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AN ALGORITHM FOR THE ANCHOR POINTS OF THE PPS OF THE FRH MODELS

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Abstract. In this paper we deal with a variant of non-convex data envelopment analysis, called free replication hull model and try to obtain their anchor points. This paper uses a variant of super-efficiency model to characterize all extreme efficient decision making units and anchor points of the free replication hull models. A necessary and sufficient conditions for a decision making unit to be anchor point of the production possibility set of the free replication hull models are stated and proved. Since the set of anchor points is a subset of the set of extreme units, a definition of extreme units and a new method for obtaining these units in non-convex technologies are given. To illustrate the applicability of the proposed model, some numerical examples are finally provided.

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1. Introduction

Data envelopment analysis (DEA), first introduced by Charnes, Cooper and Rhodes (CCR) [9], is a non-parametric method for measuring the efficiency of a set of decision making units (DMUs). DEA approaches are traditionally based on the convexity assumption, but some authors pointed out that it is not always an appropriate one [7, 14, 18, 25]. The evaluation of efficiency of DMUs in DEA is based on the construction of unobserved DMUs which usually require the convexity of the production possibility set (PPS). This needs divisibility in the input and output, which is not always possible. In the free replicability model [24] the input and output are permitted to enter in only discrete amounts (see [11, 13, 14, 25] and references therein). The non-convex counterpart of constant return to scale (CRS) is the free replication under which every integer multiple of any observed input-output bundle is feasible. This added assumption can be added in order to define a free replication hull (FRH) of the data points and obtain the associated efficiency measures. In fact, these models which have been first proposed by Tulkens [24], are a special case of the CCR models in which the convexity axiom is relaxed and the intensity variables are restricted to the whole numbers (see also [2, 13, 17]). An important set of frontier points of the DEA technologies is that of the anchor points. Anchor points in DEA are extreme efficient DMUs for which some of their inputs can increase and/or some outputs can decrease while remaining on the (weak) efficiency frontier. In other words, an anchor point in the DEA model is an extreme DMU of the PPS that lies on the intersection of some strong and weak efficient frontiers (see [3, 12]).

Keywords. Data envelopment analysis (DEA), free replication hull (FRH), integer programming.

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The concept of anchor point was used in Thanassoulis and Allen [23] for the generation of unobserved DMUs in order to extend the DEA efficient frontier. Rouse [19] utilized the anchor points for identifying the prices in health care services. Bougnol and Dulá [8] defined these points as the production possibilities which give the transition from the pareto-efficient frontier to the free-disposability portion of the PPS boundary and provided an approach to identify the anchor points of the variable returns to scale (VRS) PPS based on their geometrical properties. Mostafaee and Soleimani-damaneh [16] presented an algorithm for identifying of the anchor points by employing the sensitivity analysis techniques. The empirical applications led to somewhat surprising results in which that almost all extreme efficient units are in fact anchor points. Free disposal hull (FDH) models are non-convex DEA models which have been introduced by Deprins *et al.* [10]. Soleimani-damaneh and Mostafaee [21] investigated the extreme units and anchor points in non-convex FDH technology and provided necessary and sufficient conditions for characterizing the anchor points of the PPS of the FDH models. To the best of the authors' knowledge, no specific study has been conducted on finding extreme and anchor points of the PPS of the FRH models. Following Soleimani-damaneh and Mostafaee [21], in this paper a definition of the extreme unit notion in (non-convex) FRH models has been provided as well as a new method to identify the extreme units and anchor points of the PPS of these models by testing all FRH-efficient DMUs *via* a variety of super-efficiency models (see models (3.1) and (3.2)) (after eliminating the FRH-inefficient DMUs from the PPS). The reader is referred to Andersen and Petersen [5], Mehrabian *et al.* [15] and Seiford *et al.* [20] for the super-efficiency models. Also some useful facts associated with the models (3.1) and (3.2) have been stated and proved as along with the necessary and sufficient conditions for a FRH-efficient DMU to be an anchor point. In addition, three numerical examples have been given.

2. Background

Consider a set of $\boldsymbol{n}$ DMUs (activities) which are associated with $\boldsymbol{m}$ inputs and $\boldsymbol{s}$ outputs. Particularly, each $\mathrm{DMU}_j = (X_j, Y_j)$ $(j \in J = \{1, \ldots, n\})$ consumes amount $x_{ij}(> 0)$ of input $\boldsymbol{i}$ and produces amount $y_{rj}(> 0)$ of output $\boldsymbol{r}$. The set of feasible activities is called the production possibility set (PPS) and is defined as follows: $T = \big\{(X, Y)|X \in E^m \text{can produce } Y \in E^s, X \geq 0, Y \geq 0\big\}$. The PPS T can be made by the following properties:

(A1) The observed activities (X_j, Y_j) $(j \in J = \{1, \ldots, n\})$ belong to T.

(A2) The production possibility set is convex.

(A3) If an activity (X, Y) belongs to T, then the activity (tX, tY) belongs to T for any positive scalar t. This property is called the constant returns-to-scale assumption.

(A4) For an activity (X, Y) belongs to T, any semipositive activity $(\bar{X}, \bar{Y})$ with $(\bar{X} \geq X)$ and $\bar{Y} \leq Y$ is included in T. This property is called the free disposability assumption.

(A5) For each T' satisfying in the properties **A1** through **A4**, we have $T \subseteq T'$. This property is called the Minimal Extrapolation assumption.

The set $T = \{(X, Y)|X \geqq \sum_{j\in J} \lambda_j X_j,\ 0 \leq Y \leqq \sum_{j\in J} \lambda_j Y_j, \lambda_j \geq 0\, j \in J\}$ satisfies all the properties **A1** through **A5** and is denoted by T^{CCR}. The input radial efficiency of DMU_o is defined as:

$$\begin{aligned}&\min\ \theta\\ &\text{s.t. } (\theta X_o, Y_o) \in T^{\mathrm{CCR}}.\end{aligned}$$

Model (2.1) shows the CCR envelopment model for evaluating DMU_o in an input-oriented manner,

$$\begin{aligned}\min\ & \theta\\ \text{s.t. } & \sum_{j\in J} \lambda_j x_{ij} \leq \theta x_{io}, && i = 1, \ldots, m\\ & \sum_{j\in J} \lambda_j y_{rj} \geq y_{ro}, && r = 1, \ldots, s\\ & \lambda_j \geq 0, && j \in J\\ & \theta \in \mathbb{R}.\end{aligned} \tag{2.1}$$

Also, the CCR envelopment model for evaluating DMU_o in an output-oriented manner is as follows:

$$\begin{aligned} \max\ & \varphi \\ \text{s.t.}\ & \sum_{j\in J}\lambda_j x_{ij} \le x_{io}, && i=1,\dots,m \\ & \sum_{j\in J}\lambda_j y_{rj} \ge \varphi y_{ro}, && r=1,\dots,s \\ & \lambda_j \ge 0, && j\in J \\ & \varphi\in\mathbb{R}. \end{aligned} \tag{2.2}$$

One of the non-convex DEA models to evaluate the relative efficiency of a set of DMUs is the FDH model introduced by Deprins *et al.* [10] and further developed by Tulkens [24]. The production possibility set of FDH can be specified as

$$T^{\text{FDH}} = \left\{(X,Y)|X \geqq \sum_{j\in J}\lambda_j X_j,\ 0\le Y \leqq \sum_{j\in J}\lambda_j Y_j, \sum_{j\in J}\lambda_j = 1,\ \lambda_j\in\{0,1\}, j\in J\right\}.$$

in which X_j and Y_j are vectors of inputs and outputs of DMU_j, $j\in J$, respectively. Another non-convex DEA models is the FRH model, which was proposed by Tulkens [24]. Following properties are postulated for the PPS of the FRH, denoted by T^{FRH}:

(1) The observed activities (DMUs) belongs to T^{FRH}; *i.e.* $(x_j,y_j)\in T^{\text{FRH}}$, $j\in J$.
(2) If $(x,y)\in T^{\text{FRH}}$ then, $(tx,ty)\in T^{\text{FRH}}$ for each $t\in\{0,1,2,3,\dots\}$.
(3) For each activity $(x,y)\in T^{\text{FRH}}$, if $\overline{x}\ge x$ and $y\ge\overline{y}$ then, $(x,y)\in T^{\text{FRH}}$.
(4) T^{FRH} is closed and non-convex set.

The PPS of the FRH model can be defined as follows:

$$T^{\text{FRH}} = \left\{(X,Y)|X \geqq \sum_{j\in J}\lambda_j X_j,\ 0\le Y \leqq \sum_{j\in J}\lambda_j Y_j, \lambda_j\in\{0,1,2,3,\dots\}, j\in J\right\}.$$

in which X_j and Y_j are vectors of inputs and outputs of DMU_j, $j\in J$, respectively. Evidently, $T^{\text{FDH}}\subseteq T^{\text{FRH}}$. Unlike the FDH model, the FRH model is computationally quite challenging (see [11]).

The input-oriented FRH model corresponds to DMU_k, $k\in J$, is given by:

$$\begin{aligned} \min\ & \theta_k - \epsilon\left(\sum_{i=1}^{m}s_i^- + \sum_{r=1}^{s}s_r^+\right) \\ \text{s.t.}\ & \sum_{j\in J}\lambda_j y_{rj} - s_r^+ = y_{rk}, && r=1,\dots,s \\ & \sum_{j\in J}\lambda_j x_{ij} + s_i^- = \theta_k x_{ik}, && i=1,\dots,m \\ & \lambda_j\in\{0,1,2,3,\dots\}, && j\in J \\ & s_i^-\ge 0, && i=1,\dots,m \\ & s_r^+\ge 0, && r=1,\dots,s \\ & \theta_k\in\mathbb{R}. \end{aligned} \tag{2.3}$$

Also, the output-oriented FRH model corresponds to DMU_k, $k \in J$, is as follows:

$$
\begin{array}{lll}
\max & \varphi_k + \epsilon \left(\sum_{i=1}^{m} t_i^- + \sum_{r=1}^{s} t_r^+ \right) & \\
\text{s.t.} & \sum_{j \in J} \lambda_j y_{rj} - t_r^+ = \varphi_k y_{rk}, & r = 1, \ldots, s \\
& \sum_{j \in J} \lambda_j x_{ij} + t_i^- = x_{ik}, & i = 1, \ldots, m \\
& \lambda_j \in \{0, 1, 2, 3, \ldots\}, & j \in J \\
& t_i^- \geq 0, & i = 1, \ldots, m \\
& t_r^+ \geq 0, & r = 1, \ldots, s \\
& \varphi_k \in \mathbb{R} &
\end{array}
\tag{2.4}
$$

where ϵ is non-Archimedean small and positive number and s_i^-, s_r^+, t_i^- and t_r^+, $i = 1, \ldots, m$, $r = 1, \ldots, s$ are called slack variables belonging to $\mathbb{R}_{\geq 0}$. Note that s_i^- and t_i^- represent input excesses; also s_r^+ and t_r^+ represent output shortfalls. Inhere, θ_k and φ_k are real numbers.

At optimality of models (2.3) and (2.4), $0 < \theta_k^* \leq 1$ and $\varphi_k^* \geq 1$[1].

In evaluation of DMU_k by input(output)-oriented CCR model and input(output)-oriented FRH model, we have $\theta^*_{\mathrm{CCR}} \leq \theta^*_{\mathrm{FRH}} (\varphi^*_{\mathrm{CCR}} \geq \varphi^*_{\mathrm{FRH}})$.

DMU_k is called *FRH-inefficient* if and only if either (i) or (ii) happens:

(i) $\theta_k^* = 1$ and $(\mathbf{s}^{+*}, \mathbf{s}^{-*}) = (\mathbf{0}, \mathbf{0})$.
(ii) $\varphi_k^* = 1$ and $(\mathbf{t}^{+*}, \mathbf{t}^{-*}) = (\mathbf{0}, \mathbf{0})$.

Otherwise, DMU_k is called *FRH-inefficient i.e.*:

DMU_k is called *FRH-inefficient* if and only if either (í) and (ií) happens:

(í) $\theta_k^* < 1$ or $\Big(\theta_k^* = 1$ and $(\mathbf{s}^{+*}, \mathbf{s}^{-*}) \neq (\mathbf{0}, \mathbf{0})\Big)$.
(ií) $\varphi_k^* > 1$ or $\Big(\varphi_k^* = 1$ and $(\mathbf{t}^{+*}, \mathbf{t}^{-*}) \neq (\mathbf{0}, \mathbf{0})\Big)$.

Note 1. In the case of

$$(\theta_k^* = 1 \text{ and } (\mathbf{s}^{+*}, \mathbf{s}^{-*}) \neq (\mathbf{0}, \mathbf{0}))$$

or

$$(\varphi_k^* = 1, (\mathbf{t}^{+*} \text{ and } \mathbf{t}^{-*}) \neq (\mathbf{0}, \mathbf{0})),$$

FRH-inefficient DMU_k is called *weak* FRH-*efficient*. Also, if $\theta_k^* < 1$ and $\varphi_k^* > 1$ then DMU_k is an interior point of the T^{FRH}.

As mentioned in the previous section, the set of anchor points is a subset of the set of extreme units. A basic definition of extreme DMU of the PPS of the FRH technology is as follows (see [21]).

For each $o \in J$, define:

$$T_o^{\mathrm{FRH}} = \left\{ (X, Y) | X \geqq \sum_{j \in J \setminus \{o\}} \lambda_j X_j,\ 0 \leq Y \leqq \sum_{j \in J \setminus \{o\}} \lambda_j Y_j, \lambda_j \in \{0, 1, 2, 3, \ldots\}, j \in J \setminus \{o\} \right\},$$

as the PPS obtained by removing $\mathrm{DMU}_o = (X_o, Y_o)$ from $\{\mathrm{DMU}_1, \mathrm{DMU}_2, \ldots, \mathrm{DMU}_n\}$. The unit under evaluation, DMU_o, is called an extreme unit in T_o^{FRH}, if (X_o, Y_o) not belong to the T_o^{FRH}. In fact, $\mathrm{DMU}_o = (X_o, Y_o)$ is an extreme unit if deleting it from the set of the observed units does change the PPS.

[1] (*) is used for optimal solution.

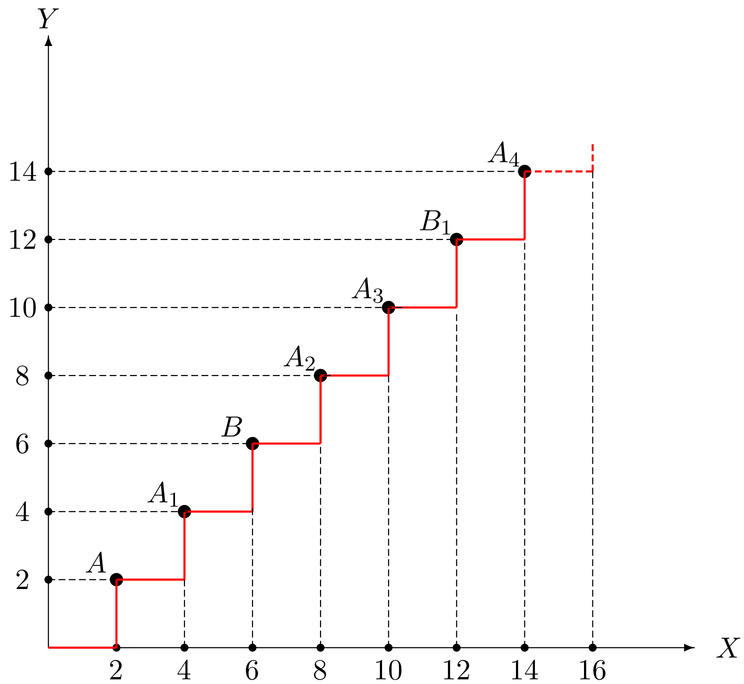

FIGURE 1. A, B are FRH-efficient DMUs and only A is extreme efficient DMUs.

Figure 1 shows the FRH production possibility set constructed from observed DMUs $A = (x_A = 2, y_A = 2)$ and $B = (x_B = 6, y_B = 6)$. Both DMUs A and B are FRH-efficient DMUs but, DMU A is just extreme DMU. The frontier of the PPS is denoted by red broken line. The necessary and sufficient conditions for DMU_k $(k \in E)$ under evaluation to be FRH extreme unit is given in the Theorem 3.9.

Hereafter, we will denote the set of FRH-efficient, extreme efficient and FRH-inefficient units by E, E' and F, respectively.

The set of extreme units, $E'(\subseteq E)$, is also called the *frame* of J. The frames are important in DEA because the PPS of the DEA models are constructed by them and the exclusion each of them alters the shape of the PPS. As an other example; Figure 2 shows the FRH production possibility set constructed from four observed DMUs $A(x_A = 4, y_A = 3)$; $B(x_B = 6, y_B = 4)$; $C(x_C = 11, y_C = 5)$ and $D(x_D = 21, y_D = 9)$. In Figure 2, $J = \{A, B, C, D\}$, $F = \{C, D\}$ and $E = E' = \{A, B\}$. Here, the point A_1 and B_1 is a twofold replication of A and B, respectively. The point L, P, S and U are the sum of the bundles (A and B), (B and A_1), (A_1 and L) and (A, B and L), respectively. The point A_2 and B_2 are a threefold replication of A and B, respectively. Also, A_3 is fourfold replication of A. The frontier of the FRH production possibility set is denoted by red broken line[2]. Also, the frontier of the T_A^{FRH} (*i.e.* the new PPS after excluding extreme unit A) is denoted by blue broken line.

In this paper, corresponding to each FRH-efficient DMU $\text{DMU}_j = (x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{sj})$ we name virtual DMUs $\text{DMU}_j^l = (x_{1j}, \ldots, x_{lj} + \alpha, \ldots, x_{mj}, y_{1j}, \ldots, y_{sj})$ and $\text{DMU}_j^q = (x_{1j}, \ldots, x_{mj}, y_{1j}, \ldots, y_{qj} - \gamma, \ldots, y_{sj})$ as FRH "Dominated Input Virtual" DMU (DIV_j^l DMU) and "Dominated Output Virtual" DMU (DOV_j^q DMU), respectively, in which $\alpha, \gamma > 0$. These virtual DMUs are either interior points of the PPS of the FRH model or lie on the some weak efficient frontiers (bounded or unbounded). In the latter case we call these virtual DMUs as "weak efficient virtual DMUs" or WEV DMUs, hereafter.

[2]For more details see Subhash [22] page 144.

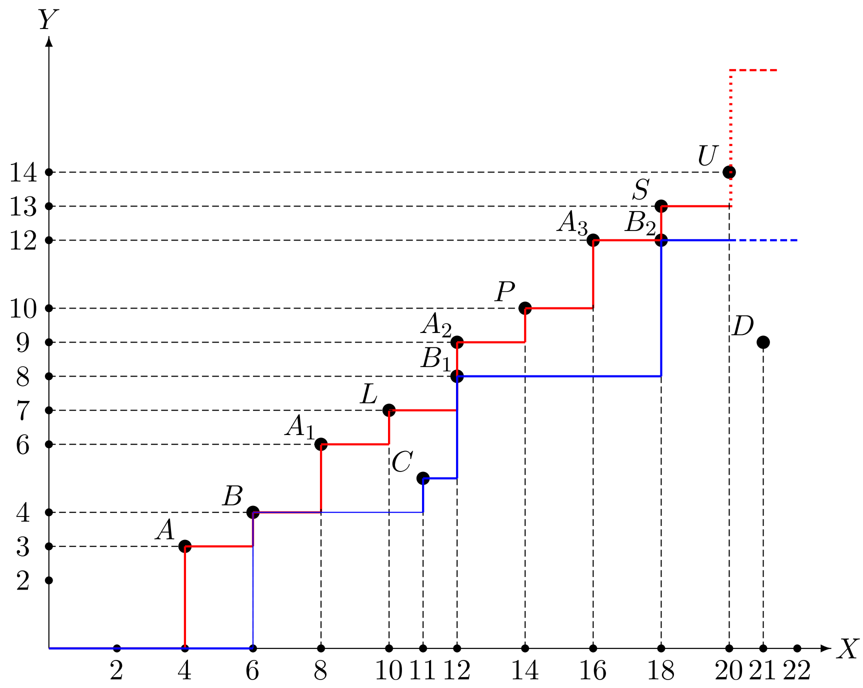

FIGURE 2. A, B are (extreme) FRH-efficient DMUs, C and D are FRH-inefficient DMUs (extracted from Subhash [22]).

Following Soleimani-damaneh and Mostafaee [21], the anchor points of the non-convex FRH models is defined as follows:

Definition 2.1. $\text{DMU}_k \in E'$ is an anchor DMU if some component of outputs can be decreased until zero or some component of inputs can be infinitely increased without penetrating the interior of T^{FRH}.

Remark 2.2. By Definition 2.1, $\text{DMU}_k \in E'$ is an anchor DMU if there exist some l (or q) so that for each $\alpha > 0$ $(0 < \gamma \leq y_{qk})$; DIV_k^l (DOV_k^q) DMU is to be WEV DMU.

3. IDENTIFYING THE ANCHOR DMUS OF THE PPS OF THE FRH MODEL

In this section, we identify the anchor points of the PPS of the FRH models in the following way. First, we evaluate each DMU_k, $(k \in J)$, using models (2.3) or (2.4). Then, we hold all FRH-efficient DMUs, *i.e.* $\text{DMU}_k \in E$, and remove other DMUs. Corresponding to each $\text{DMU}_k = (x_{1k}, \ldots, x_{mk}, y_{1k}, \ldots, y_{sk})$, $(k \in E)$, we solve the following models:

$$\begin{aligned}
\min\ & \theta_l^k \\
\text{s.t.}\ & \sum_{j \in E-\{k\}} \lambda_j^k x_{lj} \leq \theta_l^k x_{lk} \\
& \sum_{j \in E-\{k\}} \lambda_j^k x_{ij} \leq x_{ik}, \qquad i = 1, \ldots, m \ i \neq l \\
& \sum_{j \in E-\{k\}} \lambda_j^k y_{rj} \geq y_{rk}, \qquad r = 1, \ldots, s \\
& \lambda_j \in \{0, 1, 2, 3, \ldots\}, \qquad j \in E - \{k\} \\
& \theta_l^k \in \mathbb{R} \qquad\qquad l = 1, \ldots, m
\end{aligned} \tag{3.1}$$

$$
\begin{array}{lll}
\max & \varphi_q^k & \\
\text{s.t.} & \displaystyle\sum_{j\in E-\{k\}} \mu_j^k x_{ij} \leq x_{ik}, & i=1,\ldots,m \\
& \displaystyle\sum_{j\in E-\{k\}} \mu_j^k y_{qj} \geq \varphi_q^k y_{qk}, & \\
& \displaystyle\sum_{j\in E-\{k\}} \mu_j^k y_{rj} \geq y_{rk}, & r=1,\ldots,s \;\; r\neq q \\
& \mu_j \in \{0,1,2,3,\ldots\}, & j\in E-\{k\} \\
& \varphi_q^k \in \mathbb{R} & \qquad\qquad q=1,\ldots,s.
\end{array} \tag{3.2}
$$

The following theorems hold for models (3.1) and (3.2). Theorems 3.3 and 3.5 provide the necessary and sufficient conditions for a FRH-efficient DMU to be an anchor DMU in the sense that for each $\alpha > 0$; DIV_k^l DMU is a WEV DMU. Also, Theorems 3.6 and 3.8 provide the necessary and sufficient conditions for a FRH-efficient DMU to be an anchor DMU in the sense that for each $\gamma > 0$; DOV_k^q DMU is a WEV DMU.

Theorem 3.1. *In the single input case, for each* $\text{DMU}_k = (x_k, y_{1k}, \ldots, y_{sk})$*, there is a large enough* $\alpha > 0$ *so that the* DIV_k^1 $\text{DMU}_k' = (x_k + \alpha, y_{1k}, \ldots, y_{sk})$*, is an interior point of the PPS of the* FRH *model.*

Proof. Consider the model (2.4) corresponding to DMU_k' as follows:

$$
\begin{array}{lll}
\max & \varphi & \\
\text{s.t.} & \displaystyle\sum_{j\in E} \lambda_j y_{rj} + \lambda_o y_{rk} \geq \varphi y_{rk}, & r=1,\ldots,s \\
& \displaystyle\sum_{j\in E} \lambda_j x_j + \lambda_o (x_k+\alpha) \leq x_k + \alpha & \\
& \lambda_j \in \{0,1,2,3,\ldots\}, & j\in E\cup\{o\} \\
& \varphi \in \mathbb{R}. &
\end{array} \tag{3.3}
$$

For each $\overline{\lambda}_k \in \{2,3,\ldots\}$, there is large enough $\alpha > 0$ so that $\left(\overline{\lambda}_o = 0, \overline{\lambda}_j = 0 (j \in E - \{k\}), \overline{\lambda}_k, \varphi = \overline{\lambda}_k\right)$ is feasible solution of model (3.3). So, we have $\varphi^* \geq \varphi > 1$ at optimality. On the other hand, it can be shown that in efficiency evaluation of DMU_k by model (2.3), $\theta^* < 1$. Therefore, DMU_k is an interior point of the PPS of the FRH model. The proof is completed. □

Theorem 3.2. *In the single output case, for each* $\text{DMU}_k = (x_{1k}, \ldots, x_{mk}, y_k)$*, there is a large enough* $\gamma > 0$ *so that the* DOV_k^1 $\text{DMU}_k' = (x_{1k}, \ldots, x_{mk}, y_k - \gamma)$*, is an interior point of the PPS of the* FRH *model.*

Proof. The proof is similar to the proof of Theorem 3.1 except that we consider the models (2.3) and (2.4) corresponding to the DOV_k^1 DMU_k'. □

Theorem 3.3. *In the multiple inputs case, if for some* l*, model (3.1) is infeasible then,* DMU_k *is an anchor* DMU*.*

Proof. Suppose that model (3.1) is infeasible. We show that DMU_k is an anchor point. For this aim, it is enough to show that WEV DIV_k^l DMU lies on the weak efficient unbounded frontier. Consider the following output-oriented FRH model which corresponds to the DIV_k^l DMU:

$$
\begin{array}{lll}
\max & \varphi & \\
\text{s.t.} & \sum_{j\in E} \mu_j x_{lj} + \mu_o(x_{lk}+\alpha) \le x_{lk}+\alpha & \\
& \sum_{j\in E} \mu_j x_{ij} + \mu_o x_{ik} \le x_{ik}, & i=1,\dots,m \ i\ne l \\
& \sum_{j\in E} \mu_j y_{rj} + \mu_o y_{rk} \ge \varphi y_{rk}, & r=1,\dots,s \\
& \mu_j \in \{0,1,2,3,\dots\}, & j\in E\cup\{o\} \\
& \varphi\in\mathbb{R}. &
\end{array}
\tag{3.4}
$$

Now, suppose that $(\varphi^*, \mu_j^*, \mu_o^*)$ is the optimal solution of the model (3.4). Since $\varphi^* \ge 1$ and $x_{lk}+\alpha = \widehat{\theta} x_{lk}$, for some $\widehat{\theta} > 1$, the constraints of the model (3.4) can be rewritten as follows:

$$
\begin{array}{ll}
\sum_{j\in E-\{k\}} \mu_j^* x_{lj} + (\mu_k^*+\mu_o^*)x_{lk} + \mu_o^*\alpha \le \widehat{\theta} x_{lk} & \\
\sum_{j\in E-\{k\}} \mu_j^* x_{ij} + (\mu_k^*+\mu_o^*)x_{ik} \le x_{ik}, & i=1,\dots,m \ i\ne l \\
\sum_{j\in E-\{k\}} \mu_j^* y_{rj} + (\mu_k^*+\mu_o^*)y_{rk} \ge y_{rk}, & r=1,\dots,s \\
\mu_j \in \{0,1,2,3,\dots\}, & j\in E\cup\{o\}.
\end{array}
\tag{3.5}
$$

If $\mu_k^*+\mu_o^*=0$ then, $\mu_k^*=\mu_o^*=0$. Therefore:

$$
\begin{array}{ll}
\sum_{j\in E-\{k\}} \mu_j^* x_{lj} \le \widehat{\theta} x_{lk} & \\
\sum_{j\in E-\{k\}} \mu_j^* x_{ij} \le x_{ik}, & i=1,\dots,m \ \ i\ne l \\
\sum_{j\in E-\{k\}} \mu_j^* y_{rj} \ge y_{rk}, & =1,\dots,s \\
\mu_j^* \in \{0,1,2,3,\dots\}, & j\in E-\{k\}.
\end{array}
\tag{3.6}
$$

It easily follows from (3.6) that $(\theta_l^k, \lambda_j^k) = (\widehat{\theta}, \mu_j^*)$, $(j \in E-\{k\})$ is a feasible solution of model (3.1), a contradiction. On the other hand, the second constraint of (3.5) implies that $\mu_k^*+\mu_o^*=1$. Two cases are occur:

(i) $(\mu_k^*, \mu_o^*) = (1,0)$.
(ii) $(\mu_k^*, \mu_o^*) = (0,1)$.

In each of these two cases, we conclude that $\varphi^*=1$. Now, since WEV DIV_k^l DMU is inefficient, by definition of weak efficient DMU (see *Note 1.*), we conclude that DIV_k^l DMU is weak efficient and therefore, DMU_k is anchor point. The proof is completed. □

Remark 3.4. By Theorem 3.1, in the single input case; we do not need to solve the model (3.1).

The following theorem is, in fact, the converse of Theorem 3.3.

Theorem 3.5. *In a multiple inputs case, if* FRH-*efficient* DMU $\mathrm{DMU}_k = (x_{1k},\dots,x_{lk},\dots,x_{mk},y_{1k},\dots,y_{sk})$ *is an anchor* DMU *in the sense that for each* $\alpha>0$ *the* DIV_k^l DMU *is a* WEV DMU*; then model (3.1) is infeasible.*

Proof. Suppose that DMU_k is an anchor point. Consider the output-oriented FRH model (3.4). Suppose that $(\varphi^* = 1, \mu_j^*(\alpha), \mu_o^*(\alpha))(j \in E)$ is an optimal solution of model (3.4). The constraints of the model (3.4) can be written as follows:

$$\begin{array}{ll} \sum_{j\in E-\{k\}} \mu_j^*(\alpha)x_{lj} \le (1-\mu_o^*(\alpha)-\mu_k^*(\alpha))x_{lk} + (1-\mu_o^*(\alpha))\alpha & \\ \sum_{j\in E-\{k\}} \mu_j^*(\alpha)x_{ij} \le (1-\mu_o^*(\alpha)-\mu_k^*(\alpha))x_{ik}, & i=1,\ldots,m\ i\ne l \\ \sum_{j\in E-\{k\}} \mu_j^*(\alpha)y_{rj} \ge (1-\mu_o^*(\alpha)-\mu_k^*(\alpha))y_{rk}, & r=1,\ldots,s \\ \mu_j \in \{0,1,2,3,\ldots\}, & j\in E\cup\{o\} \end{array} \tag{3.7}$$

Two cases can be considered:

(i) $1-\mu_o^*(\alpha)-\mu_k^*(\alpha)=0$.
(ii) $1-\mu_o^*(\alpha)-\mu_k^*(\alpha)=1$.

In case (i), it is easy to show that $\big(\mu_j^*(\alpha)=0(j\in E-\{k\}), \mu_o^*(\alpha)=0, \mu_k^*(\alpha)=1\big)$ and $\big(\mu_j^*(\alpha)=0(j\in E-\{k\}), \mu_o^*(\alpha)=1, \mu_k^*(\alpha)=0\big)$ are optimal solutions of model (3.4). Now we show that the case (ii) can not be occurred. Case (ii) results that $\mu_o^*(\alpha)=\mu_k^*(\alpha)=0$. Consider the first constraint of (3.4). If for each j, $x_{lj} > x_{lk}$ then, one can find $\alpha > 0$ small enough so that the first constraint of (3.4) does not satisfy. These contradict the feasibility of the model (3.4). If for some j, $x_{lj} \le x_{lk}$ then, one can choose $\alpha > 0$ small enough to have $\sum_{j\in E-\{k\}} \mu_j^*(\alpha)x_{lj} \le x_{lk}$. Therefore, the following inequalities hold for small enough $\alpha > 0$:

$$\begin{array}{ll} \sum_{j\in E-\{k\}} \mu_j^*(\alpha)x_{lj} \le x_{lk} & \\ \sum_{j\in E-\{k\}} \mu_j^*(\alpha)x_{ij} \le x_{ik}, & i=1,\ldots,m\ i\ne l \\ \sum_{j\in E-\{k\}} \mu_j^*(\alpha)y_{rj} \ge y_{rk}, & r=1,\ldots,s. \end{array} \tag{3.8}$$

This implies that there is a DMU_p, $p\in E-\{k\}$, so that:

$$\begin{array}{ll} x_{lp} \le x_{lk} & \\ x_{ip} \le x_{ik}, & i=1,\ldots,m\ i\ne l \\ y_{rp} \ge y_{rk}, & r=1,\ldots,s \end{array}$$

and at least one of these inequalities strictly hold. Therefore, DMU_k is dominated by DMU_p, a contradiction. Therefore, case (ii) can not be occurred. Now, we show that model (3.1) is infeasible. By contradiction, suppose that model (3.1) is feasible. Therefore, it has an optimal solution as $\big(\theta_l^{k*}(>1), \lambda_j^{k*}(j\in E-\{k\})\big)$. So:

$$\begin{array}{ll} \sum_{j\in E-\{k\}} \lambda_j^{k*}x_{lj} = x_{lk}+\alpha^* & \\ \sum_{j\in E-\{k\}} \lambda_j^{k*}x_{ij} \le x_{ik}, & i=1,\ldots,m\ i\ne l \\ \sum_{j\in E-\{k\}} \lambda_j^{k*}y_{rj} \ge y_{rk}, & r=1,\ldots,s \\ \lambda_j \in \{0,1,2,3,\ldots\}, & j\in E-\{k\} \end{array} \tag{3.9}$$

in which $x_{lk}+\alpha^* = \theta_l^{k^*}x_{lk}$ for some $\alpha^* > 0$. It means that the case (ii) has been occurred, a contradiction. The proof is completed. ∎

Theorem 3.6. *In the multiple outputs case, if for some q, model (3.2) is infeasible, then,* DMU_k *is an anchor* DMU*.*

Proof. Suppose that model (3.2) is infeasible. We show that DMU_k is an anchor point. For this aim, it is enough to show that WEV DOV_k^q DMU lies on the weak efficient unbounded frontier. Consider the following input-oriented FRH model corresponding to the DOV_k^q DMU:

$$\begin{aligned} \min\ & \theta \\ \text{s.t.} & \sum_{j\in E}\mu_j x_{ij}+\mu_o x_{ik}\le \theta x_{ik}, && i=1,\dots,m \\ & \sum_{j\in E}\mu_j y_{qj}+\mu_o(y_{qk}-\gamma)\ge y_{qk}-\gamma, \\ & \sum_{j\in E}\mu_j y_{rj}+\mu_o y_{rk}\ge y_{rk}, && r=1,\dots,s \quad r\ne q \\ & \mu_j\in\{0,1,2,3,\dots\}, && j\in E\cup\{o\} \\ & \theta\in\mathbb{R}. \end{aligned} \tag{3.10}$$

Now, suppose that $(\theta^*,\mu_j^*,\mu_o^*)$ is the optimal solution of the model (3.10). Since $\theta^*\le 1$ and $y_{qk}-\gamma=\widehat{\varphi}y_{qk}$, for some $\widehat{\gamma}<1$, the constraints of the model (3.10) can be rewritten as follows:

$$\begin{aligned} & \sum_{j\in E-\{k\}}\mu_j^* x_{ij}+(\mu_k^*+\mu_o^*)x_{ik}\le x_{ik}, && i=1,\dots,m \\ & \sum_{j\in E-\{k\}}\mu_j^* y_{qj}+(\mu_k^*+\mu_o^*)y_{qk}\le \widehat{\varphi}y_{qk}, \\ & \sum_{j\in E-\{k\}}\mu_j^* y_{rj}+(\mu_k^*+\mu_o^*)y_{rk}\ge y_{rk}, && r=1,\dots,s \quad r\ne q \\ & \mu_j^*\in\{0,1,2,3,\dots\}, && j\in E\cup\{o\}. \end{aligned} \tag{3.11}$$

If $\mu_k^*+\mu_o^*=0$ then $\mu_k^*=\mu_o^*=0$. Therefore:

$$\begin{aligned} & \sum_{j\in E-\{k\}}\mu_j^* x_{ij}\le x_{ik}, && i=1,\dots,m \\ & \sum_{j\in E-\{k\}}\mu_j^* y_{qj}\le \widehat{\varphi}y_{qk}, \\ & \sum_{j\in E-\{k\}}\mu_j^* y_{rj}\ge y_{rk}, && r=1,\dots,s \quad r\ne q \\ & \mu_j^*\in\{0,1,2,3,\dots\}, && j\in E-\{k\}. \end{aligned} \tag{3.12}$$

It easily follows from (3.12) that $(\varphi_q^k,\mu_j^k)=(\widehat{\varphi},\mu_j^*)$, $(j\in E-\{k\})$ is a feasible solution of model (3.2), a contradiction. On the other hand, the second constraint of (3.11) implies that $\mu_k^*+\mu_o^*=1$. Two cases are occur:

(i) $(\mu_k^*,\mu_o^*)=(1,0)$.
(ii) $(\mu_k^*,\mu_o^*)=(0,1)$.

In each of these two cases, we conclude that $\theta^*=1$. Now, since WEV DOV_k^q DMU is inefficient, by definition of weak efficient DMU (see *Note 1.*), we conclude that DOV_k^q DMU is weak efficient and therefore, DMU_k is anchor point. The proof is completed. □

Remark 3.7. By Theorem 3.2, in the single output case, we do not need to solve the model (3.2).

The following theorem is, in fact, the converse of Theorem 3.6.

TABLE 1. Data for Example 4.1 and the results of evaluation FRH-efficient DMUs by models (3.2).

				q	
DMU	x	y_1	y_2	1	2
D_1	1	1	5	INFES	FES
D_2	1	4	4	FES	FES
D_3	1	5	1	FES	INFES

Theorem 3.8. *In a multiple outputs case, if* FRH-*efficient* DMU $\text{DMU}_k = (x_{1k}, \ldots, x_{lk}, \ldots, x_{mk}, y_{1k}, \ldots, y_{sk})$ *is an anchor* DMU *in the sense that for each* $\gamma > 0$ *the* DOV_k^q DMU *be a* WEV DMU*; then model (3.2) is infeasible.*

Proof. The proof is similar to the proof of Theorem 3.5; except that instead of model (3.4) we consider the input-oriented FRH model (3.10). The details are deleted. □

The necessary and sufficient condition for DMU_k ($k \in E$) under evaluation to be FRH extreme unit is given in the following theorem:

Theorem 3.9. DMU_k*,* $k \in E$*, is extreme* DMU *if and only if either for at least one* l *(or* q*), model (3.1) (or (3.2)) is infeasible or for all* l *and* q*,* $\theta_k^l > 1$ *and* $\varphi_k^q < 1$*.*

Proof. The proof is straightforward. □

To sum up, by Theorems 3.3 and 3.5 we can find all anchor DMUs for which the DIV_k^l DMUs $\text{DMU}_k^{'} = (x_{1k}, \ldots, x_{(l-1)k}, x_{lk} + \alpha, x_{(l+1)k}, \ldots, x_{mk}, y_{1k}, \ldots, y_{sk})$ are WEV DMUs, for each $\alpha > 0$, and by Theorems 3.6 and 3.8 we can find all anchor DMUs for which the DOV_k^q DMUs $\text{DMU}_k^{'} = (x_{1k}, \ldots, x_{mk}, y_{1k}, \ldots, y_{qk} - \beta, \ldots, y_{sk})$ are WEV DMUs, for each $0 < \beta \leq y_{qk}$.

Now we are in the position to put all together the ingredients of the method.

Summary of finding all anchor DMUs' algorithm

- **Step 1**. Evaluate **n** DMUs with a suitable form of models (2.3) and (2.4). Hold all FRH-efficient DMUs and remove other DMUs. Put indices of this FRH-efficient DMUs in E.
- **Step 2**. Evaluate each DMUs in E with models (3.1) and (3.2). (Note that in the single input case we don't use model (3.1) and in the single output case we don't use model (3.2)).
- **Step 3**. If for some l (or q) the model (3.1) (or (3.2)) is infeasible, then, DMU_k is an anchor DMU and DIV_k^l (or DOV_k^q) DMU_k' is WEV DMU.
- **Step 4**. If all DMUs in E evaluated by models (3.1) and (3.2), stop. Otherwise, go to step 1.

4. Numerical examples

Example 4.1 (Single input and two outputs case)**.** Three DMUs with one input and two outputs are considered in Table 1. By applying the model (2.4), $E = \{D_1, D_2, D_3\}$. Figure 3 shows the constructed PPS by these three DMUs. The last two columns of Table 1 indicates the results of applying model (3.1). In Table 1, "INFES" and "FES" denotes "infeasible" and "feasible", respectively. For instance, "INFES" in the first row means that model (3.1), corresponding to DMU D_1 ($k = 1$) with $q = 1$, is infeasible. So, by Theorem 3.5, D_1 is an anchor DMU and for each $\beta > 0$, DOV_1^1 DMU $D_1' = (1, 1 - \beta, 5)$ is a WEV DMU. Using Theorems 3.1, 3.6, 3.8 and 3.9 and the information of Table 1, DMUs D_1, D_2 and D_3 are extreme DMUs but, D_1 and D_3 are anchor DMUs. Please pay attention that by Remark 3.4, we don't need to apply the model (3.1).

TABLE 2. Example 4.2. Multiple inputs and outputs.

DMU	D_1	D_2	D_3	D_4	D_5
x_1	2	1	2	4	3
x_2	3	2	2	2	5
y_1	7	3	4	6	5
y_2	4	5	3	1	2

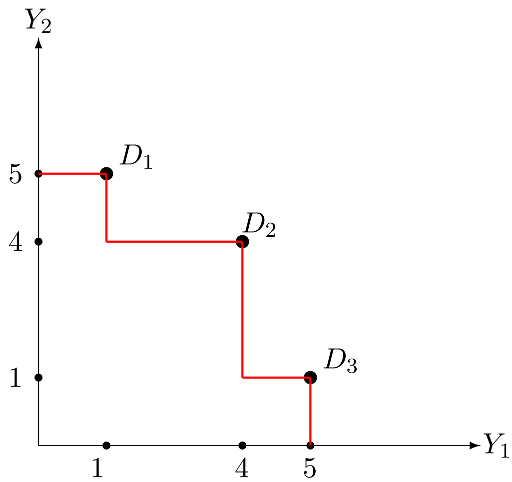

FIGURE 3. Example 4.1. D_1 and D_3 are anchor points.

Example 4.2 (Multiple outputs and inputs case)**.** Table 2 shows data for 5 DMUs with two inputs and two outputs. Running model (2.3) (or (2.4)) shows that D_2, D_3 and D_4 are FRH-efficient and other DMUs are FRH-inefficient. So, $E = \{2, 3, 4\}$. Applying models (3.1) and (3.2) to each DMU$_k$, $k \in E$, produces the results reported in Table 3. So, by Theorems 3.3 and 3.5, D_2 is an anchor DMU and for each $\alpha > 0$, DIV$_2^2$ DMU $D_2' = (1, 2 + \alpha, 3, 5)$ is a WEV DMU. Using Theorems 3.3 to 3.9 and the information of Table 3, all DMU$_k$ $k \in E$ are extreme and anchor DMUs. Also, for instance, in view of Table 3, for each $\alpha > 0$, DIV$_2^2$ DMU $D_2' = (1, 2 + \alpha, 3, 5)$ is a WEV DMU.

Example 4.3 (Real word data)**.** We evaluated the data of 20 branches of a bank in Iran using the proposed method. The data was previously analyzed by Amirteimoori *et al.* [4], (see Tab. 4). Running the DEA model (2.3) (or (2.4)) resulted in $E = \{1, 2, 3, 4, 7, 8, 9, 11, 15, 16, 17, 20\}$. Using the proposed method, all DMUs in E are found to be extreme and anchor DMUs. Also DIV$_1^{1,2}$, DIV$_2^{1,2,3}$, DIV$_3^{1,2,3}$, DIV$_4^{1,2,3}$, DIV$_7^{1,2,3}$, DIV$_8^{1,2}$, DIV$_9^{1,2,3}$, DIV$_{11}^{1,2,3}$, DIV$_{15}^{1,2,3}$, DIV$_{16}^{2,3}$, DIV$_{17}^{1,2,3}$, DIV$_{19}^{2,3}$, DIV$_{20}^{2,3}$ and also, DOV$_1^{1,2,3}$, DOV$_2^{2,3}$, DOV$_3^{1,2,3}$, DOV$_4^{1,2,3}$, DOV$_5^{2,3}$, DOV$_6^{2,3}$, DOV$_7^{1,2,3}$, DOV$_8^{1,2,3}$, DOV$_9^{1,2,3}$, DOV$_{11}^{1,2,3}$, DOV$_{15}^{2,3}$, DOV$_{16}^{2,3}$, DOV$_{17}^{1,2,3}$, DOV$_{20}^{1,2,3}$ DMUs are WEV DMUs.

5. CONCLUSIONS

Anchor points are a new category of extreme-efficient DMUs. They delineate the pareto efficient frontier from the unbounded inefficient part of the PPS. The identification and applications of these points have been studied

TABLE 3. Example 4.2. The results of evaluation FRH-efficient DMUs by models (3.1) and (3.2).

DMU	l		q	
	1	2	1	2
D_2	FES	INFES	FES	INFES
D_3	INFES	FES	INFES	FES
D_4	INFES	FES	INFES	FES

TABLE 4. Example 4.3. DMUs' data (extracted from [Amirteimoori *et al.* [4], p. 689]).

Branch	Input			output		
	Staff	Computer terminals	Space m2	Deposits	Loans	Charge
1	0.9503	0.70	0.1550	0.1900	0.5214	0.2926
2	0.7962	0.60	1.0000	0.2266	0.6274	0.4624
3	0.7982	0.75	0.5125	0.2283	0.9703	0.2606
4	0.8651	0.55	0.2100	0.1927	0.6324	1.0000
5	0.8151	0.85	0.2675	0.2333	0.7221	0.2463
6	0.8416	0.65	0.5000	0.2069	0.6025	0.5689
7	0.7189	0.60	0.3500	0.1824	0.9000	0.7158
8	0.7853	0.75	0.1200	0.1250	0.2340	0.2977
9	0.4756	0.60	0.1350	0.0801	0.3643	0.2439
10	0.6782	0.55	0.5100	0.0818	0.1835	0.0486
11	0.7112	1.00	0.3050	0.2117	0.3179	0.4031
12	0.8113	0.65	0.2550	0.1227	0.9225	0.6279
13	0.6586	0.85	0.3400	0.1755	0.6452	0.2605
14	0.9763	0.80	0.5400	0.1443	0.5143	0.2433
15	0.6845	0.95	0.4500	1.0000	0.2617	0.0982
16	0.6127	0.90	0.5250	0.1151	0.4021	0.4641
17	1.0000	0.60	0.2050	0.0900	1.0000	0.1614
18	0.6337	0.65	0.2350	0.0591	0.3492	0.0678
19	0.3715	0.70	0.2375	0.0385	0.1898	0.1112
20	0.5827	0.55	0.5000	0.1101	0.6145	0.7643

by several authors, including Bougnol [8], Thanassoulis and Allen [23] and Rouse [19]. As far as we know, no efficient study has been performed on finding anchor DMUs of the PPS of the FRH models. In this paper a method has been presented for finding all extreme and anchor DMUs of the PPS of the FRH models using two super-efficiency models (see models (3.1) and (3.2)). The necessary and sufficient conditions for a DMU to be an extreme and anchor DMU has been stated and proved. According to the proposed approach, one can determine the inputs (outputs) of the anchor DMUs that can be increased (decreased) without penetrating the interior of the PPS. The validity of the presented approach has been tested through some examples. Finally, the GAMs software has been employed to run the models (3.1) and (3.2).

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