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ALTERNATIVE APPROACH BASED ON ROOTS FOR COMPUTING THE STATIONARY QUEUE-LENGTH DISTRIBUTIONS IN $GI^X/M^{(1,b)}/1$ SINGLE WORKING VACATION QUEUE

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Abstract. The purpose of this paper is to present an alternative algorithm for computing the stationary queue-length and system-length distributions of a single working vacation queue with renewal input batch arrival and exponential holding times. Here we assume that a group of customers arrives into the system, and they are served in batches not exceeding a specific number b. Because of batch arrival, the transition probability matrix of the corresponding embedded Markov chain for the working vacation queue has no skip-free-to-the-right property. Without considering whether the transition probability matrix has a special block structure, through the calculation of roots of the associated characteristic equation of the generating function of queue-length distribution immediately before batch arrival, we suggest a procedure to obtain the steady-state distributions of the number of customers in the queue at different epochs. Furthermore, we present the analytic results for the sojourn time of an arbitrary customer in a batch by utilizing the queue-length distribution at the pre-arrival epoch. Finally, various examples are provided to show the applicability of the numerical algorithm.

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1. Introduction

From a mathematical modeling point of view, batch-arrival bulk-service queue under various vacation policies is an extension of individual arrival and single service vacation queue. For the vast majority of real-world applications, bulk arrivals and services are the only realistic assumptions. Such queues are very useful to investigate the performance measures of systems in the areas like manufacturing, production, telecommunication, and transportation. As far as vacation or batch-arrival bulk-service systems are concerned, many authors have addressed these topics during the last several decades. Readers are urged to see Doshi [19], Takagi [37], Alfa [2], and Chaudhry and Templeton [14] for a more comprehensive survey of related references. However, being limited by the method and analytic technique, the generalization of batch-arrival bulk-service queue incorporating the concept of server vacation has not been fully considered in the previous literature. Notably, due to model complexity and analytical difficulty, there is little research on the renewal input batch-arrival bulk-service

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vacation queue. Numerous theoretical and computational issues for this type of queue have been left unexplored. We think one primary reason behind this situation is that the renewal input batch-arrival queue has a more general block-structured Markov chain. The transition probability matrix associated with the renewal batch-arrival process has no requisite particular block structure, the matrix-analytic method developed by Neuts [29] does not work well in the analysis of such a queue. Thus, the practical difficulties described above motivate us to consider algorithmic aspects of this queueing model and look for a feasible way to estimate the queue size.

Objectively speaking, queues of type $GI^X/M/1$ are more challenging from a mathematical point of view than the corresponding $M^X/G/1$ type queue. The introduction of batch service and semi-vacation policy in $GI^X/M/1$ queue further makes the model more complex both from the mathematical and computational points of view. To better demonstrate this research topic's necessity, let us briefly review the work done in the field of batch-arrival bulk-service vacation queue. Earlier studies in this area have mainly focused on Poisson or Bernoulli batch arrival vacation queue with general bulk service times. Chang and Choi [11] presented a complete analytic and computational framework for handling the $Geo^X/G^Y/1/K+B$ vacation queue with variable service capacity and finite waiting space. Using similar arguments, Chang and Choi [12] also discussed algorithmic aspects of $M^X/G^Y/1/K+B$ queue with server startup time. Later, inspired by the above work, Samanta, Chaudhry and Gupta [33] further investigated discrete-time $Geo^X/G^{(a,b)}/1/N$ vacation queue with general bulk service rule. It is worth mentioning that the analysis method adopted by them differs significantly from the one by Chang and Choi, as previously stated. In their work, the queue-length distribution at an arbitrary epoch is derived using the supplementary variable technique and treating the supplementary variable as the remaining bulk service time. Additionally, by resorting to the same method, Yu *et al.* [40] firstly studied $GI^X/M^b/1/N$ bulk-service working vacation queue with general inter-batch arrival time, and developed an approach to compute the queue-length distributions at pre-arrival and arbitrary epochs. As far as we know, this is the only journal paper related to renewal batch input and bulk-service vacation queue. On the other hand, their infinite buffer size counterparts received relatively little attention among researchers. Except for a limited number of studies done by several Indian scholars and Belgian scholars (see, [3, 4, 18, 23, 25]), no work in this direction has come to our notice. Simultaneously, we note that due to the remarkable properties of the Poisson process, the existing literature presented above primarily discussed infinite buffer capacity bulk-service vacation queue with batch Poisson arrivals. While, for the case of renewal input batch-arrival bulk-service vacation queue, we still lack an efficient way to deal with them uniformly. Therefore, despite its wide potential applications in the manufacturing and production industry, some profound results on this queueing problem have not yet been systematically studied. This fact can be further confirmed from the monograph written by Tian and Zhang [38], and a series of work done by Li and Tian [26, 27], and Chae *et al.* [10]. We may quickly find that the authors concentrated their attention on the renewal input single arrival and individual service queue subject to different types of vacation mechanisms instead of the more realistic batch-arrival bulk-service vacation queue.

However, fortunately, the recent work done by Chaudhry and his collaborators [5–7, 13, 16, 17, 28, 31, 32, 34] inspires us to overcome difficulties in analyzing renewal input batch-arrival bulk-service vacation queue. Compared with other approaches, the method advocated by Chaudhry can cleverly avoid high complexity caused by the structure of the transition matrix. Moreover, under such an analytical framework, the queue-length distribution at the pre-arrival epoch can be obtained by solving a system of linear equations. It also implies that the method developed by Chaudhry is easy to implement and computationally efficient. Thus, following the idea of Chaudhry, a simple solution to determine the queue-length distribution and the sojourn time of an arbitrary customer for the $GI^X/M^{(1,b)}/1/\infty$ single working vacation (SWV) queue is present in our current work. Although our work has some similarities to the existing literature (see, [21, 22, 30]), due to considering batch-arrival and bulk-service at the same time, the mathematical discussion of the transition probability at the embedded pre-arrival epochs is much more complicated than the corresponding part of the above references. Moreover, we note that a system of linear equations that determines the unknown coefficients of the partial fractions was not explicitly reported in the literature mentioned above. Authors just stated how to obtain these unknown coefficients by solving a system of linear equations, but the specific form of the linear equations that must be used in numerical programming is not given. In our paper, the linear algebraic equations for finding the unknown coefficients are clearly presented. Meanwhile, we also note that the numerical experiments conducted in the above three articles are very limited. The arrival processes are assumed to be Markovian or phase-type (PH). Undoubtedly this does not reflect the advantages of the model in the arrival process. Thus, the conclusion of other arbitrary arrival distributions that do not belong to the class of PH distribution in our numerical experiments is also an extension of the previous work.

The concept of working vacation policy was introduced by Servi and Finn [35] for use in the analysis of a Wavelength-Division Multiplexing optical access network. It is well known that in the classical vacation queue, the server does not continue its original work during the vacation period, and this type of policy may cause the loss or dissatisfaction of the customers. Unlike standard vacation policy, during working vacation period, the server serves customers at a lower service rate rather than completely stopping service. That is to say, the server can still work during the vacation and may accomplish other assistant work simultaneously. Undoubtedly, such a policy reduces the chance of reneging of the customers compared to regular vacation policy. Also, a working vacation queue can well describe the operation mode of the original equipment manufacturer (OEM). In order to increase the production capacity, shift work, particularly work including night shifts, is often adopted by the OEM managers. The term shift work usually means that people have a work schedule outside the standard daytime from 9 AM to 5 PM. OEM can rely on the work provided by the shift workers during non-working hours to fulfill more customer orders. It is precise because of its application in many fields that a growing number of researchers have focused their attention on working vacation queues in recent years.

The remainder of this paper is organized as follows. Section 2 gives the underlying model assumptions and introduces the notations to describe the model parameters. Queue-length distributions at pre-arrival and arbitrary epochs are analyzed in Sections 3 and 4, respectively. Additionally, in Sections 5 and 6, we discuss several other quantitative measures such as system-length distributions at two different epochs and the sojourn time of a randomly chosen tagged customer in an arriving batch. Finally, numerical illustration has been done in Section 7, and conclusions and future scope are also provided in Section 8.

2. Mathematical model description

This section specifies the details of the queueing model. We consider an infinite-buffer single-server queue wherein batches of customers arrive at epochs $0 = \tau_0, \tau_1, \tau_2, \ldots, \tau_n, \ldots$, and the arrival stream forms a renewal process with group arrival rate λ. In other words, the inter-batch arrival times, denoted by T, constitute a sequence of independent and identically distributed random variables having a general distribution function $A(t)$ and mean inter-arrival time $1/\lambda$. The actual number of customers in any arriving batch is a random variable, which may take on any positive integral value k $(<+\infty)$ with probability distribution $\Pr\{X = k\} = g_k$, where X represents generic batch size with finite mean $E[X] = \bar{g}$ and associated probability generating function (p.g.f.) $X(z) = \sum_{k=1}^{\infty} g_k z^k$, $|z| \le 1$. Instead of being served individually, customers are now served in batches of maximum size b (≥ 1). That is to say, the server can process up to "b" customers at once, and the remaining customers have to wait for the next round of service. During the regular busy period, the normal service time distribution of a batch is assumed to be exponential with parameter μ_b. When the system becomes empty at a batch service completion instant, the server begins a vacation V, which is an exponentially distributed random variable with parameter θ. During the vacation, an arriving batch of customers can be served at a lower rate μ_v as compared to the normal service rate μ_b. When the working vacation period terminates, if there are customers in the system, the server changes its service rate from μ_v to μ_b, and the batch service interrupted at the end of vacation restarts from the beginning in a new regular busy period. Otherwise, the server enters an idle period, and a new regular busy period starts when a batch of customers arrive. Such type of vacation is called a single working vacation, and the model under consideration is denoted by $GI^X/M^{(1,b)}/1/SWV$. Furthermore, the traffic intensity of the system is $\rho = \lambda\bar{g}/b\mu_b$. For the stationary analysis of the model, we need the stability condition $\rho < 1$.

3. Determination of queue-length distribution at pre-arrival epoch

In this section, by constructing a Markov chain embedded within the $GI^X/M^{(1,b)}/1/SWV$ queue, the stationary distribution of the number of customers in the queue at pre-arrival epoch is numerically obtained by setting up a system of linear equations in terms of the roots of the associated characteristic equation. The embedded time instants are precisely the instants of batch arrivals since the elapsed inter-batch arrival time at these moments is exactly equal to zero. This allows us to form a transition probability matrix and to compute the distribution of customers seen by an arriving batch.

Let $N_q(t)$ denote the number of customers present in the queue (excluding customers in service) at time t. Further, the state of the server at time t can be described by a random variable $Y(t)$. Here, $Y(t)$ only takes on the value of 1 or 0. If the server is in the working vacation period, we set $Y(t)$ equal to 0. If the server is in the regular busy period, $Y(t)$ equals 1. Then it follows that a state of the system can be completely described by the pair (i,j), where i is the number of customers in the queue, and j denotes the server's current state. The pairs $(\bar{0},0)$ and $(\bar{0},1)$ refer to an empty queue, and the server is idle in the working vacation period and regular busy period, respectively. There is a clear difference between states $(\bar{0},0)$ and $(0,0)$. Specifically, the pair $(\bar{0},0)$ refers to the waiting line is empty in the working vacation period, and the number of customers receiving services is zero. While pair $(0,0)$ also refers to the waiting line is empty in working vacation period, but the number of customers receiving services is nonzero (maybe equals $1,2,\ldots$, or b). You may note here that $N_q(t)$ only records the number of customers in the waiting line, excluding the customers in service at time t. Therefore, $(\bar{0},0)$ and $(0,0)$ are two different states. Similarly, $(\bar{0},1)$ and $(0,1)$ are two completely different states. Thus, the state space of the non-Markovian process $\{(N_q(t),Y(t)):t\ge 0\}$ is $\Omega=\{(i,j):i=\bar{0},0,1,2,3,\ldots;j=0,1\}$. As described earlier, the above process can generate an embedded Markov chain at the pre-arrival epoch. Let $N_q(\tau_n^-)$ and $Y(\tau_n^-)$ represent the queue-length and the state of the server immediately prior to the nth batch arrival, respectively. Since the memoryless property of the exponential distribution, $\{(N_q(\tau_n^-),Y(\tau_n^-)):n\ge 1\}$ forms a bivariate Markov chain on the state space Ω. We shall now construct the transition probability matrix of the Markov chain embedded at arrival instants and write its element. Let us enumerate the states of the embedded Markov chain in lexicographic order. The transition probability matrix $\mathbb{P}$ of the system with state space Ω can be displayed in a block-partitioned structure as follows:

$$\mathbb{P}=\begin{array}{c} \\ (\bar{0},0)\\(\bar{0},1)\\(0,0)\\(0,1)\\(1,0)\\(1,1)\\(2,0)\\(2,1)\\ \vdots \end{array}\begin{array}{c}(\bar{0},0)\;\;(\bar{0},1)\;\;(0,0)\;\;(0,1)\;\;(1,0)\;\;(1,1)\;\;(2,0)\;\;(2,1)\;\;(3,0)\;\;(3,1)\;\;\cdots\;\;\cdots\\ \left(\begin{array}{ccccccc} \boldsymbol{P}_{\bar{0},\bar{0}} & \boldsymbol{P}_{\bar{0},0} & \boldsymbol{P}_{\bar{0},1} & \boldsymbol{P}_{\bar{0},2} & \boldsymbol{P}_{\bar{0},3} & \cdots & \cdots\\ \boldsymbol{P}_{0,\bar{0}} & \boldsymbol{P}_{0,0} & \boldsymbol{P}_{0,1} & \boldsymbol{P}_{0,2} & \boldsymbol{P}_{0,3} & \cdots & \cdots\\ \boldsymbol{P}_{1,\bar{0}} & \boldsymbol{P}_{1,0} & \boldsymbol{P}_{1,1} & \boldsymbol{P}_{1,2} & \boldsymbol{P}_{1,3} & \cdots & \cdots\\ \boldsymbol{P}_{2,\bar{0}} & \boldsymbol{P}_{2,0} & \boldsymbol{P}_{2,1} & \boldsymbol{P}_{2,2} & \boldsymbol{P}_{2,3} & \cdots & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right)\end{array},$$

where each block $\boldsymbol{P}_{i,j}$ is a two by two matrix. Let

$$p_{(i,j)(h,m)}=\Pr\left\{N_q(\tau_{n+1}^-)=h,Y(\tau_{n+1}^-)=m\,\middle|\,N_q(\tau_n^-)=i,Y(\tau_n^-)=j\right\},\quad m,j=0,1;h,i=\bar{0},0,1,2,\ldots,$$

be the transition probabilities of the chain. To explicitly express all elements of the matrix $\mathbb{P}$, we define the following probabilities of service completions under three different scenarios

$$\alpha_k=\int_0^\infty\frac{(\mu_b t)^k}{k!}e^{-\mu_b t}\,\mathrm{d}A(t),\quad k=0,1,2,\ldots;\quad \beta_k=\int_0^\infty\frac{(\mu_v t)^k}{k!}e^{-(\mu_v+\theta)t}\,\mathrm{d}A(t),\quad k=0,1,2,\ldots;$$

$$\Delta_k=\sum_{l=0}^{k}\int_0^\infty\int_0^t\theta e^{-\theta x}e^{-\mu_v x}\frac{(\mu_v x)^l}{l!}e^{-\mu_b(t-x)}\frac{(\mu_b(t-x))^{k-l}}{(k-l)!}\,\mathrm{d}x\,\mathrm{d}A(t),\quad k=0,1,2,\ldots.$$

Here, α_k is the probability that exactly k batches of customers have been served with a normal service rate during an inter-batch arrival time. β_k denotes the probability that the working vacation time is greater than an inter-batch arrival time and k batches of customers have been served with a lower service rate during the inter-batch arrival time. Similarly, Δ_k represents the probability that the ongoing working vacation ends sometime during an inter-batch arrival time, and there are k batch-service completions within this period, in which l batches complete services before the end of a single working vacation and $k-l$ batches complete services in the following normal busy period. Additionally, we introduce two special symbols for the brevity of notations and are defined as follows:

$$\phi_k = \int_0^\infty \int_0^t \theta e^{-\theta x} e^{-\mu_b(t-x)} \frac{(\mu_b(t-x))^k}{k!} \,\mathrm{d}x\,\mathrm{d}A(t), \quad k=0,1,2,\ldots;$$

$$m_k = \sum_{l=0}^{k} \int_0^\infty \int_0^t \theta e^{-\theta x} e^{-\mu_v x} \frac{(\mu_v x)^l}{l!} \int_0^{t-x} e^{-\mu_b y} \frac{\mu_b(\mu_b y)^{k-l}}{(k-l)!} e^{-\theta(t-x-y)} \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}A(t), \quad k=0,1,2,\ldots.$$

Further, we denote the Laplace–Stieltjes transform (LST) of $A(t)$ by $a^*(s)$, the p.g.f. of α_k, β_k and Δ_k are given by

$$\Lambda(z) = \sum_{k=0}^\infty \alpha_k z^k = a^*(\mu_b - \mu_b z), \quad D(z) = \sum_{k=0}^\infty \beta_k z^k = a^*\left(\theta + \mu_v - \mu_v z\right),$$

$$\text{and} \quad \Phi(z) = \sum_{k=0}^\infty \Delta_k z^k = \frac{\theta\left[\Lambda(z) - D(z)\right]}{(1-z)(\mu_v - \mu_b) + \theta},$$

respectively. With the above notations, now we will consider the transition probabilities in detail.

– First, for $i,j = 0,1,2,\ldots$, the transition from $(i,1)$ to $(j,1)$ corresponds to the case in a regular busy period, the size of an arriving batch equals $kb+j-i$ with probability g_{kb+j-i}. k batches of customers complete their services during an inter-batch arrival time, where k denotes all integers that are not less than $\max\{0, \lfloor \frac{i+1-j}{b} \rfloor + 1\}$, and $\lfloor x \rfloor$ is the greatest integer less than x, *e.g.* $\lfloor 2 \rfloor = 1$, $\lfloor 2.5 \rfloor = 2$. Hence, if $j \neq 0$, we get

$$\begin{aligned} p_{(i,1)(j,1)} &= \sum_{k=\max\{0,\lfloor \frac{i+1-j}{b} \rfloor+1\}}^{\infty} g_{kb+j-i} \Pr\left\{ \sum_{n=1}^{k} S_n \le T < \sum_{n=1}^{k+1} S_n \right\} \\ &= \sum_{k=\max\{0,\lfloor \frac{i+1-j}{b} \rfloor+1\}}^{\infty} g_{kb+j-i} \int_0^{+\infty} \frac{(\mu_b t)^k}{k!} e^{-\mu_b t}\,\mathrm{d}A(t) \\ &= \sum_{k=\max\{0,\lfloor \frac{i+1-j}{b} \rfloor+1\}}^{\infty} g_{kb+j-i}\alpha_k, \end{aligned}$$

where S_n denotes the service time of the nth batch during the regular busy period. Otherwise, we have

$$p_{(i,1)(0,1)} = \sum_{r=1}^\infty g_r \Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil} S_n \le T < \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil + 1} S_n \right\} = \sum_{r=1}^\infty g_r \int_0^\infty \frac{(\mu_b t)^{\lceil \frac{i+r}{b} \rceil}}{\lceil \frac{i+r}{b} \rceil !} e^{-\mu_b t}\,\mathrm{d}A(t) = \sum_{r=1}^\infty g_r \alpha_{\lceil \frac{i+r}{b} \rceil},$$

where $\lceil x \rceil$ represents the least integer not less than x, *e.g.* $\lceil 2 \rceil = 2$, $\lceil 2.5 \rceil = 3$. Furthermore, the transitions from state $(\bar{0},1)$ to states $(0,1)$ and $(j,1)$ $(j=1,2,\ldots)$ can be obtained in a similar way

$$p_{(\bar{0},1)(0,1)} = \sum_{r=1}^\infty g_r \Pr\left\{ \sum_{n=1}^{\lceil \frac{r}{b} \rceil - 1} S_n \le T < \sum_{n=1}^{\lceil \frac{r}{b} \rceil} S_n \right\} = \sum_{r=1}^\infty g_r \alpha_{\lceil \frac{r}{b} \rceil - 1},$$

$$p_{(\bar{0},1)(j,1)} = \sum_{k=1}^\infty g_{kb+j} \Pr\left\{ \sum_{n=1}^{k-1} S_n \le T < \sum_{n=1}^{k} S_n \right\} = \sum_{k=1}^\infty g_{kb+j}\alpha_{k-1}.$$

– Second, considering a single working vacation period, the transition from state $(i,0)$ $(i \geq 0)$ to state $(j,0)$ $(j \geq 1)$ occurs if the arriving batch contains $kb+j-i$ customers with probability g_{kb+j-i}, and the remaining vacation time is greater than an inter-batch arrival time. Meanwhile, there are k batch-service completions during the inter-batch arrival time, where k also denotes all integers that are not less than $\max\{0, \lfloor \frac{i+1-j}{b} \rfloor + 1\}$. Thus, we have

$$\begin{aligned} p_{(i,0)(j,0)} &= \sum_{k=\max\{0,\lfloor \frac{i+1-j}{b} \rfloor+1\}}^{\infty} g_{kb+j-i} \Pr\left\{ \sum_{n=1}^{k} \tilde{S}_n \leq T < \sum_{n=1}^{k+1} \tilde{S}_n, T < V \right\} \\ &= \sum_{k=\max\{0,\lfloor \frac{i+1-j}{b} \rfloor+1\}}^{\infty} g_{kb+j-i} \int_0^{\infty} \frac{(\mu_v t)^k}{k!} e^{-(\mu_v+\theta)t} \, \mathrm{d}A(t) = \sum_{k=\max\{0,\lfloor \frac{i+1-j}{b} \rfloor+1\}}^{\infty} g_{kb+j-i}\beta_k, \end{aligned}$$

in which $\tilde{S}_n$ denotes the service time of the nth batch during the working vacation period. Especially, if $j=0$, the transition from state $(i,0)$ to state $(0,0)$ is given by

$$\begin{aligned} p_{(i,0)(0,0)} &= \sum_{r=1}^{\infty} g_r \Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil} \tilde{S}_n \leq T < \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil+1} \tilde{S}_n, T < V \right\} \\ &= \sum_{r=1}^{\infty} g_r \int_0^{\infty} \frac{(\mu_v t)^{\lceil \frac{i+r}{b} \rceil}}{\lceil \frac{i+r}{b} \rceil!} e^{-(\mu_v+\theta)t} \, \mathrm{d}A(t) = \sum_{r=1}^{\infty} g_r \beta_{\lceil \frac{i+r}{b} \rceil}. \end{aligned}$$

Similarly, the transitions from state $(\bar{0},0)$ to states $(0,0)$ and $(j,0)$ $(j=1,2,\ldots)$ are given respectively as follows:

$$\begin{aligned} p_{(\bar{0},0)(0,0)} &= \sum_{r=1}^{\infty} g_r \Pr\left\{ \sum_{n=1}^{\lceil \frac{r}{b} \rceil-1} \tilde{S}_n \leq T < \sum_{n=1}^{\lceil \frac{r}{b} \rceil} \tilde{S}_n, T < V \right\} = \sum_{r=1}^{\infty} g_r \beta_{\lceil \frac{r}{b} \rceil - 1}, \\ p_{(\bar{0},0)(j,0)} &= \sum_{k=1}^{\infty} g_{kb+j} \Pr\left\{ \sum_{n=1}^{k-1} \tilde{S}_n \leq T < \sum_{n=1}^{k} \tilde{S}_n, T < V \right\} = \sum_{k=1}^{\infty} g_{kb+j}\beta_{k-1}. \end{aligned}$$

– Third, for $i = 0,1,2,\ldots$ and $j \neq \bar{0}$, the transition from state $(i,1)$ to state $(j,0)$ is an impossible event. Thus, we only need to consider the transition from state $(i,1)$ to state $(\bar{0},0)$. If the size of an arriving batch is equal to r with probability g_r, such a transition means that there are $\lceil \frac{i+r}{b} \rceil + 1$ batch-service completions during an inter-batch arrival time and then the newly started single working vacation does not end during the remaining inter-batch arrival time. So, we have

$$\begin{aligned} p_{(i,1)(\bar{0},0)} &= \sum_{r=1}^{\infty} g_r \Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil+1} S_n \leq T < \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil+1} S_n + V \right\} \\ &= \sum_{r=1}^{\infty} g_r \int_0^{\infty} \int_0^{\infty} \Pr\left\{ t-x < \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil+1} S_n \leq t \right\} \theta e^{-\theta x} \, \mathrm{d}x \, \mathrm{d}A(t) \\ &= \sum_{r=1}^{\infty} g_r \int_0^{\infty} \int_0^{\infty} \left[\Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil+1} S_n \leq t \right\} - \Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil+1} S_n \leq t-x \right\} \right] \theta e^{-\theta x} \, \mathrm{d}x \, \mathrm{d}A(t) \\ &= \sum_{r=1}^{\infty} g_r \left(\int_0^{\infty} \int_0^{t} \left[\Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil+1} S_n \leq t \right\} - \Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil+1} S_n \leq t-x \right\} \right] \theta e^{-\theta x} \, \mathrm{d}x \, \mathrm{d}A(t) \right. \end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty\int_t^\infty \Pr\left\{\sum_{n=1}^{\lceil\frac{i+r}{b}\rceil+1} S_n\le t\right\}\theta e^{-\theta x}\,\mathrm{d}x\,\mathrm{d}A(t)\Bigg)\\
&=\sum_{r=1}^\infty g_r\left(\int_0^\infty\int_0^t\left[e^{-\mu_b(t-x)}\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\frac{(\mu_b(t-x))^n}{n!}-e^{-\mu_b t}\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\frac{(\mu_b t)^n}{n!}\right]\theta e^{-\theta x}\,\mathrm{d}x\,\mathrm{d}A(t)\right.\\
&\left.+\int_0^\infty\int_t^\infty \theta e^{-\theta x}\,\mathrm{d}x\,\mathrm{d}A(t)-\int_0^\infty\int_t^\infty\theta e^{-\theta x}e^{-\mu_b t}\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\frac{(\mu_b t)^n}{n!}\,\mathrm{d}x\,\mathrm{d}A(t)\right)\\
&=a^*(\theta)-\sum_{r=1}^\infty g_r\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\alpha_n+\sum_{r=1}^\infty g_r\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\phi_n.
\end{aligned}
$$

Moreover, the transition from state $(\bar{0},1)$ to state $(\bar{0},0)$ can be analyzed in a similar manner

$$
p_{(\bar{0},1)(\bar{0},0)}=\sum_{r=1}^\infty g_r\Pr\left\{\sum_{n=1}^{\lceil\frac{r}{b}\rceil}S_n\le T<\sum_{n=1}^{\lceil\frac{r}{b}\rceil}S_n+V\right\}=a^*(\theta)-\sum_{r=1}^\infty g_r\sum_{n=0}^{\lceil\frac{r}{b}\rceil-1}\alpha_n+\sum_{r=1}^\infty g_r\sum_{n=0}^{\lceil\frac{r}{b}\rceil-1}\phi_n.
$$

– Fourth, the transition from state $(i,0)$ $(i\ge 0)$ to state $(j,1)$ $(j\ge 1)$ occurs if the arriving batch contains $kb+j-i$ customers with probability g_{kb+j-i}, and the ongoing working vacation ends sometime during an inter-batch arrival time. Meanwhile, there are k batch-service completions, in which l $(l=0,1,2,\dots,k)$ batches complete services before the end of a single working vacation and $k-l$ batches complete services after the end of a single working vacation. Here, k also takes all integers that are not less than $\max\{0,\lfloor\frac{i+1-j}{b}\rfloor+1\}$. Thus,

$$
\begin{aligned}
p_{(i,0)(j,1)}&=\sum_{k=\max\{0,\lfloor\frac{i+1-j}{b}\rfloor+1\}}^\infty g_{kb+j-i}\sum_{l=0}^k\Pr\left\{\sum_{n=1}^l\tilde{S}_n\le V<\sum_{n=1}^{l+1}\tilde{S}_n, V+\sum_{n=l+1}^k S_n\le T<V+\sum_{n=l+1}^{k+1}S_n\right\}\\
&=\sum_{k=\max\{0,\lfloor\frac{i+1-j}{b}\rfloor+1\}}^\infty g_{kb+j-i}\sum_{l=0}^k\int_0^\infty\int_0^t\theta e^{-\theta x}\frac{(\mu_v x)^l}{l!}e^{-\mu_v x}\\
&\quad\times\Pr\left\{\sum_{n=l+1}^k S_n\le t-x<\sum_{n=l+1}^{k+1}S_n\right\}\mathrm{d}x\,\mathrm{d}A(t)\\
&=\sum_{k=\max\{0,\lfloor\frac{i+1-j}{b}\rfloor+1\}}^\infty g_{kb+j-i}\sum_{l=0}^k\int_0^\infty\int_0^t\theta e^{-\theta x}e^{-\mu_v x}\frac{(\mu_v x)^l}{l!}e^{-\mu_b(t-x)}\frac{(\mu_b(t-x))^{k-l}}{(k-l)!}\,\mathrm{d}x\,\mathrm{d}A(t)\\
&=\sum_{k=\max\{0,\lfloor\frac{i+1-j}{b}\rfloor+1\}}^\infty g_{kb+j-i}\Delta_k.
\end{aligned}
$$

By using an argument similar as above, the transition probability $p_{(i,0)(0,1)}$ is given by

$$
\begin{aligned}
p_{(i,0)(0,1)}&=\sum_{r=1}^\infty g_r\sum_{l=0}^{\lceil\frac{i+r}{b}\rceil}\Pr\left\{\sum_{n=1}^l\tilde{S}_n\le V<\sum_{n=1}^{l+1}\tilde{S}_n, V+\sum_{n=l+1}^{\lceil\frac{i+r}{b}\rceil}S_n\le T<V+\sum_{n=l+1}^{\lceil\frac{i+r}{b}\rceil+1}S_n\right\}\\
&=\sum_{r=1}^\infty g_r\sum_{l=0}^{\lceil\frac{i+r}{b}\rceil}\int_0^\infty\int_0^t\theta e^{-\theta x}e^{-\mu_v x}\frac{(\mu_v x)^l}{l!}e^{-\mu_b(t-x)}\frac{(\mu_b(t-x))^{\lceil\frac{i+r}{b}\rceil-l}}{(\lceil\frac{i+r}{b}\rceil-l)!}\,\mathrm{d}x\,\mathrm{d}A(t)=\sum_{r=1}^\infty g_r\Delta_{\lceil\frac{i+r}{b}\rceil}.
\end{aligned}
$$

Additionally, the analysis of the transitions from state $(\bar{0},0)$ to states $(0,1)$ and $(j,1)$ $(j=1,2,\ldots)$ is very similar to the procedure as we have mentioned above. Thus,

$$p_{(\bar{0},0)(0,1)}=\sum_{r=1}^{\infty}g_r\sum_{l=0}^{\lceil\frac{r}{b}\rceil-1}\Pr\left\{\sum_{n=1}^{l}\tilde{S}_n\le V<\sum_{n=1}^{l+1}\tilde{S}_n, V+\sum_{n=l+1}^{\lceil\frac{r}{b}\rceil-1}S_n\le T<V+\sum_{n=l+1}^{\lceil\frac{r}{b}\rceil}S_n\right\}=\sum_{r=1}^{\infty}g_r\Delta_{\lceil\frac{r}{b}\rceil-1},$$

$$p_{(\bar{0},0)(j,1)}=\sum_{k=1}^{\infty}g_{kb+j}\sum_{l=0}^{k-1}\Pr\left\{\sum_{n=1}^{l}\tilde{S}_n\le V<\sum_{n=1}^{l+1}\tilde{S}_n, V+\sum_{n=l+1}^{k-1}S_n\le T<V+\sum_{n=l+1}^{k}S_n\right\}=\sum_{k=1}^{\infty}g_{kb+j}\Delta_{k-1}.$$

– Fifth, for $i=0,1,2,\ldots$, two mutually exclusive cases cause a transition from state $(i,0)$ to state $(\bar{0},0)$. Case 1: Suppose that the size of an arriving batch is equal to r with probability g_r. The residual working vacation is greater than the inter-batch arrival time, and $\lceil\frac{i+r}{b}\rceil+1$ batches of customers complete their services during the inter-batch arrival time. Case 2: With the same assumption as in Case 1, the residual working vacation time is less than the inter-batch arrival time, and the server goes on another working vacation after completing the services of $\lceil\frac{i+r}{b}\rceil+1$ batches of customers. Further, we assume that there are l $(l=0,1,\ldots,\lceil\frac{i+r}{b}\rceil)$ batch-service completions before the single working vacation ends and $\lceil\frac{i+r}{b}\rceil-l+1$ batch-service completions in a regular busy period. Thus, we have

$$\begin{aligned}
p_{(i,0)(\bar{0},0)}&=\sum_{r=1}^{\infty}g_r\Pr\left\{\sum_{n=1}^{\lceil\frac{i+r}{b}\rceil+1}\tilde{S}_n\le T, T<V\right\}\\
&\quad+\sum_{r=1}^{\infty}g_r\sum_{l=0}^{\lceil\frac{i+r}{b}\rceil}\Pr\left\{\sum_{n=1}^{l}\tilde{S}_n\le V<\sum_{n=1}^{l+1}\tilde{S}_n, V+\sum_{n=l+1}^{\lceil\frac{i+r}{b}\rceil+1}S_n\le T<V+\sum_{n=l+1}^{\lceil\frac{i+r}{b}\rceil+1}S_n+V\right\}\\
&=\sum_{r=1}^{\infty}g_r\int_0^{\infty}e^{-\theta t}\left[1-e^{-\mu_v t}\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\frac{(\mu_v t)^n}{n!}\right]\mathrm{d}A(t)\\
&\quad+\sum_{r=1}^{\infty}g_r\sum_{l=0}^{\lceil\frac{i+r}{b}\rceil}\int_0^{\infty}\int_0^{t}\theta e^{-\theta x}\frac{(\mu_v x)^l}{l!}e^{-\mu_v x}\Pr\left\{\sum_{n=l+1}^{\lceil\frac{i+r}{b}\rceil+1}S_n\le t-x<\sum_{n=l+1}^{\lceil\frac{i+r}{b}\rceil+1}S_n+V\right\}\mathrm{d}x\,\mathrm{d}A(t)\\
&=a^*(\theta)-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\int_0^{\infty}e^{-(\theta+\mu_v)t}\frac{(\mu_v t)^n}{n!}\mathrm{d}A(t)\\
&\quad+\sum_{r=1}^{\infty}g_r\sum_{l=0}^{\lceil\frac{i+r}{b}\rceil}\int_0^{\infty}\int_0^{t}\theta e^{-\theta x}e^{-\mu_v x}\frac{(\mu_v x)^l}{l!}\int_0^{t-x}e^{-\mu_b y}\frac{\mu_b(\mu_b y)^{\lceil\frac{i+r}{b}\rceil-l}}{(\lceil\frac{i+r}{b}\rceil-l)!}e^{-\theta(t-x-y)}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}A(t)\\
&=a^*(\theta)-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\beta_n+\sum_{r=1}^{\infty}g_r m_{\lceil\frac{i+r}{b}\rceil}.
\end{aligned}$$

Furthermore, the transition from state $(\bar{0},0)$ to state $(\bar{0},0)$ is handled similarly.

$$\begin{aligned}
p_{(\bar{0},0)(\bar{0},0)}&=\sum_{r=1}^{\infty}g_r\Pr\left\{\sum_{n=1}^{\lceil\frac{r}{b}\rceil}\tilde{S}_n\le T, T<V\right\}\\
&\quad+\sum_{r=1}^{\infty}g_r\sum_{l=0}^{\lceil\frac{r}{b}\rceil-1}\Pr\left\{\sum_{n=1}^{l}\tilde{S}_n\le V<\sum_{n=1}^{l+1}\tilde{S}_n, V+\sum_{n=l+1}^{\lceil\frac{r}{b}\rceil}S_n\le T<V+\sum_{n=l+1}^{\lceil\frac{r}{b}\rceil}S_n+V\right\}\\
&=a^*(\theta)-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{r}{b}\rceil-1}\beta_n+\sum_{r=1}^{\infty}g_r m_{\lceil\frac{r}{b}\rceil-1}.
\end{aligned}$$

– Sixth, for $i = 0, 1, 2, \ldots$, we can also distinguish two possible cases to consider the transition from state $(i, 0)$ to state $(\bar{0}, 1)$. Case 1: Assuming that the size of an arriving batch equals r with probability g_r, the residual working vacation is less than the inter-batch arrival time, and $\lceil \frac{i+r}{b} \rceil + 1$ batches of customers complete their services during the residual working vacation time. Case 2: Under the same assumption as mentioned above, the residual working vacation time is less than the inter-batch arrival time, and the server starts another new working vacation after completing the services of $\lceil \frac{i+r}{b} \rceil + 1$ batches of customers. Furthermore, during this new working vacation period, no customer arrives at the service system. We still assume that l ($l = 0, 1, \ldots, \lceil \frac{i+r}{b} \rceil$) batches of customers complete their services in the residual working vacation time and $\lceil \frac{i+r}{b} \rceil + 1 - l$ batches of customers complete their services in a regular busy period. Thus

$$
\begin{aligned}
p_{(i,0)(\bar{0},1)} &= \sum_{r=1}^{\infty} g_r \Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil + 1} \tilde{S}_n \le V, V < T \right\} \\
&\quad + \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{i+r}{b} \rceil} \Pr\left\{ \sum_{n=1}^{l} \tilde{S}_n \le V < \sum_{n=1}^{l+1} \tilde{S}_n, V + \sum_{n=l+1}^{\lceil \frac{i+r}{b} \rceil + 1} S_n + V \le T \right\} \\
&= \sum_{r=1}^{\infty} g_r \int_0^{\infty} \int_0^{t} \theta e^{-\theta x} \Pr\left\{ \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil + 1} \tilde{S}_n \le x \right\} \mathrm{d}x \, \mathrm{d}A(t) \\
&\quad + \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{i+r}{b} \rceil} \int_0^{\infty} \int_0^{t} \theta e^{-\theta x} \frac{(\mu_v x)^l}{l!} e^{-\mu_v x} \Pr\left\{ \sum_{n=l+1}^{\lceil \frac{i+r}{b} \rceil + 1} S_n + V \le t - x \right\} \mathrm{d}x \, \mathrm{d}A(t) \\
&= \sum_{r=1}^{\infty} g_r \int_0^{\infty} \int_0^{t} \theta e^{-\theta x} \left[1 - e^{-\mu_v x} \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil} \frac{(\mu_v x)^n}{n!} \right] \mathrm{d}x \, \mathrm{d}A(t) \\
&\quad + \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{i+r}{b} \rceil} \int_0^{\infty} \int_0^{t} \theta e^{-\theta x} \frac{(\mu_v x)^l}{l!} e^{-\mu_v x} \int_0^{t-x} \frac{\mu_b (\mu_b y)^{\lceil \frac{i+r}{b} \rceil - l}}{(\lceil \frac{i+r}{b} \rceil - l)!} e^{-\mu_b y} [1 - e^{-\theta(t-x-y)}] \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}A(t) \\
&= 1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=1}^{\lceil \frac{i+r}{b} \rceil} \int_0^{\infty} \int_0^{t} \theta e^{-\theta x} e^{-\mu_v x} \frac{(\mu_v x)^n}{n!} \mathrm{d}x \, \mathrm{d}A(t) \\
&\quad + \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{i+r}{b} \rceil} \int_0^{\infty} \int_0^{t} \theta e^{-\theta x} \frac{(\mu_v x)^l}{l!} e^{-\mu_v x} \left[1 - e^{-\mu_b (t-x)} \sum_{n=0}^{\lceil \frac{i+r}{b} \rceil - l} \frac{(\mu_b (t-x))^n}{n!} \right] \mathrm{d}x \, \mathrm{d}A(t) \\
&\quad - \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{i+r}{b} \rceil} \int_0^{\infty} \int_0^{t} \theta e^{-\theta x} \frac{(\mu_v x)^l}{l!} e^{-\mu_v x} \int_0^{t-x} \frac{\mu_b (\mu_b y)^{\lceil \frac{i+r}{b} \rceil - l}}{(\{ \frac{i+r}{b} \} - l)!} e^{-\mu_b y} e^{-\theta(t-x-y)} \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}A(t) \\
&= 1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{i+r}{b} \rceil} \Delta_l - \sum_{r=1}^{\infty} g_r m_{\lceil \frac{i+r}{b} \rceil}.
\end{aligned}
$$

For $i = \bar{0}$, the transition from state $(\bar{0}, 0)$ to state $(\bar{0}, 1)$ is derived in a similar way

$$p_{(\bar{0},0)(\bar{0},1)} = \sum_{r=1}^{\infty} g_r \Pr\left\{\sum_{n=1}^{\lceil \frac{r}{b}\rceil} \tilde{S}_n \le V, V < T\right\} + \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{r}{b}\rceil -1} \Pr\left\{\sum_{n=1}^{l} \tilde{S}_n \le V < \sum_{n=1}^{l+1} \tilde{S}_n, V + \sum_{n=l+1}^{\lceil \frac{r}{b}\rceil} S_n + V \le T\right\}$$
$$= 1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{r}{b}\rceil -1} \Delta_l - \sum_{r=1}^{\infty} g_r m_{\lceil \frac{r}{b}\rceil -1}.$$

Suppose that the size of an arriving batch equals r with probability g_r. The transition from state $(i,1)$ $(i \ge 0)$ to state $(\bar{0},1)$ occurs if the inter-batch arrival time is not less than the time required to serve $\lceil \frac{i+r}{b}\rceil + 1$ batches of customers in a regular busy period followed by a vacation completion duration. Therefore,

$$p_{(i,1)(\bar{0},1)} = \sum_{r=1}^{\infty} g_r \Pr\left\{\sum_{n=1}^{\lceil \frac{i+r}{b}\rceil +1} S_n + V \le T\right\} = \sum_{r=1}^{\infty} g_r \int_0^{\infty}\int_0^t \theta e^{-\theta x} \Pr\left\{\sum_{n=1}^{\lceil \frac{i+r}{b}\rceil +1} S_n \le t - x\right\} \mathrm{d}x\, \mathrm{d}A(t)$$
$$= \sum_{r=1}^{\infty} g_r \int_0^{\infty}\int_0^t \theta e^{-\theta x}\left[1 - e^{-\mu_b(t-x)} \sum_{n=0}^{\lceil \frac{i+r}{b}\rceil} \frac{(\mu_b(t-x))^n}{n!}\right] \mathrm{d}x\, \mathrm{d}A(t) = 1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{i+r}{b}\rceil} \phi_n.$$

Similar to the above case,

$$p_{(\bar{0},1)(\bar{0},1)} = \sum_{r=1}^{\infty} g_r \Pr\left\{\sum_{n=1}^{\lceil \frac{r}{b}\rceil} S_n + V \le T\right\} = 1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{r}{b}\rceil -1} \phi_n.$$

Once the transition probabilities are determined, the block sub-matrices required for finding the z-transform of the queue-length distribution just before batch arrivals are given by

$$\boldsymbol{P}_{\bar{0},\bar{0}} = \begin{pmatrix} a^*(\theta) - \sum\limits_{r=1}^{\infty} g_r \sum\limits_{n=0}^{\lceil \frac{r}{b}\rceil -1} \beta_n + \sum\limits_{r=1}^{\infty} g_r m_{\lceil \frac{r}{b}\rceil -1} & 1 - a^*(\theta) - \sum\limits_{r=1}^{\infty} g_r \sum\limits_{l=0}^{\lceil \frac{r}{b}\rceil -1} \Delta_l - \sum\limits_{r=1}^{\infty} g_r m_{\lceil \frac{r}{b}\rceil -1} \\ a^*(\theta) - \sum\limits_{r=1}^{\infty} g_r \sum\limits_{n=0}^{\lceil \frac{r}{b}\rceil -1} \alpha_n + \sum\limits_{r=1}^{\infty} g_r \sum\limits_{n=0}^{\lceil \frac{r}{b}\rceil -1} \phi_n & 1 - a^*(\theta) - \sum\limits_{r=1}^{\infty} g_r \sum\limits_{n=0}^{\lceil \frac{r}{b}\rceil -1} \phi_n \end{pmatrix},$$

$$\boldsymbol{P}_{\bar{0},0} = \begin{pmatrix} \sum\limits_{r=1}^{\infty} g_r \beta_{\lceil \frac{r}{b}\rceil -1} & \sum\limits_{r=1}^{\infty} g_r \Delta_{\lceil \frac{r}{b}\rceil -1} \\ 0 & \sum\limits_{r=1}^{\infty} g_r \alpha_{\lceil \frac{r}{b}\rceil -1} \end{pmatrix}, \quad \boldsymbol{P}_{\bar{0},j} = \begin{pmatrix} \sum\limits_{k=1}^{\infty} g_{kb+j} \beta_{k-1} & \sum\limits_{k=1}^{\infty} g_{kb+j} \Delta_{k-1} \\ 0 & \sum\limits_{k=1}^{\infty} g_{kb+j} \alpha_{k-1} \end{pmatrix}, j = 1, 2, \ldots,$$

$$\boldsymbol{P}_{i,\bar{0}} = \begin{pmatrix} a^*(\theta) - \sum\limits_{r=1}^{\infty} g_r \sum\limits_{n=0}^{\lceil \frac{i+r}{b}\rceil} \beta_n + \sum\limits_{r=1}^{\infty} g_r m_{\lceil \frac{i+r}{b}\rceil} & 1 - a^*(\theta) - \sum\limits_{r=1}^{\infty} g_r \sum\limits_{l=0}^{\lceil \frac{i+r}{b}\rceil} \Delta_l - \sum\limits_{r=1}^{\infty} g_r m_{\lceil \frac{i+r}{b}\rceil} \\ a^*(\theta) - \sum\limits_{r=1}^{\infty} g_r \sum\limits_{n=0}^{\lceil \frac{i+r}{b}\rceil} \alpha_n + \sum\limits_{r=1}^{\infty} g_r \sum\limits_{n=0}^{\lceil \frac{i+r}{b}\rceil} \phi_n & 1 - a^*(\theta) - \sum\limits_{r=1}^{\infty} g_r \sum\limits_{n=0}^{\lceil \frac{i+r}{b}\rceil} \phi_n \end{pmatrix}, \quad i = 0, 1, \ldots,$$

$$\boldsymbol{P}_{i,0} = \begin{pmatrix} \sum\limits_{r=1}^{\infty} g_r \beta_{\lceil \frac{i+r}{b}\rceil} & \sum\limits_{r=1}^{\infty} g_r \Delta_{\lceil \frac{i+r}{b}\rceil} \\ 0 & \sum\limits_{r=1}^{\infty} g_r \alpha_{\lceil \frac{i+r}{b}\rceil} \end{pmatrix}, \quad i = 0, 1, 2, \ldots,$$

$$\boldsymbol{P}_{i,j} = \begin{pmatrix} \sum\limits_{k=\max\{0,\lfloor \frac{i+1-j}{b}\rfloor +1\}}^{\infty} g_{kb+j-i} \beta_k & \sum\limits_{k=\max\{0,\lfloor \frac{i+1-j}{b}\rfloor +1\}}^{\infty} g_{kb+j-i} \Delta_k \\ 0 & \sum\limits_{k=\max\{0,\lfloor \frac{i+1-j}{b}\rfloor +1\}}^{\infty} g_{kb+j-i} \alpha_k \end{pmatrix}, i = 0, 1, \ldots, \quad j = 1, 2, \ldots.$$

Let $\boldsymbol{\pi}^- = \left(\pi^-_{\bar{0},0}, \pi^-_{\bar{0},1}, \pi^-_{0,0}, \pi^-_{0,1}, \pi^-_{1,0}, \pi^-_{1,1}, \dots\right)$ be the stationary distribution of the Markov chain $\{(N_q(\tau_n^-), Y(\tau_n^-)) : n \geq 0\}$, that is to say, the component $\pi^-_{i,j}$ $(i = \bar{0}, 0, 1, 2, \dots, j = 0, 1)$ gives the probability that an arriving batch of customers see the system in state (i, j). It provides us with the queue length distribution at pre-arrival epoch. From the balance equation $\boldsymbol{\pi}^-\mathbb{P} = \boldsymbol{\pi}^-$, the following set of difference equations can be obtained

$$\begin{aligned}\pi^-_{\bar{0},0} = {}& \pi^-_{\bar{0},0}\left(a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{r}{b}\rceil -1} \beta_n + \sum_{r=1}^{\infty} g_r m_{\lceil \frac{r}{b}\rceil -1}\right) + \pi^-_{\bar{0},1}\left(a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{r}{b}\rceil -1} \alpha_n + \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{r}{b}\rceil -1} \phi_n\right)\\ &+ \sum_{i=0}^{\infty} \pi^-_{i,0}\left(a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{i+r}{b}\rceil} \beta_n + \sum_{r=1}^{\infty} g_r m_{\lceil \frac{i+r}{b}\rceil}\right)\\ &+ \sum_{i=0}^{\infty} \pi^-_{i,1}\left(a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{i+r}{b}\rceil} \alpha_n + \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{i+r}{b}\rceil} \phi_n\right),\end{aligned} \tag{3.1}$$

$$\begin{aligned}\pi^-_{\bar{0},1} = {}& \pi^-_{\bar{0},0}\left(1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{r}{b}\rceil -1} \Delta_l - \sum_{r=1}^{\infty} g_r m_{\lceil \frac{r}{b}\rceil -1}\right) + \pi^-_{\bar{0},1}\left(1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{r}{b}\rceil -1} \phi_n\right)\\ &+ \sum_{i=0}^{\infty} \pi^-_{i,0}\left(1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{i+r}{b}\rceil} \Delta_l - \sum_{r=1}^{\infty} g_r m_{\lceil \frac{i+r}{b}\rceil}\right) + \sum_{i=0}^{\infty} \pi^-_{i,1}\left(1 - a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{n=0}^{\lceil \frac{i+r}{b}\rceil} \phi_n\right),\end{aligned} \tag{3.2}$$

$$\pi^-_{0,0} = \pi^-_{\bar{0},0} \sum_{r=1}^{\infty} g_r \beta_{\lceil \frac{r}{b}\rceil -1} + \sum_{i=0}^{\infty} \pi^-_{i,0} \sum_{r=1}^{\infty} g_r \beta_{\lceil \frac{i+r}{b}\rceil}, \tag{3.3}$$

$$\pi^-_{0,1} = \pi^-_{\bar{0},0} \sum_{r=1}^{\infty} g_r \Delta_{\lceil \frac{r}{b}\rceil -1} + \pi^-_{\bar{0},1} \sum_{r=1}^{\infty} g_r \alpha_{\lceil \frac{r}{b}\rceil -1} + \sum_{i=0}^{\infty} \pi^-_{i,0} \sum_{r=1}^{\infty} g_r \Delta_{\lceil \frac{i+r}{b}\rceil} + \sum_{i=0}^{\infty} \pi^-_{i,1} \sum_{r=1}^{\infty} g_r \alpha_{\lceil \frac{i+r}{b}\rceil}, \tag{3.4}$$

$$\pi^-_{j,0} = \pi^-_{\bar{0},0} \sum_{k=1}^{\infty} g_{kb+j} \beta_{k-1} + \sum_{i=0}^{\infty} \pi^-_{i,0} \sum_{k=\max\{0, \lfloor \frac{i+1-j}{b} \rfloor +1\}}^{\infty} g_{kb+j-i} \beta_k, \quad j = 1, 2, \dots, \tag{3.5}$$

$$\begin{aligned}\pi_{j,1} = {}& \pi^-_{\bar{0},0} \sum_{k=1}^{\infty} g_{kb+j} \Delta_{k-1} + \pi^-_{\bar{0},1} \sum_{k=1}^{\infty} g_{kb+j} \alpha_{k-1} + \sum_{i=0}^{\infty} \pi^-_{i,0} \sum_{k=\max\{0, \lfloor \frac{i+1-j}{b} \rfloor +1\}}^{\infty} g_{kb+j-i} \Delta_k\\ &+ \sum_{i=0}^{\infty} \pi^-_{i,1} \sum_{k=\max\{0, \lfloor \frac{i+1-j}{b} \rfloor +1\}}^{\infty} g_{kb+j-i} \alpha_k, \quad j = 1, 2, \dots.\end{aligned} \tag{3.6}$$

Let $\pi_0^-(z) = \sum_{i=0}^{\infty} \pi^-_{i,0} z^i$ and $\pi_1^-(z) = \sum_{i=0}^{\infty} \pi^-_{i,1} z^i$ be the z-transforms of the sequences $\{\pi^-_{i,0}\}$ and $\{\pi^-_{i,1}\}$, respectively. Using equations (3.3) and (3.5), after some straightforward but tedious algebraic manipulation, we have

$$\pi_0^-(z) = \frac{\xi_1(z)}{1 - X(z) D\left(z^{-b}\right)}, \tag{3.7}$$

where

$$\xi_1(z) = z^{-b}\pi^-_{\bar{0},0}X(z)D\left(z^{-b}\right) - \pi^-_{\bar{0},0}\sum_{k=0}^{\infty}\frac{\beta_k}{z^{(k+1)b}}\left[\sum_{j=1}^{(k+1)b-1} g_j z^j - z^{(k+1)b}\sum_{j=kb+1}^{(k+1)b-1} g_j\right]$$
$$-\sum_{k=1}^{\infty}\frac{\beta_k}{z^{kb}}\sum_{i=0}^{kb-2}\pi^-_{i,0}z^i\sum_{j=1}^{kb-i-1} g_j\left(z^j - z^{kb-i}\right) - \sum_{k=2}^{\infty}\frac{\beta_k}{z^{kb}}\sum_{i=0}^{(k-1)b-1}\pi^-_{i,0}z^i\sum_{j=1}^{(k-1)b-i} g_j z^{kb-i}.$$

Employing equations (3.4) and (3.6), $\pi_1^-(z)$ are found similarly,

$$\pi_1^-(z) = \frac{\pi_0^-(z)X(z)H\left(z^{-b}\right) + \xi_2(z) + \xi_3(z)}{1 - X(z)\Lambda\left(z^{-b}\right)}, \tag{3.8}$$

where

$$\xi_2(z) = z^{-b}\pi^-_{\bar{0},0}X(z)H\left(z^{-b}\right) - \pi^-_{\bar{0},0}\sum_{k=0}^{\infty}\frac{\Delta_k}{z^{(k+1)b}}\left(\sum_{j=1}^{(k+1)b-1} g_j z^j - z^{(k+1)b}\sum_{j=kb+1}^{(k+1)b-1} g_j\right)$$
$$-\sum_{k=1}^{\infty}\frac{\Delta_k}{z^{kb}}\sum_{i=0}^{kb-2}\pi^-_{i,0}z^i\sum_{j=1}^{kb-i-1} g_j\left(z^j - z^{kb-i}\right) - \sum_{k=2}^{\infty}\frac{\Delta_k}{z^{kb}}\sum_{i=0}^{(k-1)b-1}\pi^-_{i,0}z^i\sum_{j=1}^{(k-1)b-i} g_j z^{kb-i},$$

$$\xi_3(z) = z^{-b}\pi^-_{\bar{0},1}X(z)\Lambda\left(z^{-b}\right) - \pi^-_{\bar{0},1}\sum_{k=0}^{\infty}\frac{\alpha_k}{z^{(k+1)b}}\left(\sum_{j=1}^{(k+1)b-1} g_j z^j - z^{(k+1)b}\sum_{j=kb+1}^{(k+1)b-1} g_j\right)$$
$$-\sum_{k=1}^{\infty}\frac{\alpha_k}{z^{kb}}\sum_{i=0}^{kb-2}\pi^-_{i,1}z^i\sum_{j=1}^{kb-i-1} g_j\left(z^j - z^{kb-i}\right) - \sum_{k=2}^{\infty}\frac{\alpha_k}{z^{kb}}\sum_{i=0}^{(k-1)b-1}\pi^-_{i,1}z^i\sum_{j=1}^{(k-1)b-i} g_j z^{kb-i}.$$

Next, we will rewrite $\pi_0^-(z)$ and $\pi_1^-(z)$ as partial fractions with constant numerators. As can be seen in equation (3.7), the numerator of $\pi_0^-(z)$ contains unknown constants. These constants can be determined by invoking the convergence of the p.g.f. in $|z| \le 1$ and by using Rouché's theorem (see [24]). Due to the convergence of $\pi_0^-(z)$, the roots inside and on the unit circle must cancel in the numerator and denominator. Therefore, in making a partial fraction of $\pi_0^-(z)$, the zeros of $1 - X(z)D(z^{-b})$ whose absolute value are less than or equal to one do not play any role. Meanwhile, notice that in most practical scenarios, the size of an arriving batch is always bounded. If we assume that the maximum size of an arriving batch is $\tilde{r}$, then $X(z)$ must be a polynomial function. On the other hand, the literature on queueing theory indicates inter-batch arrival time distributions having LST as a rational function cover a wide range of distributions that arise in applications (see [8]). Even if the LST of the inter-batch arrival time distribution is in non-rational form, it still can be approximated by rational functions using the technique of Padé's approximation (see, [17,36]). For the reasons mentioned above, we can immediately say that $D(z^{-b}) = a^*(\theta + \mu_v - \mu_v z^{-b})$ is a rational function. Therefore, $1 - X(z)D(z^{-b})$ can be easily expressed as a ratio of two polynomials, namely $1 - X(z)D(z^{-b}) = d_1(z)/d_2(z)$. It means that $1 - X(z)D(z^{-b})$ is also a rational function. Thus, equation (3.7) can be rewritten as $\pi_0^-(z) = \Upsilon_1(z)/d_1(z)$, where $\Upsilon_1(z) = \xi_1(z)d_2(z)$. Similar to the work done by Chaudhry *et al.* [17], after some tedious but straightforward algebraic manipulations, we can show that $\Upsilon_1(z)$ is a polynomial in z, and $\Upsilon_1(z)$ has at most the same degrees as $d_1(z)$ or $d_1(z)$ has a degree which is at most $\tilde{r}$ greater than that of $\Upsilon_1(z)$. Clearly, the zeros of $d_1(z)$ are exactly the same as the zeros of $1 - X(z)D(z^{-b})$. To study the number of roots of the equation $1 - X(z)D(z^{-b}) = 0$ lying outside the unit circle, we first investigate the number of roots of the equation $z^{\tilde{r}} - \left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) D\left(z^b\right) = 0$ lying inside the unit circle. Here, the equation $z^{\tilde{r}} - \left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) D\left(z^b\right) = 0$ is called the characteristic equation of $\pi_0^-(z)$. Additionally, we may see that the roots of $1 - X(z)D(z^{-b}) = 0$ and the characteristic equation $z^{\tilde{r}} - \left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) D\left(z^b\right) = 0$ are reciprocals to each other. Hence, in numerical experiment, our aim is to find the roots of the characteristic equation inside the unit circle. Now we further show that the equation $z^{\tilde{r}} - \left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) D\left(z^b\right) = 0$ has exactly $\tilde{r}$ roots inside the unit circle $|z| = 1$ whenever $\rho < 1$.

Define the functions $f(z) = z^{\tilde{r}}$ and $h(z) = -\left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) D\left(z^b\right)$. Consider absolute values of $f(z)$ and $h(z)$ on the circle $|z| = 1 - \delta$, where δ is positive and sufficiently small. On the circle $|z| = 1 - \delta$, using the Taylor series expansion, we have

$$\begin{aligned}
|h(z)| &= \left| -\left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) D(z^b) \right| \le \left(\sum_{j=1}^{\tilde{r}} g_j |z|^{\tilde{r}-j}\right) D\left(|z|^b\right) = \sum_{j=1}^{\tilde{r}} g_j (1-\delta)^{\tilde{r}-j} D((1-\delta)^b) \\
&= \sum_{j=1}^{\tilde{r}} g_j [1 - (\tilde{r}-j)\,\delta + o(\delta)] \left[D(1) + \frac{D'(1)}{1!}\left((1-\delta)^b - 1\right) + \sum_{n=2}^{\infty} \frac{D^{(n)}(1)}{n!} \left((1-\delta)^b - 1\right)^n \right] \\
&= \sum_{j=1}^{\tilde{r}} g_j \left[1 - (\tilde{r}-j)\delta + o(\delta)\right] \\
&\quad \times \left[\int_0^{\infty} e^{-\theta t}\, \mathrm{d}A(t) + \mu_v \left((1-\delta)^b - 1\right) \int_0^{\infty} t e^{-\theta t}\, \mathrm{d}A(t) + \sum_{n=2}^{\infty} \frac{D^{(n)}(1)}{n!} \left((1-\delta)^b - 1\right)^n \right] \\
&\le \sum_{j=1}^{\tilde{r}} g_j \left[1 - (\tilde{r}-j)\delta + o(\delta)\right] \left[1 + \mu_b \left((1-\delta)^b - 1\right) \int_0^{\infty} t\, \mathrm{d}A(t) + \sum_{n=2}^{\infty} \frac{D^{(n)}(1)}{n!} \left((1-\delta)^b - 1\right)^n \right] \\
&= \sum_{j=1}^{\tilde{r}} g_j \left[1 - (\tilde{r}-j)\delta + o(\delta)\right] \left[1 + \frac{\mu_b}{\lambda} \left((1-\delta)^b - 1\right) + \sum_{n=2}^{\infty} \frac{D^{(n)}(1)}{n!} \left((1-\delta)^b - 1\right)^n \right] \\
&= \sum_{j=1}^{\tilde{r}} g_j \left[1 - (\tilde{r}-j)\delta + o(\delta)\right] \left[1 - \frac{b\mu_b}{\lambda}\delta + o(\delta) \right] \\
&= 1 - \tilde{r}\delta - \left(\frac{b\mu_b}{\bar{g}\lambda} - 1\right) \bar{g}\delta + o(\delta) < 1 - \tilde{r}\delta + o(\delta) = (1-\delta)^{\tilde{r}} = |f(z)|.
\end{aligned}$$

Hence from Rouché's theorem, it follows that $f(z)$ and $f(z) + h(z)$ will have the same number of zeros inside $|z| = 1 - \delta$. Since $f(z)$ has exactly $\tilde{r}$ zeros inside this circle, $f(z) + h(z)$ will also have $\tilde{r}$ zeros inside $|z| = 1 - \delta$. Letting δ tend to zero, then the equation $f(z) + h(z) = 0$ has exactly $\tilde{r}$ roots inside the unit circle $|z| = 1$. Denote these roots by $\omega_1, \omega_2, \ldots, \omega_{\tilde{r}}$, and the modulus of each root is smaller than one. Further, in a similar manner, we can prove that the characteristic equation $z^{\tilde{r}} - \left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) \Lambda\left(z^b\right) = 0$ has also $\tilde{r}$ roots η_i in the region $|z| < 1$. According to the above analysis, we know that the equations $1 - X(z)D\left(z^{-b}\right) = 0$ and $1 - X(z)\Lambda\left(z^{-b}\right) = 0$ should have exactly $\tilde{r}$ roots outside the unit circle, respectively. As $\pi_0^-(z)$ is a rational function of z for $|z| \le 1$, we proceed to form the partial fraction expansion of $\pi_0^-(z)$ as

$$\pi_0^-(z) = \sum_{j=1}^{\tilde{r}} \frac{K_j}{(1 - \omega_j z)}, \tag{3.9}$$

where $K_1, K_2, \ldots, K_{\tilde{r}}$ are the non-zero constants to be determined. To invert the z-transform we can expand equation (3.9) as a power series and pick off the terms. Observe that $1/(1 - \omega_j z)$ is equal to the sum of a geometric series: $1/(1 - \omega_j z) = \sum_{i=0}^{\infty} (z\omega_j)^i$. Therefore

$$\pi_0^-(z) = \sum_{j=1}^{\tilde{r}} K_j \sum_{i=0}^{\infty} (\omega_j z)^i = \sum_{i=0}^{\infty} \left(\sum_{j=1}^{\tilde{r}} K_j \omega_j^i \right) z^i,$$

and so the coefficient of z^i (which we previously specified to be $\pi_{i,0}^-$) is equal to $\sum_{j=1}^{\tilde{r}} K_j \omega_j^i$. This gives the final solution as

$$\pi_{i,0}^- = \sum_{j=1}^{\tilde{r}} K_j \omega_j^i. \tag{3.10}$$

Furthermore, following the similar analysis and using the dependence of $\pi_1^-(z)$ on $\pi_0^-(z)$, the partial fraction expansion of $\pi_1^-(z)$ is given by

$$\pi_1^-(z) = \sum_{j=1}^{\tilde{r}} \frac{L_j}{(1-\omega_j z)} + \sum_{j=1}^{\tilde{r}} \frac{H_j}{(1-\eta_j z)}\cdot \tag{3.11}$$

We may invert equation (3.11) by repeated use of the transform relationships $L_j/(1-\omega_j z) \Leftrightarrow L_j \omega_j^i$ and $H_j/(1-\eta_j z) \Leftrightarrow H_j \eta_j^i$, where L_j and H_j are also non-zero constants to be determined, and the double arrow symbol is used to associate a sequence with its z-transform. Thus, the equilibrium distribution of queue size when the server is in regular busy period can be computed as

$$\pi_{i,1}^- = \sum_{j=1}^{\tilde{r}} \left(L_j \omega_j^i + H_j \eta_j^i \right). \tag{3.12}$$

Remark 3.1. As demonstrated in the above analysis, we assumed that the roots of the characteristic equations are distinct, and the case of repeated roots was omitted in our research. This is based on the following reasons. Tijms [39] and Chaudhry *et al.* [15] have shown that in queueing theory, the roots of the characteristic equations are generally distinct and follow a nice pattern. Our numerical experiments also indicate that, in the vast majority of cases, the roots happen to be distinct. Even if some roots of the characteristic equations inside the unit circle are repeated, the queue-length distribution at the pre-arrival epoch can also be determined by slightly modified equations (3.9) and (3.11). Because the length of the paper is strictly limited, and such a case rarely happens in practical computations, we omitted the detailed discussion of the case of repeated roots in our current paper.

Now, it is obvious that including $\pi_{\bar{0},0}^-$ and $\pi_{\bar{0},1}^-$, we have to determine $3\tilde{r}+2$ unknowns. Having found these unknowns, all the state probabilities $\pi_{i,0}^-$ and $\pi_{i,1}^-$ $(i = 0, 1, 2, \ldots)$ can be completely derived from equations (3.10) and (3.12). Substituting equations (3.10) and (3.12) into equations (3.1) to (3.5) for $j = 1, 2, \ldots, \tilde{r}-1$ and (3.6) for $j = 1, 2, \ldots, 2\tilde{r}-2$, and then simplifying these equations gives a linear system for $\pi_{\bar{0},0}^-$, $\pi_{\bar{0},1}^-$, K_j, H_j and L_j $(j = 1, 2, \ldots, \tilde{r})$.

$$\begin{aligned} 0 = {} & \pi_{\bar{0},0}^- \left(a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{r}{b} \rceil - 1} \beta_l + \sum_{r=1}^{\infty} g_r m_{\lceil \frac{r}{b} \rceil - 1} - 1 \right) + \pi_{\bar{0},1}^- \left(a^*(\theta) - \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{r}{b} \rceil - 1} \alpha_l + \sum_{r=1}^{\infty} g_r \sum_{l=0}^{\lceil \frac{r}{b} \rceil - 1} \phi_l \right) \\ & + \sum_{h=1}^{\tilde{r}} K_h \left(\sum_{l=2}^{\infty} \beta_l \sum_{i=0}^{(l-1)b-1} \omega_h^i \sum_{j=1}^{(l-1)b-i} g_j + \sum_{j=1}^{\tilde{r}} g_j \sum_{i=0}^{\infty} \omega_h^i m_{\lceil \frac{i+j}{b} \rceil} \right) \\ & + \sum_{h=1}^{\tilde{r}} L_h \left(\sum_{l=2}^{\infty} (\alpha_l - \phi_l) \sum_{i=0}^{(l-1)b-1} \omega_h^i \sum_{j=1}^{(l-1)b-i} g_j \right) + \sum_{h=1}^{\tilde{r}} H_h \left(\sum_{l=2}^{\infty} (\alpha_l - \phi_l) \sum_{i=0}^{(l-1)b-1} \eta_h^i \sum_{j=1}^{(l-1)b-i} g_j \right), \end{aligned} \tag{3.13}$$

$$0=\pi_{\bar{0},0}^{-}\left(1-a^*(\theta)-\sum_{r=1}^{\infty}g_r\sum_{l=0}^{\lceil\frac{r}{b}\rceil-1}\Delta_l-\sum_{r=1}^{\infty}g_r m_{\lceil\frac{r}{b}\rceil-1}\right)-\pi_{\bar{0},1}^{-}\left(a^*(\theta)+\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{r}{b}\rceil-1}\phi_n\right)$$
$$+\sum_{h=1}^{\tilde{r}}K_h\left(\sum_{l=2}^{\infty}\Delta_l\sum_{i=0}^{(l-1)b-1}\omega_h^i\sum_{j=1}^{(l-1)b-i}g_j-\sum_{j=1}^{\tilde{r}}g_j\sum_{i=0}^{\infty}\omega_h^i m_{\lceil\frac{i+j}{b}\rceil}\right)$$
$$+\sum_{h=1}^{\tilde{r}}L_h\left(\sum_{l=2}^{\infty}\phi_l\sum_{i=0}^{(l-1)b-1}\omega_h^i\sum_{j=1}^{(l-1)b-i}g_j\right)+\sum_{h=1}^{\tilde{r}}H_h\left(\sum_{l=2}^{\infty}\phi_l\sum_{i=0}^{(l-1)b-1}\eta_h^i\sum_{j=1}^{(l-1)b-i}g_j\right),\tag{3.14}$$

$$0=\pi_{\bar{0},0}^{-}\sum_{r=1}^{\infty}g_r\beta_{\lceil\frac{r}{b}\rceil-1}+\sum_{h=1}^{\tilde{r}}K_h\left(\sum_{j=1}^{\tilde{r}}g_j\sum_{i=0}^{\infty}\omega_h^i\beta_{\lceil\frac{i+j}{b}\rceil}-1\right),\tag{3.15}$$

$$0=\pi_{\bar{0},0}^{-}\sum_{r=1}^{\infty}g_r\Delta_{\lceil\frac{r}{b}\rceil-1}+\pi_{\bar{0},1}^{-}\sum_{r=1}^{\infty}g_r\alpha_{\lceil\frac{r}{b}\rceil-1}+\sum_{h=1}^{\tilde{r}}K_h\left(\sum_{j=1}^{\tilde{r}}g_j\sum_{i=0}^{\infty}\omega_h^i\Delta_{\lceil\frac{i+j}{b}\rceil}\right)$$
$$+\sum_{h=1}^{\tilde{r}}L_h\left(\sum_{j=1}^{\tilde{r}}g_j\sum_{i=0}^{\infty}\omega_h^i\alpha_{\lceil\frac{i+j}{b}\rceil}-1\right)+\sum_{h=1}^{\tilde{r}}H_h\left(\sum_{j=1}^{\tilde{r}}g_j\sum_{i=0}^{\infty}\eta_h^i\alpha_{\lceil\frac{i+j}{b}\rceil}-1\right),\tag{3.16}$$

$$0=\pi_{\bar{0},0}^{-}\sum_{k=1}^{\infty}g_{kb+j}\beta_{k-1}+\sum_{h=1}^{\tilde{r}}K_h\left[\left(\sum_{l=0}^{\infty}\beta_l\sum_{r=1}^{lb+j}g_r\omega_h^{lb+j-r}\right)-\omega_h^j\right],\quad 1\le j\le\tilde{r}-1,\tag{3.17}$$

$$0=\pi_{\bar{0},0}^{-}\sum_{k=1}^{\infty}g_{kb+j}\Delta_{k-1}+\pi_{\bar{0},1}^{-}\sum_{k=1}^{\infty}g_{kb+j}\alpha_{k-1}+\sum_{h=1}^{\tilde{r}}K_h\left(\sum_{l=0}^{\infty}\Delta_l\sum_{r=1}^{lb+j}g_r\omega_h^{lb+j-r}\right)$$
$$+\sum_{h=1}^{\tilde{r}}L_h\left[\left(\sum_{l=0}^{\infty}\alpha_l\sum_{r=1}^{lb+j}g_r\omega_h^{lb+j-r}\right)-\omega_h^j\right]+\sum_{h=1}^{\tilde{r}}H_h\left[\left(\sum_{l=0}^{\infty}\alpha_l\sum_{r=1}^{lb+j}g_r\eta_h^{lb+j-r}\right)-\eta_h^j\right],$$
$$1\le j\le 2\tilde{r}-2.\tag{3.18}$$

For $j\ge\tilde{r}$, we can easily derive the following relation from equation (3.5):

$$\pi_{\bar{0},0}^{-}\sum_{k=1}^{\infty}g_{kb+j}\beta_{k-1}+\sum_{h=1}^{\tilde{r}}K_h\omega_h^{j-\tilde{r}}\left[\left(\sum_{j=1}^{\tilde{r}}g_j\omega_h^{\tilde{r}-j}\right)D(\omega_h^b)-\omega_h^{\tilde{r}}\right]=0.$$

Since $\omega_1,\omega_2,\ldots,\omega_{\tilde{r}}$ are the roots of equation $z^{\tilde{r}}-\left(\sum_{j=1}^{\tilde{r}}g_jz^{\tilde{r}-j}\right)D\left(z^b\right)=0$ inside the unit circle, it follows that

$$\left(\sum_{j=1}^{\tilde{r}}g_j\omega_h^{\tilde{r}-j}\right)D(\omega_h^b)-\omega_h^{\tilde{r}}=0.$$

At the same time, we notice that $\tilde{r}$ is the maximum size of an arriving batch, so, for $k\ge1$, we have $\sum_{k=1}^{\infty}g_{kb+j}\beta_{k-1}=0$. It also means that, if $j\ge\tilde{r}$, equation (3.5) becomes redundant. This is the reason why we ignore the values for $j\ge\tilde{r}$ of equation (3.5). To solve for $3\tilde{r}+2$ unknown variables, we need $3\tilde{r}+2$ separate and independent equations to come to a unique solution. Thus, we must use normalizing condition by letting $z=1$ in equations (3.9) and (3.11), which is given as follows

$$1=\pi_{\bar{0},0}^{-}+\pi_{\bar{0},1}^{-}+\sum_{j=1}^{\tilde{r}}\frac{K_j}{1-\omega_j}+\sum_{j=1}^{\tilde{r}}\frac{L_j}{1-\omega_j}+\sum_{j=1}^{\tilde{r}}\frac{H_j}{1-\eta_j}\cdot\tag{3.19}$$

Solving for the unknown constants $\pi^-_{\bar{0},0}$, $\pi^-_{\bar{0},1}$, K_j, L_j and $H_j (j = 1, \dots, \tilde{r})$ from above $3\tilde{r} + 2$ equations, the queue-length distribution just before the arrival epoch of a batch can be finally obtained.

To help practitioners as well as others who would like to implement the model, we describe the working procedure for evaluating the pre-arrival probabilities $\pi^-_{i,0}$ and $\pi^-_{i,1}$, $i = 0, 1, 2, \dots$. Given the values of $\tilde{r}$, b, μ_b, μ_v and the probability mass function $\Pr\{X = k\} = g_k$, $k = 1, 2, \dots, \tilde{r}$, the steps of the solution algorithm are stated as follows:

- **Step 1**: find the roots of the following two characteristic equations $z^{\tilde{r}} - \left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) D\left(z^b\right) = 0$ and $z^{\tilde{r}} - \left(\sum_{j=1}^{\tilde{r}} g_j z^{\tilde{r}-j}\right) \Lambda\left(z^b\right) = 0$ inside the unit circle, respectively. Denote these roots as ω_j and η_j, $j = 1, 2, \dots, \tilde{r}$.
- **Step 2**: insert these roots directly into equations (3.13) to (3.19), and solve a system of linear equations to find the values of the unknown variables $\pi^-_{\bar{0},0}$, $\pi^-_{\bar{0},1}$, K_j, L_j and H_j $(j = 1, \dots, \tilde{r})$.
- **Step 3**: substituting the values of K_j, L_j, H_j, ω_j and η_j into equations (3.10) and (3.12) gives $\pi^-_{i,0}$ and $\pi^-_{i,1}$, for $i = 0, 1, 2, \dots$.

4. Determination of queue-length distribution at arbitrary epoch

From our previous results regarding the stationary queue-length distributions immediately preceding the nth batch arrival, we will derive the stationary queue-length distribution at arbitrary epochs by employing the classical argument based on renewal theory and semi-Markov process (SMP). For $j = \bar{0}, 0, 1, 2, \dots$, let $\pi_{j,0}$ and $\pi_{j,1}$ denote the probability of j customers waiting in the queue at an arbitrary epoch when the server is on working vacation and in a regular busy period, respectively. Here, $j = \bar{0}$ also refers to an empty queue, and the server is in the idle state. Now, consider a new process $\left\{(\tilde{N}_q(t), \tilde{Y}(t)), t > 0\right\}$, where $\tilde{N}_q(t)$ denotes the queue size (not including those in service) right after the most recent arrival, and $\tilde{Y}(t)$ equals 0 or 1 if the most recent arrival occurs during a working vacation or during a regular service period, respectively. Then $\left\{(\tilde{N}_q(t), \tilde{Y}(t)), t > 0\right\}$ is a SMP having $\{(N_q(\tau_n^-), Y(\tau_n^-)), n \geq 1\}$ for its embedded Markov chain. Let $\gamma_{i,j}$ be the mean time that SMP $\left\{(\tilde{N}_q(t), \tilde{Y}(t)), t > 0\right\}$ spend in the state (i, j). By the definition, $\gamma_{i,j} = 1/\lambda$ for all $(i, j) \in \Omega$. Let $\nu_{i,j}$ be the steady-state probability that the SMP is in state (i, j). According to the results given by Gross and Harris [20], we have

$$\nu_{i,j} = \frac{\pi^-_{i,j}\gamma_{i,j}}{\sum\limits_{(i',j')\in\Omega} \pi^-_{i',j'}\gamma_{i',j'}}.$$

This indicates that $\nu_{i,j} = \pi^-_{i,j}$. Since the probability density function of the time back to the last transition is known as $\lambda[1 - A(t)]$, and all states (i, j) communicate with the state $(\bar{0}, 0)$ in the embedded Markov chain of the above SMP, based on the relationship between $\{(N_q(t), Y(t)), t > 0\}$ and $\left\{(\tilde{N}_q(t), \tilde{Y}(t)), t > 0\right\}$, the probability $\pi_{\bar{0},0}$ can be expressed as

$$\begin{aligned}
\pi_{\bar{0},0} = &\sum_{(i,j)\in\Omega} \nu_{i,j} \sum_{r=1}^{\infty} g_r \int_0^{\infty} \Pr\{\text{appropriate changes in the backward recurrence time} \quad t \\
&\qquad\qquad \text{to bring state from} \quad \text{(i,j)} \quad \text{to} \quad (\bar{0},0)\}\, \lambda[1 - A(t)]\, \mathrm{d}t\} \\
= &\,\pi^-_{\bar{0},0} \sum_{r=1}^{\infty} g_r \int_0^{\infty} \left[\Pr\left\{\sum_{n=1}^{\lceil \frac{r}{b} \rceil} \tilde{S}_n \leq t, t < V\right\}\right. \\
&+ \left.\sum_{l=0}^{\lceil \frac{r}{b} \rceil - 1} \Pr\left\{\sum_{n=1}^{l} \tilde{S}_n \leq V < \sum_{n=1}^{l+1} \tilde{S}_n, V + \sum_{n=l+1}^{\lceil \frac{r}{b} \rceil} S_n \leq t < V + \sum_{n=l+1}^{\lceil \frac{r}{b} \rceil} S_n + V\right\}\right] \lambda[1 - A(t)]\, \mathrm{d}t
\end{aligned}$$

$$+\pi^-_{\bar{0},1}\sum_{r=1}^{\infty}g_r\int_0^{\infty}\Pr\left\{\sum_{n=1}^{\lceil\frac{r}{b}\rceil}S_n\le t<\sum_{n=1}^{\lceil\frac{r}{b}\rceil}S_n+V\right\}\lambda\left[1-A(t)\right]\,\mathrm{d}t$$
$$+\sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{r=1}^{\infty}g_r\int_0^{\infty}\left[\Pr\left\{\sum_{n=1}^{\lceil\frac{i+r}{b}\rceil+1}\tilde{S}_n\le t,t<V\right\}\right.$$
$$\left.+\sum_{l=0}^{\lceil\frac{i+r}{b}\rceil}\Pr\left\{\sum_{n=1}^{l}\tilde{S}_n\le V<\sum_{n=1}^{l+1}\tilde{S}_n,V+\sum_{n=l+1}^{\lceil\frac{i+r}{b}\rceil+1}S_n\le t<V+\sum_{n=l+1}^{\lceil\frac{i+r}{b}\rceil+1}S_n+V\right\}\right]\lambda\left[1-A(t)\right]\,\mathrm{d}t$$
$$+\sum_{i=0}^{\infty}\pi^-_{i,1}\sum_{r=1}^{\infty}g_r\int_0^{\infty}\Pr\left\{\sum_{n=1}^{\lceil\frac{i+r}{b}\rceil+1}S_n\le t<\sum_{n=1}^{\lceil\frac{i+r}{b}\rceil+1}S_n+V\right\}\lambda\left[1-A(t)\right]\,\mathrm{d}t. \tag{4.1}$$

To simplify the above equation, the equivalent auxiliary probabilities of α_k, β_k, ϕ_k, Δ_k and m_k by considering the time back to the most recent transition looking from t, denoted as $\tilde{\alpha}_k$, $\tilde{\beta}_k$, $\tilde{\phi}_k$, $\tilde{\Delta}_k$ and $\tilde{m}_k$ are really needed. Replacing the probability density function of the inter-batch arrival time $\mathrm{d}A(t)$ with $\lambda\left(1-A(t)\right)\,\mathrm{d}t$, for $k=0,1,2,\ldots$, we have

$$\tilde{\alpha}_k=\int_0^{\infty}\frac{(\mu_b t)^k}{k!}e^{-\mu_b t}\lambda\left(1-A(t)\right)\,\mathrm{d}t;\quad \tilde{\beta}_k=\int_0^{\infty}\frac{(\mu_v t)^k}{k!}e^{-(\mu_v+\theta)t}\lambda\left(1-A(t)\right)\,\mathrm{d}t;$$
$$\tilde{\phi}_k=\int_0^{\infty}\int_0^{t}\theta e^{-\theta x}e^{-\mu_b(t-x)}\frac{(\mu_b(t-x))^k}{k!}\lambda\left(1-A(t)\right)\,\mathrm{d}x\,\mathrm{d}t;$$
$$\tilde{\Delta}_k=\sum_{l=0}^{k}\int_0^{\infty}\int_0^{t}\theta e^{-\theta x}e^{-\mu_v x}\frac{(\mu_v x)^l}{l!}e^{-\mu_b(t-x)}\frac{(\mu_b(t-x))^{k-l}}{(k-l)!}\lambda\left(1-A(t)\right)\,\mathrm{d}x\,\mathrm{d}t;$$
$$\tilde{m}_k=\sum_{l=0}^{k}\int_0^{\infty}\int_0^{t}\theta e^{-\theta x}e^{-\mu_v x}\frac{(\mu_v x)^l}{l!}\int_0^{t-x}e^{-\mu_b y}\frac{\mu_b(\mu_b y)^{k-l}}{(k-l)!}e^{-\theta(t-x-y)}\lambda\left(1-A(t)\right)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.$$

Note that $\int_0^{\infty}e^{-st}\lambda\left[1-A(t)\right]\,\mathrm{d}t=\frac{\lambda(1-a^*(s))}{s}$. Using the auxiliary probabilities defined here, equation (4.1) can be simplified as

$$\pi_{\bar{0},0}=\pi^-_{\bar{0},0}\left(\frac{\lambda\left(1-a^*\left(\theta\right)\right)}{\theta}-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{r}{b}\rceil-1}\tilde{\beta}_n+\sum_{r=1}^{\infty}g_r\tilde{m}_{\lceil\frac{r}{b}\rceil-1}\right)$$
$$+\pi^-_{\bar{0},1}\left(\frac{\lambda\left(1-a^*\left(\theta\right)\right)}{\theta}-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{r}{b}\rceil-1}\tilde{\alpha}_n+\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{r}{b}\rceil-1}\tilde{\phi}_n\right)$$
$$+\sum_{i=0}^{\infty}\pi^-_{i,0}\left(\frac{\lambda\left(1-a^*\left(\theta\right)\right)}{\theta}-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\tilde{\beta}_n+\sum_{r=1}^{\infty}g_r\tilde{m}_{\lceil\frac{i+r}{b}\rceil}\right)$$
$$+\sum_{i=0}^{\infty}\pi^-_{i,1}\left(\frac{\lambda\left(1-a^*\left(\theta\right)\right)}{\theta}-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\tilde{\alpha}_n+\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\tilde{\phi}_n\right). \tag{4.2}$$

The following equations (4.3) to (4.7) also give the relations between the distributions of the number of customers in the queue at the pre-arrival and arbitrary epochs. These equations can be derived in an analogs way as mentioned before.

$$
\begin{aligned}
\pi_{\bar{0},1} = \pi_{\bar{0},0}^{-}&\left(1-\frac{\lambda\left(1-a^{*}\left(\theta\right)\right)}{\theta}-\sum_{r=1}^{\infty}g_r\sum_{l=0}^{\lceil\frac{r}{b}\rceil-1}\tilde{\Delta}_l-\sum_{r=1}^{\infty}g_r\tilde{m}_{\lceil\frac{r}{b}\rceil-1}\right)\\
&+\pi_{\bar{0},1}^{-}\left(1-\frac{\lambda\left(1-a^{*}\left(\theta\right)\right)}{\theta}-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{r}{b}\rceil-1}\tilde{\phi}_n\right)\\
&+\sum_{i=0}^{\infty}\pi_{i,0}^{-}\left(1-\frac{\lambda\left(1-a^{*}\left(\theta\right)\right)}{\theta}-\sum_{r=1}^{\infty}g_r\sum_{l=0}^{\lceil\frac{i+r}{b}\rceil}\tilde{\Delta}_l-\sum_{r=1}^{\infty}g_r\tilde{m}_{\lceil\frac{i+r}{b}\rceil}\right)\\
&+\sum_{i=0}^{\infty}\pi_{i,1}^{-}\left(1-\frac{\lambda\left(1-a^{*}\left(\theta\right)\right)}{\theta}-\sum_{r=1}^{\infty}g_r\sum_{n=0}^{\lceil\frac{i+r}{b}\rceil}\tilde{\phi}_n\right),
\end{aligned}
\tag{4.3}
$$

$$
\pi_{0,0}=\pi_{\bar{0},0}^{-}\sum_{r=1}^{\infty}g_r\tilde{\beta}_{\lceil\frac{r}{b}\rceil-1}+\sum_{i=0}^{\infty}\pi_{i,0}^{-}\sum_{r=1}^{\infty}g_r\tilde{\beta}_{\lceil\frac{i+r}{b}\rceil},
\tag{4.4}
$$

$$
\pi_{0,1}=\pi_{\bar{0},0}^{-}\sum_{r=1}^{\infty}g_r\tilde{\Delta}_{\lceil\frac{r}{b}\rceil-1}+\pi_{\bar{0},1}^{-}\sum_{r=1}^{\infty}g_r\tilde{\alpha}_{\lceil\frac{r}{b}\rceil-1}+\sum_{i=0}^{\infty}\pi_{i,0}^{-}\sum_{r=1}^{\infty}g_r\tilde{\Delta}_{\lceil\frac{i+r}{b}\rceil}+\sum_{i=0}^{\infty}\pi_{i,1}^{-}\sum_{r=1}^{\infty}g_r\tilde{\alpha}_{\lceil\frac{i+r}{b}\rceil},
\tag{4.5}
$$

$$
\pi_{j,0}=\pi_{\bar{0},0}^{-}\sum_{k=1}^{\infty}g_{kb+j}\tilde{\beta}_{k-1}+\sum_{i=0}^{\infty}\pi_{i,0}^{-}\sum_{k=\max\{0,\lfloor\frac{i+1-j}{b}\rfloor+1\}}^{\infty}g_{kb+j-i}\tilde{\beta}_k,\quad j=1,2,\ldots,
\tag{4.6}
$$

$$
\begin{aligned}
\pi_{j,1}=\pi_{\bar{0},0}^{-}\sum_{k=1}^{\infty}g_{kb+j}\tilde{\Delta}_{k-1}&+\pi_{\bar{0},1}^{-}\sum_{k=1}^{\infty}g_{kb+j}\tilde{\alpha}_{k-1}+\sum_{i=0}^{\infty}\pi_{i,0}^{-}\sum_{k=\max\{0,\lfloor\frac{i+1-j}{b}\rfloor+1\}}^{\infty}g_{kb+j-i}\tilde{\Delta}_k\\
&+\sum_{i=0}^{\infty}\pi_{i,1}^{-}\sum_{k=\max\{0,\lfloor\frac{i+1-j}{b}\rfloor+1\}}^{\infty}g_{kb+j-i}\tilde{\alpha}_k,\quad j=1,2,\ldots.
\end{aligned}
\tag{4.7}
$$

Since the queue-length distribution at the pre-arrival epoch has been computed using the method explained in Section 3, we can easily obtain the queue-length distribution at an arbitrary epoch by substituting the values of $\pi_{i,0}^{-}$ and $\pi_{i,1}^{-}$ directly into equations (4.2) to (4.7).

5. System-length distributions at pre-arrival and arbitrary epochs

For the bulk service rule, we can further determine the system-length distributions at different epochs using the corresponding queue-length distributions. Since the system-length is the sum of the number of customers waiting in the queue and the number of customers being served, both system-length and queue-length coincide with an idle server. According to the status of the server, we derive the system-length distribution under two different cases. Let $\pi_{j,i,0}^{-}(\pi_{j,i,1}^{-})$ represent the pre-arrival epoch probability that there are i customers waiting in the queue when the server is on working vacation (in the regular busy period) with a service batch of size j $(1\leq j\leq b)$. The approach adopted to solve for the probabilities $\pi_{j,i,0}^{-}$ and $\pi_{j,i,1}^{-}$ is to enumerate the mutually exclusive events which can result in a group of size j in service just before a batch arrival epoch and then weigh these by the probability of their occurrence. For example, below are two events that cause the $(n+1)$th batch to find the system is in state $(j,0,0)$ upon its arrival. Event 1: Assume that the size of the nth batch equals $kb+j$ $(k=0,1,\ldots)$ with probability g_{kb+j}, and upon arrival it sees that the system is in state $(\bar{0},0)$. If the residual working vacation time is greater than the inter-batch arrival time, and k batches of customers complete their services during the inter-batch arrival time, the state transition from $(\bar{0},0)$ to $(j,0,0)$ can be realized. Event 2: Suppose that the nth batch finds the system is in state $(i,0)$ just before its arrival epoch, and the size of this batch equals $kb+j-i$ ($k=\max\left(0,\lfloor\frac{i+1-j}{b}\rfloor+1\right),\ldots$) with probability g_{kb+j-i}. If the residual working vacation time is greater than the inter-batch arrival time, and $k+1$ batches of customers complete their services during the inter-batch arrival time, the state transition from $(i,0)$ to $(j,0,0)$ can also be realized. Thus, we have

$$\pi^-_{j,0,0}=\pi^-_{\bar{0},0}\sum_{k=0}^{\infty}g_{kb+j}\beta_k+\sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{k=\max\left(0,\lfloor\frac{i+1-j}{b}\rfloor+1\right)}^{\infty}g_{kb+j-i}\beta_{k+1},\quad 1\le j\le b.$$

Investigating the number of customers in the system at two consecutive pre-arrival epochs during working vacation, we can further establish the following relationships

$$\pi^-_{j,n,0}=\sum_{i=0}^{n-1}\pi^-_{j,i,0}g_{n-i}\beta_0,\quad 1\le j\le b-1,\quad n\ge 1,$$

$$\pi^-_{b,n,0}=\sum_{i=0}^{n-1}\pi^-_{b,i,0}g_{n-i}\beta_0+\pi^-_{\bar{0},0}\sum_{k=1}^{\infty}g_{kb+n}\beta_{k-1}+\sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{k=\max\left(1,\lfloor\frac{i+1-n}{b}\rfloor+1\right)}^{\infty}g_{kb-i+n}\beta_k,\quad n\ge 1.$$

Similarly, the corresponding relations when the server is in regular busy period are given by

$$\begin{aligned}\pi^-_{j,0,1}=&\,\pi^-_{\bar{0},0}\sum_{k=0}^{\infty}g_{kb+j}\Delta_k+\sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{k=\max\left(0,\lfloor\frac{i+1-j}{b}\rfloor+1\right)}^{\infty}g_{kb+j-i}\Delta_{k+1}+\pi^-_{\bar{0},1}\sum_{k=0}^{\infty}g_{kb+j}\alpha_k\\&+\sum_{i=0}^{\infty}\pi^-_{i,1}\sum_{k=\max\left(0,\lfloor\frac{i+1-j}{b}\rfloor+1\right)}^{\infty}g_{kb+j-i}\alpha_{k+1},\quad 1\le j\le b,\\
\pi^-_{j,n,1}=&\sum_{i=0}^{n-1}\pi^-_{j,i,0}g_{n-i}\Delta_0+\sum_{i=0}^{n-1}\pi^-_{j,i,1}g_{n-i}\alpha_0,\quad 1\le j\le b-1,\quad n\ge 1,\\
\pi^-_{b,n,1}=&\sum_{i=0}^{n-1}\pi^-_{b,i,0}g_{n-i}\Delta_0+\sum_{i=0}^{n-1}\pi^-_{b,i,1}g_{n-i}\alpha_0+\pi^-_{\bar{0},0}\sum_{k=1}^{\infty}g_{kb+n}\Delta_{k-1}+\sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{k=\max\left(1,\lfloor\frac{i+1-n}{b}\rfloor+1\right)}^{\infty}g_{kb-i+n}\Delta_k\\&+\pi^-_{\bar{0},1}\sum_{k=1}^{\infty}g_{kb+n}\alpha_{k-1}+\sum_{i=0}^{\infty}\pi^-_{i,1}\sum_{k=\max\left(1,\lfloor\frac{i+1-n}{b}\rfloor+1\right)}^{\infty}g_{kb-i+n}\alpha_k,\quad n\ge 1.\end{aligned}$$

Further, let $\pi_{j,i,0}$ ($\pi_{j,i,1}$) denote the arbitrary epoch probability that there are i customers waiting in the queue when the server is on working vacation (in regular busy period) with a service batch of size j ($1\le j\le b$). The same argument used before is applied here to derive the system-length distribution at an arbitrary epoch. Replacing the time frame being considered above with the probabilities appropriate to the elapsed inter-batch arrival time, rather than the inter-batch arrival time, we obtain the following relationships

$$\begin{aligned}\pi_{j,0,0}=&\,\pi^-_{\bar{0},0}\sum_{k=0}^{\infty}g_{kb+j}\tilde{\beta}_k+\sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{k=\max\left(0,\lfloor\frac{i+1-j}{b}\rfloor+1\right)}^{\infty}g_{kb+j-i}\tilde{\beta}_{k+1},\quad 1\le j\le b,\\
\pi_{j,n,0}=&\sum_{i=0}^{n-1}\pi^-_{j,i,0}g_{n-i}\tilde{\beta}_0,\quad 1\le j\le b-1,\quad n\ge 1,\\
\pi_{b,n,0}=&\sum_{i=0}^{n-1}\pi^-_{b,i,0}g_{n-i}\tilde{\beta}_0+\pi^-_{\bar{0},0}\sum_{k=1}^{\infty}g_{kb+n}\tilde{\beta}_{k-1}+\sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{k=\max\left(1,\lfloor\frac{i+1-n}{b}\rfloor+1\right)}^{\infty}g_{kb-i+n}\tilde{\beta}_k,\quad n\ge 1,\end{aligned}$$

$$\pi_{j,0,1} = \pi^-_{\bar{0},0}\sum_{k=0}^{\infty} g_{kb+j}\tilde{\Delta}_k + \sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{k=\max\left(0,\lfloor\frac{i+1-j}{b}\rfloor+1\right)}^{\infty} g_{kb+j-i}\tilde{\Delta}_{k+1} + \pi^-_{\bar{0},1}\sum_{k=0}^{\infty} g_{kb+j}\tilde{\alpha}_k$$
$$+ \sum_{i=0}^{\infty}\pi^-_{i,1}\sum_{k=\max\left(0,\lfloor\frac{i+1-j}{b}\rfloor+1\right)}^{\infty} g_{kb+j-i}\tilde{\alpha}_{k+1}, \quad 1 \le j \le b,$$
$$\pi_{j,n,1} = \sum_{i=0}^{n-1}\pi^-_{j,i,0}g_{n-i}\tilde{\Delta}_0 + \sum_{i=0}^{n-1}\pi^-_{j,i,1}g_{n-i}\tilde{\alpha}_0, \quad 1 \le j \le b-1, \quad n \ge 1,$$
$$\pi_{b,n,1} = \sum_{i=0}^{n-1}\pi^-_{b,i,0}g_{n-i}\tilde{\Delta}_0 + \sum_{i=0}^{n-1}\pi^-_{b,i,1}g_{n-i}\tilde{\alpha}_0 + \pi^-_{\bar{0},0}\sum_{k=1}^{\infty} g_{kb+n}\tilde{\Delta}_{k-1} + \sum_{i=0}^{\infty}\pi^-_{i,0}\sum_{k=\max\left(1,\lfloor\frac{i+1-n}{b}\rfloor+1\right)}^{\infty} g_{kb-i+n}\tilde{\Delta}_k$$
$$+ \pi^-_{\bar{0},1}\sum_{k=1}^{\infty} g_{kb+n}\tilde{\alpha}_{k-1} + \sum_{i=0}^{\infty}\pi^-_{i,1}\sum_{k=\max\left(1,\lfloor\frac{i+1-n}{b}\rfloor+1\right)}^{\infty} g_{kb-i+n}\tilde{\alpha}_k, \quad n \ge 1.$$

The formulas above are clearly numerically tractable. However, as some expressions for $\pi^-_{j,i,0}(\pi^-_{j,i,1})$ and $\pi_{j,i,0}(\pi_{j,i,1})$ are not in explicit form, the system-length distributions at different epochs can only be computed by an iterative method.

6. Sojourn time of an arbitrary customer in an arriving batch

Let us define the random variable W_A as the equilibrium sojourn time for an arbitrary test customer in an arriving group, and denote the corresponding cumulative distribution function by $W_A(t)$. At the time of an arbitrary test customer's arrival, there will be a number of customers arriving in his batch who will be served before him. Let g^-_r $(r = 0, 1, 2, \ldots, \tilde{r}-1)$ be the probability of an arbitrary test customer being in the $(r+1)$th position of an arrived batch. Using a result in the renewal theory, Burke [9] and Chaudhry and Templeton [14] have shown that $g^-_r = \frac{1}{\bar{g}}\sum_{j=r+1}^{\infty} g_j$. Thus, considering various possible cases seen by an arriving batch, the expression of $W_A(t)$ is given below

$$W_A(t) = \Pr\{W_A \le t\}$$
$$= \pi^-_{\bar{0},0}\sum_{r=0}^{\infty} g^-_r \Pr\left\{W_A = \sum_{h=1}^{\lceil\frac{r+1}{b}\rceil}\tilde{S}_h \le t \,\middle|\, V \ge \sum_{h=1}^{\lceil\frac{r+1}{b}\rceil}\tilde{S}_h\right\}\Pr\left\{V \ge \sum_{h=1}^{\lceil\frac{r+1}{b}\rceil}\tilde{S}_h\right\}$$
$$+ \pi^-_{\bar{0},0}\sum_{r=0}^{\infty} g^-_r \sum_{k=0}^{\lceil\frac{r+1}{b}\rceil-1}\Pr\left\{W_A = V + \sum_{h=k+1}^{\lceil\frac{r+1}{b}\rceil} S_h \le t \,\middle|\, \sum_{h=1}^{k}\tilde{S}_h \le V < \sum_{h=1}^{k+1}\tilde{S}_h\right\}\Pr\left\{\sum_{h=1}^{k}\tilde{S}_h \le V < \sum_{h=1}^{k+1}\tilde{S}_h\right\}$$
$$+ \sum_{j=0}^{\infty}\pi^-_{j,0}\sum_{r=0}^{\infty} g^-_r \Pr\left\{W_A = \sum_{h=1}^{\lceil\frac{r+j+1}{b}\rceil+1}\tilde{S}_h \le t \,\middle|\, V \ge \sum_{h=1}^{\lceil\frac{r+j+1}{b}\rceil+1}\tilde{S}_h\right\}\Pr\left\{V \ge \sum_{h=1}^{\lceil\frac{r+j+1}{b}\rceil+1}\tilde{S}_h\right\}$$
$$+ \sum_{j=0}^{\infty}\pi^-_{j,0}\sum_{r=0}^{\infty} g^-_r \sum_{k=0}^{\lceil\frac{r+j+1}{b}\rceil}\Pr\left\{W_A = V + \sum_{h=k+1}^{\lceil\frac{r+j+1}{b}\rceil+1} S_h \le t \,\middle|\, \sum_{h=1}^{k}\tilde{S}_h \le V < \sum_{h=1}^{k+1}\tilde{S}_h\right\}$$
$$\times \Pr\left\{\sum_{h=1}^{k}\tilde{S}_h \le V < \sum_{h=1}^{k+1}\tilde{S}_h\right\}$$

$$
\begin{aligned}
&+\sum_{j=0}^{\infty}\pi_{j,1}^{-}\sum_{r=0}^{\infty}g_r^{-}\Pr\left\{W_A=\sum_{h=1}^{\lceil\frac{r+j+1}{b}\rceil+1}S_h\le t\right\}+\pi_{\bar{0},1}^{-}\sum_{r=0}^{\infty}g_r^{-}\Pr\left\{W_A=\sum_{h=1}^{\lceil\frac{r+1}{b}\rceil}S_h\le t\right\}\\
&=\pi_{\bar{0},0}^{-}\sum_{r=0}^{\infty}g_r^{-}\int_0^t\frac{\mu_v(\mu_v x)^{\lceil\frac{r+1}{b}\rceil-1}}{(\lceil\frac{r+1}{b}\rceil-1)!}e^{-(\theta+\mu_v)x}\,\mathrm{d}x\\
&+\pi_{\bar{0},0}^{-}\sum_{r=0}^{\infty}g_r^{-}\sum_{k=0}^{\lceil\frac{r+1}{b}\rceil-1}\int_0^t\theta e^{-(\theta+\mu_v)x}\frac{(\mu_v x)^k}{k!}\left[1-e^{-\mu_b(t-x)}\sum_{i=0}^{\lceil\frac{r+1}{b}\rceil-k-1}\frac{(\mu_b(t-x))^i}{i!}\right]\mathrm{d}x\\
&+\sum_{j=0}^{\infty}\pi_{j,0}^{-}\sum_{r=0}^{\infty}g_r^{-}\int_0^t\frac{\mu_v(\mu_v x)^{\lceil\frac{r+j+1}{b}\rceil}}{\lceil\frac{r+j+1}{b}\rceil!}e^{-(\theta+\mu_v)x}\,\mathrm{d}x\\
&+\sum_{j=0}^{\infty}\pi_{j,0}^{-}\sum_{r=0}^{\infty}g_r^{-}\sum_{k=0}^{\lceil\frac{r+j+1}{b}\rceil}\int_0^t\theta e^{-(\theta+\mu_v)x}\frac{(\mu_v x)^k}{k!}\left[1-e^{-\mu_b(t-x)}\sum_{i=0}^{\lceil\frac{r+j+1}{b}\rceil-k}\frac{(\mu_b(t-x))^i}{i!}\right]\mathrm{d}x\\
&+\sum_{j=0}^{\infty}\pi_{j,1}^{-}\sum_{r=0}^{\infty}g_r^{-}\int_0^t e^{-\mu_b x}\frac{\mu_b(\mu_b x)^{\lceil\frac{r+j+1}{b}\rceil}}{\lceil\frac{r+j+1}{b}\rceil!}\,\mathrm{d}x+\pi_{\bar{0},1}^{-}\sum_{r=0}^{\infty}g_r^{-}\int_0^t e^{-\mu_b x}\frac{\mu_b(\mu_b x)^{\lceil\frac{r+1}{b}\rceil-1}}{(\lceil\frac{r+1}{b}\rceil-1)!}\,\mathrm{d}x.
\end{aligned}
$$

Taking LST on both sides of the above equation, we get

$$
\begin{aligned}
W_A^*(s)&=\int_0^{\infty}e^{-st}\,\mathrm{d}W_A(t)\\
&=\pi_{\bar{0},0}^{-}\sum_{r=0}^{\infty}g_r^{-}\left(\frac{\mu_v}{s+\theta+\mu_v}\right)^{\lceil\frac{r+1}{b}\rceil}+\pi_{\bar{0},0}^{-}\sum_{r=0}^{\infty}g_r^{-}\sum_{k=0}^{\lceil\frac{r+1}{b}\rceil-1}\frac{\theta\mu_v^k}{(s+\theta+\mu_v)^{k+1}}\left(\frac{\mu_b}{s+\mu_b}\right)^{\lceil\frac{r+1}{b}\rceil-k}\\
&\quad+\sum_{j=0}^{\infty}\pi_{j,0}^{-}\sum_{r=0}^{\infty}g_r^{-}\left(\left(\frac{\mu_v}{s+\theta+\mu_v}\right)^{\lceil\frac{r+j+1}{b}\rceil+1}+\sum_{k=0}^{\lceil\frac{r+j+1}{b}\rceil}\frac{\theta\mu_v^k}{(s+\theta+\mu_v)^{k+1}}\left(\frac{\mu_b}{s+\mu_b}\right)^{\lceil\frac{r+j+1}{b}\rceil-k+1}\right)\\
&\quad+\sum_{j=0}^{\infty}\pi_{j,1}^{-}\sum_{r=0}^{\infty}g_r^{-}\left(\frac{\mu_b}{s+\mu_b}\right)^{\lceil\frac{r+j+1}{b}\rceil+1}+\pi_{\bar{0},1}^{-}\sum_{r=0}^{\infty}g_r^{-}\left(\frac{\mu_b}{s+\mu_b}\right)^{\lceil\frac{r+1}{b}\rceil}.
\end{aligned}
$$

Then, the expected sojourn time of an arbitrary customer in an arriving batch can be obtained as

$$
\begin{aligned}
\mathrm{E}[W_A]&=-\frac{\mathrm{d}}{\mathrm{d}s}W_A^*(s)\,|_{s=0}\\
&=\pi_{\bar{0},0}^{-}\sum_{r=0}^{\infty}g_r^{-}\frac{\lceil\frac{r+1}{b}\rceil\mu_v^{\lceil\frac{r+1}{b}\rceil}}{(\theta+\mu_v)^{\lceil\frac{r+1}{b}\rceil+1}}+\pi_{\bar{0},0}^{-}\sum_{r=0}^{\infty}g_r^{-}\sum_{k=0}^{\lceil\frac{r+1}{b}\rceil-1}\left(\frac{(k+1)\theta\mu_v^k}{(\theta+\mu_v)^{k+2}}+\frac{(\lceil\frac{r+1}{b}\rceil-k)\theta\mu_v^k}{\mu_b\,(\theta+\mu_v)^{k+1}}\right)\\
&\quad+\sum_{j=0}^{\infty}\pi_{j,0}^{-}\sum_{r=0}^{\infty}g_r^{-}\frac{(\lceil\frac{r+j+1}{b}\rceil+1)\mu_v^{\lceil\frac{r+j+1}{b}\rceil+1}}{(\theta+\mu_v)^{\lceil\frac{r+j+1}{b}\rceil+2}}\\
&\quad+\sum_{j=0}^{\infty}\pi_{j,0}^{-}\sum_{r=0}^{\infty}g_r^{-}\sum_{k=0}^{\lceil\frac{r+j+1}{b}\rceil}\left(\frac{(k+1)\theta\mu_v^k}{(\theta+\mu_v)^{k+2}}+\frac{(\lceil\frac{r+j+1}{b}\rceil-k+1)\theta\mu_v^k}{\mu_b\,(\theta+\mu_v)^{k+1}}\right)\\
&\quad+\sum_{j=0}^{\infty}\pi_{j,1}^{-}\sum_{r=0}^{\infty}g_r^{-}\frac{(\lceil\frac{r+j+1}{b}\rceil+1)}{\mu_b}+\pi_{\bar{0},1}^{-}\sum_{r=0}^{\infty}g_r^{-}\frac{\lceil\frac{r+1}{b}\rceil}{\mu_b}. \qquad (6.1)
\end{aligned}
$$

Remark 6.1. With just a slight modification of equation (6.1), the mean waiting time of an arbitrary customer in the queue (excluding service time) is given by

$$\begin{aligned}\mathrm{E}\left[W_{qA}\right] &= \pi_{\bar{0},0}^{-}\sum_{r=b}^{\infty} g_r^{-}\frac{\left(\lceil \frac{r+1}{b}\rceil - 1\right)\mu_v^{\lceil \frac{r+1}{b}\rceil - 1}}{(\theta+\mu_v)^{\lceil \frac{r+1}{b}\rceil}} \\ &\quad + \pi_{\bar{0},0}^{-}\sum_{r=b}^{\infty} g_r^{-}\sum_{k=0}^{\lceil \frac{r+1}{b}\rceil - 2}\left(\frac{(k+1)\theta\mu_v^k}{(\theta+\mu_v)^{k+2}} + \frac{\left(\lceil \frac{r+1}{b}\rceil - k - 1\right)\theta\mu_v^k}{\mu_b(\theta+\mu_v)^{k+1}}\right) + \sum_{j=0}^{\infty}\pi_{j,0}^{-}\sum_{r=0}^{\infty} g_r^{-}\frac{\lceil \frac{r+j+1}{b}\rceil \mu_v^{\lceil \frac{r+j+1}{b}\rceil}}{(\theta+\mu_v)^{\lceil \frac{r+j+1}{b}\rceil + 1}} \\ &\quad + \sum_{j=0}^{\infty}\pi_{j,0}^{-}\sum_{r=0}^{\infty} g_r^{-}\sum_{k=0}^{\lceil \frac{r+j+1}{b}\rceil - 1}\left(\frac{(k+1)\theta\mu_v^k}{(\theta+\mu_v)^{k+2}} + \frac{\left(\lceil \frac{r+j+1}{b}\rceil - k\right)\theta\mu_v^k}{\mu_b(\theta+\mu_v)^{k+1}}\right) \\ &\quad + \sum_{j=0}^{\infty}\pi_{j,1}^{-}\sum_{r=0}^{\infty} g_r^{-}\frac{\lceil \frac{r+j+1}{b}\rceil}{\mu_b} + \pi_{\bar{0},0}^{-}\sum_{r=b}^{\infty} g_r^{-}\frac{\lceil \frac{r+1}{b}\rceil - 1}{\mu_b}. \end{aligned} \tag{6.2}$$

In the next section, from the third numerical example, we can see that equation (6.2) can provide us an effective way to validate the correctness of our analysis results.

7. Numerical results and discussion

In this section, we demonstrate the applicability of the algorithm based on roots *via* numerical experiments. All the calculations are performed on a PC having Intel Corei5 processor at 2.6 GHz with 4 GB DDR3 RAM using Matlab, Mathematica and Maple software packages. Though all the numerical results were carried out in high precision, they are reported here in six decimal places due to lack of space. Moreover, for testing the procedure discussed in this article, during the computational work, numerical results have been presented in some self-explanatory tables, but due to the same reason, only a few of them are appended in this section. Various performance measures such as the average number of customers in the queue at arbitrary epoch $\mathrm{E}[L_q]$, the mean sojourn time of a random customer $\mathrm{E}[W_A]$ and the average waiting time of an arbitrary customer in the queue $\mathrm{E}[W_{qA}]$ are also given at the bottom of the tables.

Example 7.1. The general Poisson process is one of the most important models used in queueing theory. Often the arrival process of customers can be described by a Poisson process. Here we consider an $M^X/M^{(1,b)}/1/\infty$ single working vacation queue in which customers arrive at the service facility as a batch Poisson process with rate $\lambda = 4$. The maximum batch size is $\tilde{r} = 6$, and the sizes of successive arriving groups are independent and identically distributed random variables with probability mass function $g_1 = 0.15$, $g_2 = 0.25$, $g_3 = 0.3$, $g_4 = 0.2$, $g_5 = 0.05$, $g_6 = 0.05$. Customers are served in a batch with a maximum batch size $b = 4$. The service rates in the regular busy period and working vacation period are $\mu_b = 4$ and $\mu_v = 2.5$ so that the traffic intensity $\rho = 0.725$. For fixed parameter $\theta = 1$, we compute the stationary probability distributions of the number of customers in the queue and system at different epochs like pre-arrival and arbitrary.

Replacing the finite support distribution with an appropriate PH distribution can greatly facilitate Matlab programming. According to work done by Alfa [1], the probability mass function of an arriving batch can be represented as a discrete PH distribution. We denote by $(\boldsymbol{g}, \boldsymbol{T})$ the PH representation of batch size X, where $\boldsymbol{g}$ is a row vector of length six, and $\boldsymbol{T}$ is a square matrix of order six, which have the following forms:

$$\boldsymbol{g} = (g_1, g_2, \ldots, g_6) = (0.15, 0.25, 0.3, 0.2, 0.05, 0.05)\,, \quad \boldsymbol{T} = \begin{pmatrix} \mathbf{0}_{1\times 5} & 0 \\ \boldsymbol{I}_5 & \mathbf{0}_{5\times 1}\end{pmatrix},$$

in which $\boldsymbol{I}_m$ stands for an identity matrix of dimension m, and $\mathbf{0}$ denotes a zero matrix of appropriate dimension (when needed, the dimension of zero matrix will be identified with a suffix). For evaluating the queue-length

TABLE 1. The roots of the characteristic equations (7.1) and (7.2) with modulus less than one.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
ω_i	-0.546431	$-0.238168+0.471339\mathbf{i}$	$-0.238168-0.471339\mathbf{i}$	$0.141009+0.442665\mathbf{i}$	$0.141009-0.442665\mathbf{i}$	0.871586
η_i	-0.541868	$-0.236955+0.464936\mathbf{i}$	$-0.236955-0.464936\mathbf{i}$	$0.141909+0.439888\mathbf{i}$	$0.141909-0.439888\mathbf{i}$	0.917900

TABLE 2. The numerical results for the coefficients K_j, L_j and H_j in $M^X/M^{(1,4)}/1/SWV$ queue.

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
K_j	0.010554	$0.009141-0.000812\mathbf{i}$	$0.009141+0.000812\mathbf{i}$	$0.004986+0.001215\mathbf{i}$	$0.004986-0.001215\mathbf{i}$	0.022813
L_j	0.028787	$0.016663+0.001973\mathbf{i}$	$0.016663-0.001973\mathbf{i}$	$0.010648+0.001018\mathbf{i}$	$0.010648-0.001018\mathbf{i}$	-0.062395
H_j	-0.017702	$-0.006032-0.003757\mathbf{i}$	$-0.006032+0.003757\mathbf{i}$	$-0.005298+0.000732\mathbf{i}$	$-0.005298-0.000732\mathbf{i}$	0.093299

distributions at various epochs, we first need to calculate the roots of the following characteristic equations inside the unit circle

$$z^6-\left(\sum_{j=1}^{6} g_j z^{6-j}\right) D\left(z^4\right)=z^6-\frac{4\left(0.15z^5+0.25z^4+0.3z^3+0.2z^2+0.05z+0.05\right)}{7.5-2.5z^4}=0, \tag{7.1}$$

$$z^6-\left(\sum_{j=1}^{6} g_j z^{6-j}\right) \Lambda\left(z^4\right)=z^6-\frac{0.15z^5+0.25z^4+0.3z^3+0.2z^2+0.05z+0.05}{2-z^4}=0. \tag{7.2}$$

Using Mathematica software, we can get six roots of equations (7.1) and (7.2) in the region $|z|<1$, respectively. Numerical results for these roots are presented in Table 1, where $\mathbf{i}$ is the imaginary unit. Substituting ω_i $(i=1,2,\ldots,6)$ and η_i $(i=1,2,\ldots,6)$ into equations (3.13) to (3.19), the corresponding $\pi^-_{\bar{0},0}$, $\pi^-_{\bar{0},1}$, K_j, L_j and H_j $(j=1,2,\ldots,6)$ are calculated by solving systems of linear equations in Matlab. The calculation results for K_j, L_j and H_j are given in Table 2. Furthermore, employing the algorithm based on roots, Table 3 also gives a few queue-length distributions at pre-arrival epoch for the current model.

As the queue-length distributions at pre-arrival epochs are known, one can easily determine the arbitrary epoch probabilities $\pi_{j,0}$ and $\pi_{j,1}$ using equations (4.2) to (4.7). The values are presented in Table 4. Due to the PASTA (Poisson Arrivals See Time Averages) property of Poison process, the queue-length distributions seen by an arriving batch are the same as those at an arbitrary instant. Such a relationship is very useful for debugging programs and checking accuracy for computations. In what follows, by the calculation of the system-length distribution at pre-arrival epoch, we can further numerically verify the correctness of our analysis. If our analysis turns out to be correct, we would expect to have $\sum_{j=1}^{4}\pi^-_{j,i,0}=\pi^-_{i,0}$ and $\sum_{j=1}^{4}\pi^-_{j,i,1}=\pi^-_{i,1}$. From the numerical results shown in Tables 5 and 6, we may find that the above relationships always hold. It indicates that our theoretical analysis is reliable and accurate.

Example 7.2. Since any distribution can be arbitrarily well approximated by a PH distribution, in this example, we consider the inter-batch arrival time distribution to be of phase type with representation $(\boldsymbol{\epsilon},\boldsymbol{U})$ which is given by $\boldsymbol{\epsilon}=(1,0)$, $\boldsymbol{U}=\begin{pmatrix}-7 & 7\\ 0 & -7\end{pmatrix}$. Therefore, the LST of the inter-batch arrival time distribution is $a^*(s)=\boldsymbol{\epsilon}\left(s\boldsymbol{I}-\boldsymbol{U}\right)^{-1}\boldsymbol{U}^0=\frac{49}{(s+7)^2}$, where $\boldsymbol{U}^0=-\boldsymbol{U}\boldsymbol{e}$ and $\boldsymbol{e}$ is a column vector of ones with appropriate dimension. Additionally, we further assume that the maximum service batch size is $b=3$, and the maximum size of an arriving batch is $\hat{r}=4$. The batch size distribution of successive batch arrivals is taken as $g_1=0.3$, $g_2=0.4$, $g_3=0.2$ and $g_4=0.1$. Since the above distribution has a finite support, it can be viewed as a discrete PH

Table 3. Stationary queue-length distribution at pre-arrival epoch for $M^X/M^{(1,4)}/1/SWV$ queue.

j	$\pi^-_{j,0}$	j	$\pi^-_{j,0}$	j	$\pi^-_{j,0}$	j	$\pi^-_{j,1}$	j	$\pi^-_{j,1}$	j	$\pi^-_{j,1}$
$\bar{0}$	0.089971	–	–	–	–	$\bar{0}$	0.022493	–	–	–	–
0	0.061621	17	0.002205	34	0.000213	0	0.073951	17	0.015714	34	0.004486
1	0.010858	18	0.001922	35	0.000186	1	0.021627	18	0.014705	35	0.004145
2	0.015034	19	0.001676	36	0.000162	2	0.027904	19	0.013741	36	0.003828
3	0.015346	20	0.001460	37	0.000141	3	0.031384	20	0.012825	37	0.003534
4	0.014066	21	0.001273	38	0.000123	4	0.031337	21	0.011957	38	0.003262
5	0.010588	22	0.001109	39	0.000107	5	0.028322	22	0.011136	39	0.003010
6	0.010649	23	0.000967	40	0.000093	6	0.029194	23	0.010363	40	0.002776
7	0.008509	24	0.000844	41	0.000082	7	0.027153	24	0.009634	41	0.002560
8	0.007571	25	0.000734	42	0.000071	8	0.026196	25	0.008950	42	0.002360
9	0.006631	26	0.000640	43	0.000062	9	0.025057	26	0.008308	43	0.002175
10	0.005805	27	0.000558	44	0.000054	10	0.023868	27	0.007707	44	0.002005
11	0.005001	28	0.000486	45	0.000047	11	0.022566	28	0.007145	45	0.001847
12	0.004397	29	0.000424	46	0.000041	12	0.021399	29	0.006620	46	0.001701
13	0.003818	30	0.000369	47	0.000036	13	0.020180	30	0.006130	47	0.001567
14	0.003331	31	0.000322	48	0.000031	14	0.019010	31	0.005674	48	0.001443
15	0.002903	32	0.000281	49	0.000027	15	0.017871	32	0.005249	49	0.001328
16	0.002532	33	0.000245	$\geq$50	$\leq$0.000024	16	0.016773	33	0.004853	$\geq$50	$\leq$0.001223
Sum			0.295803			Sum			0.704197		

Notes. $\mathrm{E}[W_A] = 1.166747$.

Table 4. Stationary queue-length distribution at arbitrary epoch for $M^X/M^{(1,4)}/1/SWV$ queue.

j	$\pi_{j,0}$	j	$\pi_{j,0}$	j	$\pi_{j,0}$	j	$\pi_{j,1}$	j	$\pi_{j,1}$	j	$\pi_{j,1}$
$\bar{0}$	0.089971	–	–	–	–	$\bar{0}$	0.022493	–	–	–	–
0	0.061621	17	0.002205	34	0.000213	0	0.073951	17	0.015714	34	0.004486
1	0.010858	18	0.001922	35	0.000186	1	0.021627	18	0.014705	35	0.004145
2	0.015034	19	0.001676	36	0.000162	2	0.027904	19	0.013741	36	0.003828
3	0.015346	20	0.001460	37	0.000141	3	0.031384	20	0.012825	37	0.003534
4	0.014066	21	0.001273	38	0.000123	4	0.031337	21	0.011957	38	0.003262
5	0.010588	22	0.001109	39	0.000107	5	0.028322	22	0.011136	39	0.003010
6	0.010649	23	0.000967	40	0.000093	6	0.029194	23	0.010363	40	0.002776
7	0.008509	24	0.000844	41	0.000082	7	0.027153	24	0.009634	41	0.002560
8	0.007571	25	0.000734	42	0.000071	8	0.026196	25	0.008950	42	0.002360
9	0.006631	26	0.000640	43	0.000062	9	0.025057	26	0.008308	43	0.002175
10	0.005805	27	0.000558	44	0.000054	10	0.023868	27	0.007707	44	0.002005
11	0.005001	28	0.000486	45	0.000047	11	0.022566	28	0.007145	45	0.001847
12	0.004397	29	0.000424	46	0.000041	12	0.021399	29	0.006620	46	0.001701
13	0.003818	30	0.000369	47	0.000036	13	0.020180	30	0.006130	47	0.001567
14	0.003331	31	0.000322	48	0.000031	14	0.019010	31	0.005674	48	0.001443
15	0.002903	32	0.000281	49	0.000027	15	0.017871	32	0.005249	49	0.001328
16	0.002532	33	0.000245	$\geq$50	$\leq$0.000024	16	0.016773	33	0.004853	$\geq$50	$\leq$0.001223
Sum			0.295803			Sum			0.704197		

Notes. $\mathrm{E}[L_q] = 10.586747$.

TABLE 5. System-length distribution at pre-arrival epoch in working vacation period for $M^X/M^{(1,4)}/1/SWV$ queue.

i	$j=1$	$j=2$	$j=3$	$j=4$	Sum
0	0.010817	0.017007	0.019511	0.014286	$\sum_{j=1}^4 \pi_{j,0,0}^- = \pi_{0,0}^- = 0.061621$
1	0.000865	0.001361	0.001560	0.007072	$\sum_{j=1}^4 \pi_{j,1,0}^- = \pi_{1,0}^- = 0.010858$
2	0.001512	0.002376	0.002726	0.008420	$\sum_{j=1}^4 \pi_{j,2,0}^- = \pi_{2,0}^- = 0.015034$
3	0.001967	0.003093	0.003548	0.006738	$\sum_{j=1}^4 \pi_{j,3,0}^- = \pi_{3,0}^- = 0.015346$
4	0.001651	0.002596	0.002978	0.006841	$\sum_{j=1}^4 \pi_{j,4,0}^- = \pi_{4,0}^- = 0.014066$
5	0.001017	0.001599	0.001834	0.006138	$\sum_{j=1}^4 \pi_{j,5,0}^- = \pi_{5,0}^- = 0.010588$
6	0.001089	0.001712	0.001964	0.005884	$\sum_{j=1}^4 \pi_{j,6,0}^- = \pi_{6,0}^- = 0.010649$
7	0.000760	0.001195	0.001371	0.005183	$\sum_{j=1}^4 \pi_{j,7,0}^- = \pi_{7,0}^- = 0.008509$
8	0.000638	0.001002	0.001150	0.004781	$\sum_{j=1}^4 \pi_{j,8,0}^- = \pi_{8,0}^- = 0.007571$
9	0.000532	0.000836	0.000959	0.004304	$\sum_{j=1}^4 \pi_{j,9,0}^- = \pi_{9,0}^- = 0.006631$
10	0.000437	0.000686	0.000787	0.003895	$\sum_{j=1}^4 \pi_{j,10,0}^- = \pi_{10,0}^- = 0.005805$
11	0.000345	0.000543	0.000622	0.003491	$\sum_{j=1}^4 \pi_{j,11,0}^- = \pi_{11,0}^- = 0.005001$
12	0.000288	0.000453	0.000520	0.003136	$\sum_{j=1}^4 \pi_{j,12,0}^- = \pi_{12,0}^- = 0.004397$
13	0.000233	0.000366	0.000420	0.002799	$\sum_{j=1}^4 \pi_{j,13,0}^- = \pi_{13,0}^- = 0.003818$
14	0.000190	0.000299	0.000343	0.002499	$\sum_{j=1}^4 \pi_{j,14,0}^- = \pi_{14,0}^- = 0.003331$
15	0.000155	0.000244	0.000280	0.002224	$\sum_{j=1}^4 \pi_{j,15,0}^- = \pi_{15,0}^- = 0.002903$
16	0.000127	0.000200	0.000228	0.001977	$\sum_{j=1}^4 \pi_{j,16,0}^- = \pi_{16,0}^- = 0.002532$
17	0.000103	0.000162	0.000186	0.001754	$\sum_{j=1}^4 \pi_{j,17,0}^- = \pi_{17,0}^- = 0.002205$
18	0.000084	0.000132	0.000152	0.001554	$\sum_{j=1}^4 \pi_{j,18,0}^- = \pi_{18,0}^- = 0.001922$
19	0.000069	0.000108	0.000124	0.001375	$\sum_{j=1}^4 \pi_{j,19,0}^- = \pi_{19,0}^- = 0.001676$
≥20	≤0.000055	≤0.000088	≤0.000101	≤0.001216	$\sum_{i=20}^{\infty}\sum_{j=1}^4 \pi_{j,i,0}^- = \sum_{i=20}^{\infty} \pi_{i,0}^- \leq 0.001460$

distribution of order 4 with representation $(\boldsymbol{g}, \boldsymbol{T})$, where $\boldsymbol{g}$ is a row vector of length four and $\boldsymbol{T}$ is a square matrix of order four, which have the following forms:

$$\boldsymbol{g} = (g_1, g_2, g_3, g_4) = (0.3, 0.4, 0.2, 0.1), \quad \boldsymbol{T} = \begin{pmatrix} \mathbf{0}_{1\times 3} & 0 \\ \boldsymbol{I}_3 & \mathbf{0}_{3\times 1} \end{pmatrix}.$$

Also, we set $\mu_b = 3$, $\mu_v = 1.5$ and $\theta = 1$. According to these input parameters, we see that $\lambda = 3.5$, $\bar{g} = 2.1$, and hence $\rho = 0.816667$. The numerical experiment is based on the roots of the following characteristic equations lie inside the unit circle. These roots are given in Table 7.

$$z^4 - \left(\sum_{j=1}^{4} g_j z^{4-j}\right) D\left(z^3\right) = z^4 - \frac{49(0.3z^3 + 0.4z^2 + 0.2z + 0.1)}{\left(9.5 - 1.5z^3\right)^2} = 0, \tag{7.3}$$

$$z^4 - \left(\sum_{j=1}^{4} g_j z^{4-j}\right) \Lambda\left(z^3\right) = z^4 - \frac{49(0.3z^3 + 0.4z^2 + 0.2z + 0.1)}{\left(10 - 3z^3\right)^2} = 0. \tag{7.4}$$

Substituting these values of ω_j and η_j into equations (3.13) to (3.19) and solving a set of simultaneous linear equations, the fourteen unknowns, namely $\pi_{0,0}^-$, $\pi_{0,1}^-$, K_j, L_j and H_j $(j = 1, 2, 3, 4)$ can be determined. Table 8 presents the values for K_j, L_j and H_j. Then, employing the coefficients mentioned above and using the equations (3.10) and (3.12), the stationary probability distribution of the number of customers in queue at pre-arrival epoch are shown in Table 9. Inserting $\pi_{j,0}^-$ and $\pi_{j,1}^-$ into equations (4.2) to (4.7), Table 10 exhibits of queue size distribution at arbitrary epoch for the system.

TABLE 6. System-length distribution at pre-arrival epoch in regular busy period for $M^X/M^{(1,4)}/1/SWV$ queue.

i	$j=1$	$j=2$	$j=3$	$j=4$	Sum
0	0.013853	0.018890	0.021505	0.019704	$\sum_{j=1}^{4}\pi^-_{j,0,1}=\pi^-_{0,1}=0.073951$
1	0.001147	0.001587	0.001808	0.017085	$\sum_{j=1}^{4}\pi^-_{j,1,1}=\pi^-_{1,1}=0.021627$
2	0.002007	0.002777	0.003164	0.019956	$\sum_{j=1}^{4}\pi^-_{j,2,1}=\pi^-_{2,1}=0.027904$
3	0.002618	0.003627	0.004132	0.021007	$\sum_{j=1}^{4}\pi^-_{j,3,1}=\pi^-_{3,1}=0.031384$
4	0.002211	0.003071	0.003499	0.022556	$\sum_{j=1}^{4}\pi^-_{j,4,1}=\pi^-_{4,1}=0.031337$
5	0.001382	0.001931	0.002201	0.022808	$\sum_{j=1}^{4}\pi^-_{j,5,1}=\pi^-_{5,1}=0.028322$
6	0.001485	0.002076	0.002367	0.023266	$\sum_{j=1}^{4}\pi^-_{j,6,1}=\pi^-_{6,1}=0.029194$
7	0.001051	0.001479	0.001686	0.022937	$\sum_{j=1}^{4}\pi^-_{j,7,1}=\pi^-_{7,1}=0.027153$
8	0.000888	0.001253	0.001429	0.022626	$\sum_{j=1}^{4}\pi^-_{j,8,1}=\pi^-_{8,1}=0.026196$
9	0.000746	0.001055	0.001204	0.022052	$\sum_{j=1}^{4}\pi^-_{j,9,1}=\pi^-_{9,1}=0.025057$
10	0.000617	0.000876	0.001000	0.021375	$\sum_{j=1}^{4}\pi^-_{j,10,1}=\pi^-_{10,1}=0.023868$
11	0.000493	0.000701	0.000801	0.020571	$\sum_{j=1}^{4}\pi^-_{j,11,1}=\pi^-_{11,1}=0.022566$
12	0.000414	0.000591	0.000675	0.019719	$\sum_{j=1}^{4}\pi^-_{j,12,1}=\pi^-_{12,1}=0.021399$
13	0.000338	0.000483	0.000551	0.018808	$\sum_{j=1}^{4}\pi^-_{j,13,1}=\pi^-_{13,1}=0.020180$
14	0.000277	0.000398	0.000454	0.017881	$\sum_{j=1}^{4}\pi^-_{j,14,1}=\pi^-_{14,1}=0.019010$
15	0.000228	0.000328	0.000374	0.016941	$\sum_{j=1}^{4}\pi^-_{j,15,1}=\pi^-_{15,1}=0.017871$
16	0.000187	0.000270	0.000309	0.016007	$\sum_{j=1}^{4}\pi^-_{j,16,1}=\pi^-_{16,1}=0.016773$
17	0.000153	0.000222	0.000253	0.015086	$\sum_{j=1}^{4}\pi^-_{j,17,1}=\pi^-_{17,1}=0.015714$
18	0.000126	0.000183	0.000209	0.014187	$\sum_{j=1}^{4}\pi^-_{j,18,1}=\pi^-_{18,1}=0.014705$
19	0.000104	0.000150	0.000172	0.013315	$\sum_{j=1}^{4}\pi^-_{j,19,1}=\pi^-_{19,1}=0.013741$
≥20	≤0.000085	≤0.000124	≤0.000141	≤0.012475	$\sum_{i=20}^{\infty}\sum_{j=1}^{4}\pi^-_{j,i,1}=\sum_{i=20}^{\infty}\pi^-_{i,1}\le 0.012825$

TABLE 7. The roots of the characteristic equations (7.3) and (7.4) with modulus is less than one.

	$i=1$	$i=2$	$i=3$	$i=4$
ω_i	-0.429286	$-0.081904+0.399455\mathbf{i}$	$-0.081904-0.399455\mathbf{i}$	0.822984
η_i	-0.416969	$-0.078361+0.393892\mathbf{i}$	$-0.078361-0.393892\mathbf{i}$	0.912747

TABLE 8. The numerical results for the coefficients K_j, L_j and H_j in $PH^X/M^{(1,3)}/1/SWV$ queue.

	$j=1$	$j=2$	$j=3$	$j=4$
K_j	0.006758	$0.005965-0.000285\mathbf{i}$	$0.005965+0.000285\mathbf{i}$	0.018920
L_j	0.010924	$0.012911-0.003086\mathbf{i}$	$0.012911+0.003086\mathbf{i}$	-0.056291
H_j	0.001545	$-0.003724+0.003232\mathbf{i}$	$-0.003724-0.003232\mathbf{i}$	0.096470

As is known, the system-length is the sum of queue-length and service batch size (number of customers in the service batch). Thus, we can check the correctness of the computed values using the relations $\sum_{j=1}^{3}\pi^-_{j,i,0}=\pi^-_{i,0}$ and $\sum_{j=1}^{3}\pi^-_{j,i,1}=\pi^-_{i,1}$ at pre-arrival epoch, and $\sum_{j=1}^{3}\pi_{j,i,0}=\pi_{i,0}$ and $\sum_{j=1}^{3}\pi_{j,i,1}=\pi_{i,1}$ at arbitrary epoch. The system-length distributions at pre-arrival and arbitrary epochs are given respectively in Tables 11 and 12. The computational results presented in the fifth and tenth columns of Tables 11 and 12 suggest that the above relations always hold true in our numerical experiments.

TABLE 9. Stationary queue-length distribution at pre-arrival epoch for $PH^X/M^{(1,3)}/1/SWV$ queue.

j	$\pi^-_{j,0}$	j	$\pi^-_{j,0}$	j	$\pi^-_{j,0}$	j	$\pi^-_{j,1}$	j	$\pi^-_{j,1}$	j	$\pi^-_{j,1}$
$\bar{0}$	0.055750	–	–	–	–	$\bar{0}$	0.011567	–	–	–	–
0	0.037608	11	0.002219	22	0.000260	0	0.071022	11	0.028735	22	0.012170
1	0.011921	12	0.001827	23	0.000214	1	0.034780	12	0.026821	23	0.011178
2	0.012199	13	0.001503	24	0.000176	2	0.041683	13	0.024968	24	0.010260
3	0.010441	14	0.001237	25	0.000145	3	0.041738	14	0.023191	25	0.009412
4	0.009148	15	0.001018	26	0.000119	4	0.041911	15	0.021498	26	0.008629
5	0.006934	16	0.000838	27	0.000098	5	0.039501	16	0.019894	27	0.007908
6	0.005899	17	0.000690	28	0.000081	6	0.038337	17	0.018382	28	0.007245
7	0.004842	18	0.000568	29	0.000067	7	0.036524	18	0.016962	29	0.006634
8	0.003989	19	0.000467	30	0.000055	8	0.034640	19	0.015634	30	0.006073
9	0.003270	20	0.000384	31	0.000045	9	0.032656	20	0.014395	31	0.005558
10	0.002699	21	0.000316	$\geq$32	$\leq$0.000037	10	0.030697	21	0.013241	$\geq$32	$\leq$0.005085
Sum			0.177238			Sum			0.822762		

Notes. $\mathrm{E}[W_A] = 1.853188$.

TABLE 10. Stationary queue-length distribution at arbitrary epoch for $PH^X/M^{(1,3)}/1/SWV$ queue.

j	$\pi_{j,0}$	j	$\pi_{j,0}$	j	$\pi_{j,0}$	j	$\pi_{j,1}$	j	$\pi_{j,1}$	j	$\pi_{j,1}$
$\bar{0}$	0.040484	–	–	–	–	$\bar{0}$	0.007585	–	–	–	–
0	0.040621	11	0.002483	22	0.000291	0	0.058226	11	0.029764	22	0.012743
1	0.013069	12	0.002044	23	0.000240	1	0.032400	12	0.027831	23	0.011709
2	0.013635	13	0.001682	24	0.000197	2	0.041279	13	0.025948	24	0.010751
3	0.011673	14	0.001384	25	0.000162	3	0.041721	14	0.024133	25	0.009866
4	0.010263	15	0.001139	26	0.000134	4	0.042413	15	0.022397	26	0.009048
5	0.007745	16	0.000938	27	0.000110	5	0.040048	16	0.020747	27	0.008295
6	0.006602	17	0.000772	28	0.000091	6	0.039133	17	0.019187	28	0.007600
7	0.005417	18	0.000635	29	0.000075	7	0.037427	18	0.017719	29	0.006961
8	0.004464	19	0.000523	30	0.000061	8	0.035620	19	0.016343	30	0.006374
9	0.003658	20	0.000430	31	0.000050	9	0.033673	20	0.015056	31	0.005834
10	0.003020	21	0.000354	$\geq$32	$\leq$0.000042	10	0.031731	21	0.013858	$\geq$32	$\leq$0.005338
Sum			0.171687			Sum			0.828313		

Notes. $\mathrm{E}[L_q] = 11.046892$.

The advantage of this algorithm is that it can efficiently deal with computational problems in queueing systems with group renewal arrival process. But in the first two numerical examples, we consider only cases when inter-arrival times of the groups have exponential distribution or Erlangian distribution of order 2. For such arrival processes that are the particular cases of the batch Markovian arrival process (BMAP), the numerical results can be much easier obtained without using the technique presented in our paper. To justify our efforts, another example with an inter-batch arrival time that does not belong to the class of PH distribution is demonstrated below.

Example 7.3. In this example we examine the $D^X/M^{(1,5)}/1/SWV$ queue, which are characterized by batch deterministic arrivals. Specifically, we assume that customers arrive at the service facility in batches according to a deterministic process with constant inter-batch arrival times, equal to 0.25. The group size X is a random variable with probability mass function $g_1 = 0.15$, $g_2 = 0.25$, $g_3 = 0.3$, $g_4 = 0.2$ and $g_5 = 0.1$. It also means

TABLE 11. Stationary system-length distribution at pre-arrival epoch for $PH^X/M^{(1,3)}/1/SWV$ queue.

	$\pi^-_{j,i,0}$					$\pi^-_{j,i,1}$			
i	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^{3}\pi^-_{j,i,0}$	i	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^{3}\pi^-_{j,i,1}$
0	0.012617	0.015787	0.009204	$0.037608=\pi^-_{0,0}$	0	0.020720	0.026251	0.024051	$0.071022=\pi^-_{0,1}$
1	0.002055	0.002572	0.007294	$0.011921=\pi^-_{1,0}$	1	0.003447	0.004360	0.026973	$0.034780=\pi^-_{1,1}$
2	0.003075	0.003847	0.005277	$0.012199=\pi^-_{2,0}$	2	0.005167	0.006537	0.029979	$0.041683=\pi^-_{2,1}$
3	0.002317	0.002899	0.005225	$0.010441=\pi^-_{3,0}$	3	0.003918	0.004953	0.032867	$0.041738=\pi^-_{3,1}$
4	0.001954	0.002444	0.004750	$0.009148=\pi^-_{4,0}$	4	0.003322	0.004200	0.034389	$0.041911=\pi^-_{4,1}$
5	0.001267	0.001585	0.004082	$0.006934=\pi^-_{5,0}$	5	0.002179	0.002752	0.034570	$0.039501=\pi^-_{5,1}$
6	0.001049	0.001313	0.003537	$0.005899=\pi^-_{6,0}$	6	0.001813	0.002289	0.034235	$0.038337=\pi^-_{6,1}$
7	0.000784	0.000981	0.003077	$0.004842=\pi^-_{7,0}$	7	0.001364	0.001721	0.033439	$0.036524=\pi^-_{7,1}$
8	0.000600	0.000750	0.002641	$0.003989=\pi^-_{8,0}$	8	0.001049	0.001323	0.032268	$0.034640=\pi^-_{8,1}$
9	0.000451	0.000564	0.002255	$0.003270=\pi^-_{9,0}$	9	0.000794	0.001001	0.030861	$0.032656=\pi^-_{9,1}$
10	0.000346	0.000432	0.001921	$0.002699=\pi^-_{10,0}$	10	0.000612	0.000772	0.039313	$0.030697=\pi^-_{10,1}$
11	0.000262	0.000327	0.001630	$0.002219=\pi^-_{11,0}$	11	0.000466	0.000588	0.027681	$0.028735=\pi^-_{11,1}$
12	0.000200	0.000249	0.001378	$0.001827=\pi^-_{12,0}$	12	0.000357	0.000449	0.026015	$0.026821=\pi^-_{12,1}$
13	0.000151	0.000189	0.001163	$0.001503=\pi^-_{13,0}$	13	0.000272	0.000343	0.024353	$0.024968=\pi^-_{13,1}$
14	0.000115	0.000144	0.000978	$0.001237=\pi^-_{14,0}$	14	0.000208	0.000262	0.022721	$0.023191=\pi^-_{14,1}$
15	0.000087	0.000110	0.000821	$0.001018=\pi^-_{15,0}$	15	0.000159	0.000200	0.021139	$0.021498=\pi^-_{15,1}$
16	0.000067	0.000083	0.000688	$0.000838=\pi^-_{16,0}$	16	0.000121	0.000153	0.019620	$0.019894=\pi^-_{16,1}$
17	0.000051	0.000063	0.000576	$0.000690=\pi^-_{17,0}$	17	0.000093	0.000116	0.018173	$0.018382=\pi^-_{17,1}$
18	0.000039	0.000048	0.000481	$0.000568=\pi^-_{18,0}$	18	0.000071	0.000089	0.016803	$0.016962=\pi^-_{18,1}$
19	0.000029	0.000037	0.000401	$0.000467=\pi^-_{19,0}$	19	0.000054	0.000068	0.015512	$0.015634=\pi^-_{19,1}$
≥20	≤0.000022	≤0.000028	≤0.000334	≤0.000384	≥20	≤0.000041	≤0.000052	≤0.014302	≤0.014395

that the maximum group size is $\tilde{r}=5$. For computational purposes, the default parameters are fixed as $\mu_b=3$, $\mu_v=2$, $\theta=1$, $b=5$ such that $\rho=0.76$. To facilitate code writing, we can rewrite the distribution of X as a PH distribution with representation $(\boldsymbol{g},\boldsymbol{T})$, where the stationary probability vector $\boldsymbol{g}=(0.15,0.25,0.3,0.2,0.1)$ and the matrix $\boldsymbol{T}$ is given by $\boldsymbol{T}=\begin{pmatrix}\boldsymbol{0}_{1\times 4} & 0\\ \boldsymbol{I}_4 & \boldsymbol{0}_{4\times 1}\end{pmatrix}$. Next, we need to calculate the roots of the following characteristic equations lying inside the unit circle.

$$z^5-\left(\sum_{j=1}^{5}g_j z^{5-j}\right)D\left(z^5\right)=z^5-(0.15z^4+0.25z^3+0.3z^2+0.2z+0.1)e^{-0.25(3-2z^5)}=0, \tag{7.5}$$

$$z^5-\left(\sum_{j=1}^{5}g_j z^{5-j}\right)\Lambda\left(z^5\right)=z^5-(0.15z^4+0.25z^3+0.3z^2+0.2z+0.1)e^{-0.25(3-3z^5)}=0. \tag{7.6}$$

Since the deterministic inter-batch arrival time does not have a rational LST, the above equations cannot be directly solved by using the standard Mathematica commands. Through the Padé's approximation [7/8], we approximate the LST of the inter-batch arrival time distribution $e^{-0.25z}$ with a rational function of the type $Q_1(z)/Q_2(z)$:

$$e^{-0.25z}=\frac{Q_1(z)}{Q_2(z)}=\frac{\begin{array}{c}1.0-0.116667z+0.00625z^2-0.000200321z^3+4.17334\times 10^{-6}z^4\\ -5.69092\times 10^{-8}z^5+4.74244\times 10^{-10}z^6-1.88192\times 10^{-12}z^7\end{array}}{\begin{array}{c}1.0+0.133333z+0.00833333z^2+0.000320513z^3+8.34669\times 10^{-6}z^4\\ +1.51758\times 10^{-7}z^5+1.89697\times 10^{-9}z^6+1.50554\times 10^{-11}z^7+5.881\times 10^{-14}z^8\end{array}},$$

TABLE 12. Stationary system-length distribution at arbitrary epoch for $PH^X/M^{(1,3)}/1/SWV$ queue.

	$\pi_{j,i,0}$					$\pi_{j,i,1}$			
i	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^{3}\pi_{j,i,0}$	i	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^{3}\pi_{j,i,1}$
0	0.013593	0.017299	0.009729	$0.040621=\pi_{0,0}$	0	0.016806	0.021817	0.019603	$0.058226=\pi_{0,1}$
1	0.002422	0.003031	0.007616	$0.013069=\pi_{1,0}$	1	0.004038	0.005111	0.023251	$0.032400=\pi_{1,1}$
2	0.003624	0.004534	0.005477	$0.013635=\pi_{2,0}$	2	0.006055	0.007662	0.027562	$0.041279=\pi_{2,1}$
3	0.002731	0.003417	0.005525	$0.011673=\pi_{3,0}$	3	0.004591	0.005808	0.031322	$0.041721=\pi_{3,1}$
4	0.002302	0.002881	0.005080	$0.010263=\pi_{4,0}$	4	0.003895	0.004925	0.033593	$0.042413=\pi_{4,1}$
5	0.001493	0.001868	0.004384	$0.007745=\pi_{5,0}$	5	0.002555	0.003228	0.034265	$0.040048=\pi_{5,1}$
6	0.001237	0.001547	0.003818	$0.006602=\pi_{6,0}$	6	0.002127	0.002686	0.034320	$0.039133=\pi_{6,1}$
7	0.000924	0.001156	0.003337	$0.005417=\pi_{7,0}$	7	0.001600	0.002020	0.033807	$0.037427=\pi_{7,1}$
8	0.000706	0.000884	0.002874	$0.004464=\pi_{8,0}$	8	0.001231	0.001553	0.032836	$0.035620=\pi_{8,1}$
9	0.000531	0.000664	0.002463	$0.003658=\pi_{9,0}$	9	0.000932	0.001175	0.031566	$0.033673=\pi_{9,1}$
10	0.000407	0.000510	0.002103	$0.003020=\pi_{10,0}$	10	0.000719	0.000906	0.030106	$0.031731=\pi_{10,1}$
11	0.000308	0.000386	0.001789	$0.002483=\pi_{11,0}$	11	0.000547	0.000690	0.028527	$0.029764=\pi_{11,1}$
12	0.000235	0.000294	0.001515	$0.002044=\pi_{12,0}$	12	0.000419	0.000528	0.026884	$0.027831=\pi_{12,1}$
13	0.000178	0.000223	0.001281	$0.001682=\pi_{13,0}$	13	0.000320	0.000403	0.025225	$0.025948=\pi_{13,1}$
14	0.000136	0.000170	0.001078	$0.001384=\pi_{14,0}$	14	0.000244	0.000308	0.023581	$0.024133=\pi_{14,1}$
15	0.000103	0.000129	0.000907	$0.001139=\pi_{15,0}$	15	0.000187	0.000235	0.021975	$0.022397=\pi_{15,1}$
16	0.000079	0.000098	0.000761	$0.000938=\pi_{16,0}$	16	0.000142	0.000179	0.020426	$0.020747=\pi_{16,1}$
17	0.000060	0.000075	0.000637	$0.000772=\pi_{17,0}$	17	0.000109	0.000137	0.018941	$0.019187=\pi_{17,1}$
18	0.000045	0.000057	0.000533	$0.000635=\pi_{18,0}$	18	0.000083	0.000104	0.017532	$0.017719=\pi_{18,1}$
19	0.000035	0.000043	0.000445	$0.000523=\pi_{19,0}$	19	0.000063	0.000080	0.016200	$0.016343=\pi_{19,1}$
≥20	≤0.000026	≤0.000033	≤0.000371	≤0.000430	≥20	≤0.000048	≤0.000061	≤0.014947	≤0.015056

TABLE 13. The roots of the characteristic equations (7.3) and (7.4) with modulus is less than one.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
ω_i	$-0.393314-0.258734\mathbf{i}$	$-0.393314+0.258734\mathbf{i}$	0.831129	$0.035767-0.522451\mathbf{i}$	$0.035767+0.522451\mathbf{i}$
η_i	$-0.393764-0.258909\mathbf{i}$	$-0.393764+0.258909\mathbf{i}$	0.901270	$0.034864-0.522547\mathbf{i}$	$0.034864+0.522547\mathbf{i}$

TABLE 14. The numerical results for the coefficients K_j, L_j and H_j in $D^X/M^{(1,5)}/1/SWV$ queue.

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
K_j	$0.010116+0.000313\mathbf{i}$	$0.010116-0.000313\mathbf{i}$	0.020150	$0.009755-0.002584\mathbf{i}$	$0.009755+0.002584\mathbf{i}$
L_j	$-0.422553-0.113889\mathbf{i}$	$-0.422553+0.113889\mathbf{i}$	0.050809	$-0.144839-0.211377\mathbf{i}$	$-0.144839+0.211377\mathbf{i}$
H_j	$0.441446+0.114892\mathbf{i}$	$0.441446-0.114892\mathbf{i}$	0.099902	$0.162853+0.207599\mathbf{i}$	$0.162853-0.207599\mathbf{i}$

where the symbol [7/8] stands for a rational function with degree of numerator polynomial 7 and degree of denominator polynomial 8. Letting $z=3-2z^5$ and $z=3-3z^5$ in the above equation, respectively, and plugging them into equations (7.5) and (7.6), Mathematica calculates all distinct roots of the characteristic equations inside the unit circle which are presented in Table 13.

Substitution of ω_i and η_i $(i=1,2,\ldots,5)$ into equations (3.13) to (3.19) gives the values of $\pi^-_{\bar{0},0}$, $\pi^-_{\bar{0},1}$, K_j, L_j and H_j $(j=1,2,3,4,5)$, in which the numerical results for K_j, L_j and H_j are presented in Table 14.

Based on the roots method described in this paper, the stationary queue-length distributions at different epochs (pre-arrival and arbitrary) have been displayed in Tables 15 and 16, respectively. The notations used

TABLE 15. Stationary queue-length distribution at pre-arrival epoch for $D^X/M^{(1,5)}/1/SWV$ queue.

j	$\pi^-_{j,0}$	j	$\pi^-_{j,0}$	j	$\pi^-_{j,0}$	j	$\pi^-_{j,1}$	j	$\pi^-_{j,1}$	j	$\pi^-_{j,1}$
$\bar{0}$	0.081604	–	–	–	–	$\bar{0}$	0.009094	–	–	–	–
0	0.059892	11	0.002621	22	0.000344	0	0.122907	11	0.025170	22	0.009280
1	0.006950	12	0.002198	23	0.000286	1	0.030197	12	0.023198	23	0.008425
2	0.010074	13	0.001822	24	0.000238	2	0.038593	13	0.021281	24	0.007643
3	0.012157	14	0.001510	25	0.000198	3	0.045093	14	0.019492	25	0.006931
4	0.010428	15	0.001256	26	0.000164	4	0.043214	15	0.017838	26	0.006282
5	0.008516	16	0.001045	27	0.000137	5	0.040369	16	0.016301	27	0.005690
6	0.006026	17	0.000869	28	0.000114	6	0.035506	17	0.014877	28	0.005153
7	0.005532	18	0.000722	29	0.000094	7	0.034368	18	0.013560	29	0.004664
8	0.004696	19	0.000600	30	0.000078	8	0.032160	19	0.012349	30	0.004220
9	0.003822	20	0.000499	31	0.000065	9	0.029597	20	0.011236	31	0.003817
10	0.003151	21	0.000414	$\geq$32	$\leq$0.000321	10	0.027295	21	0.010215	$\geq$32	$\leq$0.035540
Sum			0.228442			Sum			0.771558		

Notes. $\mathrm{E}[W_{qA}] = 0.837883$.

TABLE 16. Stationary queue-ength distribution at arbitrary epoch for $D^X/M^{(1,5)}/1/SWV$ queue.

j	$\pi_{j,0}$	j	$\pi_{j,0}$	j	$\pi_{j,0}$	j	$\pi_{j,1}$	j	$\pi_{j,1}$	j	$\pi_{j,1}$
$\bar{0}$	0.036378	–	–	–	–	$\bar{0}$	0.002818	–	–	–	–
0	0.070692	11	0.003496	22	0.000459	0	0.073804	11	0.028307	22	0.010701
1	0.008013	12	0.002935	23	0.000382	1	0.025439	12	0.026200	23	0.009726
2	0.012843	13	0.002432	24	0.000317	2	0.037682	13	0.024110	24	0.008833
3	0.016268	14	0.002015	25	0.000264	3	0.047962	14	0.022146	25	0.008017
4	0.014038	15	0.001676	26	0.000219	4	0.046798	15	0.020320	26	0.007272
5	0.011453	16	0.001395	27	0.000182	5	0.044108	16	0.018614	27	0.006593
6	0.007951	17	0.001159	28	0.000151	6	0.038450	17	0.017023	28	0.005974
7	0.007380	18	0.000963	29	0.000126	7	0.037851	18	0.015546	29	0.005411
8	0.006283	19	0.000800	30	0.000105	8	0.035717	19	0.014181	30	0.004899
9	0.005102	20	0.000665	31	0.000087	9	0.033001	20	0.012923	31	0.004434
10	0.004201	21	0.000553	$\geq$32	$\leq$0.000428	10	0.030565	21	0.011765	$\geq$32	$\leq$0.041400
Sum			0.221411			Sum			0.778589		

Notes. $\mathrm{E}[W_{qA}]_{\text{Little}} = 0.837883$.

in the tables are the same as those defined earlier in this paper except $\mathrm{E}[W_{qA}]_{\text{Little}}$ which denotes the average waiting time in the queue of an arbitrary customer evaluated through Little's rule.

Using the data in Tables 15 and 16, we can give another way to check the validity of our numerical as well as analytical results. The formula for calculating the average waiting time in the queue of an arbitrary customer has been obtained in Section 6 (see Eq. (6.2)). Moreover, it is to be noted here that we can also obtain the above performance measure from Little's rule, $\mathrm{E}[W_{qA}] = L_q/\lambda\bar{g}$, where L_q denotes the mean number of customers waiting in the queue at an arbitrary epoch. By our numerical computation (see the bottom of Tabs. 15 and 16), we find that the average waiting time in the queue of an arbitrary customer evaluated through equation (6.2) exactly matches with the one obtained from Little's rule. It also shows that the theoretical analysis and numerical experiments in this paper are reliable and accurate.

8. Conclusions and future scope

In this paper, we have successfully analyzed $GI^X/M^{(1,b)}/1$ single working vacation queue with infinite buffer space. Although this model is very complicated, we did not back away from the complex algebraic manipulation and provide a meticulous derivation of the various formulas. Based on the roots of the characteristic equation, a procedure to obtain the numerical solutions of the stationary queue-length distribution at different epochs has been provided. In the problem-solving process, the most critical step is to solve a system of non-homogeneous linear equations. From this point of view, the roots method is more easily accepted and implemented by practitioners. Meanwhile, in order to ensure the reliability of the analytical approach, some numerical experiments have been performed, and the calculation results indicate that our method is valid and accurate. Furthermore, in numerical experiments, we even give information about established tools that facilitate computation. By resorting to the queue-length distributions at the pre-arrival epoch, the amount of time spent in the system by a random customer of an incoming batch is also given. We think the model studied here can be extended to include the correlation structure among successive batch-service times. We leave this problem as a future extension of the current research.

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