RAIRO-Oper. Res. 55 (2021) S1647–S1655
https://doi.org/10.1051/ro/2020122

RAIRO Operations Research
www.rairo-ro.org

ON THE DOMINATOR CHROMATIC NUMBER OF THE GENERALIZED CATERPILLARS FOREST

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Abstract. A dominator coloring is a proper coloring of the vertices of a graph such that each vertex of the graph dominates all vertices of at least one color class (possibly its own class). The dominator chromatic number of a graph G is the minimum number of color classes in a dominator coloring of G. In this paper, we determine the exact value of the dominator chromatic number of a subclass of forests which we call, generalized caterpillars forest, where every vertex of degree at least three is a support vertex.

Mathematics Subject Classification. 05C15, 05C69.

Received March 20, 2020. Accepted October 24, 2020.

1. Introduction

Throughout the paper, we consider finite, simple and undirected graphs. Let G be a graph with *vertex-set* $V(G)$ and *edge-set* $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u|uv \in E\}$. The *degree* of a vertex v is $d_G(v) = |N(v)|$. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. Given a subset $A \subseteq V(G)$, we denote by $G[A]$ (or sometimes $G \setminus A$) the *subgraph* of G induced by A.

Recall that a *tree* is a connected acyclic graph, and a *forest* is an acyclic graph. A *caterpillar* is a tree such that the removal of all its leaves produces a path.

A *coloring* of the vertices of G is a mapping $c : V(G) \to \mathbb{N}$, where for every vertex v the integer $c(v)$ is called the *color* of v. A coloring c is *proper* if for any two adjacent vertices u and v, $c(v) \neq c(u)$. The *chromatic number* $\chi(G)$ of graph G is the smallest integer k such that G admits a proper coloring with k colors. Let X be a color class of a proper coloring of G. Then we say that a vertex x of G *sees* X if x is adjacent to all vertices in X, and x *misses* X otherwise. In particular, if $X = \{x\}$, then we say that x sees its own class.

A *dominator coloring* of G is a proper coloring of the vertices of G such that each vertex in G sees at least one color class (possibly its own class). The *dominator chromatic number* $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of G. A dominator coloring of G with $\chi_d(G)$ colors will be called a χ_d-coloring of G. The concept of dominator coloring was introduced by Gera *et al.* [6] and studied further by Gera [4,5], Chellali and Maffray [3] and Boumediene and Chellali [1,2]. In particular, in [2] the authors gave a polynomial

Keywords. Dominator coloring, dominator chromatic number, trees.

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time algorithm computing the dominator chromatic number for every nontrivial tree. It is worth noting that the decision problem corresponding to the dominator coloring is NP-complete for arbitrary graphs [6]. Therefore, it is natural to look for graph classes where the value of the dominator chromatic number is given either exactly or can be computed in polynomial time.

Given a χ_d-coloring c of G, we denote by Ω_c the set of color classes of c containing a single vertex, and let Π_c be the set of the remaining color classes of c. Let us also define the following sets.

– Let Π_c^1 be the subset of color classes that are missed, that is $\Pi_c^1 = \{X \in \Pi_c |$ no vertex of G sees $X\}$, and let $\Pi_c^2 = \Pi_c \diagdown \Pi_c^1$.
– For $i \in \{1,2\}$, let B_c^i be the set of all vertices belonging to color classes in Π_c^i, and let $A_c = V(G) \diagdown (B_c^1 \cup B_c^2)$.

Clearly A_c, B_c^1, B_c^2 are disjoint sets and $V(G) = A_c \cup B_c^1 \cup B_c^2$. Also, $|A_c| = |\Omega_c|$ and $\left|B_c^1 \cup B_c^2\right| \geq 2\,|\Pi_c|$.

It has been shown in [1] that for every χ_d-coloring c of a nontrivial tree either each support vertex belongs to A_c or its unique leaf neighbor belongs to A_c. Moreover, they proved the following.

Proposition 1.1 ([1])**.** *Every tree of order at least three admits a χ_d-coloring c such that each support vertex belongs to A_c and all leaves of G have the same color.*

In this paper, we are interested in determining the exact value of the dominator chromatic number for a more general class of caterpillars which we call generalized caterpillars. A *generalized caterpillar* is a tree such that each vertex of degree at least three is a support vertex. A *generalized caterpillars forest* is a forest such that each component is a generalized caterpillar. A *stalk* in a generalized caterpillar forest G is a path whose endvertices are support vertices in G and whose inner vertices are not. Clearly, each stalk (if any) has order at least two. Also, if G is a generalized caterpillar forest without stalks, then each component of G is a star or a single vertex.

It is worth mentioning that every tree T of order at least three is a subtree of a generalized caterpillar. Indeed, it is enough to add for any vertex of degree at least 3 that is not a support vertex a new vertex attached to it. Clearly, in this way the supertree obtained, which will denoted by G_T, is a generalized caterpillar. In this context, if T is a tree of order $n \geq 3$, then I_T will denote the set of vertices of degree at least three that are not support vertices. Obviously, if T is a tree with $I_T = \emptyset$, then $G_T = T$. Our next observation gives a relationship between $\chi_d\,(T)$ and $\chi_d\,(G_T)$ for every nontrivial tree T.

Observation 1.2. If T is a nontrivial tree, then $\chi_d\,(G_T) - |I_T| \leq \chi_d\,(T) \leq \chi_d\,(G_T)$.

Proof. Clearly, if T has order two or I_T is empty, then $G_T = T$ and the result is valid. Hence we can assume that T has order at least three and $I_T \neq \emptyset$. The upper bound follows from the fact that the restriction of any χ_d-coloring of G_T to T is a dominator coloring of T. Now to prove the lower bound, consider a χ_d-coloring c of T satisfying Proposition 1.1. Let $M = I_T \diagdown A_c$ and π be a coloring of G_T obtained from c as follows. Color each vertex of M with a new, different color; and color the new vertices in G_T with the color used by the leaves in T. The remaining vertices of G_T keep their colors already given by coloring c. It is easy to see that π is a dominator coloring of G_T with $\chi_d\,(T) + |M|$ colors, and thus $\chi_d\,(G_T) \leq \chi_d\,(T) + |M| \leq \chi_d\,(T) + |I_T|$. □

The sharpness of the bounds in Observation 1.2 is given by the following result.

Observation 1.3. For every integer $j \geq 0$, there exists a tree T_j such that $\left|I_{T_j}\right| = j$ and $\chi_d(G_{T_j}) = \chi_d(T_j) + \left|I_{T_j}\right|$.

Proof. Clearly, if $j = 0$, then for any caterpillar T we have $G_T = T$ and thus $\chi_d(G_T) = \chi_d(T)$. Hence let $j \geq 1$ be an integer. Let H_i be a tree obtained from a star $K_{1,3}$ centered at u_i by subdividing each edge exactly once, and let v_i be a support vertex of H_i. Let T_j be a tree obtained from $H_1, H_2, \ldots, H_j$ by adding $j-1$ edges connecting v_i's so that they induce a path P_j. For example, the tree T_3 is illustrated in Figure 1. A tree T_3 and its corresponding generalized caterpillar G_{T3}. Note that T_j has $3j$ support vertices, and since the remaining vertices of T_j that are independent, we deduce from Proposition 1.1 that $\chi_d\,(T_j) = 3j + 1$. Moreover, the generalized caterpillar G_{T_j} constructed from T_j by adding for each u_i a new vertex attached to it by an edge contains $4j$ support vertices. One can easily see that $\chi_d(G_{T_j}) = 4j + 1 = \chi_d(T_j) + j$. □

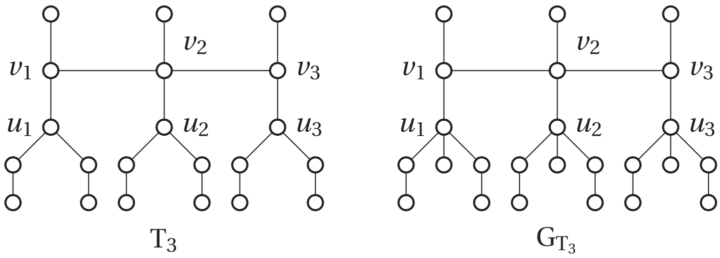

FIGURE 1. A tree T_3 and its corresponding generalized caterpillar G_{T_3}

By using a similar proof to that presented in [1], we can see that a generalized caterpillar forest G admits a χ_d-coloring c such that each support vertex belongs to A_c and all leaves of G have the same color. Hence we have the following.

Corollary 1.4. *Proposition 1.1 is still valid for generalized caterpillar forest.*

Observation 1.5. Let G be a generalized caterpillar forest with $p \geq 0$ single vertices and $q \geq 0$ nontrivial stars, and let H be the subgraph of G containing all components that are neither single vertices nor stars. Then $\chi_d(G) = p + q + \chi_d(H) + i$, where $i = 1$ if $q \geq 1$ and $V(H)$ is empty, and $i = 0$ otherwise.

Proof. If G contains no edge, then clearly $\chi_d(G) = p$. Hence assume that G contains at least one edge. If $V(H)$ is empty, then it is easy to show that $\chi_d(G) = p + q + 1$.

From now on, we can assume that $V(H)$ is non-empty. Let $G_1, G_2, \ldots, G_r$ be the components of $G - H$ (if any). Clearly $\chi_d(H) \geq 2$, and $\chi_d(G_i) \leq 2$ since G_i is a single vertex or a nontrivial star. This means that $\chi_d(H) \geq \chi_d(G_i)$ for every i. Moreover, each component in $G - H$ needs at least one new color, since a vertex in such component must see an entire color class. Hence $\chi_d(G) \geq p + q + \chi_d(H)$.

The equality follows by exhibiting a dominator coloring of G with $p + q + \chi_d(H)$ colors. According to Corollary 1.4, H admits a χ_d-coloring c such that all leaves have the same color, say 1. Let π be a coloring of G defined as follows. For every $x \in V(H)$, let $\pi(x) = c(x)$, and for each leaf v in $G - H$, let $\pi(v) = 1$, unless v belongs to a component of order 2, in which case v is one of the two leaves. Color the remaining vertices of G differently using $(p + q)$ new colors. Clearly, π is a dominator coloring using $p + q + \chi_d(H)$ colors, and thus $\chi_d(G) \leq p + q + \chi_d(H)$. □

According to Observation 1.5, we can assume in the remainder of this paper that each component of a generalized caterpillar forest is nontrivial and different from a star. Our aim is to prove the following result.

Theorem 1.6. *Let G be a generalized caterpillar forest with s support vertices and p connected components, each is nontrivial and different from a star. Let $n_i \geq 2$ be the order of the i^{th} stalk of G. Then*

$$\chi_d(G) = \alpha + s + \sum_{i=1}^{s-p} \left\lfloor \frac{n_i - 2}{3} \right\rfloor,$$

where $\alpha = \begin{cases} 1 & \text{if each } n_i \in \{2, 3, 5\}, \\ 2 & \text{otherwise.} \end{cases}$

2. Proof of Theorem 1.6

The proof of Theorem 1.6 is based on the following preliminary results.

Observation 2.1. Let G be a generalized caterpillar forest such that each component is nontrivial and different from a star. Then G admits a χ_d-coloring c such that the following properties hold.

(i) All leaves of G are in B_c^1 and hence every vertex in $A_c \cup B_c^2$ has degree at least two.
(ii) Every vertex in B_c^1 has degree at most two.
(iii) Every vertex in B_c^2 has degree exactly two.

Proof. Let c be a χ_d-coloring satisfying Corollary 1.4.

(i) Let X be the color class containing all leaves of G. Since G has at least one stalk, each vertex in G misses X, and the first part of item (i) follows. The second part of item (i) is obvious.
(ii) If B_c^1 has a vertex of degree at least three, then such a vertex would be a support vertex, contradicting the choice of c.
(iii) Using the fact that every vertex of degree at least three is a support vertex, the desired result follows from item (i).

□

Lemma 2.2. *Let c be a χ_d-coloring of a graph G and let $\mu_c = \left|\Pi_c^1\right|$. Then*

(i) $\mu_c \le \chi(G)$. *In particular, if G is bipartite, then $\mu_c \in \{0, 1, 2\}$.*
(ii) *Moreover, if G is a generalized caterpillar forest such that each component is nontrivial and different from a star, then $\mu_c \in \{1, 2\}$.*

Proof. (i) Suppose to the contrary that $\mu_c \ge \chi(G) + 1$, that is, at least $\chi(G) + 1$ colors appear in B_c^1. Without loss of generality, we may assume that vertices in B_c^1 use colors $1, 2, \ldots, \mu_c$. Since no color can appear in both B_c^1 and $V(G)\setminus B_c^1$, vertices in $V(G)\setminus B_c^1$ must use the remaining colors, that is, colors $\mu_c + 1, \ldots, \chi_d(G)$. Define a new coloring π of G as follows. Recolor properly all vertices of B_c^1 with $\chi(G[B_c^1])$ colors among $\{1, 2, \ldots, \mu_c\}$ (this is possible since $\chi(G[B_c^1]) \le \chi(G) \le \mu_c - 1$), while vertices of $V(G)\setminus B_c^1$ keep their colors already given by c. Clearly π is a dominator coloring of G with $\chi_d(G) - \mu_c + \chi(G[B_c^1]) < \chi_d(G)$ colors, a contradiction. The second part of item (i) follows from the fact that $\chi(G) \le 2$ for bipartite graphs.

(ii) Note first that $\mu_c \ne 0$, since B_c^1 is nonempty (by Observation 2.1(i)). This together with Lemma 2.2(i) yield the desired result.

□

Lemma 2.3. *Let G be a generalized caterpillar forest such that each component is nontrivial and different from a star. Then G admits a χ_d-coloring π such that Π_π^2 is empty and Corollary 1.4 is fulfilled for π.*

Proof. Among all χ_d-colorings of G fulfilling Corollary 1.4, let c be chosen so that

(C1) $|\Omega_c|$ is maximized.
(C2) Subject to (C1), $\left|\Pi_c^1\right|$ is maximized.

Put $k = \chi_d(G)$ and $\mu_c = \left|\Pi_c^1\right|$. Let $X_1, X_2, \ldots, X_k$ be the color classes of c in which vertices in X_i get color i for $i \in \{1, \ldots, k\}$. Assume to the contrary that Π_c^2 is nonempty. Pick t from $\{1, \ldots, k\}$ such that $X_t \in \Pi_c^2$ and let $X_t = \{x_1, x_2, \ldots, x_p\}$. By definition, $p \ge 2$ and each $x_i \in B_c^2$. Also, by Observation 2.1(iii), we have

$$d_G(x_i) = 2 \text{ for every } i \in \{1, \ldots, p\}. \tag{2.1}$$

Let us denote by $W(X_t)$ the set of vertices of G that see X_t. Then, since G is acyclic, we have

$$|W(X_t)| = 1. \tag{2.2}$$

Thus, let $W(X_t) = \{w_t\}$. Clearly,

$$d_G(w_t) \geq p \geq 2. \tag{2.3}$$

In view of (2.1), let $N_G(x_i) = \{w_t, y_i\}$ for $i \in \{1, \ldots, p\}$ and put $Y_t = \{y_1, y_2, \ldots, y_p\}$. Obviously $N_G(X_t) = Y_t \cup \{w_t\}$. Observe that, since G is acyclic, $N_G[X_t]$ induces a subdivided star of order $2p+1$ centered at w_t.

Recall that Lemma 2.2(ii) shows that $\mu_c \in \{1, 2\}$. Without loss of generality, we can assume that $\Pi_c^1 = \{X_1\}$ when $\mu_c = 1$ and $\Pi_c^1 = \{X_1, X_2\}$ when $\mu_c = 2$. In this case, we have $t \geq \mu_c + 1$ (since $X_t \in \Pi_c^2$). Let π be a χ_d-coloring of G obtained from c and defined according to the following cases.

Case 1. $w_t \in B_c^1 \cup B_c^2$.

Items (ii) and (iii) of Observation 2.1 together with (2.3) yield $d_G(w_t) = 2$. Then $p = 2$ and $N_G(X_t) = \{w_t, y_1, y_2\}$. Thus $N_G(X_t)$ induces a path $P_5 : y_1 - x_1 - w_t - x_2 - y_2$. Next, we will show that y_1 and y_2 either are both in A_c or one of them is in A_c and the other one is in B_c^2 having a neighbor in A_c. In each case, we will recolor some vertices of G and increase the number of color classes of Ω_c or Π_c^1. To this end, consider the following situations whether w_t is in B_c^1 or B_c^2.

Subcase 1. $w_t \in B_c^1$. Note that since $d_G(x_1) = d_G(x_2) = 2$, neither x_1 nor x_2 can see a color class of Π_c^2, for otherwise, one of them will be of degree at least 3, which contradicts (2.1). Therefore, both y_1 and y_2 must be in A_c, for otherwise, one of x_1 and x_2 misses all color classes of c, which contradicts the definition of c. Define π as follows: assign color 1 to x_1 and x_2, and assign color t to w_t. The remaining vertices of G keep their colors (given by c). It is easy to see that π is a χ_d-coloring of G fulfilling Corollary 1.4 such $\Omega_\pi = \Omega_c \cup \{w_t\}$ (that is, Ω_π contains more color classes than Ω_c), which contradicts the choice of c.

Subcase 2. $w_t \in B_c^2$. Let $X_s = \{x'_1, x'_2, \ldots, x'_q\}$ be the color class containing $w_t = x'_1$. Clearly $t \neq s \geq \mu_c + 1$. Since $X_s \in \Pi_c^2$, (2.2) says that $W(X_s) = \{w_s\}$. If $w_s \notin \{x_1, x_2\}$, then $d_G(w_t) \geq 3$, contradicting the fact that $d_G(w_t) = 2$. Hence, $w_s \in \{x_1, x_2\}$, say $w_s = x_1$. Since w_s and X_s play the same role as w_t and X_t, respectively, we conclude that $q = 2$, and each vertex in $X_t \cup X_s$ has degree 2. Thus $x'_2 = y_1$ and $X_s = \{w_t, y_1\}$. Also, since $d_G(y_1) = 2$, there is a vertex $z_1 \neq x_1$ such that $y_1 z_1 \in E(G)$. Hence $N_G(X_t \cup X_s)$ induces a path $P_6 : z_1$-y_1-x_1-w_t-x_2-y_2. Using the same argument as in Subcase 1, we can see that $y_2, z_1 \in A_c$. Now assigning color 1 to x_2 and y_1 and keep the colors already given to the remaining vertices (under c) provides a χ_d-coloring π of G fulfilling Corollary 1.4 such that $\Omega_\pi = \Omega_c \cup \{\{x_1\}, \{w_t\}\}$, which contradicts again the choice of c.

Case 2. $w_t \in A_c$.

Then $c(w_t) \neq 1$. We claim that $Y_0 = Y_t \cap B_c^1$ is nonempty. Suppose to the contrary that $Y_0 = \emptyset$. Then each vertex in Y_t has color different from 1. By recoloring all vertices of X_t with color 1 (this is possible since $t \geq \mu_c + 1$), we would obtain a dominator coloring of G with $\chi_d(G) - 1$, a contradiction, which proved the claim. Now, let y_{i_0} be any vertex in Y_0. By Observation 2.1(ii), y_{i_0} has degree at most two. If $d_G(y_{i_0}) = 1$, then y_{i_0} misses all color classes of c, which is impossible. So $d_G(y_{i_0}) = 2$ and let $N_G(y_{i_0}) = \{x_{i_0}, z_{i_0}\}$. A similar argument as to the previous cases shows that vertex z_{i_0} is in A_c. In this case, we define π by interchanging colors of x_{i_0} and y_{i_0} and keeping the same colors for the remaining vertices of G. Clearly, π is a χ_d-coloring of G fulfilling Corollary 1.4. In addition, $\Omega_\pi = \Omega_c$, but $\Pi_\pi^1 = \Pi_c^1 \cup \{(X_t \setminus \{x_{i_0}\}) \cup \{y_{i_0}\}\}$, which contradicts the choice of c. □

In the rest of this paper, we denote by T_i the i^{th} stalk of order $n_i \geq 2$ in the generalized caterpillar forest G, where $V(T_i) = \{x_1^i, x_2^i, \ldots, x_{n_i-1}^i, x_{n_i}^i\}$ and $x_j^i x_{j+1}^i \in E(G)$ for every $j \in \{1, 2, \ldots, n_i - 1\}$. We also denote by $I_i = \{x_2^i, x_3^i, \ldots, x_{n_i-1}^i\}$ (possibly empty) the set of inner vertices in T_i.

Lemma 2.4. *Let G be a generalized caterpillar forest such that each component is nontrivial and different from a star. If c is a χ_d-coloring fulfilling the statement of Lemma 2.3, with $\mu_c = |\Pi_c^1|$, then*

(i) *If $|I_i| \geq 3$, then for every three consecutive vertices of I_i, one of them belongs to B_c^1 and another to A_c.*
(ii) *If $\mu_c = 1$ and $|I_i| \geq 2$, then one of any two consecutive vertices of I_i belongs to A_c.*
(iii) *$x_1^i, x_{n_i}^i \in A_c$.*

Proof. We first observe that by Lemma 2.3, each vertex of G is in $B_c^1 \cup A_c$.

(i) Let $|I_i| \geq 3$, and suppose to the contrary, that for some $j_0 \in \{2, \ldots, n_i - 3\}$ all of $x^i_{j_0}, x^i_{j_0+1}, x^i_{j_0+2}$ belong to A_c. Then, recoloring $x^i_{j_0+1}$ with a color used by the leaves provides a dominator coloring of G with $\chi_d(G) - 1$ colors, a contradiction. Therefore, for every $j \in \{2, \ldots, n_i - 3\}$, one of $x^i_j, x^i_{j+1}, x^i_{j+2}$ belongs to B_c^1. Moreover, one of $x^i_{j_0}, x^i_{j_0+1}, x^i_{j_0+2}$ must be in A_c, for otherwise, these three vertices will be all in B_c^1 and thus $x^i_{j_0+1}$ misses all color classes of c, which is impossible.

(ii) Follows from the fact that each vertex of G is in $B_c^1 \cup A_c$.

(iii) Follows from the definition of the stalk T_i and Corollary 1.4.

□

Lemma 2.5. *Let G be a generalized caterpillar forest with $r \geq 1$ stalks $T_1, \ldots, T_r$, s support vertices and p components each is nontrivial and different from a star. Consider a χ_d-coloring c of G satisfying Lemma 2.3 and μ_c be the number of colors of c appearing in $G[B_c^1]$. Then*

(i) $\chi_d(G) = s + \mu_c + \sum_{i=1}^r \left\lfloor \frac{|V(T_i)|-2}{3} \right\rfloor$ *with $r = s - p$.*

(ii) *If $|V(T_i)| \in \{2, 3, 5\}$ for each $i \in \{1, \ldots, r\}$, then $\mu_c = 1$. Otherwise, G admits a χ_d-coloring φ such that $\mu_\varphi = 2$.*

Proof. According to Lemma 2.3, Π_c^2 is empty and Corollary 1.4 is fulfilled for c. Hence $\chi_d(G) = \left|\Pi_c^1\right| + |\Omega_c|$. Also, since $\left|\Pi_c^1\right| = \mu_c$ and $|\Omega_c| = |A_c|$, we get

$$\chi_d(G) = \mu_c + |A_c|. \tag{2.4}$$

Let $n_i = |V(T_i)|$ and recall that $V(T_i) = \{x_1^i, x_2^i, \ldots, x_{n_i}^i\}$ and $I_i = V(T_i) \setminus \{x_1^i, x_{n_i}^i\}$. Let S_G be the set of support vertices of G and $|S_G| = s$. Clearly,

$$A_c = S_G \cup (\cup_{i=1}^r I_i \cap A_c) \tag{2.5}$$

and

$$n_i = 2 + \left|I_i \cap B_c^1\right| + |I_i \cap A_c|. \tag{2.6}$$

(i) It is easy to cheek that $r = s - p$. Hence, combining this together with (2.4) and (2.5), we obtain

$$\chi_d(G) = \mu_c + s + \sum_{i=1}^{s-p} |I_i \cap A_c|. \tag{2.7}$$

In the sequel, we shall show that

$$|I_i \cap A_c| = \left\lfloor \frac{n_i - 2}{3} \right\rfloor \text{ for all } i. \tag{2.8}$$

To do this, we need to prove the following three claims. Let λ_c^i denote the number of colors of c appearing in $G[I_i \cap B_c^1]$.

Claim 2.6. If $n_i = 2$, then $\lambda_c^i = 0$, while if $n_i \geq 3$, then $1 \leq \lambda_c^i \leq \mu_c \leq 2$.

Proof. Clearly, by (2.6), $n_i \geq 2$. If $n_i = 2$, then $I_i = \emptyset$ and thus $\lambda_c^i = 0$. Hence, assume that $n_i \geq 3$. Then $I_i \neq \emptyset$. If $\lambda_c^i = 0$, then all vertices of I_i are in A_c. In this case, recoloring one of these vertices with a color used by the leaves provides a dominator coloring of G with $\chi_d(G) - 1$ colors, a contradiction. Therefore $\lambda_c^i \geq 1$. Now, using the fact that $\mu_c \in \{1, 2\}$ (by Lem. 2.2(ii)), and the definition of λ_c^i we obtain $\lambda_c^i \leq \mu_c \leq 2$. This achieves the proof of Claim 2.6. □

In what follows, we can assume, without loss of generality, that vertices in B_c^1 use either colors 1 and 2 when $\mu_c = 2$ or only color 1 when $\mu_c = 1$. We will additionally assume that all leaves are colored with color 1.

Claim 2.7. If $\mu_c = 2$, then for all $i \in \{1, 2, \dots, r\}$ either $n_i \in \{2, 3, 5\}$ or $\lambda_c^i = 2$.

Proof. Suppose, to the contrary, that there is an integer $i_0 \in \{1, 2, \dots, r\}$ such that $n_{i_0} \notin \{2, 3, 5\}$ and $\lambda_c^{i_0} \neq 2$. By Claim 2.6, $\lambda_c^{i_0} = 1$. Let $I_{i_0} = \{x_2^{i_0}, x_3^{i_0}, \dots, x_{n_{i_0}-1}^{i_0}\}$, and observe that every vertex of I_{i_0} not colored with 1 uses a color belonging to A_c (because of $\lambda_c^{i_0} = 1$), and thus

$$|I_{i_0} \cap A_c| = \left\lfloor \frac{n_{i_0} - 2}{2} \right\rfloor. \tag{2.9}$$

Since $\mu_c = 2$, we can define a dominator coloring φ of G obtained from c as follows. For each $x \notin I_{i_0}$, $\varphi(x) = c(x)$; for each $i \equiv 2(\text{mod}3)$, let $\varphi(x_i^{i_0}) = 1$; for each $i \equiv 0(\text{mod}3)$, let $\varphi(x_i^{i_0}) = 2$; for the remaining vertices of I_{i_0}, we color them differently among colors used by $I_{i_0} \cap A_c$. Clearly now under φ, each vertex of I_{i_0} not colored with 1 or 2 uses a color belonging to A_φ, and thus

$$|I_{i_0} \cap A_\varphi| = \left\lfloor \frac{n_{i_0} - 2}{3} \right\rfloor. \tag{2.10}$$

Now, since $|I_{i_0} \cap A_\varphi| < |I_{i_0} \cap A_c|$, φ is a dominator coloring of G using less colors than c, which leads to a contradiction. This achieves the proof of Claim 2.7. ☐

Claim 2.8. If $n_{i_0} \notin \{2, 3, 5\}$ for some $i_0 \in \{1, 2, \dots, r\}$, then G admits a χ_d-coloring with $\mu = 2$.

Proof. If $\mu_c = 2$, we are done. Hence assume that $\mu_c = 1$ and thus, by Claim 2.6, $\lambda_c^{i_0} = 1$. First, assume that $n_i = 4$ for each i. Define a new dominator coloring π as follows: color each support vertex with a new color starting from 3, and color all its neighbors by colors 1 or 2 so that both colors appear in each stalk. Clearly, $|\pi| = |c|$ and $\mu_\pi = 2$. For the next, we can assume that $n_i \geq 6$ for at least some i. Without loss of generality, we can assume that color 2 appears in $I_{i_0} \cap A_c$. Let φ be the dominator coloring of G defined as in Claim 2.7 with $|\varphi|$ colors. A similar argument as in Claim 2.6 shows that (2.9) and (2.10) remain valid. In addition, we have

$$\mu_\varphi = \mu_c + 1 \text{ and } |I_i \cap A_\varphi| = |I_i \cap A_c| \text{ for all } i \neq i_0 \tag{2.11}$$

and

$$|\varphi| = \mu_\varphi + s + |I_{i_0} \cap A_\varphi| + \sum_{i=1 (i \neq i_0)}^{s-p} |I_i \cap A_\varphi|. \tag{2.12}$$

By replacing the expressions of (2.11) together with (2.10) and (2.9) in (2.12), we obtain

$$|\varphi| = \mu_c + s + 1 + \left\lfloor \frac{n_{i_0} - 2}{3} \right\rfloor - \left\lfloor \frac{n_{i_0} - 2}{2} \right\rfloor + \sum_{i=1}^{s-p} |I_i \cap A_c|.$$

Using (2.7), we get

$$|\varphi| = 1 + \left\lfloor \frac{n_{i_0} - 2}{3} \right\rfloor - \left\lfloor \frac{n_{i_0} - 2}{2} \right\rfloor + \chi_d(G).$$

Now, since $\left\lfloor \frac{n_{i_0}-2}{3} \right\rfloor \leq \left\lfloor \frac{n_{i_0}-2}{2} \right\rfloor - 1$, it follows that $|\varphi| \leq \chi_d(G)$. That is φ is χ_d-coloring of G with $\mu_\varphi = 2$. This achieves the proof of Claim 2.8. ☐

Now we turn our attention to prove equality (2.8). Pick i_0 from $\{1, 2, \dots, r\}$ and consider the following cases.

Case 1. $n_{i_0} \in \{2, 3, 5\}$.
If $n_{i_0} = 2$, then $I_{i_0} = \emptyset$ and thus (2.8) holds since $|I_{i_0} \cap A_c| = 0$. If $n_{i_0} = 3$, then clearly $|I_{i_0}| = 1$. In this case, again $|I_{i_0} \cap A_c| = 0$, for otherwise recoloring the vertex of I_{i_0} with a color used by the leaves provides

a dominator coloring of G with $\chi_d(G)-1$ colors, a contradiction. Thus (2.8) holds for $n_{i_0}=3$. Finally, let $n_{i_0}=5$. Then $|I_{i_0}|=3$, where two vertices of I_{i_0} do not belong to A_c (for otherwise recoloring the two non-adjacent vertices of I_{i_0} with the color used by the leaves and the other vertex with a color already used by $I_{i_0}\cap A_c$ provides a dominator coloring of G with $\chi_d(G)-1$ colors, a contradiction). Thus $|I_{i_0}\cap A_c|\leq 1$. This together with Lemma 2.4(i) yield $|I_{i_0}\cap A_c|=1$ and thus (2.8) holds for $n_{i_0}=5$.

Case 2. $n_{i_0}\notin\{2,3,5\}$.
By Claim 2.8, we may assume that $\mu_c=2$ and thus by Claim 2.7, we have $\lambda_c^{i_0}=2$. Moreover, we have $\left|I_{i_0}\cap B_c^1\right|\geq 2$. This together with (2.6) imply that $n_{i_0}\geq 4$. Consider the dominator coloring φ of G as defined in the proof of Claim 2.7. Note that (2.10) and (2.12) remain valid. In addition, we have

$$\mu_\varphi=\mu_c \text{ and } |I_i\cap A_\varphi|=|I_i\cap A_c| \text{ for all } i\neq i_0. \tag{2.13}$$

Now, by substituting the expressions of (2.10) and (2.13) in formula (2.12), we obtain

$$|\varphi|=\mu_c+s+\left\lfloor\frac{n_{i_0}-2}{3}\right\rfloor+\sum_{i=1(i\neq i_0)}^{s-p}|I_i\cap A_c|\,. \tag{2.14}$$

Now, since $|\varphi|\geq\chi_d(G)$, (2.14) and (2.7) together yield $|I_{i_0}\cap A_c|\leq\left\lfloor\frac{n_{i_0}-2}{3}\right\rfloor$.
On the other hand, Lemma 2.4(i) yields $|I_{i_0}\cap A_c|\geq\left\lfloor\frac{n_{i_0}-2}{3}\right\rfloor$, and thus equality (2.8) follows. This achieves the proof of Item (i).

(ii) Assume that $n_i\in\{2,3,5\}$ for all $i\in\{1,2,\ldots,r\}$ and let l be the number of stalks of order 5. In this case, we have $\sum_{i=1}^{s-p}\left\lfloor\frac{n_i-2}{3}\right\rfloor=l$. By replacing this in the expression of item (i), we get

$$\chi_d(G)=\mu_c+s+l. \tag{2.15}$$

Now, let ψ be a coloring of G obtained from c as follows. For each I_i, let $\psi(x_j^i)=1$ if j is even, and color the vertices x_j^i with j odd differently among colors used by $I_i\cap A_c$. The remaining vertices of G keep their colors given by coloring c. It is easy to show that ψ is dominator coloring using at least $\chi_d(G)$ colors such that $\mu_\psi=1$ and $|I_i\cap A_\psi|=\left\lfloor\frac{n_i-2}{2}\right\rfloor$ for each i, where

$$|\psi|=1+s+\sum_{i=1}^{s-p}\left\lfloor\tfrac{n_i-2}{2}\right\rfloor. \tag{2.16}$$

Using the fact that $n_i\in\{2,3,5\}$, (2.16) becomes

$$|\psi|=1+s+l. \tag{2.17}$$

Since $|\psi|\geq\chi_d(G)$ and $\mu_c\in\{1,2\}$, (2.15) and (2.17) together yield $\mu_c=1$.
The second part follows by Claim 2.8. This completes the proof of Lemma 2.5.

□

Now, we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. If $n_i\in\{2,3,5\}$ for all i, then by items (i) and (ii) of Lemma 2.5, we have $\mu_c=1$ and thus $\chi_d(G)=s+1+\sum_{i=1}^{s-p}\left\lfloor\frac{n_i-2}{3}\right\rfloor$. Now if $n_i\notin\{2,3,5\}$ for some i, then by the second part of Lemma 2.5(ii), we can take $\mu_c=2$. Using Lemma 2.5(i), we obtain $\chi_d(G)=s+2+\sum_{i=1}^{s-p}\left\lfloor\frac{n_i-2}{3}\right\rfloor$. □

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