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ON THE GEODETIC HULL NUMBER FOR COMPLEMENTARY PRISMS II

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Abstract. In the geodetic convexity, a set of vertices S of a graph G is *convex* if all vertices belonging to any shortest path between two vertices of S lie in S. The *convex hull* $H(S)$ of S is the smallest convex set containing S. If $H(S) = V(G)$, then S is a *hull set*. The cardinality $h(G)$ of a minimum hull set of G is the *hull number* of G. The *complementary prism* $G\overline{G}$ of a graph G arises from the disjoint union of the graph G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. A graph G is *autoconnected* if both G and $\overline{G}$ are connected. Motivated by previous work, we study the hull number for complementary prisms of autoconnected graphs. When G is a split graph, we present lower and upper bounds showing that the hull number is unlimited. In the other case, when G is a non-split graph, it is limited by 3.

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1. Introduction

For a finite and simple graph G with vertex set $V(G)$, a *graph convexity* on $V(G)$ is a collection $\mathcal{C}$ of subsets of $V(G)$ such that $\emptyset, V(G) \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. The sets in $\mathcal{C}$ are called *convex sets* and the *convex hull* $H_{\mathcal{C}}(S)$ in $\mathcal{C}$ of a set S of vertices of G is the smallest set in $\mathcal{C}$ containing S. Some natural convexities in graphs are defined by a set $\mathcal{P}$ of paths in G, in a way that a set S of vertices of G is convex if and only if for every path $P = v_0v_1 \dots v_l$ in $\mathcal{P}$ such that if v_0 and v_l belong to S, then all vertices of P belong to S.

The definition of convex sets in graphs originally come from Euclidean geometry, in which a set S is convex if every line segment between two points of S remains in S. The concepts of convexity in graphs can be applied to model contexts involving some disseminating processes between entities, *e.g.* marketing strategies [7], spread of disease and opinion [13], and distributed computing [20].

In this paper we study the convexity related to shortest paths in graphs, the *geodetic convexity* $\mathcal{C}$. Given a graph G, the *closed interval* $I[u, v]$ of a pair $u, v \in V(G)$ consists of all vertices lying in any shortest (u, v)-path in G. For a set $S \subseteq V(G)$, the *closed interval* $I[S]$ is the union of all sets $I[u, v]$ for $u, v \in S$. If $I[S] = S$, then S is a *convex set*. The *convex hull* $H_{\mathcal{C}}(S)$ of S is the smallest convex set containing S. Since a graph G uniquely determines its geodetic convexity $\mathcal{C}$, we may write $H(S)$, instead of $H_{\mathcal{C}}(S)$. If $H(S) = V(G)$ we say that S is a *hull set* of G. The cardinality $h(G)$ of a minimum hull set of G is called the *(geodetic) hull number* of G.

Keywords. Geodetic convexity, hull set, hull number, complementary prism.

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Everett and Seidman [15] introduced the concept of hull number in the geodetic convexity. For some later results see, *e.g.* [3,10,18]. The computation of hull number is NP-hard for bipartite graphs [3], partial cubes [1], and P_9-free graphs [12], but it can be computed in polynomial time for cographs, split graphs [9], $(q, q-4)$-graphs [3], $\{paw, P_5\}$-free graphs [12], and distance-hereditary graphs [19].

The hull number is also studied in other graph convexities. It can be determined in polynomial time for general graphs in *triangle path convexity* [8], related to paths which allow only short chords, and *monophonic convexity* [11], related to induced paths. In P_3-*convexity*, that considers paths with three vertices, computing the hull number for general graphs is NP-complete [5]. However, Duarte *et al.* [14] showed that the hull number of complementary prisms can be determined efficiently in P_3-convexity, more specifically, it is limited by 5 when G and $\overline{G}$ are connected.

In 2007, Haynes *et al.* [17] introduced the *complementary product*, an operation that generalizes the *Cartesian product* of two graphs. In their same work, they introduce a particular case of complementary product called *complementary prism*. The *complementary prism* $G\overline{G}$ of a graph G arises from the disjoint union of the graph G and its complement $\overline{G}$ by adding the edges of a perfect matching between vertices with the same label in G and $\overline{G}$. Related to geodetic convexity, we mention that some works were conducted for graph operations like join, composition, and Cartesian product [4], lexicographic product [2], and strong product [21].

Considering the geodetic hull number in complementary prisms $G\overline{G}$, in our previous paper [6], we presented bounds when G is a disconnected graph or G is a tree. Here, we present lower and upper bounds on the hull number of complementary prisms $G\overline{G}$ when both G and $\overline{G}$ are connected. In particular, we show that $h(G\overline{G}) \leq 3$ when G is a non-split graph. Otherwise, we characterize convex sets in split graphs G that allow the hull number for $G\overline{G}$ be unlimited.

This paper is divided into three more sections. In Section 2, we define the fundamental concepts. In Section 3, we present our main results.

2. Preliminaries

Before we discuss our contributions, we present some relevant definitions. All graphs will be finite, simple, and undirected, and we use standard terminology and notation.

Let G be a graph. Given a vertex $v \in V(G)$, its open neighborhood is denoted by $N_G(v)$, and its *closed neighborhood*, denoted by $N_G[v]$, is the set $N_G[v] = N_G(v) \cup \{v\}$. For a set $U \subseteq V(G)$, let $N_G(U) = \bigcup_{v \in U} N_G(v) \setminus U$, and $N_G[U] = N_G(U) \cup U$. We denote the degree of a vertex $v \in V(G)$ by $deg_G(v)$. If $deg_G(v) = 0$, then we say that v is an *isolated vertex*.

A K_n (resp. C_n) in a graph G denotes an induced complete subgraph (resp. cycle) on n vertices. A *clique* (resp. *independent set*) is a set of pairwise adjacent (resp. non-adjacent) vertices. A vertex of a graph G is *simplicial* in G if its neighborhood is a clique.

The *distance* $d_G(u, v)$ of two vertices u and v in G is the minimum number of edges of a path in G between u and v. The greatest distance between any two vertices in G is the *diameter* of G, denoted by $diam(G)$. We say that H is an *isometric subgraph* of G if H is a subgraph of G such that $d_H(u, v) = d_G(u, v)$ for any pair $u, v \in V(H)$.

A graph G is called *connected* if any two of its vertices are linked by a path in G. Otherwise, G is called *disconnected*. A maximal connected subgraph of G is called a *connected component* or *component* of G. A component G_i of a graph G is *trivial* if $|V(G_i)| = 1$, and *non-trivial* otherwise. If a graph G is connected and its complement $\overline{G}$ is also connected, we say that G is an *autoconnected* graph. Let G be a graph and $\overline{G}$ its complement. For every vertex $v \in V(G)$, we denote $\overline{v} \in V(\overline{G})$ as its *corresponding vertex*, and for a set $X \subseteq V(G)$, we let $\overline{X}$ be the *corresponding set* of vertices in $V(\overline{G})$. The set of positive integers $\{1, \ldots, k\}$ is denoted by $[k]$.

For a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$, the *complementary prism* of G is the graph denoted by $G\overline{G}$ with vertex set $V(G\overline{G}) = \{v_1, \ldots, v_n\} \cup \{\overline{v}_1, \ldots, \overline{v}_n\}$ and edge set $E(G\overline{G}) = E(G) \cup \{\overline{v}_i\overline{v}_j : 1 \leq i < j \leq n \text{ and } v_iv_j \notin E(G)\} \cup \{v_1\overline{v}_1, \ldots, v_n\overline{v}_n\}$. Let $S \subseteq V(G\overline{G})$. Throughout this paper we consider the convex hull $H(S)$ most of times on the graph $G\overline{G}$. If we need to indicate the convex hull on another graph, say G, we add a subscript to the notation, *e.g.* $H_G(S)$.

A *split graph* G is one whose vertex set admits a partition $V(G) = C \cup I$ into a clique C and an independent set I. If G is a split graph, we consider the partition of $V(G)$ such that C is a maximum clique.

3. Results

3.1. General and non-split graphs

We begin by showing three fundamental lemmas and a proposition that we use in the sequel.

Lemma 3.1 (Dourado *et al.* [9])**.** *Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G) \setminus S$ is convex then every hull set of G contains at least one vertex of S.*

Lemma 3.2 (Dourado *et al.* [9])**.** *Let G be a graph and H an isometric subgraph of G. Then for every hull set S of H it holds that $V(H) \subseteq H_G(S)$.*

Lemma 3.3 (Coelho *et al.* [6])**.** *Let G be a graph. If u is a simplicial vertex in G and $\overline{u}$ is a simplicial vertex in $\overline{G}$, then every hull set S of $G\overline{G}$ intersects $\{u, \overline{u}\}$.*

Proposition 3.4. *Let G be a graph, $S \subseteq V(G\overline{G})$, and $v_1 \dots v_k$ be a path in G, for $k \geq 2$. If $\{v_1, \overline{v}_2, \dots, \overline{v}_k\} \subseteq H(S)$, then $v_k \in H(S)$.*

Proof. The proof is by induction on k. First, let $k = 2$. Since $v_1v_2 \in E(G)$ and $v_1, \overline{v}_2 \in H(S)$, $v_2 \in I[v_1, \overline{v}_2]$. Now, let $k > 2$. Let $v_1 \dots v_{k-1}v_k$ be a path in G and suppose that $\{v_1, \overline{v}_2, \dots, \overline{v}_{k-1}, \overline{v}_k\} \subseteq H(S)$. By induction hypothesis $v_{k-1} \in H(S)$, which implies that $v_k \in I[v_{k-1}, \overline{v}_k]$. Therefore, it follows that $v_k \in H(S)$, for $k \geq 2$. □

Next we state our contributions. We first show that for complementary prisms of non-split graphs the geodetic hull number is limited by three.

Theorem 3.5. *Let G be a non-split autoconnected graph. Then $h(G\overline{G}) \leq 3$.*

Proof. Suppose that G is a non-split graph. According to Foldes and Hammer [16], G is split if and only if G does not have an induced subgraph isomorphic to one of the three forbidden graphs, C_4, C_5, or $2K_2$. To show the upper bound $h(G\overline{G}) \leq 3$ we construct hull sets on three cases: (1) C_4 or (2) C_5 or (3) $2K_2$ are induced subgraphs of G.

Case 1. Suppose that C_4 is an induced subgraph of G.

Let $V(C_4) = \{u_1, \dots, u_4\}$, $E(C_4) = \{u_iu_{i+1} : 1 \leq i \leq 3\} \cup \{u_4u_1\}$, and $S = \{u_1, \overline{u}_3, \overline{u}_4\}$. We show that $H(S) = V(G\overline{G})$.

First, we show that $V(C_4) \cup \overline{V(C_4)} \subseteq H(S)$. We have that $u_4 \in I[u_1, \overline{u}_4]$ and $\overline{u}_1 \in I[u_1, \overline{u}_3]$. Consequently $u_3 \in I[u_4, \overline{u}_3]$ and $u_2 \in I[u_1, u_3]$, then $\overline{u}_2 \in I[u_2, \overline{u}_4]$. Hence $V(C_4) \cup \overline{V(C_4)} \subseteq H(S)$.

Let $A = (N_G(u_1) \cap N_G(u_3)) \cup (N_G(u_2) \cap N_G(u_4))$ and $v \in A$. Since $u_1u_3 \notin E(G)$ (resp. $u_2u_4 \notin E(G)$), then $v \in I[u_1, u_3]$ (resp. $v \in I[u_2, u_4]$), hence $A \subseteq H(S)$.

Let $B = V(G) \setminus (A \cup V(C_4))$. For every $b \in B$, there exist $i, j \in [4]$, $i \neq j$, such that $u_iu_j \in E(G)$ and $b \notin N_G(\{u_i, u_j\})$. Then $\overline{u}_i\overline{u}_j \notin E(\overline{G})$, consequently $\overline{b} \in I[\overline{u}_i\overline{u}_j]$, hence $\overline{B} \subseteq H(S)$.

Since G is connected, there exists a path in G joining every vertex from B to a vertex in $A \cup V(C_4)$. Since $\overline{B} \subseteq H(S)$, then Proposition 3.4 implies that $B \subseteq H(S)$. Hence $V(G) \subseteq H(S)$. Since $\overline{G}$ is connected, also by Proposition 3.4, $\overline{A} \subseteq H(S)$. Therefore S is a hull set of $G\overline{G}$.

Case 2. Suppose that C_5 is an induced subgraph of G.

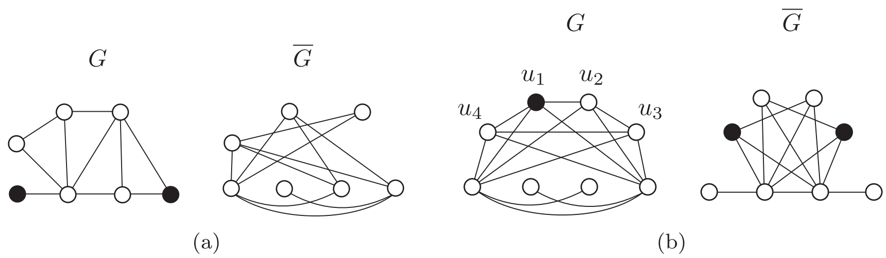

FIGURE 1. Graphs satisfying $h(G\overline{G}) \leq 3$. (a) $h(G\overline{G}) = 2$. (b) $h(G\overline{G}) = 3$.

Let $V(C_5) = \{u_1, \ldots, u_5\}$, $E(C_5) = \{u_iu_{i+1} : 1 \leq i \leq 4\} \cup \{u_5u_1\}$, and $S = \{u_1, u_4, \overline{u}_3\}$. We show that $H(S) = V(G\overline{G})$.

First, we show that $V(C_5) \cup \overline{V(C_5)} \subseteq H(S)$. We have that $u_5 \in I[u_1, u_4]$, $\overline{u}_1 \in I[u_1, \overline{u}_3]$ and $u_3 \in I[u_4, \overline{u}_3]$. Then $u_2 \in I[u_1, u_3]$. By Proposition 3.4 (dual), we have that $\overline{u}_5, \overline{u}_4, \overline{u}_2 \in H(S)$. Hence $V(C_5) \cup \overline{V(C_5)} \subseteq H(S)$.

Let

$$A = \bigcup_{i,j \in [5], i \neq j:\, u_iu_j \notin E(G)} (N_G(u_i) \cap N_G(u_j)).$$

By the definition of A, for every $v \in A$, there exists $i, j \in [5]$, $i \neq j$, such that $u_iu_j \notin E(G)$, which implies that $v \in I[u_i, u_j]$. Hence $A \subseteq H(S)$.

Now, let $B = V(G) \setminus (A \cup V(C_5))$. By the definition of A is b a neighbor of at most two vertices from C_5 in G. Therefore is $\overline{b}$ a neighbor of at least three vertices from $\overline{C_5}$. Moreover, at least two of the mentioned three vertices are not adjacent. Clearly, $\overline{b}$ is a common neighbor of both of them and we have $\overline{b} \in H(S)$. Hence $\overline{B} \subseteq H(S)$.

Since G is connected, there exists a path in G joining every vertex from B to a vertex in $A \cup V(C_5)$. Since $\overline{B} \subseteq H(S)$, then Proposition 3.4 implies that $B \subseteq H(S)$. Hence $V(G) \subseteq H(S)$. Since $\overline{G}$ is connected, also by Proposition 3.4, $\overline{A} \subseteq H(S)$. Therefore S is a hull set of $G\overline{G}$.

Case 3. Suppose that $2K_2$ is an induced subgraph of G.

Since $2K_2$ (resp. $G\overline{G}$) is isomorphic to $\overline{C}_4$ (resp. $\overline{G}G$), we consider $\overline{G}G$ and the proof follows by Case 1.

Since there exists a hull set of order three in all cases the upper bound $h(G\overline{G}) \leq 3$ holds for a non-split graph G. □

For an illustration of the bound of Theorem 3.5 see Figure 1b. The black vertices represent a hull set of each complementary prism $G\overline{G}$. For convenience, the edges joining corresponding vertices from G to $\overline{G}$ are not depicted in the figure.

We show a lower bound by restricting the diameter of the graphs G and $\overline{G}$. That result follows in Theorem 3.6.

Theorem 3.6. *Let G be an autoconnected graph. If $diam(G) = diam(\overline{G}) = 2$, then $h(G\overline{G}) \geq 3$.*

Proof. For contradiction, suppose that $h(G\overline{G}) < 3$. We show that every set $S \subseteq V(G\overline{G})$ of order 2 is not a hull set of $G\overline{G}$. First, let $u, v \in V(G)$. Consider $S_1 = \{u, v\}$, and $S_2 = \{\overline{u}, \overline{v}\}$.

Since $diam(G) = diam(\overline{G}) = 2$, $H(S_1) \cap V(\overline{G}) = \emptyset$ (resp. $H(S_2) \cap V(G) = \emptyset$), then S_1 (resp. S_2) is not a hull set of $G\overline{G}$.

Now, let $u \in V(G)$, $\overline{v} \in V(\overline{G})$, and $S_3 = \{u, \overline{v}\}$. If $uv \in E(G)$ (resp. $uv \notin E(G)$), then $H(S_3) = \{u, v, \overline{v}\}$ (resp. $H(S_3) = \{u, \overline{u}, \overline{v}\}$). Since $\overline{u} \notin H(S_3)$ (resp. $v \notin H(S_3)$), then S_3 is not a hull set of $G\overline{G}$. □

The two previous results imply in Corollary 3.7.

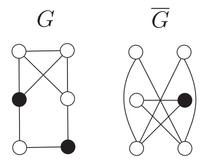

FIGURE 2. Non-split graph G with $diam(G) = diam(\overline{G}) = 2$.

Corollary 3.7. *Let G be a non-split autoconnected graph. If $diam(G) = diam(\overline{G}) = 2$, then $h(G\overline{G}) = 3$.*

Proof. It follows directly from Theorems 3.5 and 3.6. □

For an illustration of Corollary 3.7 see in Figure 2 an example of non-split autoconnected graph G with $diam(G) = diam(\overline{G}) = 2$ and $h(G\overline{G}) = 3$. The black vertices represent a hull set of $G\overline{G}$.

We consider in Theorem 3.8 graphs G with diameter greater than three. Notice that the condition $diam(G) > 3$ implies that G is not a split graph.

Theorem 3.8. *Let G be an autoconnected graph. If $diam(G) > 3$, then $h(G\overline{G}) = 2$.*

Proof. Since $diam(G) > 3$, there exist at least two vertices $x, y \in V(G)$ such that $d_G(x, y) > 3$. Thus, we can define an induced path $P = u_1u_2u_3u_4u_5$ in G such that $d_G(u_1, u_5) = 4$. Let $S = \{u_1, u_4\}$. We show that S is a hull set of $G\overline{G}$.

Let $U = \{u_1, u_2, u_3, u_4\}$. Since $u_1u_2u_3u_4$, and $u_1\overline{u}_1\overline{u}_4u_4$ are shortest (u_1, u_4)-paths, we have that $U \cup \{\overline{u}_1, \overline{u}_4\} \subseteq I[S]$. Consequently $\overline{u}_2 \in I[u_2, \overline{u}_4]$, and $\overline{u}_3 \in I[u_3, \overline{u}_1]$, which implies that $U \cup \overline{U} \subseteq H(S)$.

To complete the proof, we show that for every vertex $z \in V(G) \setminus U$, its corresponding vertex $\overline{z}$ belongs to $H(S)$. In view of the number of neighbors that z has in U, we deal with three cases. Notice that, since $d_{G\overline{G}}(u_1, u_4) = 3$, z is not adjacent to both u_1 and u_4.

Case 1. $|N_G(z) \cap U| \leq 1$.

In this case, $\overline{z}$ is neighbor of two non-adjacent vertices in $\{\overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4\}$, say $\overline{u}_i$ and $\overline{u}_j$, for $i, j \in [4]$, $i \neq j$. Then $\overline{z} \in I[\overline{u}_i, \overline{u}_j]$. Notice that by this case we obtain that $\overline{u}_5 \in I[\overline{u}_1, \overline{u}_2]$ and also $u_5 \in I[u_4, \overline{u}_5]$.

Case 2. $|N_G(z) \cap U| = 2$.

Suppose that z has two neighbors $u_i, u_j \in U$, for $i, j \in [4]$, $i \neq j$. If $i, j \in \{1, 2\}$ (resp. $i, j \in \{3, 4\}$), $i \neq j$, we fall back on Case 1. If $i, j \in \{2, 3\}$, $i \neq j$, then $\overline{z} \in I[\overline{u}_4, \overline{u}_5]$. If $i, j \in \{1, 3\}$, (resp. $i, j \in \{2, 4\}$) $i \neq j$, we obtain that $z \in I[u_1, u_4]$. Since $z \in H(S)$, then $\overline{z} \in I[z, \overline{u}_k]$, for $k = 2$ (resp. $k = 3$).

Case 3. $|N_G(z) \cap U| = 3$.

Suppose that z has three neighbors $u_i, u_j, u_k \in U$, for $i, j, k \in [4]$, $i \neq j \neq k$. Since $d_{G\overline{G}}(u_1, u_4) = 3$, $\{i, j, k\} = \{1, 2, 3\}$ or $\{2, 3, 4\}$. Then, we have that $z \in I[u_1, u_4]$. Since $z \in H(S)$, then $\overline{z} \in I[z, \overline{u}_k]$, for $k = 4$ or $k = 1$.

By all cases, we conclude that $z \in V(G) \setminus U$, $\overline{z} \in H(S)$. This way, we have that $V(\overline{G}) \subseteq H(S)$. Since G is connected, and $U \cup V(\overline{G}) \subseteq H(S)$, Proposition 3.4 implies that $V(G) \subseteq H(S)$. Therefore S is a hull set of $G\overline{G}$, and $h(G\overline{G}) = 2$, which completes the proof. □

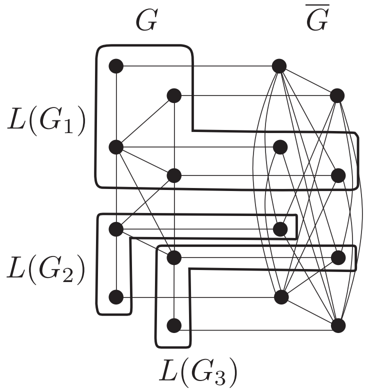

FIGURE 3. Sets $L(G_i)$ of G' on the graph $G\overline{G}$, for $i \in [3]$.

3.2. Split graphs

The following definitions and lemma characterize some convex sets in split graphs.

Let $G = (C \cup I, E)$ be a split graph. Let the graph G' arise from G by removing the edges of the graph induced by C. We call the graph G' as the *component graph* of G. We denote by $c(G')$ the number of connected components of G'. We denote by G_i, $1 \leq i \leq c(G')$, the connected components of G'. Furthermore, we refer to $nt(G')$ and $t(G')$ as the number of non-trivial and trivial components of G', respectively. Definition 3.9 expresses formally the idea of a component graph.

Definition 3.9. *Let $G = (C \cup I, E)$ be a split graph. We define the component graph G' of G as*

$$G' = G \setminus E(G[C]) = \bigcup_{i=1}^{c(G')} G_i.$$

Definition 3.10. *Let $G = (C \cup I, E)$ be a split autoconnected graph. We define the set $L(G_i)$ of a component G_i of G' as*

$$L(G_i) = V(G_i) \cup (V(\overline{G}_i) \cap \overline{C}),$$

for every $i \in [c(G')]$.

See in Figure 3 examples of sets $L(G_i)$ of G', for $i \in [3]$.

Lemma 3.11. *The set $V(G\overline{G}) \setminus L(G_i)$ is convex in $G\overline{G}$, for every $i \in [c(G')]$.*

Proof. Let $i \in [c(G')]$. We show that the closed interval of any two vertices of $V(G\overline{G}) \setminus L(G_i)$ does not intersect $L(G_i)$. We have three cases:

(1) $w, w' \in V(G) \setminus V(G_i)$;
(2) $\overline{w}, \overline{w}' \in V(\overline{G}) \setminus (V(\overline{G}_i) \cap \overline{C})$;
(3) $w \in V(G) \setminus V(G_i)$ and $\overline{w}' \in V(\overline{G}) \setminus (V(\overline{G}_i) \cap \overline{C})$.

Case 1. Let $w, w' \in V(G) \setminus V(G_i)$, and $v \in V(G_i)$.

Let P be a shortest (w, w')-path passing through v. Since G is a split graph and by definition of G_i, P contain two vertices $x, y \in C$. Then $xy \in E(G)$ and thus P cannot be a shortest path, a contradiction.

Case 2. Let $\bar{w}, \bar{w}' \in V(\overline{G}) \setminus (V(\overline{G}_i) \cap \overline{C})$, and $v \in L(G_i)$.

Let P be a shortest $(\bar{w},\bar{w}')$-path passing through v. Clearly, $\bar{w}\bar{w}' \notin E(\overline{G})$ and therefore $ww' \in E(G)$.

If $v \in V(G_i)$, this implies that $v = w$ or $v = w'$, but this yields a contradiction since both w, w' cannot belong to I and one of w, w' must belong to $V(G_i) \cap C$. If $v \in V(\overline{G}_i) \cap \overline{C}$, $N_{\overline{G}}(v)$ is a clique, and thus P cannot be a shortest $(\bar{w},\bar{w}')$-path, a contradiction.

Case 3. Let $w \in V(G) \setminus V(G_i)$, and $\overline{w}' \in V(\overline{G}) \setminus (V(\overline{G}_i) \cap \overline{C})$.

If $\overline{w}' = \overline{w}$, then $I[w,\overline{w}'] = \{w,\overline{w}'\}$. Now suppose that $\overline{w}' \neq \overline{w}$. If $ww' \notin E(G)$, then $I[w,\overline{w}'] = \{w,\overline{w}',\overline{w}\}$. Otherwise, $ww' \in E(G)$, then $I[w,\overline{w}'] = \{w,\overline{w}',w'\}$. Hence $I[w,\overline{w}'] \cap L(G_i) = \emptyset$.

By all cases, we obtain that for all $w,w' \in V(G\overline{G}) \setminus L(G_i)$, $I[w,w'] \cap L(G_i) = \emptyset$. Therefore, the set $V(G\overline{G}) \setminus L(G_i)$ is convex in $G\overline{G}$, for every $i \in [c(G')]$. □

Since $\overline{G}$ is also split, Lemma 3.11 holds for $\overline{G}G$, that is, considering $\bigcup_{j=1}^{c(\overline{G}')} G_j$ be the components of $\overline{G}'$, we have that the set $V(\overline{G}G) \setminus L(\overline{G}_j)$ is convex in $\overline{G}G$, for every $j \in [c(\overline{G}')]$. In view of that, we achieve the lower bound result.

Theorem 3.12. *Let G be a split autoconnected graph. It holds that $h(G\overline{G}) \geq \max\{c(G'), c(\overline{G}'), 2\}$.*

Proof. By Lemma 3.11, $V(G\overline{G}) \setminus L(G_i)$, for every $i \in [c(G')]$, and $V(G\overline{G}) \setminus L(\overline{G}_j)$, for every $j \in [c(\overline{G}')]$, are convex sets. Thus, Lemma 3.1 implies that every hull set of $G\overline{G}$ must contain at least one vertex from $L(G_i)$ and $L(\overline{G}_j)$. Since the components G_i and $G_{i'}$, (resp. G_j and $G_{j'}$) are disjoints for all $i, i' \in [c(G')]$, $i \neq i'$, (resp. $j, j' \in [c(\overline{G}')]$, $j \neq j'$), each vertex $v \in V(G)$ (resp. $\overline{v} \in V(\overline{G})$) intersects exactly one $L(G_i)$ (resp. $L(G_j)$). This implies that $h(G\overline{G}) \geq \max\{c(G'), c(\overline{G}')\}$. Since $|V(G)| \geq 2$, $h(G\overline{G}) \geq 2$, hence the result $h(G\overline{G}) \geq \max\{c(G'), c(\overline{G}'), 2\}$ holds. □

Proceeding on the upper bound results, first we present some useful lemmas. Let S be a subset of the vertices of a graph G. In Lemma 3.13 we show that for two vertices u, v in $V(G)$ at distance three from each other, if $u, v \in H(S)$, then their closed neighborhoods $N_G[\{u,v\}]$ as well as $\overline{N_G[\{u,v\}]}$ belong to $H(S)$. Furthermore, we deal with a split graph G that has only one trivial component in G'. In Lemma 3.14 we show that if G' has only one trivial component, then $\overline{G}'$ also has only one trivial component, and *vice versa*.

Lemma 3.13. *Let $G = (C \cup I, E)$ be a split autoconnected graph, $u, v \in I$ and $S \subseteq V(G\overline{G})$. If $d_G(u,v) = 3$ and $u, v \in H(S)$, then $N_G[\{u,v\}] \cup \overline{N_G[\{u,v\}]} \subseteq H(S)$.*

Proof. Let $u, v \in I$ such that $d_G(u,v) = 3$. Since $u\bar{u}\bar{v}v$ and $uxyv$, for $x \in N_G(u)$ and $y \in N_G(v)$, are shortest paths between u and v, and $N_G(u) \cup N_G(v) \subseteq C$ we have that $N_G(u) \cup N_G(v) \cup \{\overline{u},\overline{v}\} \subseteq I[u,v]$. Since $uy, vx \notin E(G)$, for $x \in N_G(u)$ and $y \in N_G(v)$, we have that $\bar{u}\bar{y}, \bar{v}\bar{x} \in E(\overline{G})$. Consequently $\overline{x} \in I[x,\overline{v}]$ and $\overline{y} \in I[y,\overline{u}]$. Therefore $N_G[\{u,v\}] \cup \overline{N_G[\{u,v\}]} \subseteq H(S)$. □

Lemma 3.14. *Let $G = (C \cup I, E)$ be a split autoconnected graph. Then $t(G') = 1$ if and only if $t(\overline{G}') = 1$.*

Proof. Suppose that $t(G') = 1$. Let v be the trivial component of G'. Since $N_G[v] = C$, $N_{\overline{G}}(\overline{v}) = \overline{I}$, hence $\overline{I} \cup \{\overline{v}\}$ is a clique of $\overline{G}$. Since $\overline{C}$ is an independent set, at most one vertex from $\overline{C}$ can belong to a clique in $\overline{G}$. Hence $\overline{I} \cup \{\overline{v}\}$ is a maximum clique. Since $t(G') = 1$, for every $u \in C \setminus \{v\}$, $|N_G(u) \cap I| \geq 1$. Thus, for every $\overline{u} \in \overline{C} \setminus \{\overline{v}\}$, $|N_{\overline{G}}(\overline{u})| \leq |\overline{I}| - 1$. Then $|N_{\overline{G}}(\overline{u})|$ implies that $\overline{I} \cup \{\overline{v}\}$ is the unique maximum clique of $\overline{G}$.

Since C is maximum, for every $y \in I$, we have that $|N_G(y)| < |C|$ and since C is unique, there exists no $y \in I$ such that y is adjacent to $|C| - 1$ vertices. Then, for every $y \in I$, $|N_G(y)| \leq |C| - 2$. This implies that, for every $\overline{y} \in \overline{I}$, $|N_{\overline{G}}(\overline{y}) \cap \overline{C}| \geq 2$.

Since, for every $\overline{y} \in \overline{I}$, $N_{\overline{G}}(\overline{y}) \cap \overline{C} \neq \emptyset$, it follows that $\overline{v}$ is the unique trivial component of $\overline{G}'$, therefore $t(\overline{G}') = 1$.

Reciprocally, suppose that $t(\overline{G}') = 1$. Since $\overline{G}$ is also split, by the same previous arguments, now applied for $\overline{G}'$, we conclude that $t(G') = 1$. □

Let G be a split autoconnected graph. As we have done with the lower bounds, we establish a relation of the upper bounds of $h(G\overline{G})$ with the number of components of G'. To show that we divided the proof concerning about the number of trivial and non-trivial components of G'. The case of at least one of the parameters $nt(G'), t(G'), nt(\overline{G}'), t(\overline{G}')$ or $t(\overline{G}')$ be greater than one is provided by Theorem 3.15, and the remaining cases follow in Theorems 3.18 and 3.19.

Theorem 3.15. *Let* $G = (C \cup I, E)$ *be a split autoconnected graph. If* $\max\{nt(G'),\ t(G'), nt(\overline{G}'), t(\overline{G}')\} \geq 2$, *then* $h(G\overline{G}) \leq \max\{c(G'), c(\overline{G}')\}$.

Proof. To proceed with the proof we consider two cases (1) $nt(G') \geq 2$ or $nt(\overline{G}') \geq 2$, and (2) $t(G') \geq 2$ or $t(\overline{G}') \geq 2$. By definition of complementary prism, we know that $G\overline{G}$ is isomorphic to $\overline{G}G$. Then, without loss of generality, we may consider $c(G') \geq c(\overline{G}')$. In view of that, we show a hull set of $G\overline{G}$ in each case: (1) $nt(G') \geq 2$ and (2) $t(G') \geq 2$.

Case 1. $nt(G') \geq 2$.

Let $\mathfrak{I}_1 = \{i \in [c(G')] : |V(G_i)| = 1\}$ and $\mathfrak{I}_2 = [c(G')] \setminus \mathfrak{I}_1$.

For every $k \in [c(G')]$, we obtain the set S by choosing $u_k \in V(G_k)$ such that $V(G_k) = \{u_k\}$ if $k \in \mathfrak{I}_1$ and $u_k \in V(G_k) \setminus C$ if $k \in \mathfrak{I}_2$. Notice that $|S| = c(G')$. We show that $H(S) = V(G\overline{G})$.

By hypothesis, $|\mathfrak{I}_2| \geq 2$. For all $i, j \in \mathfrak{I}_2$, $i \neq j$, since $d_G(u_i, u_j) = 3$, Lemma 3.13 implies that $N_G[\{u_i, u_j\}] \cup \overline{N_G[\{u_i, u_j\}]} \subseteq H(S)$.

Let $u \in N_G(u_i)$, $u' \in N_G(u_j)$, for some $i, j \in \mathfrak{I}_2$, $A = I \setminus N_G(u)$ and $B = I \setminus N_G(u')$. We have that $\overline{A} \subseteq N_{\overline{G}}(\overline{u})$, then $\overline{A} \subseteq I[\overline{u}, \overline{u}_i]$. Similarly, $\overline{B} \subseteq N_{\overline{G}}(\overline{u}')$, then $\overline{B} \subseteq I[\overline{u}', \overline{u}_j]$. Since $A \cup B = I$, $\overline{I} \subseteq H(S)$.

Only vertices of I that are not yet in $H(S)$ are those who are at distance 2 to all the other vertices from I. However such vertices cannot exists because $nt(G') \geq 2$. Hence $I \subseteq H(S)$, which implies that $C \subseteq H(G)$ and finally, $\overline{C} \subseteq H(S)$. Therefore S is a hull set of $G\overline{G}$, and $h(G\overline{G}) \leq c(G')$.

Case 2. $t(G') \geq 2$.

If $nt(G') \geq 2$ we fall back on Case 1. Thus, consider that $nt(G') = 1$. Let G_1 be the non-trivial component of G' and G_i, for $i \in [c(G')] \setminus \{1\}$, be the trivial components of G'. For some $x \in V(G_1) \cap C$, let $S = V(\overline{G_i}) \cup \{x\}$, for every $i \in [c(G')] \setminus \{1\}$. Notice that $|S| = c(G')$. We show that S is a hull set of $G\overline{G}$.

By hypothesis, $t(G') \geq 2$. Then there exist $i, j \in [c(G')] \setminus \{1\}$ such that G_i and G_j are trivial components in G'. Let v_i (resp. v_j) be the single vertex in $V(G_i)$ (resp. $V(G_j)$). Since $v_i v_j \in E(G)$, $\overline{v}_i \overline{v}_j \notin E(\overline{G})$. Consequently, $\overline{v}_i \overline{y} \overline{v}_j$ is a shortest $(\overline{v}_i, \overline{v}_j)$-path, for every $\overline{y} \in \overline{I}$. Then $\overline{I} \subseteq I[\overline{v}_i, \overline{v}_j]$.

For every $i \in [c(G')] \setminus \{1\}$, $d_G(x, v_i) = 1$, then $v_i \in I[x, \overline{v}_i]$. Furthermore, vertices in $y \in I$ and $w \in C \setminus \{x, v_2, \ldots, v_{c(G')}\}$ belong subsequently to $H(S)$, by the closed intervals $y \in I[x, \overline{y}]$ and $w \in I[y, v_2]$. Hence $V(G) \subseteq H(S)$.

Given that $V(G) \cup \overline{I} \subseteq H(S)$ it is easy to see that $V(\overline{G}) \subseteq H(S)$.

By all cases there exists a hull set of $G\overline{G}$ of order $c(G')$. Since $c(G') \geq c(\overline{G}')$, the result $h(G\overline{G}) \leq \max\{c(G'), c(\overline{G}')\}$ holds. □

By Theorems 3.12 and 3.15 we obtain the equality stated in Corollary 3.16. That result evidences that for complemetary prisms of split graphs the hull number can be unlimited. The class of complementary prisms $K_n\overline{K}_n\overline{K_n\overline{K}_n}$ is an example of $h(K_n\overline{K}_n\overline{K_n\overline{K}_n}) = n$.

Corollary 3.16. *Let* $G = (C \cup I, E)$ *be a split autoconnected graph. If* $\max\{nt(G'),\ t(G'), nt(\overline{G}'), t(\overline{G}')\} \geq 2$, *then* $h(G\overline{G}) = \max\{c(G'), c(\overline{G}')\}$.

Proof. It follows directly from Theorems 3.12 and 3.15. □

Now, it remains the cases in which $nt(G') < 2$, $nt(\overline{G}') < 2$, $t(G') < 2$ and $t(\overline{G}') < 2$. Since we disregard the trivial graphs, we have that $nt(G') = 1$ and $nt(\overline{G}') = 1$. By Lemma 3.14 if $t(G') = 1$, then $t(\overline{G}') = 1$. Otherwise, for $t(G') \neq 1$, we have that $t(\overline{G}') = 0$ or $t(\overline{G}') \geq 2$. If $t(\overline{G}') \geq 2$ the proof follows by Theorem 3.15 Case 2. So, it remains the cases $t(G') = 0$, $t(\overline{G}') = 0$, and $t(G') = 1$, $t(\overline{G}') = 1$, that follows in Theorems 3.18 and 3.19, respectively.

We first show a proposition that will be useful in the proof of Theorem 3.18. The next proposition is a slight modification of Proposition 3.4 now considering a split graph $G = (C \cup I, E)$. For a set of vertices S of $G\overline{G}$, Proposition 3.17 shows that every vertex lying on a path in the component graph G' of G will belong to $H(S)$ depending if some specific vertices in $C \cup \overline{I}$ belong to $H(S)$.

Proposition 3.17. *Let $G = (C \cup I, E)$ be a split autoconnected graph. For $k \geq 2$, let $w_0 v_1 w_1 \dots v_k w_k$ (or $w_0 v_1 w_1 \dots v_k$) be a path in G', such that $v_i \in I$, for every $i \in [k]$, and $w_j \in C$, for every $j \in [k] \cup \{0\}$. Consider that, for every $i \in [k]$, there exists $z_i \in C$ such that $v_i z_i \notin E(G)$, and let $S \subseteq V(G\overline{G})$. If $\{w_0, z_1, \dots z_k, \overline{v}_1, \dots, \overline{v}_k\} \subseteq H(S)$, then $w_k \in H(S)$ (resp. $v_k \in H(S)$).*

Proof. The proof is by induction on k. Let $k = 1$. Since $w_0, \overline{v}_1 \in H(S)$, and $v_1\overline{v}_1 \in E(G\overline{G})$, we obtain that $v_1 \in I[w_0, \overline{v}_1]$. Since $z_i \in H(S)$, and $v_1 z_1 \notin E(G)$, then $w_1 \in I[v_1, z_1]$.

Let $k > 1$. By induction hypothesis, $w_{k-1} \in H(S)$. Then, since $\overline{v}_k \in H(S)$, we have that $v_k \in I[w_{k-1}, \overline{v}_k]$. Since $z_k \in H(S)$, and $v_k z_k \notin E(G)$, it follows that $w_k \in I[v_k, z_k]$. ☐

Theorem 3.18. *Let $G = (C \cup I, E)$ be a split autoconnected graph such that $nt(G') = 1$, $nt(\overline{G}') = 1$, $t(G') = 0$ and $t(\overline{G}') = 0$. Then $h(G\overline{G}) \leq 3$.*

Proof. To prove that $h(G\overline{G}) \leq 3$ we must construct a hull set of $G\overline{G}$ of order three. We first show the existence of three vertices that are the basis of our construction. Consider $d = \max\{deg_G(v) : v \in I\}$, and $y \in I$ such that $deg_G(y) = d$.

Claim 1. There exist $x \in C \setminus N_G(y)$, and $y' \in N_G(x) \cap I$.

Proof of Claim 1. Suppose, by contradiction, that there exists no $x \in C \setminus N_G(y)$. Then, for every $x \in C$, $x \in N_G(y)$. This implies that $C \cup \{y\}$ is a clique of G, a contradiction, since we consider a partition of $V(G)$ such that C is a maximum clique. Now, since $t(G') = 0$ and $xy \notin E(G)$, x is not an isolated vertex in G', then there exists $y' \in N_G(x) \cap I$. ☐

Let $x \in C \setminus N_G(y)$, and $y' \in N_G(x) \cap I$. To avoid repetitive notation when considering the operations of union, intersection and symmetric difference on the sets $N_G(y)$ and $N_G(y')$, we write $U = N_G(y) \cup N_G(y')$, $A = N_G(y) \cap N_G(y')$, and $D = N_G(y) \triangle N_G(y')$, respectively.

We consider a partition $R \cup L$ of $I \setminus \{y, y'\}$ such that $R = \{v \in I \setminus \{y, y'\} : \forall u \in D,\ uv \in E(G)\}$, and $L = I \setminus (R \cup \{y, y'\})$. Furthermore, we let $R_1 = \{v \in R : N_G(v) \cap (C \setminus U) \neq \emptyset\}$, and $R_2 = R \setminus R_1$. We show hull sets of $G\overline{G}$ considering R_1 be empty or not.

Case 1. $R_1 = \emptyset$.

Let $S = \{x, y, \overline{y}'\}$. See Figure 4 for an example. First, we have that $N_G(y) \subseteq I[x, y]$, $y' \in I[x, \overline{y}']$ and $\overline{y} \in I[y, \overline{y}']$.

Since $deg_G(y)$ is maximum in I, there exists $z \in N_G(y)$ such that $y'z \notin E(G)$. Thus $N_G(y') \subseteq I[y', z]$, which implies that $U \subseteq H(S)$. Since $D \subseteq U \subseteq H(S)$, and, for every $w \in N_G(y') \setminus A$, $\overline{w}\overline{y} \in E(\overline{G})$ (resp. for every $w' \in N_G(y) \setminus A$ $\overline{w}'\overline{y}' \in E(\overline{G})$), we obtain that $\overline{D} \subseteq I[D \cup \{\overline{y}, \overline{y}'\}]$.

Let $v \in L$. By definition of L, there exists $z \in D$ such that $vz \notin E(G)$, which implies that $\overline{v}\overline{z} \in E(\overline{G})$. Since $\overline{z}$ is either adjacent to $\overline{y}$ or $\overline{y}'$, and, for every $\overline{v} \in \overline{L}$, $\overline{v}\overline{y}, \overline{v}\overline{y}' \in E(\overline{G})$, it follows that $\overline{v} \in I[\{\overline{y}, \overline{y}', \overline{z}\}]$. Hence $\overline{L} \subseteq H(S)$. By definition of L we have also that $\overline{L} \cup \{\overline{y}, \overline{y}'\} = \overline{I} \setminus \overline{R}_2 \subseteq H(S)$.

Now, we show that $L \cup (N_G(L) \setminus U) \subseteq H(S)$. For that, we are interested in the paths of minimum length between $L \cup (N_G(L) \setminus U)$ and U.

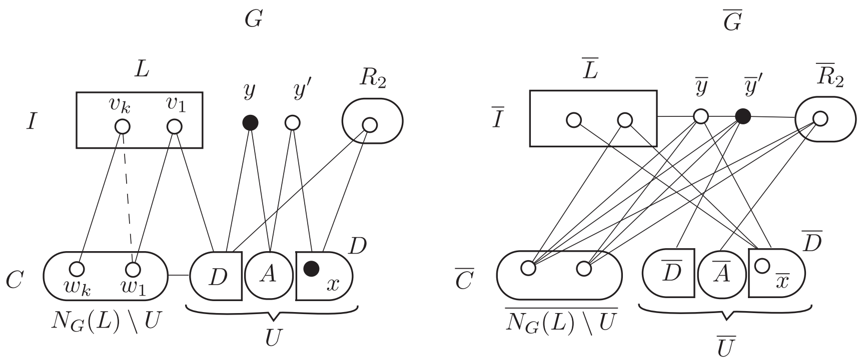

FIGURE 4. Example of hull set for Case 1 of Theorem 3.18.

Since G' is connected, for $k \geq 2$, there exists a shortest path $w_0 v_1 w_1 \dots v_k w_k$ (or $w_0 v_1 w_1 \dots v_k$) in G' between $w_0 \in U$ and $w_k \in N_G(L) \setminus U$ (resp. $v_k \in L$), where $v_i \in I$, $w_i \in C$, for every $i \in [k]$.

Recall that, up to this point $(R_2 \cup \overline{R}_2) \cap H(S) = \emptyset$. By definition of R_2, $N_G(R_2) \subseteq U$. Then the shortest path $w_0 v_1 w_1 \dots v_k w_k$ (or $w_0 v_1 w_1 \dots v_k$) in G' does not contain vertices from R_2. From this observation we can proceed to show that $v_1, w_1, \dots, v_k, w_k$ will be successively included to $H(S)$.

By the maximality of $deg_G(y)$ we have that $deg_G(v_i) < deg_G(y) + 1 = |N_G(y) \cup \{x\}|$, for every $i \in [k]$. Hence, there exists $z_i \in N_G(y) \cup \{x\}$ such that $v_i z_i \notin E(G)$ for every $i \in [k]$. Since $w_0, z_1, \dots, z_k \in U \subseteq H(S)$, and $\overline{v}_i \in \overline{I} \setminus \overline{R}_2 \subseteq H(S)$, for every $i \in [k]$, it follows from Proposition 3.17 that $w_k \in H(S)$ (resp. $v_k \in H(S)$). Consequently $L \cup (N_G(L) \setminus U) \subseteq H(S)$.

It remains to show that $R_2 \cup \overline{R}_2 \cup \overline{A} \cup \overline{N_G(L) \setminus U} \subseteq H(S)$.

Since $\overline{G}'$ is connected, for $k \geq 2$, there exists a shortest path $\overline{w}_0 \overline{v}_1 \overline{w}_1 \dots \overline{v}_k \overline{w}_k$ (or $\overline{w}_0 \overline{v}_1 \overline{w}_1 \dots \overline{v}_k$) in $\overline{G}'$ between $\overline{w}_0 \in \overline{I} \setminus \overline{R}_2$ and $\overline{w}_k \in \overline{R}_2$ (resp. $\overline{v}_k \in \overline{A} \cup \overline{N_G(L) \setminus U}$), where $\overline{v}_i \in \overline{C}$, and $\overline{w}_i \in \overline{I}$, for every $i \in [k]$.

We show that, for every $i \in [k]$, there exists $\overline{z} \in \overline{I} \setminus \overline{R}_2$ such that $\overline{v}_i \overline{z} \notin E(\overline{G})$. It is clear that $\overline{z} = \overline{y}$ for every $\overline{v} \in \overline{A}$, since $\overline{v}\overline{y} \notin E(\overline{G})$. Now, let $\overline{v} \in \overline{N_G(L) \setminus U}$. We know that $N_G(R_2 \cup \{y, y'\}) \cap (N_G(L) \setminus U) = \emptyset$. Since G' is connected, there exists $z \in L$ such that $vz \in E(G)$, which implies that $\overline{v}\overline{z} \notin E(\overline{G})$, with $\overline{z} \in \overline{L} \subseteq \overline{I} \setminus \overline{R}_2$.

Since $\overline{w}_0, \overline{z} \in \overline{I} \setminus \overline{R}_2 \subseteq H(S)$, and $v_i \in A \cup N_G(L) \subseteq H(S)$, for every $i \in [k]$, Proposition 3.17 (dual) implies that $\overline{w}_k \in H(S)$ (resp. $\overline{v}_k \in H(S)$). Hence, $\overline{R}_2 \cup \overline{A} \cup \overline{N_G(L) \setminus U} \subseteq H(S)$. Finally, since $D \cup \overline{R}_2 \subseteq H(S)$, we obtain that $R_2 \subseteq I[D \cup \overline{R}_2]$, which completes the proof of Case 1.

Case 2. $R_1 \neq \emptyset$.

Let $y'' \in R_1$ such that $N_G(y'') \setminus U$ is maximum, and $S = \{x, y, \overline{y}''\}$. First, we have that $N_G(y) \subseteq I[x, y]$, $y'' \in I[x, \overline{y}'']$, and $\overline{y} \in I[y, \overline{y}'']$. Since $deg_G(y)$ is maximum, y'' is not adjacent to every vertex in $N_G(y) \cup \{x\}$, then $N_G(y'') \subseteq I[N_G(y) \cup \{x, y''\}]$.

Let $x'' \in N_G(y'') \setminus U$. Since $x'' \in H(S)$, and $\overline{x}''\overline{y} \in E(\overline{G})$, we obtain that $\overline{x}'' \in I[x'', \overline{y}]$. Hence $\overline{N_G(y'') \setminus U} \subseteq H(S)$. Since $\overline{x}''\overline{y}' \in E(\overline{G})$, and $\overline{x}''\overline{y}'' \notin E(\overline{G})$, it follows that $\overline{y}' \in I[\overline{x}'', \overline{y}'']$. Consequently $y' \in I[x, \overline{y}']$. This implies that $\overline{D} \in I[D \cup \{\overline{y}, \overline{y}'\}]$. Since x'' is not adjacent to any vertex in R_2, we have that $\overline{x}''$ is adjacent to every vertex in $\overline{R}_2$. Thus, $\overline{R}_2 \subseteq I[\overline{x}'', \overline{y}'']$. Then, $R_2 \subseteq I[D \cup \overline{R}_2]$.

Let $v \in L$. By definition of L, there exists $z \in D$ such that $vz \notin E(G)$, which implies that $\overline{v}\overline{z} \in E(\overline{G})$. Since $\overline{z}$ is either adjacent to $\overline{y}$ or $\overline{y}'$, and, for every $\overline{v} \in \overline{L}$, $\overline{v}\overline{y}, \overline{v}\overline{y}' \in E(\overline{G})$, it follows that $\overline{v} \in I[\{\overline{y}, \overline{y}', \overline{z}\}]$. Hence $\overline{L} \subseteq H(S)$.

We consider $X \cup Y$ a partition of $R_1 \setminus \{y''\}$ such that $X = \{v \in R_1 \setminus \{y''\} : \forall u \in (A \setminus N_G(y'')) \cup (N_G(y'') \setminus U), uv \in E(G)\}$, and $Y = R_1 \setminus (X \cup \{y''\})$.

Let $a \in A \setminus N_G(y'')$. Since $\overline{a}\overline{y}'' \in E(\overline{G})$, we obtain that $\overline{a} \in I[a, \overline{y}'']$. Thus, $\overline{A \setminus N_G(y'')} \subseteq H(S)$. Let $v \in Y$. By definition of Y, and since we selected $y'' \in R_1$ such that $N_G(y'') \setminus U$ is maximum, there exists $u \in (N_G(y'') \setminus U)$

such that $uv \notin E(G)$, then $\bar{u}\bar{v} \in E(\overline{G})$. Since $\bar{u}\bar{y}'' \notin E(\overline{G})$, and $\bar{u} \in \overline{N_G(y'') \setminus U} \subseteq H(S)$, we have that $\overline{v} \in I[\overline{u}, \overline{y}'']$. It follows that $\overline{Y} \subseteq H(S)$, and $Y \subseteq I[D \cup \overline{Y}]$. Hence, $N_G(Y) \subseteq H(S)$.

Now, we show that $L \cup (N_G(L) \setminus (N_G(Y \cup y'') \cup U)) \subseteq H(S)$.

Since G' is connected, for $k \geq 2$, there exists a shortest path $w_0v_1w_1 \dots v_kw_k$ (or $w_0v_1w_1 \dots v_k$) in G' between $w_0 \in N_G(Y \cup y'') \cup U$ and $w_k \in N_G(L) \setminus (N_G(Y \cup y'') \cup U)$ (resp. $v_k \in L$), where $v_i \in I$, $w_i \in C$, for every $i \in [k]$.

By definition of X, no vertex in X is adjacent to some vertex in $N_G(L)\setminus(N_G(Y \cup y'') \cup U)$, then $w_0v_1w_1 \dots v_kw_k$ (resp. $w_0v_1w_1 \dots v_k$) does not contain vertices from X.

By the maximality of $deg_G(y)$ we have that $deg_G(v_i) < deg_G(y) + 1 = |N_G(y) \cup \{x\}|$, for every $i \in [k]$. Hence, there exists $z_i \in N_G(y) \cup \{x\}$ such that $v_iz_i \notin E(G)$, for every $i \in [k]$. Since $w_0, z_1, \dots, z_k \in N_G(Y \cup y'') \cup U \subseteq H(S)$, and $\overline{v}_i \in \overline{I} \setminus \overline{X} \subseteq H(S)$, for every $i \in [k]$, it follows from Proposition 3.17 that $w_k \in H(S)$ (resp. $v_k \in H(S)$). Consequently $L \cup (N_G(L) \setminus (N_G(Y \cup y'') \cup U)) \subseteq H(S)$.

Similarly to Case 1, since $\overline{G}'$ is connected, Proposition 3.17 (dual) can be used to show that $\overline{X} \cup \overline{A \cap N_G(y'')} \cup \overline{N_G(L) \setminus (N_G(Y \cup y'') \cup U)} \subseteq H(S)$. Since $\overline{X} \subseteq H(S)$, we obtain that $X \subseteq I[\overline{X} \cup U]$, therefore $H(S) = V(G\overline{G})$.

By all cases we have that there exists a hull set of $G\overline{G}$ of order three, which completes the proof. □

Figure 4 shows an example of complementary prism $G\overline{G}$ illustrating Case 1 of Theorem 3.18. The black vertices represent a hull set of $G\overline{G}$.

Theorem 3.19. *Let $G = (C \cup I, E)$ be a split autoconnected graph such that $nt(G') = 1$, $nt(\overline{G}') = 1$, $t(G') = 1$ and $t(\overline{G}') = 1$. Then $h(G\overline{G}) \leq 3$.*

Proof. Let v be the trivial component of G'. Let G_1 be the graph induced by $V(G) \setminus \{v\}$.

Since G_1 is a subgraph of G that satisfies $nt(G_1') = 1$, $nt(\overline{G}_1') = 1$, $t(G_1') = 0$ and $t(\overline{G}_1') = 0$, we proceed in a similar way as in the proof of Theorem 3.18. Consider $d = \max\{deg_G(w) : w \in I\}$, and $y \in I$ such that $deg_G(y) = d$. Let $x \in C \setminus (N_G(y) \cup \{v\})$, and $y' \in N_G(x) \cap I$. Claim 1 implies the existence of x, y, y' in $V(G_1) \subseteq V(G)$.

Let $U = N_G(y) \cup N_G(y')$, $A = N_G(y) \cap N_G(y')$, and $D = N_G(y) \triangle N_G(y')$. Consider $R = \{v \in I \setminus \{y, y'\} : \forall u \in D,\ uv \in E(G)\}$, and $R_1 = \{v \in R : N_G(v) \cap (C \setminus U) \neq \emptyset\}$. We show hull sets of $G\overline{G}$ considering R_1 be empty or not.

Case 1. $R_1 = \emptyset$.

Let $S = \{y, y', \overline{v}\}$. We have that $A = N_G(y) \cap N_G(y') \subseteq I[y, y']$. Since $nt(G') = 1$, we have that $d_G(y, y') < 3$, hence $A \neq \emptyset$. Since v is adjacent to every vertex in A, we obtain that $v \in I[A \cup \{\overline{v}\}]$. Consequently $U = N_G(y) \cup N_G(y') \subseteq I[\{v, y, y'\}]$. Since $vy, vy' \notin E(G)$, $\overline{v}\overline{y}, \overline{v}\overline{y}' \in E(\overline{G})$, then $\overline{y}, \overline{y}' \in I[\{y, y', \overline{v}\}]$.

At this point, we have that $x, y, \overline{y}' \in H(S)$. Since G_1 is a subgraph of G with $nt(G_1') = 1$, $nt(\overline{G}_1') = 1$, $t(G_1') = 0$ and $t(\overline{G}_1') = 0$, Theorem 3.18 Case 1 implies that $V(G_1) \cup V(\overline{G}_1) \subseteq H_{G_1\overline{G}_1}(\{x, y, \overline{y}'\})$. Since $G_1\overline{G}_1$ is an isometric subgraph of $G\overline{G}$, Lemma 3.2 implies that $V(G_1) \cup V(\overline{G}_1) \subseteq H(\{x, y, \overline{y}'\})$, consequently $V(G\overline{G}) = H(S)$.

Case 2. $R_1 \neq \emptyset$.

Let $y'' \in R_1$ such that $N_G(y'') \setminus U$ is maximum, and $S = \{y, y'', \overline{v}\}$. We have that $N_G(y) \cap N_G(y'') \subseteq I[y, y'']$. Since $nt(G') = 1$, we have that $d_G(y, y'') < 3$, hence $N_G(y) \cap N_G(y'') \neq \emptyset$. Let $z \in N_G(y) \cap N_G(y'')$. Since $vz \in E(G)$, we have that $v \in I[z, \overline{v}]$. It follows that $N_G(y) \cup N_G(y'') \subseteq I[\{v, y, y''\}]$. Since $vy, vy'' \notin E(G)$, $\overline{v}\overline{y}, \overline{v}\overline{y}'' \in E(\overline{G})$, then $\overline{y}, \overline{y}'' \in I[\{y, y'', \overline{v}\}]$.

So far, we have that $x, y, \overline{y}'' \in H(S)$. Since G_1 is a subgraph of G with $nt(G_1') = 1$, $nt(\overline{G}_1') = 1$, $t(G_1') = 0$ and $t(\overline{G}_1') = 0$, Theorem 3.18 Case 2 implies that $V(G_1) \cup V(\overline{G}_1) \subseteq H_{G_1\overline{G}_1}(\{x, y, \overline{y}''\})$. Since $G_1\overline{G}_1$ is an isometric subgraph of $G\overline{G}$, Lemma 3.2 implies that $V(G_1) \cup V(\overline{G}_1) \subseteq H(\{x, y, \overline{y}''\})$, consequently $V(G\overline{G}) = H(S)$.

By all cases we have that S is a hull set of $G\overline{G}$, and the result $h(G\overline{G}) \leq 3$ holds. □

Still considering a split autoconnected graph G, we show in Corollary 3.20 an equality for the case $nt(G') = 1$, $nt(\overline{G}') = 1$, $t(G') = 1$, and $t(\overline{G}') = 1$.

Corollary 3.20. *Let $G = (C \cup I, E)$ be a split autoconnected graph. If $nt(G') = nt(\overline{G}') = t(G') = t(\overline{G}') = 1$, then $h(G\overline{G}) = 3$.*

Proof. Let v be the trivial component of G'. The upper bound follows by Theorem 3.19. So, remains to show the lower bound $h(G\overline{G}) \geq 3$.

Since v is a simplicial vertex in G, and $\overline{v}$ is a simplicial vertex in $\overline{G}$, Lemma 3.3 implies that $S \cap \{v, \overline{v}\} \neq \emptyset$. Let $x \in \{v, \overline{v}\}$, and suppose that $S = \{u, x\}$ is a hull set of $G\overline{G}$. Since $d_{G\overline{G}}(u, x) < 2$, for every $u \in V(G\overline{G}) \setminus \{v, \overline{v}\}$, we have that $H(S) = N_{G\overline{G}}[u] \cap N_{G\overline{G}}[x]$. This implies that $H(S) \neq V(G\overline{G})$, a contradiction. Therefore, $h(G\overline{G}) \geq 3$. □

We close our contributions with the result expressed in Corollary 3.21. We consider split autoconnected graphs G such that $diam(G) = diam(\overline{G}) = 2$. If $nt(G') \geq 2$ or $nt(\overline{G}') \geq 2$, then $diam(G) = 3$ or $diam(\overline{G}) = 3$, which does not belong to the current case. If $t(G') \geq 2$ or $t(\overline{G}') \geq 2$ we fall back on Theorem 3.15, hence $h(G\overline{G}) = max\{c(G'), c(\overline{G}')\}$. Thus, we consider $nt(G') = 1$, $nt(\overline{G}') = 1$, $t(G') \leq 1$, and $t(\overline{G}') \leq 1$.

Corollary 3.21. *Let $G = (C \cup I, E)$ be a split autoconnected graph such that $diam(G) = diam(\overline{G}) = 2$. If $max\{nt(G'), t(G'), nt(\overline{G}'), t(G')\} < 2$, then $h(G\overline{G}) = 3$.*

Proof. It follows from Theorems 3.6 and 3.15. □

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