1}^{m} g_{di} + 2\sum_{k=2}^{m}\sum_{i=1}^{k-1} g_{di}g_{dk}\right) + 2\sum_{i=1}^{m} g_{di}\gamma_d b_1\right\} + \beta_d^{(2)}\psi_d\varphi_d\right); \quad 0 \le d \le l$$

$$\chi_5 = -\left[2(\theta+1)\lambda b_3 cv + 2b_3v(\lambda c)^2\left(\beta_0\psi_0 + \sum_{d=1}^{l} r_d\beta_d\psi_d\right) + v^{(2)}(\lambda b_3 c)^2 + v\lambda b_3 c_{(2)}\right]$$

$$\chi_6 = \left(\beta_0\psi_0 + \sum_{d=1}^{l} r_d\beta_d\psi_d\right), \chi_7 = \left(\beta_0 + \sum_{d=1}^{l} r_d\beta_d\right).$$

Proof. L_q can be obtained by using $L_q = \left(\frac{\mathrm{d}P(z)}{\mathrm{d}z}\right)_{z=1}$. □

Theorem 5.4. *The mean orbit size (L_o) at arbitrary epoch is*

$$L_o = \varepsilon_1(\varepsilon_2)^{-1}\Bigg[\overline{\widetilde{M}(\lambda b)}\{-\varepsilon_1\chi_4 - (\bar{\theta} - \kappa_1 - c)S''(1)\}[2\varepsilon_1^2]^{-1} + b\widetilde{M}(\lambda b)\Bigg\{\lambda c\Bigg(M_0 + \sum_{d=1}^{l}\{r_d\lambda c\beta_0\beta_d\psi_0\varphi_d + M_d\}\Bigg)(\varepsilon_1)^{-1} + \chi_6 \times \left(\varepsilon_1(\lambda c_{(2)} + 2\lambda c) + \lambda cS''(1)\right)[2\varepsilon_1^2]^{-1}\Bigg\} + bp\widetilde{M}(\lambda b)b_3^{-1}\{-\varepsilon_1\chi_5 + \lambda b_3vcS''(1)\}[2\varepsilon_1^2]^{-1}\Bigg]. \tag{5.15}$$

Proof. Using the expression $L_o = \left(\frac{\mathrm{d}O(z)}{Oz}\right)_{z=1}$, we get result given in equation (5.15). □

Remark. Using Little's formula, we get mean waiting time

(i) at departure epoch $W_{\text{dep}} = L_{\text{dep}}(\lambda_e c)^{-1}$ (5.16)

(ii) at arbitrary epoch $W_q = L_q(\lambda_e c)^{-1}$. (5.17)

5.3. Reliability indices

Reliability indices for unreliable server queueing system can be used for the estimation availability and failure frequency of the system. These indices facilitate the imperative features required for the improved designing and upgrading of the system. Reliability indices under the steady state conditions are obtained as follows:

(i) Availability

Theorem 5.5. *The availability of the server is*

$$A_v = b[\varepsilon_1 + \lambda c\widetilde{M}(\lambda b)\chi_7][\varepsilon_2]^{-1}. \tag{5.18}$$

Proof. The availability given in equation (5.18) is obtained by using the following relation:

$$A_v = P_0 + \sum_{d=0}^{l}\int_0^\infty P^d(x,1)\,\mathrm{d}x = P_0 + \lim_{z\to 1}\left[\sum_{d=0}^{l} P^d(z)\right].$$

□

(ii) Failure frequency

Theorem 5.6. *The failure frequency of the server is*

$$F_f = \chi_2 \left\{ \alpha_0 \beta_0 + \sum_{d=1}^{l} \alpha_d r_d \beta_d \right\}. \tag{5.19}$$

Proof. The equation (5.19) can be obtained by using

$$F_f = \sum_{d=0}^{l} \alpha_d \int_0^\infty P^d(x,1) = \lim_{z\to 1} \left[\sum_{d=0}^{l} \alpha_d P^d(z) \right].$$

□

5.4. Cost analysis

Cost function per unit time can be expressed in terms of the following cost elements incurred per unit time on different activities:

C_h	Holding cost of unit joining the system.
C_S	Start up cost.
C_{B_d}	Cost incurred on the server while rendering dth phase.
$C_{D_d}(C_{R_d})$	Cost incurred on server when waiting for repair/under repair in case when the server breakdown occurs in dth phase.
C_v	Cost for keeping server on vacation.

Expected total cost (TC) per unit time:

$$\mathrm{TC} = C_h L_q + [E(C)]^{-1} \left[C_S + \sum_{d=0}^{l} \{ C_{B_d} E(B_d) + C_{D_d} E(D_d) + C_{R_d} E(R_d) \} + C_v E(V) \right]. \tag{5.20}$$

6. Maximum entropy principle

In this section, we employ the maximum entropy principle to obtain approximate waiting time. The entropy function is framed as follows:

$$Z = -P_0 \log P_0 - \sum_{n\geq 1} \left\{ A_n \log A_n - \sum_{d=0}^{l} \left(P_n^d \log P_n^d + \log D_n^d + R_n^d \log R_n^d \right) - V_n \log V_n \right\}. \tag{6.1}$$

The maximum entropy results are obtained by maximizing the entropy function (6.1) using Lagrange's multiplier method, subject to the following constraints.

$$P_0^0 + \sum_{n\geq 1} \left\{ A_n + \sum_{d=0}^{l} \left(P_n^d + D_n^d + R_n^d \right) + V_n \right\} = 1 \tag{6.2a}$$

$$\sum_{n\geq 1} A_n = \eta_1 \equiv A(1) \tag{6.2b}$$

$$\sum_{n\geq 1} P_n^d = \eta_{2d} \equiv P^d(1); \quad d \in W_l \cup \{0\} \tag{6.2c}$$

$$\sum_{n\geq 1} D_n^d = \eta_{3d} \equiv D^d(1); \quad d \in W_l \cup \{0\} \tag{6.2d}$$

$$\sum_{n\geq 1} R_n^d = \eta_{4d} \equiv R^d(1); \quad d \in W_l \cup \{0\} \tag{6.2e}$$

$$\sum_{n\geq 1} V_n = \eta_5 \equiv V(1) \tag{6.2f}$$

$$\sum_{n\geq 1}\left(nA_n + \sum_{d=0}^{l}\{nP_n^d + nD_n^d + nR_n^d\} + nV_n\right) = L_q. \tag{6.2g}$$

Theorem 6.1. *The MEP approximate results for various system state probabilities of the system are*

$$\hat{A}_n = \eta_1\Gamma \tag{6.3a}$$

$$\hat{P}_n^d = \eta_{2d}\Gamma; \quad d \in W_l \cup \{0\} \tag{6.3b}$$

$$\hat{D}_n^d = \eta_{3d}\Gamma; \quad d \in W_l \cup \{0\} \tag{6.3c}$$

$$\hat{R}_n^d = \eta_{4d}\Gamma; \quad d \in W_l \cup \{0\} \tag{6.3d}$$

$$\hat{V}_n = \eta_5\Gamma \tag{6.3e}$$

where, $\delta = \eta_1 + \sum_{d=0}^{l}(\eta_{2d} + \eta_{3d} + \eta_{4d}) + \eta_5, \Gamma = \delta^{\frac{(L_q-\delta)}{(L_q)}}$.

Proof. Following the Lagrange's multipliers approach, the entropy function (6.1) and constraints (6.2a–6.2g), we get

$$\begin{aligned} Z = & -P_0 \log P_0 - \sum_{n\geq 1}\left\{A_n \log A_n + \sum_{d=0}^{l}\left(P_n^d \log P_n^d + D_n^d \log D_n^d + R_n^d \log R_n^d\right) + V_n \log V_n\right\} \\ & - \alpha_1\left(P_0^0 + \sum_{n\geq 1}\left\{A_n + \sum_{d=0}^{l}(P_n^d + D_n^d + R_n^d) + V_n\right\} - 1\right) - \alpha_2\left(\sum_{n\geq 1}^{\infty} A_n - \eta_1\right) \\ & - \sum_{d=0}^{l}\left[\alpha_{3d}\left(\sum_{n\geq 1} P_n^d - \eta_{2d}\right) + \alpha_{4d}\left(\sum_{n\geq 1} D_n^d - \eta_{3d}\right) + \alpha_{5d}\left(\sum_{n\geq 1} R_n^d - \eta_{4d}\right)\right] - \alpha_6\left(\sum_{n\geq 1} V_n - \eta_5\right) \\ & - \alpha_7\left(\sum_{n\geq 1}\left(nA_n + \sum_{d=0}^{l}\{nP_n^d + nD_n^d + nR_n^d\} + nV_n\right) - L_q\right), \end{aligned} \tag{6.4}$$

where $\alpha_1, \alpha_2, \alpha_{3d}, \alpha_{4d}, \alpha_{5d}, \alpha_6$ and α_7 are Lagrange's multipliers for respective constraints given as (6.2a–6.2g).

Now differentiating partially equation (6.4) with respect to $P_0, A_n, P_n^d, D_n^d, R_n^d, V_n$ and equating the outcomes to zero, we obtain

$$-(1 + \log P_0) - \alpha_1 = 0 \Rightarrow P_0 = e^{-(1+\alpha_1)} \tag{6.5a}$$

$$-(1 + \log A_n) - \alpha_1 - \alpha_2 - n\alpha_7 = 0 \Rightarrow A_n = e^{-(1+\alpha_1+\alpha_2+n\alpha_7)}; \quad n \in W_\infty \tag{6.5b}$$

$$-(1 + \log P_n^d) - \alpha_1 - \alpha_{3d} - n\alpha_7 = 0 \Rightarrow P_n^d = e^{-(1+\alpha_1+\alpha_{3d}+n\alpha_7)}; \quad n \in W_\infty, d \in W_l \cup \{0\} \tag{6.5c}$$

$$-(1 + \log D_n^d) - \alpha_1 - \alpha_{4d} - n\alpha_7 = 0 \Rightarrow D_n^d = e^{-(1+\alpha_1+\alpha_{4d}+n\alpha_7)}; \quad n \in W_\infty, d \in W_l \cup \{0\} \tag{6.5d}$$

$$-(1 + \log R_n^d) - \alpha_1 - \alpha_{5d} - n\alpha_7 = 0 \Rightarrow R_n^d = e^{-(1+\alpha_1+\alpha_{5d}+n\alpha_7)}; \quad n \in W_\infty, d \in W_l \cup \{0\} \tag{6.5e}$$

$$-(1 + \log V_n) - \alpha_1 - \alpha_6 - n\alpha_7 = 0 \Rightarrow V_n = e^{-(1+\alpha_1+\alpha_6+n\alpha_7)}; \quad n \in W_\infty, d \in W_l \cup \{0\}. \tag{6.5f}$$

Denote $\xi_1 = e^{-(1+\alpha_1)}, \xi_2 = e^{-\alpha_2}, \xi_{3d} = e^{-\alpha_{3d}}, \xi_{4d} = e^{-\alpha_{4d}}, \xi_{5d} = e^{-\alpha_{5d}}, \xi_6 = e^{-\alpha_6}, \xi_7 = e^{-\alpha_7}$ and substituting these values in equations (6.5a–6.5f), we obtain

$$P_0 = \xi_1, A_n = \xi_1\xi_2\xi_7^n, P_n^d = \xi_1\xi_{3d}\xi_7^n, D_n^d = \xi_1\xi_{4d}\xi_7^n, R_n^d = \xi_1\xi_{5d}\xi_7^n, V_n = \xi_1\xi_6\xi_7^n; \quad 0 \le d \le l. \tag{6.6}$$

Using the expressions given in equations (6.6) and (6.2b–6.2f), we get

$$\xi_1\xi_2\xi_7 = \eta_1(1-\xi_7) \tag{6.7a}$$
$$\xi_1\xi_{3d}\xi_7 = \eta_{2d}(1-\xi_7) \tag{6.7b}$$
$$\xi_1\xi_{4d}\xi_7 = \eta_{3d}(1-\xi_7) \tag{6.7c}$$
$$\xi_1\xi_{5d}\xi_7 = \eta_{4d}(1-\xi_7) \tag{6.7d}$$
$$\xi_1\xi_6\xi_7 = \eta_5(1-\xi_7). \tag{6.7e}$$

Utilizing equations (6.6) and (6.7a–6.7e) in equations (6.2a) and (6.2g), we get

$$\xi_1 = 1-\delta, \xi_7 = \frac{L_q - \delta}{L_q}\cdot \tag{6.8}$$

□

Theorem 6.2. *The approximate expected waiting time of the units is*

$$W_q^* = \chi_7 2^{-1}\left(c_{(2)}(c)^{-1} - 1\right) + \sum_{n\ge1}\left[\sum_{d=0}^{l}\left\{\left(\gamma_d^{(2)}(2\gamma_d)^{-1} + g_d\right)\hat{D}_n^d + \left(g_d^{(2)}(2g_i)^{-1}\right)\hat{R}_n^d\right\}\right.$$
$$\left. + \left(v^{(2)}(2v)^{-1}\right)\hat{V}_n\right] + \chi_7\left(\sum_{n\ge1}\left\{\hat{A}_n + \sum_{d=0}^{l}\left\{n\hat{P}_n^d + n\hat{D}_n^d + n\hat{R}_n^d\right\} + n\hat{V}_n\right\}\right). \tag{6.9}$$

Proof. The elaborated proof is given in Appendix B. □

Remark. The deviation between W_q and W_q^* is obtained using

$$\text{Dev}(\%) = \frac{|W_q - W_q^*|}{W_q} \times 100. \tag{6.10}$$

7. Special cases

Certain results of existing queueing models available in literature can be established by fixing some specific parameters.

Case (i). Bulk queue with server breakdown, retrial, optional service and vacation.

By fixing, $l = 1$ and $\theta = 0$ in equation (4.18), we obtain

$$\omega(z) = \left[\varepsilon_1\overline{X(z)}\widetilde{B}_0(\tau_0(z))\left\{r_0 + r_1\widetilde{B}_1(\tau_1(z))\right\}\left\{\bar{p} + p\widetilde{V}(\phi_5(z))\right\}\right][cS(z)]^{-1}. \tag{7.1}$$

Above result is same as obtained in the model given by Singh and Kaur [27].

Case (ii). Unreliable server retrial queue with vacation.

By letting $P(X=1)=1$, $b=b_1=b_2=b_3=1, r_i=0; 1\le i\le l, \theta=0, m=1$, the equation (4.18) provides

$$\omega(z)= \left(\widetilde{M}(\lambda)-\rho_h\right)\bar{z}\widetilde{B}_0(\tau_0(z))\left\{\bar{p}+p\widetilde{V}(\lambda\bar{z})\right\}\left[\widetilde{M}(\lambda)\bar{z}\widetilde{B}_0(\tau_0(z))\left\{q+p\widetilde{V}(\lambda\bar{z})\right\}\right.$$
$$\left. - z\left[1-\left\{\bar{p}+p\widetilde{V}(\lambda(\bar{z})\right\}\widetilde{B}_0(\tau_0(z))\right]\right]^{-1}; \tag{7.2}$$

where

$$\rho_h=\lambda\left[\beta_0(1+\alpha_0(\gamma_0+g_{01}))+pv\right], \tau_0(z)=\lambda\bar{z}+\alpha_0\left(1-\widetilde{D}_0(\lambda\bar{z})\widetilde{G}_{0,1}(\lambda\bar{z})\right).$$

The above outcome is same as established by Choudhury and Ke [7].

Case (iii). Queue with server breakdown and delay in repair.

By fixing $P(X=1)=1$, $m=1, \theta=0, l=1, b=b_1=b_2=b_3=1, p=1, \widetilde{M}(\lambda b)=1$, expression (4.18) reduces to

$$\omega(z)=\left[(1-\rho_h)\bar{z}\Theta(z)\right]\left[\Theta(z)-(z)\right]^{-1}; \quad \rho_h=\lambda c\left[\beta_0\left(1+\alpha_0(\gamma_0+g_{01})\right)+r_1\beta_1\left(1+\alpha_1(\gamma_1+g_{i1})\right)\right],$$
$$\tau_i(z)=\lambda\bar{z}+\alpha_i\left(1-\widetilde{D}_i(\lambda\bar{z})\widetilde{G}_{i,1}(\lambda\bar{z})\right); \quad i=0,1 \quad \text{and} \quad \Theta(z)=\widetilde{B}_0(\tau_0(z))\left\{r_0+r_1\widetilde{B}_1(\tau_1(z))\right\}. \tag{7.3}$$

The above result (7.3) corresponds to the result derived in [10].

8. Numerical illustrations

In this section, we examine the impact of varying parameters on various operational characteristics of queueing model. For numerical illustration, we take the distributions for batch arrival as geometric with parameter e and for service time as k-Erlangian with parameter μ_d. The repair time is considered to follow Gamma distribution with parameters $(h_{dj_1}, h_{dj_2}); 0\le d\le l, 1\le j\le m$. Furthermore the delay time for repair and retrial time follow the exponential distributions with corresponding parameters $\sigma_d(0\le d\le l), \varpi$ respectively. The vacation is assumed to follow deterministic distribution with parameter ϑ. To compute the numerical results, we fix parameters related to distributions as $\sigma_d=6\mu_d, h_{dj_1}=h_{dj_2}(\mu_d 3)^{-1}; 0\le d\le l, 1\le j\le m$.

Following are the values of default parameters to examine the behaviour of different performance measures:

$$c=2, l=2, m=3, \mu=1.8, \mu_0=\mu, \mu_1=1.5\mu, \mu_2=1.5\mu, \lambda=1.2, k=2, \vartheta=40, p=0.2, \theta=0.02,$$
$$b=0.4, b_1=0.25, b_2=0.3,\ b_3=0.35; \alpha=0.01, \alpha_0=\alpha, \alpha_1=0.1\alpha, \alpha_2=0.2\alpha, \varpi=20, h_{dj_1}=2.$$

8.1. Steady state probabilities

Tables 1–3 display the effect of various parameters on probabilities of various system states. From Table 1, it can be inspected that the long run probabilities $P(I), P(V)$ and $P(N)$ increase (decrease) while $P(B), P(D), P(R)$ decrease (increase) with the increment in $\mu(\lambda)$. Tables 2 and 3 summarize the probabilities $P(I), P(N), P(B)$ which seems to reduce on increasing the parameters α and p. On the other side, the probabilities $P(R), P(D)$ show an increment for growing values of (α, ϖ) while no significant change are noticed in the value $P(R), P(D)$ by increasing the parameters (θ, p). The effects of the parameters α and p on $P(V)$ are also displayed in Tables 2 and 3. We notice that the probabilities $P(V)$ and $P(B)$ increase for the increment in (θ, ϖ). Also, $P(I)$ decreases (increases) with an increase in $\theta(\varpi)$, on the contrary $P(N)$ increases (decreases) with the growth in the values of $\theta(\varpi)$.

8.2. Reliability indices

Table 4 exhibits the impact of failure rate α on the reliability indices of the server for increasing the number of phases (l) of optional services. The decreasing trends noticed in A_v and F_f with the increased values of α and l also match with experience in real time systems.

TABLE 1. The probabilities of server's status on varying λ and μ.

λ	μ	$P(I)$	$P(B)$	$P(D)$	$P(R)$	$P(V)$	$P(N)$
1.2	1.6	0.03979	0.90183	0.00070	0.00419	0.00519	0.04831
	1.7	0.06386	0.87767	0.00064	0.00383	0.00537	0.04863
	1.8	0.08665	0.85476	0.00059	0.00353	0.00554	0.04893
	1.9	0.10828	0.83301	0.00054	0.00326	0.00570	0.04922
	2.0	0.12882	0.81232	0.00050	0.00302	0.00585	0.04949
1.3	1.6	0.00664	0.93120	0.00072	0.00432	0.00536	0.05176
	1.7	0.03040	0.90732	0.00066	0.00396	0.00555	0.05210
	1.8	0.05296	0.88463	0.00061	0.00365	0.00573	0.05242
	1.9	0.07440	0.86303	0.00056	0.00337	0.00590	0.05273
	2.0	0.09481	0.84245	0.00052	0.00313	0.00607	0.05303

TABLE 2. The probabilities of server's status on varying Θ and α.

Θ	α	$P(I)$	$P(B)$	$P(D)$	$P(R)$	$P(V)$	$P(N)$
0.02	0.01	0.08665	0.85476	0.00059	0.00353	0.00554	0.04893
	0.02	0.08471	0.85269	0.00117	0.00704	0.00553	0.04886
	0.03	0.08277	0.85063	0.00175	0.01053	0.00551	0.04880
	0.04	0.08085	0.84858	0.00233	0.01400	0.00550	0.04873
	0.05	0.07893	0.84654	0.00291	0.01746	0.00549	0.04867
0.04	0.01	0.07738	0.86292	0.00059	0.00356	0.00559	0.04996
	0.02	0.07544	0.86081	0.00118	0.00710	0.00558	0.04989
	0.03	0.07351	0.85871	0.00177	0.01063	0.00556	0.04982
	0.04	0.07159	0.85662	0.00236	0.01414	0.00555	0.04975
	0.05	0.06967	0.85454	0.00294	0.01763	0.00554	0.04968

TABLE 3. The probabilities of server's status on varying ϖ and p.

ϖ	p	$P(I)$	$P(B)$	$P(D)$	$P(R)$	$P(V)$	$P(N)$
10	0.1	0.06577	0.83234	0.00057	0.00343	0.0027	0.09518
	0.2	0.06466	0.83083	0.00057	0.00343	0.00538	0.09512
	0.3	0.06356	0.82932	0.00057	0.00342	0.00806	0.09506
	0.4	0.06246	0.82782	0.00057	0.00342	0.01073	0.09500
	0.5	0.06137	0.82633	0.00057	0.00341	0.01339	0.09494
20	0.1	0.08781	0.85633	0.00059	0.00353	0.00277	0.04896
	0.2	0.08665	0.85476	0.00059	0.00353	0.00554	0.04893
	0.3	0.08550	0.85320	0.00059	0.00352	0.00829	0.04890
	0.4	0.08436	0.85164	0.00059	0.00351	0.01104	0.04887
	0.5	0.08322	0.85008	0.00058	0.00351	0.01377	0.04884

8.3. Mean queue length

From Table 5, we have examined that for all service phases, mean queue lengths L_{dep}, L_q and L_o become long for increasing the parameters θ and λ. But for fixed values of θ and λ, the queue length metrics L_{dep}, L_q and L_o decrease as the number of service phases for (k) of the Erlangian distribution increases

TABLE 4. A_v and F_f on varying α and l.

α	$l=0$		$l=1$		$l=2$	
	A_v	F_f	A_v	F_f	A_v	F_f
0.01	0.9398	0.0072	0.9425	0.0065	0.9428	0.0065
0.02	0.9363	0.0144	0.9393	0.0130	0.9397	0.013
0.03	0.9327	0.0215	0.9361	0.0194	0.9365	0.0194
0.04	0.9292	0.0286	0.9329	0.0258	0.9334	0.0258
0.05	0.9257	0.0357	0.9298	0.0322	0.9303	0.0322

TABLE 5. L_{dep}, L_q and L_o on varying Θ and λ for different distributions.

Θ	λ	$M/M/1$			$M/E_2/1$			$M/D/1$		
		L_{dep}	L_q	L_o	L_{dep}	L_q	L_o	L_{dep}	L_q	L_o
0.02	1.0	7.92	8.30	7.52	7.73	8.03	7.25	7.54	7.76	6.98
	1.1	10.19	10.70	9.87	9.90	10.31	9.49	9.60	9.93	9.11
	1.2	14.06	14.68	13.83	13.59	14.11	13.25	13.11	13.53	12.68
	1.3	22.17	22.91	22.02	21.31	21.93	21.05	20.45	20.96	20.07
	1.4	50.00	50.87	49.96	47.78	48.53	47.61	45.57	46.18	45.27
0.04	1.0	8.32	8.73	7.94	8.11	8.45	7.65	7.91	8.16	7.37
	1.1	10.90	11.44	10.61	10.58	11.03	10.20	10.26	10.62	9.79
	1.2	15.52	16.17	15.31	14.99	15.54	14.68	14.45	14.91	14.04
	1.3	26.20	26.99	26.09	25.17	25.84	24.94	24.14	24.69	23.79
	1.4	78.06	79.05	78.13	74.55	75.40	74.48	71.04	71.76	70.84

8.4. Mean waiting time

For our queueing model, waiting time is also an imperative performance measure. In Table 6, we have provided the numerical results for the mean waiting time W_q and approximate waiting time W_q^* evaluated from analytical and MEP approaches, respectively. By varying parameters, W_q, W_q^* and absolute percentage error are obtained; the maximum error is noticed to be 10.43%.

Figures 1–4 show the impact of λ, μ, θ and ϖ on W_q for distinct values of number of optional services. In Figures 1–4, it is observed that W_q attains higher values with the growth in $\lambda(\theta)$, while decreases with growth in $\mu(\varpi)$. It can be viewed from these figures that there is notable increase in W_q for higher values of l.

8.5. Cost analysis

In this section, we provide the sensitivity analysis of cost function and optimal cost *via* quasi Newton method by varying different parameters.

To compute the cost function (TC) for varying specific parameters, we consider the following set of default cost components:

$$C_h = \$5/\text{day}, C_s = \$500/\text{day}, C_{b_i} = \$35/\text{unit}, C_{d_i} = \$20/\text{day}, C_{r_i} = \$40/\text{unit}, C_V = \$30/\text{day}.$$

(i) Sensitivity analysis

From Tables 7 and 8, it can be noticed that for increasing the arrival rate (λ), total cost (TC) first decreases then increases for fixed values of l, θ and k. By fixing λ and l, TC seems to lower down with the growth in θ and k. The noteworthy trend of TC on varying μ is shown in Table 9; the convex nature of cost function with respect to μ in the feasible range of μ is observed. For exact values of μ and l, TC increases on raising the value of θ. In Tables 7–9, we observe that TC reveals significant change for $l = 1$ and 2.

TABLE 6. W_q and W_q^* for varying parameters.

Parameters		$l=1$			$l=2$		
		W_q	W_q^*	error (%)	W_q	W_q^*	error (%)
λ	1.10	8.73	8.00	8.41	9.88	9.10	7.85
	1.15	9.72	8.93	8.13	11.2	10.33	7.76
	1.20	11.00	10.11	8.08	13.00	11.94	8.10
	1.25	12.71	11.64	8.47	15.56	14.10	9.39
ϖ	17.00	11.47	10.44	9.00	13.64	12.39	9.14
	19.00	11.14	10.20	9.33	13.18	12.07	8.41
	21.00	10.87	10.02	7.84	12.83	11.83	7.83
	23.00	10.67	9.88	7.41	12.55	11.63	7.35
Θ	0.01	10.54	9.79	7.14	12.37	11.50	7.03
	0.02	11.00	10.11	8.08	13.00	11.94	8.10
	0.03	11.50	10.46	9.05	13.69	12.43	9.23
	0.04	12.05	10.84	10.06	14.48	12.97	10.43
μ	1.70	14.16	12.96	8.47	17.58	15.84	9.92
	1.80	11.00	10.11	8.08	13.00	11.94	8.10
	1.90	9.04	8.28	8.45	10.38	9.54	8.04
	2.00	7.70	7.01	9.08	8.68	7.94	8.47
p	0.10	10.87	10.03	7.79	12.83	11.83	7.79
	0.30	11.12	10.19	8.37	13.16	12.06	8.42
	0.50	11.39	10.37	8.95	13.52	12.29	9.05
	0.70	11.66	10.55	9.53	13.88	12.54	9.69

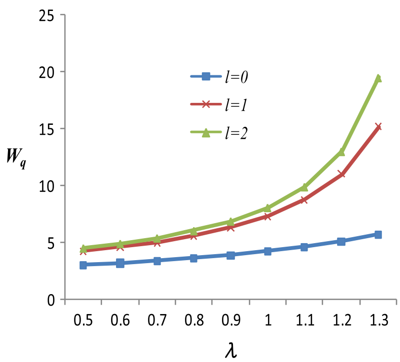

FIGURE 1. W_q for varying different values of λ.

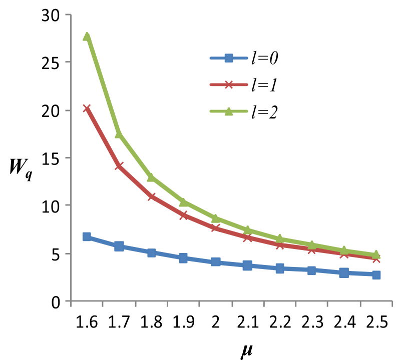

FIGURE 2. W_q for varying different values of μ.

(ii) Quasi Newton method to find optimal cost

In order to find optimum cost for our present model, we consider cost as a function of service rate (μ) and mean vacation time (v) with all other parameters keeping as fixed. Optimal values of service rate μ and mean vacation time (v) which minimize the cost $\mathrm{TC}(\mu^*, v^*)$ are denoted by (μ^*, v^*). Then unconstraint cost minimization problem can be expressed in terms of μ and v as

$$\mathrm{TC}(\mu^*, v^*) = \operatorname*{Min}_{\mu,v} \mathrm{TC}(\mu, v). \tag{8.1}$$

For some fixed initial values of (μ, v), quasi Newton method is used for searching the optimal value of (μ, v) in the feasible range so as to minimize the total cost $\mathrm{TC}(\mu, v)$ *i.e.* to evaluate $\mathrm{TC}(\mu^*, v^*)$. Quasi Newton method

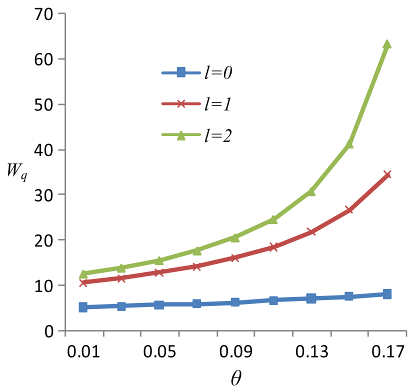

FIGURE 3. W_q for varying different values of θ.

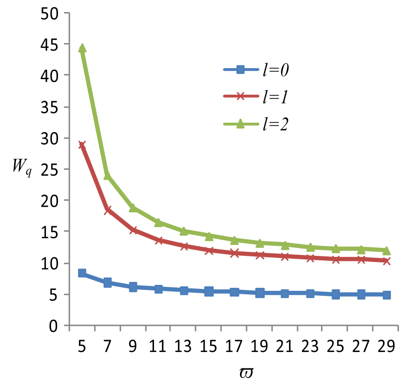

FIGURE 4. W_q for varying different values of ϖ.

TABLE 7. TC on varying (λ, l) for different service phases.

λ	k					
	$l=1$			$l=2$		
	$M/M/1$	$M/E_2/1$	$M/D/1$	$M/M/1$	$M/E_2/1$	$M/D/1$
0.90	165.49	164.55	163.61	158.43	157.45	156.48
0.95	161.25	160.15	159.05	154.51	153.37	152.22
1.00	157.30	156.02	154.73	151.09	149.74	148.39
1.05	153.86	152.36	150.85	148.45	146.85	145.25
1.10	151.23	149.45	147.68	147.01	145.09	143.18
1.15	149.80	147.69	145.58	147.41	145.08	142.76
1.20	150.20	147.66	145.11	150.69	147.81	144.93
1.25	153.43	150.30	147.16	158.75	155.09	151.43
1.30	161.29	157.33	153.37	175.58	170.70	165.82
1.35	177.46	172.25	167.03	211.09	204.08	197.08
1.40	210.87	203.51	196.14	299.36	287.64	275.92

is used in the direction of finding minimum value of cost with some fixed tolerance (say ε). The steps of quasi Newton method algorithm are as follows:

Algorithm

Step 1. For fixed values of other parameters, let $\Psi_j = [\mu, \nu]^T$.
Step 2. Start with some initial trial value Ψ_j for $j = 0$ and evaluate $\mathrm{TC}(\Psi_0)$.
Step 3. Compute cost gradient $\nabla \mathrm{TC}(\Psi_j) = [\partial \mathrm{TC}/\partial\mu, \partial \mathrm{TC}/\partial v]^T \big|_{\Psi_j}$ and the cost Hessian matrix is

$$H(\Psi_j) = \begin{bmatrix} \partial^2\mathrm{TC}/\partial\mu^2 & \partial^2\mathrm{TC}/\partial\mu\partial\nu \\ \partial^2\mathrm{TC}/\partial\nu\partial\mu & \partial^2\mathrm{TC}/\partial\nu^2 \end{bmatrix}\Bigg|_{\Psi_j}.$$

Step 4. Find the new trial solution $\Psi_{j+1} = \Psi_j - (H(\Psi_j))^{-1}\nabla \mathrm{TC}(\Psi_j)$.
Step 5. Set $j = j+1$ and repeat the steps 3 and 4 until $\mathrm{Max}\left(|\partial \mathrm{TC}/\partial\mu|, |\partial \mathrm{TC}/\partial v|\right) < \varepsilon$ where ε is minimum accepted tolerance, say $\varepsilon = 10^{-7}$.
Step 6. Find the global minimum value $\mathrm{TC}\,(\mu^*, v^*) = \mathrm{TC}(\Psi_j)$.

Implementing the quasi Newton method in Table 10, we display the optimal solution (μ^*, v^*) along with optimal cost $\mathrm{TC}\,(\mu^*, v^*)$ for $M/M/1, M/E_2/1$ and $M/D/1$ models taking $l = 1$ and 2 and varying values of the

TABLE 8. TC on varying (λ, l) for different values of Θ.

λ	$l = 1$			$l = 2$		
	$\Theta = 0.02$	$\Theta = 0.04$	$\Theta = 0.06$	$\Theta = 0.02$	$\Theta = 0.04$	$\Theta = 0.06$
0.90	164.55	160.59	156.71	157.45	153.8	150.27
0.95	160.15	156.35	152.67	153.37	149.95	146.72
1.00	156.02	152.46	149.11	149.74	146.69	143.92
1.05	152.36	149.20	146.34	146.85	144.36	142.29
1.10	149.45	146.88	144.76	145.09	143.47	142.49
1.15	147.69	146.01	145.03	145.08	144.8	145.61
1.20	147.66	147.35	148.20	147.81	149.76	153.66
1.25	150.30	152.25	156.25	155.09	161.08	171.00
1.30	157.33	163.31	173.33	170.70	185.05	209.09
1.35	172.25	186.36	210.12	204.08	240.25	311.33
1.40	203.51	237.97	305.01	287.64	416.92	923.23

TABLE 9. TC on varying (μ, l) for different values of Θ.

μ	$l = 1$			$l = 2$		
	$\Theta = 0.02$	$\Theta = 0.04$	$\Theta = 0.06$	$\Theta = 0.02$	$\Theta = 0.04$	$\Theta = 0.06$
1.55	186.04	211.21	255.96	245.31	321.85	521.82
1.60	165.38	178.09	198.84	193.79	224.62	281.96
1.65	154.85	161.57	172.47	169.69	185.15	210.84
1.70	149.66	152.99	158.81	157.22	165.56	179.04
1.75	147.65	148.82	151.63	150.76	155.16	162.57
1.80	147.66	147.35	148.20	147.81	149.76	153.66
1.85	149.04	147.66	147.13	147.09	147.39	149.04
1.90	151.40	149.20	147.65	147.87	146.98	147.07
1.95	154.46	151.61	149.27	149.71	147.92	146.87
2.00	158.05	154.67	151.70	152.32	149.82	147.91
2.05	162.04	158.22	154.73	155.50	152.43	149.82

parameters $p, \lambda, \theta, \alpha$ and ϖ. In order to find the optimum solution using quasi Newton algorithm the feasible ranges of μ has chosen to lie in the range [2, 6] and ν in range [1, 2].

The trends of TC for varying different sensitive parameters are depicted in Figures 5–8. In Figures 5 and 6, optimal total cost TC* are attained as \$141.67 and \$143.22, respectively, along with corresponding optimal parameter values ($\lambda^* = 1.05, \mu^* = 1.75$) and ($\lambda^* = 1.1, p^* = 0.9$), respectively. From Figures 7 and 8, it is also seen that TC* are \$141.45 and \$145.84 with respective optimal parameter values ($\lambda^* = 1.05, \theta^* = .07$) and ($\mu^* = 2.2, \theta^* = 0.16$).

From numerical illustrations above, we observe the impact of parameters on performance indices in the system and know that the results are coincident with the practical situations.

9. Conclusion

The unreliable server $M^X/G/1$ retrial queue investigation includes many realistic features altogether *viz.* optional feedback, vacation, multi-optional service, balking, server breakdown, delay in repair and multi phase etc. The assumptions of general distributions for phase services, delay in repair/repair, retrial processes, vacation etc. make our model practically plausible to deal with non-Markovian model and depict many queueing scenarios in more appropriate manner. The applications of retrial bulk model studied can be found in many real

TABLE 10. Optimal (μ^*, v^*) and TC on varying $(l, p, \lambda, \theta, \alpha, \varpi)$ for different phases and optional services.

		$M/M/1$			$M/E_2/1$			$M/D/1$		
l	p	μ^*	v^*	TC	μ^*	v^*	TC	μ^*	v^*	TC
1	0.2	3.3540	1.9634	142.99	3.0375	1.7502	142.19	2.7217	1.4878	41.20
	0.3	6.2160	1.9848	139.34	5.3460	1.8625	139.10	4.4772	1.6928	138.77
2	0.2	3.4258	1.9230	142.84	3.1204	1.7188	142.08	2.8155	1.4707	141.14
	0.3	6.2887	1.9626	139.29	5.4489	1.8434	139.06	4.6104	1.6808	138.75
l	λ	μ^*	v^*	TC	μ^*	v^*	TC	μ^*	v^*	TC
1	1.3	3.5786	1.8021	147.26	3.243	1.6029	146.42	2.9081	1.3582	145.37
	1.4	3.8051	1.6644	151.35	3.4502	1.4777	150.47	3.0960	1.2484	149.36
2	1.3	3.6558	1.7644	147.10	3.3318	1.5737	146.29	3.0085	1.3422	145.30
	1.4	3.8876	1.6291	151.18	3.5451	1.4503	150.34	3.2031	1.2334	149.29
l	θ	μ^*	v^*	TC	μ^*	v^*	TC	μ^*	v^*	TC
1	0.04	3.4613	1.9402	142.79	3.1353	1.7338	142.04	2.8098	1.4798	141.09
	0.06	3.5755	1.9169	142.59	3.2392	1.7172	141.86	2.9035	1.4715	140.96
2	0.04	3.5356	1.9012	142.65	3.2208	1.7034	141.92	2.9067	1.4631	141.02
	0.06	3.6524	1.8791	142.45	3.3277	1.6878	141.76	3.0037	1.4554	140.90
l	α	μ^*	v^*	TC	μ^*	v^*	TC	μ^*	v^*	TC
1	0.02	3.3106	1.9321	142.87	2.9948	1.7126	142.05	2.6799	1.4419	141.02
	0.04	3.2258	1.8682	142.62	2.9119	1.6357	141.75	2.5992	1.3475	140.65
2	0.02	3.3842	1.8932	142.72	3.0795	1.6833	141.94	2.7754	1.4277	140.97
	0.04	3.3033	1.8323	142.49	3.0000	1.6107	141.66	2.6977	1.3398	140.62
l	ϖ	μ^*	v^*	TC	μ^*	v^*	TC	μ^*	v^*	TC
1	5	4.2590	1.7596	140.32	3.8682	1.5952	139.80	3.4780	1.3941	139.16
	10	3.6221	1.8922	142.05	3.2836	1.6963	141.36	2.9459	1.4557	140.50
2	5	4.3526	1.7284	140.22	3.9754	1.5711	139.72	3.5988	1.3810	139.12
	10	3.7004	1.8551	141.92	3.3738	1.6675	141.26	3.0476	1.4399	140.44

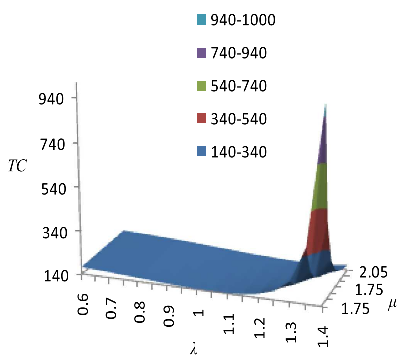

FIGURE 5. TC for varying (λ, μ).

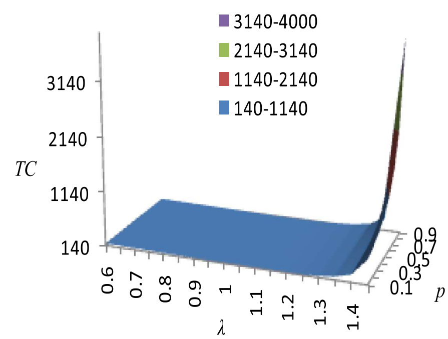

FIGURE 6. TC for varying (λ, p).

time systems such as in manufacturing and assembly organizations, telecommunication network, health care system, banking and computer network, etc. The supplementary variable technique (SVT) used facilitates various performance metrics explicitly which are computationally tractable also. MEP approach has also employed to demonstrate the scope of evaluating the operational characteristics of complex systems for which explicit analytical results cannot be derived. The present model can be modified by including admission control policy, working vacation policy, etc.

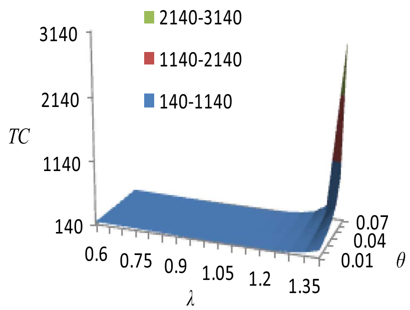

FIGURE 7. TC for varying (λ, θ).

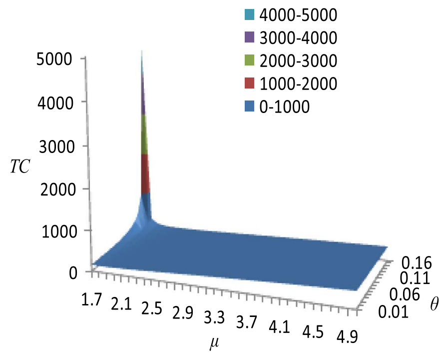

FIGURE 8. TC for varying (μ, θ).

APPENDIX A.

Proof of Lemma 4.2. In order to analyze the model mathematically, the technique of probability generating function is used for solving governing equations. Upon multiplying the equations (3.2), (3.4–3.6) by appropriate power of z and then summing for all values of n, we have

$$A(x,z) = A(0,z)(1-M(x))\exp\{-\lambda bx\} \tag{A.1}$$

$$D^d(x,y,z) = D^d(x,0,z)[1-D_d(y)]\exp\{-\phi_3(z)y\}; \quad 0 \le d \le l \tag{A.2}$$

$$R_k^d(x,y,z) = R_k^d(x,0,z)[1-G_{d,k}(y)]\exp\{-\phi_4(z)y\}; \quad 0 \le d \le l; 1 \le k \le m \tag{A.3}$$

$$V(y,z) = V(0,z)(1-V(y))\exp\{-\phi_5(z)y\}. \tag{A.4}$$

In the similar manner, from equation (3.10), we obtain

$$D^d(x,0,z) = \alpha_d P^d(x,z); \quad 0 \le d \le l. \tag{A.5}$$

Now, using (3.11) and (3.12) in similar manner and using (A.3), we obtain

$$R_1^d(x,0,z) = D^d(x,0,z)\widetilde{D}_d(\phi_3(z)); \quad 0 \le d \le l \tag{A.6}$$

$$R_k^d(x,0,z) = R_{k-1}^d(x,0,z)\widetilde{G}_{d,k-1}(\phi_4(z)); \quad 0 \le d \le l; 2 \le k \le m. \tag{A.7}$$

Putting $k = 2$ in equation (A.7) and using (A.6), then continuing recursively for $k = 3, 4, \dots, m$, we have

$$R_k^d(x,0,z) = D^d(x,0,z)\widetilde{D}_d(\phi_3(z))\prod_{j=1}^{k-1}\widetilde{G}_{i,j}(\phi_4(z)); \quad 0 \le d \le l, 2 \le k \le m. \tag{A.8}$$

Also, by utilizing (A.5) in (A.6) and (A.8), we get

$$R_1^d(x,0,z) = \alpha_d P^d(x,z)\widetilde{D}_d(\phi_3(z)); \quad 0 \le d \le l \tag{A.9}$$

$$R_k^d(x,0,z) = \alpha_d P^d(x,z)\widetilde{D}_d(\phi_3(z))\prod_{j=1}^{k-1}\widetilde{G}_{i,j}(\phi_4(z)); \quad 0 \le d \le l, 2 \le k \le m. \tag{A.10}$$

On solving the equation (3.3) and using equations (A.10), we have

$$P^d(x,z) = P^d(0,z)[1-B_d(x)]\exp\{-\tau_d(z)x\}; \quad 0 \le d \le l. \tag{A.11}$$

Using (3.8), (3.13) and (A.11), we obtain

$$P^d(0,z) = r_d P^0(0,z)\widetilde{B}_0(\tau_0(z)); \quad 1 \le d \le l \tag{A.12}$$

$$V(0,z) = p(\theta z + \bar{\theta})\left\{r_0 + \sum_{d=1}^{l} r_d \widetilde{B}_d(\tau_d(z))\right\} P^0(0,z)\widetilde{B}_0(\tau_0(z)). \tag{A.13}$$

Using (3.7) and (3.1), (A.11–A.13), we get

$$A(0,z) = z^{-1}(\theta z + \bar{\theta})P^0(0,z)\widetilde{B}_0(\tau_0(z))\left\{r_0 + \sum_{d=1}^{l} r_d \widetilde{B}_d(\tau_d(z))\right\}\left\{q + p\widetilde{V}(\phi_5(z))\right\} - \lambda P_0. \tag{A.14}$$

Similarly, in equation (3.8) using the equation (A.1), we have

$$P^0(0,z) = \lambda P_0 X(z) + A(0,z)\left[\widetilde{M}(\lambda b) + X(z)\left(1 - \widetilde{M}(\lambda b)\right)\right]. \tag{A.15}$$

Using equation (A.14) in (A.15), we get

$$P^0(0,z) = zP_0\phi_1(z)\widetilde{M}(\lambda b)(S(z))^{-1}. \tag{A.16}$$

By using equation (A.16) in equation (A.15), we get

$$A(0,z) = \lambda P_0\left[z - X(z)\left(\theta z + \bar{\theta}\right)\widetilde{B}_0(\tau_0(z))\left\{r_0 + \sum_{d=1}^{l} r_d \widetilde{B}_d(\tau_d(z))\right\}\left\{\bar{p} + p\widetilde{V}(\phi_5(z))\right\}\right](S(z))^{-1}. \tag{A.17}$$

By utilizing equations (A.5) in (A.2) and (A.9), (A.10) in (A.3), we get

$$D^d(x,y,z) = \alpha_d P^d(x,z)[1 - D_d(y)]\exp\{-\phi_3(z)y\}; \quad 0 \le d \le l. \tag{A.18}$$

$$R_1^d(x,y,z) = \alpha_d P^d(x,z)\widetilde{D}_d(\phi_3(z))[1 - G_{d,k}(y)]\exp\{-\phi_4(z)y\}; \quad 0 \le d \le l. \tag{A.19}$$

$$R_k^d(x,y,z) = \alpha_d P^d(x,z)\widetilde{D}_d(\phi_3(z))\prod_{j=1}^{k-1}\widetilde{G}_{i,j}(\phi_4(z))[1 - G_{d,k}(y)]\exp\{-\phi_4(z)y\}; \quad 0 \le d \le l, 2 \le k \le m. \tag{A.20}$$

Using (A.1), (A.4), (A.11), (A.18–A.20) in normalizing condition (3.14) and further using (A.12), (A.13) and (A.16), (A.17), we can obtain value of P_0 as

$$P_0 = b\varepsilon_1[\varepsilon_2]^{-1}. \tag{A.21}$$

Now, combining (A.1) and (A.17), and using the value of P_0 we have result given in equation (4.2).

Similarly, utilizing P_0 in equation (A.16) and further substituting the outcome in (A.11), we can easily obtain equation (4.3) and hence equations (4.4–4.6). □

Appendix B.

Proof of Theorem 6.2. In order to find the approximate expected waiting time of test unit U, for different server status is obtained as follows:

(a) Idle state

Upon arrival of unit U, if the server is in idle state I, then the server immediately turned on to busy state. In this case, the unit U, has to wait due to that unit preceding him in the same group. Thus, the mean waiting time of the unit U, in idle state is given as

$$W_I = \left(\beta_0 + \sum_{d=1}^{l} r_d\beta_d\right) 2^{-1}\left(c_{(2)}(c)^{-1} - 1\right). \tag{B.1}$$

(b) Busy state

In the busy state, while the server is rendering the essential/optional service, test unit U has to wait for (i) the service time of those n units in front of him, (ii) additional waiting time due to the unit preceding him in the same group. Thus mean waiting time in busy state while rendering essential/optional service is

$$W_B = \left(\beta_0 + \sum_{d=1}^{l} r_d\beta_d\right)\left[n + 2^{-1}\left(c_{(2)}(c)^{-1} - 1\right)\right]. \tag{B.2}$$

(c) Delayed repair state

Upon arrival of test unit U, if the server is broken down while rendering the essential/optional service of the unit and waiting for repair, then the test unit U will wait (i) residual delay time, (ii) mean repair time, (iii) service time of those n units in front of him and (iv) additional waiting time due to the unit preceding him in same group. Thus mean waiting time for unit in delay repair state $D_d(d = 1, 2 \ldots l)$ is

$$W_{D_d} = \gamma_d^{(2)}(2\gamma_d)^{-1} + g_d + \left(\beta_0 + \sum_{d=1}^{l} r_d\beta_d\right)\left[n + 2^{-1}\left(c_{(2)}(c)^{-1} - 1\right)\right]. \tag{B.3}$$

(d) Repair state

If server is failed while rendering the essential/optional service of the unit, the test unit U will wait for the (i) residual repair time, (ii) service time of n units in front of him and (iii) additional waiting time due to unit preceding him in same group. Thus mean waiting time for $R_d(d = 1, 2 \ldots l)$ state is

$$W_{R_d} = g_d^{(2)}(2g_d)^{-1} + \left(\beta_0 + \sum_{d=1}^{l} r_d\beta_d\right)\left[n + 2^{-1}\left(c_{(2)}(c)^{-1} - 1\right)\right]. \tag{B.4}$$

(e) Vacation state

During the vacation state, test unit U will wait for (i) residual vacation time, (ii) for the service time of those n units in front of him and (ii) additional waiting time due to the unit preceding him in same group. Thus mean waiting time in vacation state is

$$W_v = v^{(2)}(2v)^{-1} + \left(\beta_0 + \sum_{d=1}^{l} r_d\beta_d\right)\left[n + 2^{-1}\left(c_{(2)}(c)^{-1} - 1\right)\right]. \tag{B.5}$$

On utilizing equations (B.1–B.5) and summing up waiting time of test unit in idle state, busy state, delayed to repair state, repair and vacation state, we obtain the approximate mean waiting time in the service system given in equation (6.9). □

Acknowledgements. The authors are thankful to the editor of the journal and the reviewers for their valuable suggestions for the improvement of this article.

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