SPLIT VARIATIONAL INCLUSIONS FOR BREGMAN MULTIVALUED MAXIMAL MONOTONE OPERATORS

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Abstract. We introduce a new algorithm to approximate a solution of split variational inclusion problems of multivalued maximal monotone operators in uniformly convex and uniformly smooth Banach spaces under the Bregman distance. A strong convergence theorem for the above problem is established and several important known results are deduced as corollaries to it. As application, we solve a split minimization problem and provide a numerical example to support better findings of our result.

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1. Introduction

Censor [8] imposed the well known split feasibility problem (SFP), which is formulated as finding a point $x^* \in C$ such that $Ax^* \in Q$, where $C$ and $Q$ are nonempty closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, where $A$ is an $m \times n$ matrix. Byrne [3], defined CQ-algorithm as follows:

$$x_{n+1} = P_C(x_n + \gamma A^T(P_Q - I)Ax_n), \quad n \geq 0,$$

where $x_0 \in \mathbb{R}^n$ is an initial value, $\gamma \in (0, \frac{2}{\|A\|^2})$ and $P_C$ and $P_Q$ denote the metric projections onto $C$ and $Q$, respectively. The split feasibility problem has been considered by many authors and in many aspects [1–3, 5, 8, 9, 13, 16, 25, 26, 30]. In practice, SFP serves as a model in the intensity-modulation radiation therapy (IMRT) treatment planning [2, 5]. Censor et al. [10] introduced a concept of Split Variational Inequality Problem (SVIP), which is a problem of finding a point $x^* \in H_1$ solves

$$\langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C,$$

and the point $y^* = Ax^* \in H_2$ such that

$$\langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q,$$

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where $C$ and $Q$ are closed and convex subsets of Hilbert spaces $H_1$ and $H_2$, respectively, $A : H_1 \to H_2$ is a bounded linear operator and $A^* : H_2 \to H_1$ is adjoint of $A$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given operators. Furthermore, they proposed the following algorithm. Let $\lambda > 0$ and $x_1 \in H_1$ be arbitrary chosen. Define the sequence $\{x_n\}$ by

$$x_{n+1} = P_{C}^{f,\lambda}(x_n + \gamma A^*(P_{Q}^{g,\lambda} - I)Ax_n), \forall n \geq 0,$$

(1.1)

where $\gamma \in (0, \frac{1}{\|A\|_2^2})$, and denoted by $P_{C}^{f,\lambda}$ and $P_{Q}^{g,\lambda}$ the expressions $P_{C}(I - \lambda f)$ and $P_{Q}(I - \lambda g)$, respectively. By some assumptions imposed on the operators $f$ and $g$, they proved weak convergence result for the sequence $\{x_n\}$ to a solution point of split variational inequality problem.

Let $E$ be a real normed space with dual $E^*$ and $J(x) = \{x^* \in E^*; \langle x, x^* \rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|\}$ be the normalized duality. A map $B : E \to E^*$ is called monotone if for each $x, y \in E$, the following inequality holds: $\langle \eta - \nu, x - y \rangle \geq 0 \forall \eta \in Bx, \nu \in By$. It is called maximal monotone if, in addition, the graph of $B$ is not properly contained in the graph of any other monotone operator. Also, $B$ is maximal monotone if and only if it is monotone and for all $t > 0$, $R(J + tB) = E^*$, where $R(J + tB)$ is the range of $(J + tB)$; see [4]. By using maximal monotone mappings, Moudafi [15] introduced the following Split Monotone Variational Inclusion (SMVI).

$$\begin{cases}
\text{find } x^* \in H_1 : 0 \in f(x^*) + B_1(x^*), \\
y^* = Ax^* \in H_2 : 0 \in g(y^*) + B_2(y^*),
\end{cases}$$

(1.2)

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings on Hilbert spaces, $H_1$ and $H_2$, respectively, and $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given single-valued operators. When $f$ and $g$ are zero functions in (1.2), we have the usual Split Variational Inclusion Problem (SVIP). The algorithm introduced by Schöpf er et al. [20] involves computations in terms of Bregman distance in the setting of $p$-uniformly convex and uniformly smooth real Banach spaces. Their iterative algorithm given below, converges weakly under some suitable conditions.

$$x_{n+1} = \Pi_{C} J^{-1}(Jx_n + \gamma A^* J(P_{Q} - I)Ax_n), \text{ } n \geq 0,$$

(1.3)

where $\Pi_{C}$ denotes the Bregman Projection and $A^*$ the adjoint operator of $A$. It is obvious that, strong convergence is more useful than the weak convergence in some applications. Recently, strong convergence theorems for SFP have been studied in the setting of $p$-uniformly convex and uniformly smooth real Banach spaces; see for example [11, 17, 22, 23].

In this paper, inspired by the above cited works, we use a modified version of (1.1) and (1.3) to approximate a solution of the problem proposed here. Both the iterative methods and the underlying space used here are improvements of those employed in [6, 7, 10, 11, 13, 17, 20, 22, 23, 28] and the references therein.

**Definition 1.1.** For each $p > 1$, let $g : \mathbb{R}^+ \to \mathbb{R}^+$ given by $g(t) = t^{p-1}$ be a gauge function such that $g(0) = 0$ and $\lim_{t \to -\infty} g(t) = \infty$. We define the generalized duality map $J^p : E \to 2^{E^*}$ given by

$$J^p(x) = \{x^* \in E^*; \langle x, x^* \rangle = \|x\|\|x^*\|, \|x^*\| = g(\|x\|) = \|x\|^{p-1}\}.$$

**Definition 1.2.** Let $E$ be a smooth Banach space, the Bregman distance $\Delta_p$ of $x$ to $y$, with respect to the convex continuous function $f : E \to \mathbb{R}$ given by $f(x) = \frac{1}{p}\|x\|^p$, is defined as

$$\Delta_p(x, y) = \frac{1}{q}\|x\|^p - \langle J^p(x), y \rangle + \frac{1}{p}\|y\|^p,$$

for all $x, y \in E$ and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. 
Definition 1.3. Let $E$ be a smooth Banach space and $E^*$ its dual, the bifunctional $V_p$ with respect to the convex continuous function $f : E \to \mathbb{R}$ given by $f(x) = \frac{1}{q}||x||^q$, is defined by

$$V_p(x^*, x) = \frac{1}{q}||x^*||^q - \langle x^*, x \rangle + \frac{1}{p}||x||^p,$$

for all $x \in E$, $x^* \in E^*$ and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 1.4. A Banach space $E$ is said to be uniformly convex, if for $x, y \in E$, $0 < \delta_E(\epsilon) \leq 1$, where $\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}(x + y)||; ||x|| = ||y|| = 1, ||x - y|| \geq \epsilon, \text{ where } 0 \leq \epsilon \leq 2\}.$

Definition 1.5 ([19]). A Banach space $E$ is said to be uniformly smooth, if for $x, y \in E$ and $r > 0, \lim_{r \to 0}(\frac{\rho_E(r)}{r}) = 0$ where $\rho_E(r) = \frac{1}{2} \sup\{|x + y| + |x - y| - 2; |x|| = 1, |y|| \leq r\}$. Moreover,

1) $\rho_E$ is continuous, convex and nondecreasing with $\rho_E(0) = 0$ and $\rho_E(r) \leq r$.
2) The function $r \mapsto \frac{\rho_E(r)}{r}$ is nondecreasing and fulfills $\frac{\rho_E(r)}{r} > 0$ for all $r > 0$.

Lemma 1.6 ([19]). Let $\{x_n\}$ be a sequence in a smooth Banach space $E$. Consider the following assertions;

1) $\lim_{n \to \infty} \|x_n - x\| = 0$
2) $\lim_{n \to \infty} \|x_n\| = \|x\|$ and $\lim_{n \to \infty} \langle J^p(x_n), x \rangle = \langle J^p(x), x \rangle$
3) $\lim_{n \to \infty} \Delta_p(x_n, x) = 0$.

The assertions (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are valid. If $E$ is also uniformly convex, then the assertions are equivalent.

Lemma 1.7. Let $E$ be a reflexive and smooth Banach space and $E^*$ its dual. Let $\Delta_p$ and $V_p$ be the mappings defined as above and $J^p_E$ the generalized duality map on $E$. Then $\Delta_p(x, y) = V_p(J^p_E x, y)$ for all $x, y \in E$.

Proof. For $p, q \in (1, \infty)$ let $J^p_E : E^* \to E$ and $J^q_E : E \to E^*$ be duality mappings, where $J^q_E J^p_E = I$. It follows from $\frac{1}{p} + \frac{1}{q} = 1$ that $p(q - 1) = q$. So, we have that

$$\Delta_p(x, y) = \frac{1}{q}||x||^q - \langle J^p_E x, y \rangle + \frac{1}{p}||y||^p$$

$$= \frac{1}{q}||J^p_E x||^q - \langle J^p_E x, y \rangle + \frac{1}{p}||y||^p$$

$$= \frac{1}{q}||J^q_E x||^{p(q - 1)} - \langle J^p_E x, y \rangle + \frac{1}{p}||y||^p$$

$$= \frac{1}{q}||J^q_E x||^q - \langle J^p_E x, y \rangle + \frac{1}{p}||y||^p$$

$$= V_p(J^p_E x, y).$$

Lemma 1.8 ([19]). Let $E$ be a reflexive, strictly convex and smooth Banach space and $J^p$ be the duality mapping of $E$. Then

(i) for every closed and convex subset $C \subset E$ and $x \in E$, there exists a unique element $\Pi^p_C(x) \in C$ such that $\Delta_p(x, \Pi^p_C(x)) = \min_{y \in C} \Delta_p(x, y)$; $\Pi^p_C(x)$ is called the Bregman projection of $x$ onto $C$, with respect to the function $f(x) = \frac{1}{p}||x||^p$. Moreover, $x_0 \in C$ is the Bregman projection of $x$ onto $C$ if

$$\langle J^p(x_0 - x), y - x_0 \rangle \geq 0$$

or equivalently

$$\Delta_p(x_0, y) \leq \Delta_p(x, y) - \Delta_p(x, x_0)$$

for every $y \in C$. 

(ii) the Bregman projection and the metric projection are related via $P_C(x) - x = \Pi^p_{-x}(0)$, $\forall x \in E$. Especially, we have $P_C(0) = \Pi^p_C(0)$ and thus $\|\Pi^p_C(0)\| = \min_{y \in C} \|y\|$.

The uniform convexity of $E$ implies that $E$ is reflexive and $E^*$ is uniformly smooth. Therefore, Theorem 2 in [27], for $x, y \in E$ and $x^*, y^* \in E^*$ and $\|x + y\|^p$ replaced by $\|x^* - y^*\|^q$ gives the following technical result.

**Lemma 1.9.** For the uniformly smooth space $E^*$, with the duality map $J^q$, $\forall x^*, y^* \in E^*$, we have

$$
\|x^* - y^*\|^q \leq \|x^*\|^q - q\langle J^q(x^*), y^* \rangle + \sigma_q(x^*, y^*)
$$

where

$$
\sigma_q(x^*, y^*) = qG_q \int_0^1 \left( \|x^* - ty^*\| \vee \|x^*\| \right)^q \rho_{E^*} \left( \frac{t\|y^*\|}{2(\|x^* - ty^*\| \vee \|x^*\|)} \right) \, dt
$$

(1.4)

and $G_q = 8 \vee 64cK_q^{-1}$ with $c, K_q > 0$.

**Lemma 1.10** ([19]). Let $E$ be a reflexive, strictly convex and smooth Banach space. We write

$$
\Delta_q(x, y) = \frac{1}{q}\|x^*\|^q - \langle J^q_E(x^*), y^* \rangle + \frac{1}{q}\|y^*\|^q
$$

for $x^* = J^q_E(x)$, $y^* = J^q_E(y)$ for the Bregman distance on the dual space $E^*$ with respect to the function $f_q(x^*) = \frac{1}{q}\|x^*\|^q$. Then we have $\Delta_p(x, y) = \Delta_q(x, y^*)$.

**Lemma 1.11.** Let $E$ be a reflexive, smooth and strictly convex Banach space. Then for all $x, y, z \in E$ and $x^* = J^p_E(x)$, $z^* = J^p_E(z)$, the following hold:

1. $\Delta_p(x, y) \geq 0$ and $\Delta_p(x, y) = 0$ if $x = y$;
2. $\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle x^* - z^*, z - y \rangle$.

Proof. The property (1) is proved in [19]. For (2) we have that

$$
\Delta_p(x, z) + \Delta_p(z, y) = \frac{1}{q}\|x\|^p - \langle x^*, z \rangle + \frac{1}{p}\|z\|^p + \frac{1}{q}\|z\|^p - \langle z^*, y \rangle + \frac{1}{p}\|y\|^p
$$

$$
= \frac{1}{q}\|x\|^p - \langle x^*, z \rangle + \|z\|^p - \langle z^*, y \rangle + \frac{1}{p}\|y\|^p + \langle z^*, y \rangle - \langle x^*, y \rangle
$$

$$
= \left( \frac{1}{q}\|x\|^p - \langle x^*, y \rangle + \frac{1}{p}\|y\|^p \right) + \langle z^*, z \rangle - \langle z^*, y \rangle + \langle x^*, y \rangle - \langle x^*, z \rangle
$$

$$
\Leftrightarrow \Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle x^* - z^*, z - y \rangle.
$$

□

If $E$ is smooth and $f(x) = \frac{1}{p}\|x\|^p$, then the following result holds (cf. Prop. 5 in [18]).

**Lemma 1.12.** Let $E$ be a smooth Banach space and $f : E \to R$ be a continuous convex function given by $f(x) = \frac{1}{p}\|x\|^p$. If $x_0 \in E$ and the sequence $\{\Delta_p(x_n, x_0)\}_{n=1}^\infty$ is bounded, then the sequence $\{x_n\}$ is also bounded.

2. Main results

Let $E_1$ and $E_2$ be uniformly convex and uniformly smooth Banach spaces and $E_1^*$ and $E_2^*$ be their duals, respectively. Let $U : E_1 \to 2^{E_1^*}$ and $T : E_2 \to 2^{E_2^*}$ be multi-valued maximal monotone operators. For $K \subset E_1$, closed and convex, $\delta > 0$ and $p, q \in (1, \infty)$, let $A : E_1 \to E_2$ be a bounded and linear operator, $A^*$ denotes the adjoint of $A$ and $AK$ be closed and convex. Suppose that $\Pi^p_{AK} : E_2 \to AK$ is the Bregman projection onto a closed and convex subset $AK$. Let $B^U_\delta : E_1 \to E_1$ be the generalized resolvent operator defined by $B^U_\delta = (J^p_{E_1^*} + \delta U)^{-1}J^p_{E_1}$ and $B^T_\delta : E_2 \to E_2$ be another generalized resolvent operator defined by $B^T_\delta = (J^p_{E_2^*} + \delta T)^{-1}J^p_{E_2}$. Let us denote the solutions of variational inclusion problem with respect to $U$ and $T$ by SOLVIP($U$) and SOLVIP($T$), respectively. Let the set of solutions of split variational inclusion problem be
given by \( \Omega = \{x^* \in SOLVIP(U); Ax^* \in SOLVIP(T)\} \neq \emptyset \). Let \( x_1 \in E_1 \) be chosen arbitrarily and the sequence \( \{x_n\} \subset E_1 \) be defined as follows:

\[
\begin{cases}
  u_n = B_{\delta_n}^T \left( J_{E_1}^p \left( J_{E_2}^p (x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n) \right) \right), \\
  K_{n+1} = \{v \in K_n : \triangle_p (u_n, v) \leq \triangle_p (x_n, v)\}, \\
  x_{n+1} = \Pi_{K_{n+1}}(x_1), n \geq 1,
\end{cases}
\]

(2.1)

where \( \delta_n \in (0, \infty) \). It is remarked that we have replaced the gradient algorithm in (1.1) [the projection maps in (1.3), respectively] with the resolvent operators and used the generalized duality map in our algorithm.

We shall strictly employ the above terminology in the sequel.

**Lemma 2.1.** Suppose that \( \bar{\sigma}_q \) is the function in (1.4) for the characteristic inequality of the uniformly smooth space \( E_1^* \). For the sequence \( \{x_n\} \subset E_1 \) defined by (2.1), let \( 0 \neq x_n \in E_1, 0 \neq A \) and \( 0 \neq J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n \in E_2 \). Let \( \lambda_n > 0 \) and \( \mu_n > 0 \) be defined, respectively, by

\[
\lambda_n = \frac{1}{\|A\| \|J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n\|} \text{ and } \mu_n = \frac{1}{\|x_n\|^{p-1}}.
\]

(2.2)

Then

\[
\frac{1}{q} \bar{\sigma}_q (J_{E_1}^p x_n, \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n) \leq \begin{cases}
  2^q G_q \|J_{E_1}^p x_n\|^q \rho_{E_1} (\mu_n) & \text{if } \mu_n \in (0, 1], \\
  2^q G_q \rho_{E_1} (\mu_n) & \text{if } \mu_n \in (1, \infty),
\end{cases}
\]

(2.3)

where \( G_q \) is the constant defined in Lemma 1.9 and \( \rho_{E_1} \) is the modulus of smoothness of \( E_1^* \).

**Proof.** By Lemma 1.9, we have

\[
\frac{1}{q} \bar{\sigma}_q (J_{E_1}^p x_n, \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n) = G_q \int_0^1 \left( \frac{\|J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n\| \vee \|J_{E_1}^p x_n\|}{t} \right)^q dt,
\]

(2.4)

for every \( t \in [0, 1] \).

We note that

\[
\|J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n\| \leq \|x_n\|^{p-1} + \|\lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n\|.
\]

By (2.2), with \( x_n \neq 0 \)

\[
\lambda_n = \frac{\mu_n}{\|A\| \|J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n\|}
\]

(2.5)

and so we have that

\[
\|J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n\| \leq (1 + \mu_n) \|x_n\|^{p-1}
\]

and

\[
\begin{cases}
  \|x_n\|^{p-1} \leq \|J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n\| \vee \|J_{E_1}^p x_n\| \leq 2 \|x_n\|^{p-1} & \text{if } \mu_n \in (0, 1], \\
  \|x_n\|^{p-1} \leq \|J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_n}^T) A x_n\| \vee \|J_{E_1}^p x_n\| \leq 2 & \text{if } \mu_n \in (1, \infty).
\end{cases}
\]

(2.6)
By (2.6), (2.5) and Definition 1.5(2), we get
\[
\rho_{E_1} \left( \frac{t \| \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \|}{\| J_{E_2}^p x_n - \Pi_{AK}^\nabla B_{b_n}^T A_{x_n} \|} \right) \leq \rho_{E_1} \left( \frac{t \| \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \|}{\| x_n \|^{p-1}} \right) = \rho_{E_1} (t \mu_n).
\]
(2.7)
Substituting (2.7) and (2.6) into (2.4), and using nondecreasingness of \( \rho_{E_1} \), we get (2.3) as required.

**Lemma 2.2.** For the sequence \( \{ x_n \} \subset E_1 \) defined by (2.1), let \( 0 \neq x_n, 0 \neq J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \in E_2^\nabla \), and \( \lambda_n > 0 \) and \( \mu_n > 0 \) be defined by (2.2) and \( \lambda_n \) and \( \mu_n \) are chosen such that
\[
\rho_{E_1} (\mu_n) = \begin{cases} 
\frac{1}{2G_\|A\|} \times \frac{\| J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \|^q}{\| J_{E_2}^p x_n \|^q \| J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \|^p}, & \text{if } \mu_n \in (0, 1], \\
\frac{1}{2G_\|A\|} \times \frac{\| J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \|^q}{\| J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \|^p}, & \text{if } \mu_n \in (1, \infty),
\end{cases}
\]
where \( q \in (0, 1) \). Then, for all \( v \in \Omega \), we get
\[
\triangle_p (u_n, v) \leq \triangle_p (x_n, v) - [1 - l] \frac{\langle J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n}, (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \rangle}{\| A \| \| J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \|^p}.
\]
(2.9)

**Proof.** For \( v = B_n^U x_n ) \) and \( B_n^V A_{x_n} \), by Lemma 1.7, we have that
\[
\triangle_p (u_n, v) = \triangle_p \left( B_n^U \left( J_{E_1}^p \left( J_{E_2}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \right) \right), v \right) = \triangle_p \left( B_n^U \left( J_{E_1}^p \left( J_{E_2}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \right) \right), B_n^V v \right) \leq V_p \left( J_{E_2}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n}, v \right) \leq \frac{1}{q} \| J_{E_2}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} \|^q + \frac{1}{p} \| v \|^p - \langle J_{E_1}^p x_n, v \rangle + \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n}, v \rangle,
\]
(2.10)
where
\[
\langle \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n}, v \rangle = \langle \lambda_n J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n}, v \rangle - \langle \lambda_n J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} - A_{x_n}, v \rangle + \langle \lambda_n J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n} - A_{x_n}, v \rangle
\]
As \( AK \) is closed and convex so by Lemma 1.8(i) and the variational inequality for the Bregman projection of zero onto \( AK - Ax_n \), as in Lemma 1.8(ii), we arrive at
\[
\langle \lambda_n J_{E_2}^p (\Pi_{AK}^\nabla B_{b_n}^T - I) A_{x_n}, (Av - Ax_n) - (\Pi_{AK}^\nabla B_{b_n}^T - I) A_{x_n} \rangle \geq 0
\]
and therefore, we obtain
\[
\langle \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n}, v \rangle \leq - \langle \lambda_n J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n}, v \rangle + \langle \lambda_n J_{E_2}^p (I - \Pi_{AK}^\nabla B_{b_n}^T) A_{x_n}, A_{x_n} \rangle.
\]
(2.11)
In addition, by Lemma 2.1, we have that
\[
\frac{1}{q} \left\| J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \right\|^q \leq \frac{1}{q} \left\| J_{E_1}^p x_n \right\|^q - \lambda_n \langle A x_n, J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \rangle \\
+ \frac{1}{q} \sigma_q \left( J_{E_1}^p x_n, \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \right) (2.12)
\]

By Lemma 2.1 and (2.12), we have that
\[
\frac{1}{q} \left\| J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \right\|^q \leq \frac{1}{q} \left\| J_{E_1}^p x_n \right\|^q - \lambda_n \langle A x_n, J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \rangle \\
+ 2^q G_q \| J_{E_1}^p x_n \|^q \rho_{E_1^p} (\mu_n). (2.13)
\]

Substituting (2.13) and (2.11) into (2.10), we have that
\[
\triangle_p(u_n, v) \leq \frac{1}{q} \left\| J_{E_1}^p x_n \right\|^q + \frac{1}{p} \| v \|^p - \langle J_{E_1}^p x_n, v \rangle + 2^q G_q \| J_{E_1}^p x_n \|^q \rho_{E_1^p} (\mu_n) \\
- \langle \lambda_n J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n, (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \rangle \\
= \triangle_p (x_n, v) + 2^q G_q \| J_{E_1}^p x_n \|^q \rho_{E_1^p} (\mu_n) \\
- \langle \lambda_n J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n, (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \rangle. (2.14)
\]

Substituting (2.2) and (2.8) into (2.14), we have that
\[
\triangle_p(u_n, v) \leq \triangle_p (x_n, v) + \frac{\langle J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n, (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \rangle}{\| A \| \left\| J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A u_n \right\|} \\
- \frac{\langle J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n, (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \rangle}{\| A \| \left\| J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \right\|} \\
= \triangle_p (x_n, v) - [1 - v] \frac{\langle J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n, (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \rangle}{\| A \| \left\| J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \right\|}.
\]

Thus, (2.9) holds. \(\square\)

We now prove our main result.

**Theorem 2.3.** For \(\delta > 0\) and \(p, q \in (1, \infty)\), let \((I - \Pi_{AK}^p B_{\delta_0}^T)\) be demiclosed at zero. Let \(x_1 \in E_1\) be chosen arbitrarily and the sequence \(\{x_n\}\) be defined by (2.4), where
\[
\lambda_n = \begin{cases} 
\frac{1}{\| A \| \left\| J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \right\|}, & x_n \neq 0 \\
\frac{1}{\| A \|^p \left\| J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \right\|^p}, & x_n = 0
\end{cases}
\quad x_n \neq 0 \quad \text{and} \quad \mu_n = \frac{1}{\| x_n \|^{p-1}}, \quad x_n = 0 (2.15)
\]
are chosen such that equation (2.8) holds. If \(\Omega = \{x^* \in SOLVIP(U); Ax^* \in SOLVIP(T)\} \neq \emptyset\), then \(x_n\) converges strongly to \(x^* \in \Omega\), where \(\Pi_{AK}^p B_{\delta_0}^T (Ax^*) = B_{\delta_0}^T (Ax^*)\).

**Proof.** We will divide the proof into two steps.

**Step one.** We show that \(\{x_n\}\) is a bounded sequence.

Assume that \(\| J_{E_2}^p (I - \Pi_{AK}^p B_{\delta_0}^T) A x_n \| = 0\). Then from \(v = B_{\delta_0}^T v\), Lemma 1.7 and \(v \in \Omega\), we get
\[
\triangle_p(u_n, v) = \triangle_p \left( B_{\delta_0}^T (J_{E_1}^p (J_{E_1}^p x_n)), B_{\delta_0}^T v \right) \leq V_p (J_{E_1}^p x_n, v) = \triangle_p (x_n, v). (2.16)
\]
Next assume that $\|J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n\| \neq 0$ and $x_n \neq 0$. Then for $v \in \Omega$, by Lemma 2.2, we get

$$\triangle_p(u_n, v) \leq \triangle_p(x_n, v) - |1 - i| \frac{\langle J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n, (I - \Pi^p_{AK}B^T_{\delta_n})Ax_n \rangle}{\|A\| \|J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n\|}$$

(2.17)

$$\leq \triangle_p(x_n, v).$$

(2.18)

For $x_n = 0$, we have

$$\triangle_p(x_n, v) = \frac{1}{p} \|v\|^p$$

(2.19)

and so by (2.19), we have that

$$\triangle_p(u_n, v) = \frac{1}{q} \|\lambda_n A^*J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n\|^q$$

$$+ \triangle_p(x_n, v) + \lambda_n \langle J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n, Av \rangle.$$

(2.20)

Substituting (2.11) in (2.20), we have that

$$\triangle_p(u_n, v) \leq \frac{1}{q} \|\lambda_n A^*J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n\|^q$$

$$+ \triangle_p(x_n, v) + \lambda_n \langle J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n, Ax_n \rangle$$

$$- \lambda_n \langle J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n, (I - \Pi^p_{AK}B^T_{\delta_n})Ax_n \rangle.$$

(2.21)

By (2.15), we have that

$$\frac{1}{q} \|\lambda_n A^*J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n\|^q = \frac{1}{q} \frac{\langle J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n, (I - \Pi^p_{AK}B^T_{\delta_n})Ax_n \rangle^p}{\|J^p_{E_2}(I - \Pi^p_{AK}B^T_{\delta_n})Ax_n\|^p}.$$
We know from (2.1), \( x_n = \Pi_{K_n} x_1 \). Then, by Lemma 1.8, we have
\[
\Delta_p(x_n, x_1) = \Delta_p(\Pi_{K_n} x_1, x_1) \leq \Delta_p(v, x_1) - \Delta_p(x_n, x_1) \Rightarrow \Delta_p(x_n, x_1) \leq \Delta_p(v, x_1) \forall v \in \Omega \subset K_n.
\] (2.25)
By (2.25), the sequence \( \{\Delta_p(x_n, x_1)\} \) is bounded and therefore by Lemma 1.12, \( \{x_n\} \) is bounded. Hence, \( \{u_n\} \) is also bounded. Consequently, there exists a subsequence \( x_{n_j} \) such that \( x_{n_j} \rightharpoonup x^* \) as \( j \to \infty \) (\( \rightharpoonup \) stands for weak convergence).

**Step two.** We show that \( x_n \to x^* \in \Omega \).

Since \( x_{n+1} = \Pi_{K_{n+1}} x_1 \subset K_{n+1} \subset K_n \) and \( J^p \) is weakly sequentially continuous, we have by Lemma 1.11
\[
\Delta_p(u_n, x_n) = \Delta_p(u_n, x_{n+1}) + \Delta_p(x_{n+1}, x_n) + \langle u_n - x_{n+1}, J^p_{E_1} x_{n+1} - J^p_{E_1} x_n \rangle \\
\leq \Delta_p(x_{n+1}, x_n) + \Delta_p(x_{n+1}, x_n) + \langle u_n - x_{n+1}, J^p_{E_1} x_{n+1} - J^p_{E_1} x_n \rangle \\
\to 0 \text{ as } n \to \infty.
\] (2.26)
It follows from (2.1) that
\[
\frac{(J^p_{E_1} x_n - J^p_{E_1} u_n) - \lambda_n A^*(J^p_{E_2} (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n)}{\delta_n} \in U(u_n).
\] (2.27)
By (2.17), we have that
\[
\Delta_p(u_n, v) \leq \Delta_p(x_n, v) - |1 - t| \frac{\langle J^p_{E_2} (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n, (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n \rangle}{\|A\| \|J^p_{E_2} (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n\|},
\]
and
\[
\| (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n \| \leq \left[ \frac{\Delta_p(x_n, v) - \Delta_p(u_n, v)}{\|A\|^{-1}[1 - t]} \right] \to 0 \text{ as } n \to \infty.
\] (2.28)
By (2.23), we have that
\[
\Delta_p(u_n, v) \leq \Delta_p(x_n, v) - \frac{1}{p} \|A\|^p \frac{\langle J^p_{E_2} (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n, (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n \rangle}{\|J^p_{E_2} (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n\|^p}
\]
and therefore
\[
\| (I - \Pi^p_{A^* B^T_{\delta_n}}) Ax_n \| \leq \left[ \frac{\Delta_p(x_n, v) - \Delta_p(u_n, v)}{\|A\|^{-1}} \right]^p \to 0 \text{ as } n \to \infty.
\] (2.29)
By (2.26) to (2.29) and weak sequential continuity property of \( J^p \), we have that \( 0 \in U(x^*) \). This means that \( x^* \in SOLVIP(U) \). But, since \( \Delta_p(\cdot, x) \) is lower semi continuous and convex and thus weakly lower semi continuous on \( \text{int(domf)} \) then from the fact that \( x_{n_j} \rightharpoonup x^* \) as \( j \to \infty \), we see that
\[
\Delta_p(x^*, x_1) \leq \liminf_{j \to \infty} \Delta_p(x_{n_j}, x_1) \leq \Delta_p(v, x_1).
\]
From the definition of \( v \), that is \( v = B^T_{\delta_n}(v) \), we can conclude that \( x^* = v \) and the sequence \( x_n \to x^* \). In addition, it is clear that \( Ax_n \to Ax^* \). So by using (2.28), (2.29) and applying the demicloseness of \( (I - \Pi^p_{A^* B^T_{\delta_n}}) \) at zero, we have that \( 0 \in T(Ax^*) \) as \( \Pi^p_{A^* B^T_{\delta_n}}(Ax^*) = B^T_{\delta_n}(Ax^*) \). Therefore \( Ax^* \in SOLVIP(T) \). Hence, \( x^* \in \Omega \).
Finally, by Lemma 1.11, we have
\[
\limsup_{n \to \infty} \Delta_p(x_n, x^*) = \limsup_{n \to \infty} \left[ \Delta_p(x_n, x_1) + \Delta_p(x_1, x^*) + \langle x_n - x_1, J_{E_1}^p x_1 - J_{E_1}^p x^* \rangle \right]
\leq \limsup_{n \to \infty} \left[ \Delta_p(x^*, x_1) + \Delta_p(x_1, x^*) + \langle x_n - x_1, J_{E_1}^p x_1 - J_{E_1}^p x^* \rangle \right]
= \limsup_{n \to \infty} \langle x^* - x_n, J_{E_1}^p x^* - J_{E_1}^p x_1 \rangle = 0.
\]

Thus, we obtain \( \lim_{n \to \infty} \Delta_p(x_n, x^*) = 0 \). Hence by Lemma 1.6 we get \( x_n \to x^* \) as \( n \to \infty \).

If \( U : E_1 \to E_1 \) and \( T : E_2 \to E_2 \) are nonexpansive in Theorem 2.3, then we get:

**Corollary 2.4.** For \( \delta > 0 \) and \( p, q \in (1, \infty) \), let \( (I - \Pi_{\Pi_A K}^p T) \) be demiclosed at zero. Let \( x_1 \in E_1 \) be chosen arbitrarily and the sequence \( \{x_n\} \) be defined as follows;

\[
\begin{align*}
\{ u_n \} &= U_n \left( J_{E_1}^p \left( J_{E_2}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{\Pi_A K}^p T) A x_n \right) \right), \\
K_{n+1} &= \{ v \in K_n : \Delta_p(u_n, v) \leq \Delta_p(x_n, v) \}, \\
x_{n+1} &= \Pi_{\Pi_A K}^p (x_1), n \geq 1,
\end{align*}
\]

where
\[
\lambda_n = \begin{cases} 
\frac{1}{\|x_n\|^p} \left( \frac{1}{\|J_{E_1}^p (I - \Pi_{\Pi_A K}^p T) A x_n\|^p} + \frac{1}{\|J_{E_2}^p (I - \Pi_{\Pi_A K}^p T) A x_n\|^p} \right), & x_n \neq 0 \\
\frac{1}{\|x_n\|^p}, & x_n = 0,
\end{cases}
\]

and \( \mu_n = \frac{1}{\|x_n\|^{p-\delta}} \) are chosen such that
\[
\rho_{E_1}^\delta(\mu_n) = \begin{cases} 
\frac{\epsilon}{2\|x_n\|^{\delta}} \frac{\langle J_{E_2}^p (I - \Pi_{\Pi_A K}^p T) A x_n, (I - \Pi_{\Pi_A K}^p T) A x_n \rangle}{\|J_{E_1}^p x_n\|^p \|J_{E_2}^p (I - \Pi_{\Pi_A K}^p T) A x_n\|^p}, & \text{if } \mu_n \in (0, 1], \\
\frac{\epsilon}{2\|x_n\|^{\delta}} \frac{\langle J_{E_2}^p (I - \Pi_{\Pi_A K}^p T) A x_n, (I - \Pi_{\Pi_A K}^p T) A x_n \rangle}{\|J_{E_2}^p (I - \Pi_{\Pi_A K}^p T) A x_n\|^p}, & \text{if } \mu_n \in (1, \infty),
\end{cases}
\]

where \( \epsilon \in (0, 1) \). If \( F(U) \) and \( F(\Pi_{\Pi_A K}^p T) \) denote the fixed point set of \( U \) and \( \Pi_{\Pi_A K}^p T \), respectively, and \( \Omega = \{ x^* \in F(U) : Ax^* \in F(\Pi_{\Pi_A K}^p T) \} \neq \emptyset \), then \( \{x_n\} \) converges strongly to \( x^* \in \Omega \), where \( \Pi_{\Pi_A K}^p T(Ax^*) = T(Ax^*) \).

**Remark 2.5.** Corollary 2.4 generalizes the corresponding results in [6, 7, 11, 15–17, 22, 23, 28]. In particular, it improves and extends the main result in [11] in the following aspects:

1. we use a simpler algorithm,
2. our split variational inclusion problem contains, as special case, their split feasibility problem,
3. we work in a more general Banach space than \( p \)-uniformly convex.

In Theorem 2.3, let \( \Pi_{\Pi_A K}^p = \Pi_{\Pi_A B_3^p} \) and \( \Pi_K = B_3^U \), where \( \Pi_K : E_1 \to K \) is the Bregman projection from \( E_1 \) onto \( K \). Then we get the following result.

**Corollary 2.6.** For \( \delta > 0 \) and \( p, q \in (1, \infty) \), let \( \Pi_K : E_1 \to K \) be the Bregman projection from \( E_1 \) onto \( K \) and \( (I - \Pi_{\Pi_A K}^p) \) be demiclosed at zero. Let \( x_1 \in E_1 \) be chosen arbitrarily and the sequence \( \{x_n\} \) be defined as follows;

\[
\begin{align*}
\{ u_n \} &= J_{E_1}^p \left( J_{E_2}^p x_n - \lambda_n A^* J_{E_2}^p (I - \Pi_{\Pi_A K}^p) A x_n \right), \\
K_{n+1} &= \{ v \in K_n : \Delta_p(u_n, v) \leq \Delta_p(x_n, v) \}, \\
x_{n+1} &= \Pi_{\Pi_K}^p (x_1), n \geq 1,
\end{align*}
\]
where

$$\lambda_n = \begin{cases} \frac{1}{\|A\|} \|J_{E_2}^p (I-\Pi_{AK}^{p})Ax_n\|, & x_n \neq 0 \\ \frac{1}{\|A\|} \left\langle J_{E_2}^p (I-\Pi_{AK}^{p})Ax_n, (I-\Pi_{AK}^{p})Ax_n \right\rangle, & x_n = 0, \end{cases}$$

and $\mu_n = \frac{1}{\|x_n\|^{p-1}}$ are chosen such that

$$\rho_{E_1^*}(\mu_n) = \begin{cases} \frac{2^p \|A\|}{\mu_n} \times \frac{\left\langle J_{E_2}^p (I-\Pi_{AK}^{p})Ax_n, (I-\Pi_{AK}^{p})Ax_n \right\rangle}{\|J_{E_1}^p (I-\Pi_{AK}^{p})Ax_n\|}, & \text{if } \mu_n \in (0, 1], \\ \frac{2^p \|A\|}{\mu_n} \times \frac{\left\langle J_{E_2}^p (I-\Pi_{AK}^{p})Ax_n, (I-\Pi_{AK}^{p})Ax_n \right\rangle}{\|J_{E_1}^p (I-\Pi_{AK}^{p})Ax_n\|}, & \text{if } \mu_n \in (0, \infty), \end{cases}$$

where $\epsilon \in (0, 1)$. If $\Omega = \{x^* \in K; Ax^* \in AK\} \neq \emptyset$, then $\{x_n\}$ converges strongly to $x^* \in \Omega$, where $\Pi_{AK}^p T(Ax^*) = T(Ax^*)$.

**Remark 2.7.** Corollary 2.6 generalizes split feasibility problem result of Chen et al. [11] in the sense of Remark 2.5 (1) and (3). Moreover, this result, holds in a broader framework than a Hilbert space, so it generalizes the main result in [13].

Let $E = E_1 = E_2$ be a Hilbert space, $I = J_{E_1}^p = J_{E_2}^p = J_{E_2^*}^q = A^*$, $p = q = 2$, and let $U, T : E \to E$ be nonexpansive mappings. Suppose $F(U) \neq \emptyset$ and $F(T) \neq \emptyset$. The so-called hierarchical variational inequality problem for nonexpansive mapping $U$ with respect to a nonexpansive mapping $T$ is to find a point $x^* \in F(U)$ such that

$$\langle x^* - Tx^*, x^* - x \rangle \leq 0, \forall x \in F(U). \quad (2.30)$$

It is easy to see that (2.30) is equivalent to the following fixed point problem: find $x^* \in F(U)$ such that $Ax^* \in F(P_{F(T)}T)$, where $P_{F(T)} : E \to F(T)$ is the metric projection from $E$ onto $F(T)$. Hence by Theorem 2.3, we deduce the following:

**Corollary 2.8.** For $\delta > 0$, let $(I - P_{F(T)}T)$ be demiclosed at zero. Let $x_1 \in E$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;

$$\begin{cases} u_n = U_n \left( x_n - \lambda_n (I - P_{F(T)}T_n)x_n \right), \\ K_{n+1} = \{ v \in K_n : \|u_n, v\| \leq \|x_n, v\| \}, \\ x_{n+1} = P_{K_{n+1}}(x_1), \quad n \geq 1, \end{cases}$$

where

$$\lambda_n = \begin{cases} \frac{1}{\|I - P_{F(T)}T_n\|}, & x_n \neq 0 \\ 1, & x_n = 0, \end{cases}$$

and $\mu_n = \frac{1}{\|x_n\|}$ are chosen such that

$$\rho_{E}(\mu_n) = \begin{cases} \frac{\|I - P_{F(T)}T_n\|}{4G_2 \|x_n\|^2}, & \text{if } \mu_n \in (0, 1], \\ \frac{\|I - P_{F(T)}T_n\|}{4G_2}, & \text{if } \mu_n \in (0, \infty), \end{cases}$$

where $\epsilon \in (0, 1)$. If $F(U) \neq \emptyset$ and $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a solution of the hierarchical variational inequality problem (2.30), where $P_{F(T)}T(x^*) = T(x^*)$. 
The split minimization problem is to find:

\[ x^* \in E_1 \text{ such that } h(x^*) \leq h(x) \forall x \in E_1 \]

and

\[ Ax^* \in E_2 \text{ such that } h'(Ax^*) \leq h'(Ax) \forall Ax \in E_2 \]

where \( h : E_1 \to R \) and \( h' : E_2 \to R \) are convex lower semicontinuous functions. Now let the subdifferential of \( h \) and \( h' \), \( \partial h : E_1 \to 2^{E_1} \) and \( \partial h' : E_2 \to 2^{E_2} \) be defined by

\[ (\partial h)x = \{ x^* \in E_1^* : h(y) - h(x) \geq \langle y - x, x^* \rangle \forall y \in E_1 \} \]

and

\[ (\partial h')Ax = \{ Ax^* \in E_2^* : h'(Ay) - h'(Ax) \geq \langle Ay - Ax, Ax^* \rangle \forall Ay \in E_2 \}, \]

respectively.

It is well known that \( \partial h \) and \( \partial h' \) are maximal monotone on \( E_1 \) and \( E_2 \) and that 0 \( \in (\partial h)x \) and 0 \( \in (\partial h')Ax \) if \( x \) and \( Ax \) are minimizers of \( h \) and \( h' \), respectively. Hence

\[ B^{\partial h}_h = \text{prox}_{\delta h} \text{ and } B^{\partial h'}_h = \text{prox}_{\delta h'}. \]

In Theorem 2.3, \( U = \partial h \) and \( T = \partial h' \), give the following result.

**Theorem 3.1.** Let the mapping of \( \partial h, \partial h', \Pi^p_{\partial h}, \Pi^p_{\partial h'} \) and \( \text{prox}_{\delta h}, \text{prox}_{\delta h'} \) be defined as above. For \( \delta > 0 \) and \( p, q \in (1, \infty) \), let \( (I - \Pi^p_{\partial h}) \) be demiclosed at zero. Let \( x_1 \in E_1 \) be chosen arbitrarily and the sequence \( \{ x_n \} \) be defined as follows:

\[
\begin{align*}
K_{n+1} &= \{ v \in K_n : \Delta_p(u_n, v) \leq \Delta_p(x_n, v) \}, \\
x_{n+1} &= \Pi_{K_{n+1}}(x_1), n \geq 1,
\end{align*}
\]

where

\[
\lambda_n = \begin{cases} 
\frac{1}{\| A \|} \frac{\| J^{p}_{E_1}(I - \Pi^p_{\partial h})Ax_n \|}{\| J^{p}_{E_1}(I - \Pi^p_{\partial h})Ax_n \|^p}, & x_n \neq 0 \\
\frac{1}{\| A \|^p} \frac{\langle J^{p}_{E_2}(I - \Pi^p_{\partial h})Ax_n, (I - \Pi^p_{\partial h})Ax_n \rangle}{\| J^{p}_{E_2}(I - \Pi^p_{\partial h})Ax_n \|^p}, & x_n = 0,
\end{cases}
\]

and \( \mu_n = \frac{1}{\| x_n \|^p} \) are chosen such that

\[
\rho_{E_1}(\mu_n) = \begin{cases} 
\frac{\| J^{p}_{E_2}(I - \Pi^p_{\partial h})Ax_n, (I - \Pi^p_{\partial h})Ax_n \|}{\| J^{p}_{E_2}(I - \Pi^p_{\partial h})Ax_n \|^p}, & \text{if } \mu_n \in (0, 1], \\
\frac{\| J^{p}_{E_2}(I - \Pi^p_{\partial h})Ax_n, (I - \Pi^p_{\partial h})Ax_n \|}{\| J^{p}_{E_2}(I - \Pi^p_{\partial h})Ax_n \|^p}, & \text{if } \mu_n \in (0, \infty),
\end{cases}
\]

where \( \iota \in (0, 1) \). If \( \Omega = \{ x^* \in E_1 : h(x^*) \leq h(x) \text{ and } h'(Ax^*) \leq h'(Ax), \forall x \in E_1 \} \neq \emptyset \), then \( \{ x_n \} \) converges strongly to \( x^* \in \Omega \), where \( \Pi^p_{\partial h}(Ax^*) = \text{prox}_{\delta h'}(Ax^*) \).
4. A NUMERICAL EXAMPLE

Let $E_1 = E_2 = \mathbb{R}$, $K = AK = [0, \infty)$ and $Ax = x \forall x \in E_1$. Define

$$U, T : \mathbb{R} \rightarrow \mathbb{R} \text{ by } U(x) = T(Ax) = \begin{cases} [0, 1], & x \geq 0 \\ \{1\}, & x < 0, \end{cases}$$

$$P_{[0, \infty)} : \mathbb{R} \rightarrow [0, \infty) \text{ by } P_{[0, \infty)}(Ax) = \begin{cases} 0, & Ax \in (-\infty, 0) \\ Ax, & Ax \in [0, \infty), \end{cases}$$

$$(I + \delta U)^{-1} = (I + \delta T)^{-1} : \mathbb{R} \rightarrow \mathbb{R}$$

by $$(I + \delta T)^{-1}(Ay) = (I + \delta U)^{-1}(y) = \begin{cases} \frac{y}{1 + [0, \delta]}, & y \geq 0 \\ \frac{y}{1 + \delta}, & y < 0, \end{cases}$$

$$P_{[0, \infty)}(I + \delta T)^{-1} : \mathbb{R} \rightarrow [0, \infty) \text{ by } P_{[0, \infty)}(I + \delta T)^{-1}(Ay) = \begin{cases} Ay, & Ay \geq 0 \\ 0, & Ay < 0. \end{cases}$$

It is clear that $U$ and $T$ are multi-valued maximal monotone mappings such that $0 \in SOLVIP(U)$ and $0 \in SOLVIP(T)$. For $\delta_n = 2^n$,

$$\lambda_n = \begin{cases} \frac{|1 + [0, 2^n]|}{|x_n(1 + [0, 2^n]) - x_n|}, & x_n > 0, \\ 1, & x_n = 0, \\ \frac{1}{|x_n|}, & x_n < 0, \end{cases}$$

we get that

$$u_n = \begin{cases} \frac{x_n}{1 + [0, 2^n]}(x_n - 1), & x_n > 0, \\ 0, & x_n = 0, \\ \frac{x_n}{2^n + 1}(x_n + 1), & x_n < 0, \end{cases}$$

$$K_{n+1} = \left\{ v \in K_n : v \leq \frac{x_n - u_n}{2} \right\},$$

$$x_{n+1} = P_{K_{n+1}}x_1 = \begin{cases} \frac{x_n - \frac{x_n}{1 + [0, 2^n]}(x_n - 1)}{2}, & x_n > 0, \\ 0, & x_n = 0, \\ \frac{x_n - \frac{x_n}{2^n + 1}(x_n + 1)}{2}, & x_n < 0. \end{cases}$$

In particular,

$$x_{n+1} = \begin{cases} \frac{x_n - \frac{x_n}{1 + [0, 2^n]}(x_n - 1)}{2}, & x_n > 0, \\ 0, & x_n = 0, \\ \frac{x_n - \frac{x_n}{2^n + 1}(x_n + 1)}{2}, & x_n < 0. \end{cases}$$

Now by Theorem 2.3, the sequence $\{x_n\}$ converges strongly to $0 \in \Omega$. The Figures 1 and 2 below obtained by \textit{MATLAB} software indicate convergence of $\{x_n\}$ given by (2.1) with $x_1 = 1.0$ and $x_1 = -1.0$, respectively.
Figure 1. Convergence behavior of the sequence $\{x_n\}$ in (2.1) with $x_1 = 1.0$.

Figure 2. Convergence behavior of the sequence $\{x_n\}$ in (2.1) with $x_1 = -1.0$.

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