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AN OPTIMIZATION OF SOLID TRANSPORTATION PROBLEM WITH STOCHASTIC DEMAND BY LAGRANGIAN FUNCTION AND KKT CONDITIONS

Anjana Kuiri*, Barun Das and Sanat Kumar Mahato

Abstract. In this paper, a stochastic solid transportation problem (SSTP) is constructed where the demand of the item at the destinations are randomly distributed. Such SSTP is formulated with profit maximization form containing selling revenue, transportation cost and holding/shortage cost of the item. The proposed SSTP is framed as a nonlinear transportation problem which is optimized through Karush–Kuhn–Tucker (KKT) conditions of the Lagrangian function. The primary model is bifurcated into three different models for continuous and discrete demand patterns. The concavity of the objective functions is also presented here very carefully. Finally, a numerical example is illustrated to stabilize the models.

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1. Introduction

Transportation problem (TP) is one type of Linear Programming Problem (LPP) that involves selection of most economical shipping routes and quantities for transfer of any homogeneous commodity from a number of origins/sources to a number of destinations. The solid transportation problem (STP) was first presented by Haley [10], wherein addition with sources and destinations another parameter, the mode of transportation also is considered for the transportation of heterogeneous commodities. Keeping the requirements in mind, here three kinds of constraints are taken into account, that is, source constraint in terms of availability, destination constraint in terms of demand and conveyance's capacity constraint. The STP is of much use in public distribution systems. The STP degenerates into the classical transportation problem if the number of conveyance is only one. In recent years, there have been numerous papers in this area. Most of the papers only minimize the total transportation cost. Ojha *et al.* [29] considered a STP for an item with fixed charge, vehicle cost and price discounted varying charge. Interested readers may consult Bit *et al.* [3], Jimenez and Verdegay [16] and so on. But in reality maximization of profit become a more essential objective to the decision-makers.

In many practical situations, the vagueness appeared in the parameters of TP. This is due to the lack of information about the system, insufficiency in the transportation policy, different types of unexpected factors such as lack of evidence, fluctuation in the market, artificial crisis in the market, unstable political situation,

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etc. In such situation the stochastic nature appears to the TP and STP. The stochastic transportation problem was discussed by several researchers such as Williams [37], Holmberg [11–13], Wilson [38] and Cooper [5]. This is a variant of the ordinary (linear) transportation problem with random demand. A STP with one or more random parameters is termed as a stochastic solid transportation problem (SSTP). Yang and Feng [39] solved a bicriteria STP in stochastic environment. Quddoos *et al.* [30] presented a multi-choice stochastic transportation problem involving a general form of distribution. Halder *et al.* [9] solved a solid transportation problem through fuzzy ranking. Ojha *et al.* [28] designed a multi-objective SSTP for breakable items using analytic hierarchy process. Habiba and Quddoos [8] discussed a multi-objective stochastic transportation problem with interval cost coefficients. Yang *et al.* [40] introduced the reduction methods of type-2 uncertain variables to STP. Jana *et al.* [14] described the profit maximization STP with Gaussian type-2 fuzzy environments. Anukokila *et al.* [2] presented a goal programming approach for solving multi-objective fractional transportation problem with fuzzy parameters. Ojha *et al.* [27] framed multi-objective STP as a geometric programming problem. Das *et al.* [7] proposed a new approach for solving fully fuzzy linear fractional programming problems using the multi-objective linear programming method/technique. Jahanshahloo *et al.* [15] presented a solution procedure for multi-objective linear fractional programming problem based on goal programming and data envelopment analysis. Bhurjee and Panda [4], Roy and Midya [32], Roy *et al.* [35] developed a multi-objective transportation problem with variety of conceptions. Many researchers [6, 18, 19] described STP in different environments. In 2010, Nagarjan and Jeyaraman [26] solved a solid transportation problem by using fuzzy approach under stochastic environment. Due to the random nature of the demand, two situations can occur-either demand is less than the total commodity received or demand is more than that. Such respective situations lead to holding and shortage of the commodity at the destinations.

Because of flexibility and uni-modality, the logistic distribution has a wide application in any real life decision making problem. It can be used to model the data exhibiting having some skewness property. The probability density function of a logistic distribution with location and scale parameters α and β for a random variable t is described by

$$\phi(t) = \frac{1}{\beta}\frac{e^{-(\frac{t-\alpha}{\beta})}}{\left[1+e^{-(\frac{t-\alpha}{\beta})}\right]^2},\ 0 \le t < \infty,\ 0 \le \alpha < \infty, \text{ and } \beta > 0.$$

The logistic distribution is more popular for the economical modeling of those items whose demand function is concave in shape and the item is of high-risk management. The similarity of Poisson distribution finds in the exponential distribution. It occurs in the situation of time until the failure of a part and separation between random events happen. The probability density function of an exponential distribution is described as

$$\phi(t) = \lambda e^{-\lambda t},\ t > 0,\ \lambda > 0.$$

These types of functions present by reverse-J shaped. It is often termed as compound Poisson distribution. In real-life situations, it is observed that the demand of an item is not known precisely. To stimulate demand, we have considered uncertain demand with stochastic nature. Such type of demand is found for those items whose demand slowly decreasing. The TP with stochastic demand is a special version of the stochastic linear programming problem. It has many economic applications. The stochastic transportation problem has been discussed in many papers [21, 23, 25, 34] in several ways and solved by different methods. Mahapatra *et al.* [20,22] and Roy *et al.* [31,33] presented a procedure to solve the multi-choice transportation problem where they have converted the multi-choice transportation problem into a standard mathematical programming through the selection of binary variables, bounds for binary codes, and restriction of binary codes using auxiliary constraints. The optimality of the constrained optimization problem has been found out in descent directions. The KKT conditions based on Lagrangian function are the gradient search direction condition which are necessary for the optimization of the constrained problem. These conditions also may transform into sufficient conditions if the objective function satisfies the convexity/concavity condition. Kim *et al.* [17] make a note on second-order

TABLE 1. Comparisons of the problem with existing transportation problem (TP).

References	Type of TP	Demand of the item	Formulation type	Solution procedure
Williams [37]	2D	Stochastic	Cost minimization	Lingo Software
Das *et al.* [6]	3D	Constant	do	Fuzzy programming
Bit *et al.* [3]	3D	do	do	do
Nagarjan and Jeyaraman [26]	3D	do	do	Chance programming
Ojha *et al.* [29]	3D	do	do	Genetic algorithm
Mahapatra [22]	2D	Stochastic	do	Lingo Software
Mahapatra [20]	2D	do	do	do
Holemberg and Tuy [13]	2D	do	do	Branch and bound procedure
Jana *et al.* [14]	3D	Constant	Profit maximization	Genetic algorithm
Wilson [38]	2D	Stochastic	Cost minimization	Primal-dual algorithm
This paper	3D	Stochastic	Profit maximization in Lagrangian form	KKT conditions then by Lingo Software

Karush–Kuhn–Tucker necessary optimality conditions for smooth vector optimization problems. Recently Maity *et al.* [24] and Samanta *et al.* [36] presented the transportation problem in a spectacular direction.

The major contributions of the paper are as follows.

- Here, a STP is considered with random demand. Due to the randomness of the demand, there may arise two mutually exclusive events (situations). Demand is either less than the total receiving amount (this leads to holding situation) or greater than that (leads to shortage situation).
- The STP is constructed here on the objective of profit maximization in the decision maker's point of view, containing the terms of selling revenue, transportation cost, expected holding cost or expected shortage cost.
- The randomness of the demand also felt an effect on the STP constraints.
- To fulfil the random criteria of the demand, here logistic and exponential distributions are taken for a continuous case and a discrete probability distribution is also taken into consideration. The logistic demand met the scenario when demand varies with time of use, season and socio-economic pattern of the consumers. And the exponential distribution pattern seems to commodities whose demand starts from the non-negative quantity and tends to zero at the end of the period.
- Here, STP is optimized by the formulation of Lagrangian function with Lagrange multiplier and then by Kuhn Tuckher's optimality conditions.

The research gap of this article with the existing literatures on the TP is shown in the following Table 1.

2. Non-linear optimization method (KKT approach)

Consider the constrained maximization problem as

$$\begin{aligned} \text{Max} \quad & f(x_1, x_2, \ldots, x_n) \\ \text{subject to} \quad & g_i(x_1, x_2, \ldots, x_n) \geq 0; \quad i = 1, 2, \ldots, m \\ & h_j(x_1, x_2, \ldots, x_n) = 0; \quad j = 1, 2, \ldots, n. \end{aligned}$$

Then the Lagrangian function of the above problem is

$$L(x, \mu, \lambda) = f(x) + \sum_{i=1}^{m} \mu_i g_i(x) + \sum_{j=1}^{n} \lambda_j h_j(x).$$

Assume that f, g_i, h_j are differentiable. If the function $f(x^*)$ attains local maximum at point x^*, then the Lagrange multipliers μ_i, λ_j satisfied the following conditions:

(i) $\frac{\partial L}{\partial x^*} = \frac{\partial f}{\partial x^*} + \sum_{i=1}^{m} \mu_i \frac{\partial g_i}{\partial x^*} + \sum_{j=1}^{n} \lambda_j \frac{\partial h_j}{\partial x^*} = 0$ (stationarity condition).
(ii) $\mu_i g_i(x_1^*, x_2^*, \ldots, x_n^*) = 0,\ i = 1, 2, \ldots, m$ (complementary slackness condition).
(iii) $\lambda_j h_j(x_1^*, x_2^*, \ldots, x_n^*) = 0,\ j = 1, 2, \ldots, n$ (complementary slackness condition).
(iv) $g_i(x_1^*, x_2^*, \ldots, x_n^*) \geq 0,\ i = 1, 2, \ldots, m$ (primal feasibility condition).
(v) $h_j(x_1^*, x_2^*, \ldots, x_n^*) \geq 0,\ j = 1, 2, \ldots, n$ (primal feasibility condition).
(vi) μ_i, $\lambda_j \geq 0$ (dual feasibility condition).

In other words, the conditions (i)–(vi) are necessary conditions for a local maximum of the problem. Also, conditions (i)–(vi) are called the Karush–Kuhn–Tucker (KKT) conditions.

In the particular case of $m = 0$, *i.e.*, when there are no inequality constraints, the KKT conditions turn into the Lagrangian conditions, and the KKT multipliers are called Lagrange multipliers.

Karush–Kuhn–Tucker conditions are first-order necessary conditions for an optimal solution of nonlinear programming, provided that some regularity conditions are satisfied.

3. Model description and formulation

3.1. Assumption

In this SSTP the following assumptions and notations are made:

(a) Availability (a_i) of the ith origin is finite and known.
(b) The demand d_j of the item at the jth destination is random in nature and followed probabilities density function $\phi_j(d_j)$.
(c) The actual received quantity y_j by the jth destination may be lower or higher than the expected demand $E(d_j)$. Depending on that situation their may be shortage or holding of the item at the destination j. In this regard, shortage and holding cost are taken here.
(d) Capacity (e_k) of the kth conveyance is finite and known.
(e) The cost (c_{ijk}) for transporting one unit item from sources i to destination j by conveyance k is finite and known.
(f) The amount of transportation (x_{ijk}) from sources i to destination j with the aid of conveyance k is finite and decision variable.
(g) The unit holding cost and shortage cost of the item at the jth destination are h_j and p_j respectively, which are finite and known.
(h) The unit selling price of the item at the jth destination (S_j) is finite and known.

3.2. Formulation of the model

We consider a STP with m supply node, n demand node and k capacity conveyances. If x_{ijk} is the shifted amount from ith supply to jth destination through the kth conveyance and y_j be the total amount that is shipped to the demand point j from all supply node then, $y_j = \sum_{i=1}^{m} \sum_{k=1}^{l} x_{ijk}$. Then either $y_j \geq E(d_j)$ or $y_j < E(d_j)$.

There may be two cases:

Case I. Total transported amount is greater than or equal to the random demand.

In this case, there will be a surplus and hence holding cost arises. So the expected holding charge is paid at the destination j, which is equal to $h_j \int_0^{y_j} (y_j - t)\phi_j(t)\mathrm{d}t$ or $h_j \sum_{t<y_j} (y_j - t)P(d_j = t)$, $h_j > 0$, where $\phi_j(t)$ is the probability density function (p.d.f) of the random variable d_j.

Case II. Total transported amount is less than the random demand.

In this case, there will be a shortage, so the destination manager need to pay the expected shortage cost $p_j \int_{y_j}^{\infty}(t-y_j)\phi_j(t)\mathrm{d}t$ or $p_j \sum_{t>y_j}(t-y_j)P(d_j=t)$, $p_j>0$.

Since the above two events are mutually exclusive *i.e.*, cannot occur simultaneously, so probabilitically, the expected cost consists the sum of the individual.

For the whole system, the total holding/shortage cost

$$\sum_{j=1}^{n}\left[h_j\int_{0}^{y_j}(y_j-t)\phi_j(t)\mathrm{d}t+p_j\int_{y_j}^{\infty}(t-y_j)\phi_j(t)\mathrm{d}t\right]$$

or

$$\sum_{j=1}^{n}\left[h_j\sum_{t<y_j}(y_j-t)P(d_j=t)+p_j\sum_{t>y_j}(t-y_j)P(d_j=t)\right]$$

is a convex function (*cf.* [13]).

So the expected profit function (Z) of the decision maker is given by:

$$\begin{aligned}\langle\text{Expected Profit}\rangle &= \langle\text{Selling revenue}\rangle - \langle\text{Transportation cost}\rangle\\ &\quad - \langle\text{Expected holding cost}\rangle - \langle\text{Expected shortage cost}\rangle.\end{aligned}$$

The aim of this problem is to determine the optimum expected profit of the transportation system for delivering homogenous/heterogeneous commodities from various sources to different destinations in different conveyances. Mathematically,

$$\begin{aligned}\text{Max } Z = \sum_{j=1}^{n}S_jy_j-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l}c_{ijk}x_{ijk}-\sum_{j=1}^{n}\left[h_j\int_{0}^{y_j}(y_j-t)\phi_j(t)\mathrm{d}t\right]\\ -\sum_{j=1}^{n}\left[p_j\int_{y_j}^{\infty}(t-y_j)\phi_j(t)\mathrm{d}t\right]\qquad\text{(for continuous case)}\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\text{Max } Z = \sum_{j=1}^{n}S_jy_j-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l}c_{ijk}x_{ijk}-\sum_{j=1}^{n}\left[h_j\sum_{t<y_j}(y_j-t)P(d_j=t)\right]\\ -\sum_{j=1}^{n}\left[p_j\sum_{t>y_j}(t-y_j)P(d_j=t)\right]\qquad\text{(for discrete case).}\end{aligned}\tag{3.2}$$

After simplification and neglecting the constant term [1], equations (3.1) and (3.2) are equivalent to

$$\text{Max } Z = \sum_{j=1}^{n}S_jy_j-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l}c_{ijk}x_{ijk}-\sum_{j=1}^{n}(h_j+p_j)\int_{0}^{y_j}(y_j-t)\phi_j(t)\mathrm{d}t+\sum_{j=1}^{n}p_jy_j\tag{3.3a}$$

$$\text{Max } Z = \sum_{j=1}^{n}S_jy_j-\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l}c_{ijk}x_{ijk}-\sum_{j=1}^{n}(h_j+p_j)\sum_{r<y_j}(y_j-r)\mu_j^r+\sum_{j=1}^{n}p_jy_j.\tag{3.3b}$$

And the constraints of the proposed stochastic solid transportation problem (SSTP) are

$$\text{Capacity constraints of the origins: }\sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk}\le a_i\ for\ i=1,2,\ldots,m\tag{3.4}$$

$$\text{Demand constraints at the destination: } \sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} = y_j \ \textit{for}\ j = 1,2,\ldots,n \tag{3.5}$$

$$\text{Capacity constraints of the conveyances: } \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k \ \textit{for}\ k = 1,2,\ldots,l \tag{3.6}$$

$$\text{And feasibility constraints amount of transportation: } x_{ijk} \ge 0 \ \forall\, i,j,k. \tag{3.7}$$

Any set of allocations which satisfied the equations (3.4)–(3.7) is called a feasible solution of the solid transportation problem (without boundedness of the demand). A feasible solution to that problem is said to be basic, *i.e.*, BFS if the set of allocations are independent. The BFS which maximizes equations (3.3a) and (3.3b) is called an optimal basic feasible solution. Again, if the number of non zero variables is less than $m+n+l-2$, then the problem is called degenerate.

4. Model with distribution functions

Here, the model is bifurcated into different sub-models, based on different types (continuous and discrete) of a probability distribution of the random demand.

– **Sub-Model 1.** Let the random demands b_j $(j = 1,2,\ldots,n)$ of the jth destinations followed logistic distribution with the following probability density function $\phi(b_j)$. The logistic distribution is the "normal"-shaped pattern of its cumulative distribution function (the logistic function) and quantile function (the logit function) have been extensively used in different areas.

$$\phi(b_j) = \frac{1}{\beta_j}\,\frac{e^{-\left(\frac{b_j-\alpha_j}{\beta_j}\right)}}{\left[1+e^{-\left(\frac{b_j-\alpha_j}{\beta_j}\right)}\right]^2},\ 0 \le b_j < \infty,\ 0 \le \alpha_j < \infty, \text{ and } \beta_j > 0.$$

Then equation (3.3a) becomes

$$\begin{aligned}\text{Max}\, Z = {} & \sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} \\ & - \sum_{j=1}^{n}(h_j+p_j)\left\{ y_j \frac{e^{\frac{\alpha_j}{\beta_j}}}{1+e^{\frac{\alpha_j}{\beta_j}}} + \beta_j \log \frac{1+e^{\frac{(\alpha_j-y_j)}{\beta_j}}}{1+e^{\frac{\alpha_j}{\beta_j}}} \right\} + \sum_{j=1}^{n} p_j y_j\end{aligned} \tag{4.1}$$

subject to the constraints (3.4)–(3.7).

The Lagrange's function of the above problem is

$$\begin{aligned}L(X,u,v,z) = {} & \sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} \\ & - \sum_{j=1}^{n}(h_j+p_j)\left\{ y_j \frac{e^{\frac{\alpha_j}{\beta_j}}}{1+e^{\frac{\alpha_j}{\beta_j}}} + \beta_j \log \frac{1+e^{\frac{(\alpha_j-y_j)}{\beta_j}}}{1+e^{\frac{\alpha_j}{\beta_j}}} \right\} + \sum_{j=1}^{n} p_j y_j \\ & + \sum_{i=1}^{m} u_i \sum_{j=1}^{n}\sum_{k=1}^{l}(a_i - x_{ijk}) + \sum_{k=1}^{l} v_k \sum_{i=1}^{m}\sum_{j=1}^{n}(e_k - x_{ijk}) \\ & + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} z_{ijk}x_{ijk}\end{aligned} \tag{4.2}$$

where the variable u_i, v_k, z_{ijk} are Lagrange multipliers and $u_i, v_k, z_{ijk} \geq 0$. The KKT conditions for the problem (4.2) are

$$\begin{cases} \dfrac{\partial L}{\partial x_{ijk}} = c_{ijk} - u_i - v_k + z_{ijk} = 0 \\ \displaystyle\sum_{i=1}^{m} u_i\Bigg(\sum_{j=1}^{n}\sum_{k=1}^{l}(a_i - x_{ijk})\Bigg) = 0 \\ \displaystyle\sum_{k=1}^{l} v_k\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n}(e_k - x_{ijk})\Bigg) = 0 \\ \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} z_{ijk}x_{ijk} = 0 \end{cases}$$

Now, the KKT conditions, lead the optimization problem to

$$\begin{cases} \text{Max}\, Z = \displaystyle\sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} - \sum_{j=1}^{n}(h_j + p_j)\left\{ y_j \frac{e^{\frac{\alpha_j}{\beta_j}}}{1 + e^{\frac{\alpha_j}{\beta_j}}} + \beta_j log \frac{1 + e^{\frac{(\alpha_j - y_j)}{\beta_j}}}{1 + e^{\frac{\alpha_j}{\beta_j}}} \right\} + \sum_{j=1}^{n} p_j y_j \\ \text{subject to} \\ c_{ijk} - u_i - v_k \leq 0 \\ x_{ijk}(c_{ijk} - u_i - v_k) = 0 \\ \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \leq a_i \text{ for } i = 1, 2, \ldots, m \\ \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k \text{ for } k = 1, 2, \ldots, l \\ x_{ijk} \geq 0\ \forall i, j, k \end{cases}.$$

Proposition 4.1. *The objective function Z is concave with respect to y_j and x_{ijk}.*

Proof. To show the objective function Z is concave, we have differentiate the function Z (presence in Eq. (4.1)), partially with respect to y_j and x_{ijk} respectively. Then,

$$\frac{\partial Z}{\partial y_j} = S_j - (h_j + p_j)\left\{ \frac{e^{\frac{\alpha_j}{\beta_j}}}{1 + e^{\frac{\alpha_j}{\beta_j}}} - \frac{e^{\frac{(\alpha_j - y_j)}{\beta_j}}}{1 + e^{\frac{(\alpha_j - y_j)}{\beta_j}}} \right\} - p_j$$

$$\frac{\partial^2 Z}{\partial y_j^2} = -(h_j + p_j)\frac{e^{\frac{(\alpha_j - y_j)}{\beta_j}}}{\beta_j\left[1 + e^{\frac{(\alpha_j - y_j)}{\beta_j}}\right]^2}.$$

Similarly,

$$\frac{\partial^2 Z}{\partial x_{ijk}^2} = -(h_j+p_j)\left\{\left(\frac{e^{\frac{\alpha_j}{\beta_j}}}{1+e^{\frac{\alpha_j}{\beta_j}}} - \frac{e^{\frac{(\alpha_j-y_j)}{\beta_j}}}{1+e^{\frac{(\alpha_j-y_j)}{\beta_j}}}\right)\frac{\partial^2 y_j}{\partial x_{ijk}^2} - \frac{e^{\frac{\alpha_j-y_j}{\beta_j}}}{\beta_j\left[1+e^{\frac{(\alpha_j-y_j)}{\beta_j}}\right]^2}\left(\frac{\partial y_j}{\partial x_{ijk}}\right)^2\right\} - (p_j-S_j)\frac{\partial^2 y_j}{\partial x_{ijk}^2}.$$

As h_j, p_j, β_j are positive parameters. From the expression, it is clear that $\frac{\partial^2 Z}{\partial y_j^2} < 0$. Hence we conclude that Z is concave with respect to y_j. By similar arguments $\frac{\partial^2 Z}{\partial x_{ijk}^2} < 0$. So Z is concave function with respect to x_{ijk} also. Hence the objective function Z is concave with respect to y_j and x_{ijk}. □

– **Sub-Model 2.** When demands $b_j\,(j=1,2,\ldots,n)$ follow exponential distribution with probability density function

$$\phi(b_j) = \lambda_j e^{-\lambda_j b_j},\ b_j > 0,\ \lambda_j > 0.$$

The exponential distribution is one of the widely used distribution function as a demand function in the decision making problems.

Then the objective function (3.3a) becomes

$$\text{Max}\, Z = \sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} - \sum_{j=1}^{n}(h_j+p_j)\left\{y_j + \frac{e^{-\lambda_j y_j}-1}{\lambda_j}\right\} + \sum_{j=1}^{n} p_j y_j \tag{4.3}$$

subject to the constraints (3.4)–(3.7).

The Lagrange's function of the above problem is

$$\begin{aligned} L(X,u,v,z) = &\sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} - \sum_{j=1}^{n}(h_j+p_j)\left\{y_j + \frac{e^{-\lambda_j y_j}-1}{\lambda_j}\right\} + \sum_{j=1}^{n} p_j y_j \\ &+ \sum_{i=1}^{m} u_i \sum_{j=1}^{n}\sum_{k=1}^{l}(a_i - x_{ijk}) + \sum_{k=1}^{l} v_k \sum_{i=1}^{m}\sum_{j=1}^{n}(e_k - x_{ijk}) \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} z_{ijk}x_{ijk} \end{aligned} \tag{4.4}$$

where the variable u_i, v_k, z_{ijk} are Lagrange multipliers and $u_i, v_k, z_{ijk} \geq 0$. The KKT conditions for the problem (4.4) are

$$\begin{cases} \dfrac{\partial L}{\partial x_{ijk}} = c_{ijk} - u_i - v_k + z_{ijk} = 0 \\ \displaystyle\sum_{i=1}^{m} u_i\left(\sum_{j=1}^{n}\sum_{k=1}^{l}(a_i - x_{ijk})\right) = 0 \\ \displaystyle\sum_{k=1}^{l} v_k\left(\sum_{i=1}^{m}\sum_{j=1}^{n}(e_k - x_{ijk})\right) = 0 \\ \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} z_{ijk}x_{ijk} = 0 \end{cases}.$$

Now, the KKT conditions, lead the optimization problem to

$$\begin{cases} \text{Max } Z = \sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} - \sum_{j=1}^{n}(h_j+p_j)\left\{y_j + \frac{e^{-\lambda_j y_j}-1}{\lambda_j}\right\} + \sum_{j=1}^{n} p_j y_j \\ \text{subject to} \\ c_{ijk} - u_i - v_k \le 0 \\ x_{ijk}(c_{ijk} - u_i - v_k) = 0 \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le a_i \text{ for } i = 1,2,\dots,m \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_k \text{ for } k = 1,2,\dots,l \\ x_{ijk} \ge 0 \ \forall i,j,k \end{cases}.$$

Proposition 4.2. *The objective function Z is concave with respect to y_j and x_{ijk}.*

Proof. To show the objective function Z is concave with respect to y_j and x_{ijk}, we have differentiate Z partially twice with respect to y_j and x_{ijk} respectively, as

$$\frac{\partial Z}{\partial y_j} = S_j - (h_j+p_j)\{1 - e^{-\lambda_j y_j}\} - p_j$$
$$\frac{\partial^2 Z}{\partial y_j^2} = -(h_j+p_j)\lambda_j e^{-\lambda_j y_j}.$$

Similarly,

$$\frac{\partial^2 Z}{\partial x_{ijk}^2} = -(h_j+p_j)\left\{(1-e^{-\lambda_j y_j})\frac{\partial^2 y_j}{\partial x_{ijk}^2} - \lambda_j e^{-\lambda_j y_j}\left(\frac{\partial y_j}{\partial x_{ijk}}\right)^2\right\} - (p_j - S_j)\frac{\partial^2 y_j}{\partial x_{ijk}^2}.$$

As h_j, p_j are positive parameters. Obviously $\frac{\partial^2 Z}{\partial y_j^2} < 0$. Hence we conclude that Z is concave with respect to y_j. By similar arguments, $\frac{\partial^2 Z}{\partial x_{ijk}^2} < 0$, so Z is concave function with respect to x_{ijk} also. Hence the objective function Z is concave in nature with respect to y_j and x_{ijk}. □

– **Sub-Model 3.** Let the demands b_j of the jth destination has a discrete distribution, *i.e.*, demand of the item are frequently accept some real values (r) with certain probabilities (μ_j^r), *i.e.*, $P(d_j = r) = \mu_j^r$, $r = 1,2,\dots,K$. Where $1,2,\dots,K$ denotes the event space of the demand d_j at jth destination and $\sum_{r=1}^{K}\mu_j^r = 1$.

Then the equation (3.3b) becomes

$$\text{Max } Z = \sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} - \sum_{j=1}^{n}(h_j+p_j)\sum_{r<y_j}(y_j - r)\mu_j^r + \sum_{j=1}^{n} p_j y_j \tag{4.5}$$

subject to the constraints (3.4)–(3.7).

Now, the Lagrangian function of (4.5) is

$$L(X,u,v,z) = \sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} - \sum_{j=1}^{n}(h_j+p_j)\sum_{r<y_j}(y_j-r)\mu_j^r + \sum_{j=1}^{n} p_j y_j$$
$$+\sum_{i=1}^{m} u_i \sum_{j=1}^{n}\sum_{k=1}^{l}(a_i - x_{ijk}) + \sum_{k=1}^{l} v_k \sum_{i=1}^{m}\sum_{j=1}^{n}(e_k - x_{ijk})$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} z_{ijk}x_{ijk} \tag{4.6}$$

where the variable u_i, v_k, z_{ijk} are Lagrange multipliers and $u_i, v_k, z_{ijk} \geq 0$. The KKT conditions for the problem (4.5) are

$$\begin{cases} \dfrac{\partial L}{\partial x_{ijk}} = c_{ijk} - u_i - v_k + z_{ijk} = 0 \\ \sum_{i=1}^{m} u_i \left(\sum_{j=1}^{n}\sum_{k=1}^{l}(a_i - x_{ijk})\right) = 0 \\ \sum_{k=1}^{l} v_k \left(\sum_{i=1}^{m}\sum_{j=1}^{n}(e_k - x_{ijk})\right) = 0 \\ \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} z_{ijk}x_{ijk} = 0 \end{cases}.$$

Now, the KKT conditions, lead the optimization problem to

$$\begin{cases} \text{Max } Z = \sum_{j=1}^{n} S_j y_j - \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{l} c_{ijk}x_{ijk} - \sum_{j=1}^{n}(h_j+p_j)\sum_{r<y_j}(y_j-r)\mu_j^r + \sum_{j=1}^{n} p_j y_j \\ \text{subject to} \\ c_{ijk} - u_i - v_k \leq 0 \\ x_{ijk}(c_{ijk} - u_i - v_k) = 0 \\ \sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \leq a_i \text{ for } i = 1,2,\ldots,m \\ \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \leq e_k \text{ for } k = 1,2,\ldots,l \\ x_{ijk} \geq 0 \ \forall i,j,k \end{cases}.$$

5. Numerical example

In order to show the application of the proposed model, we shall present the phenomenon of coal transportation problem. Coal is a kind of important energy source in the development of economy and society. Accordingly, how to transport coal from mines to different areas is an important issue in coal transportation. For the simplicity of description, we summarize the problem as follows. Suppose that there are three coal mines (*i.e.*, $m = 3$) to supply the coal in two companies (*i.e.*, $n = 2$). During the process of transportation, two kinds of conveyances to be selected, *i.e.*, train and cargo ship ($k = 2$). Now, the task for the decision-maker is to make the transportation

TABLE 2. Input data.

Unit transportation cost		Holding cost	Shortage cost	Availability	Convaynce	Selling price
c_{ij1}	c_{ij2}	h_j	p_j	a_i	e_k	S_j
$c_{111}=5$, $c_{211}=5$, $c_{311}=7$	$c_{212}=8$, $c_{112}=11$, $c_{312}=8$	$h_1=7$	$p_1=18$	$a_1=50, a_2=53$	$e_1=100$	$S_1=39$
$c_{121}=6$, $c_{221}=12$, $c_{321}=4$	$c_{122}=9$, $c_{222}=9$, $c_{322}=7$	$h_2=9$	$p_2=12$	$a_3=72$	$e_2=75$	$S_2=42$

TABLE 3. Resulting solutions of Sub-Model-1 for different α_j, β_j.

Demand parameters	Conveyance K	1		2		Lagrange	Multipliers	Z
	i/j	1	2	1	2			
$\alpha_1=62$, $\alpha_2=74$	1	0	0	6	38	$u_1=4.39$	$v_1=3.97$	5703.79
$\beta_1=12$, $\beta_2=18$	2	0	49	0	0	$u_2=8.39$	$v_2=7.97$	
	3	43	0	25	0			
$\alpha_1=78$, $\alpha_2=84$	1	0	0	47	0	$u_1=4.39$	$v_1=3.97$	6157.87
$\beta_1=16$, $\beta_2=10$	2	0	53	0	0	$u_2=8.39$	$v_2=7.97$	
	3	40.15	6.84	0	25			
$\alpha_1=79$, $\alpha_2=85$	1	0	0	0	47	$u_1=4.39$	$v_1=3.97$	6477.34
$\beta_1=17$, $\beta_2=11$	2	14.65	38.34	0	0	$u_2=8.39$	$v_2=7.97$	
	3	47	0	25	0			

TABLE 4. Resulting solutions of Sub-Model-2 for different λ_j.

Demand parameters	Conveyance K	1		2		Lagrange	Multipliers	Z
	i/j	1	2	1	2			
$\lambda_1=0.018$, $\lambda_2=0.012$	1	34.56	0	0	0	$u_1=4.67$	$v_1=9.32$	4575.41
	2	0	49.34	0	3.23	$u_2=4.67$	$v_2=6.32$	
	3	0	0	39	31.14			
$\lambda_1=0.015$, $\lambda_2=0.012$	1	43.19	0	0	6	$u_1=4.67$	$v_1=9.32$	5209.39
	2	0	49.75	0	0	$u_2=4.67$	$v_2=6.32$	
	3	0	0	31	37			
$\lambda_1=0.014$, $\lambda_2=0.018$	1	32.25	0	0	0	$u_1=4.67$	$v_1=9.32$	5914.08
	2	0	34	18.25	0	$u_2=4.67$	$v_2=6.32$	
	3	32.5	0	0	32.8			

plan in order to maximize the profit. At the beginning of this problem, the decision maker needs the basic data, such as availability of the origin, demand of the destination, capacity of the conveyances, transportation cost of unit product, and so on. In this regard, the following input values are taken into consideration, common to all the three Sub-Models (Tab. 2).

For the above forecast input data, each sub-model, random demand parameters are presented in the first column of the respective Tables 3–5.

From the above illustration, it is seen that the optimal solutions are basic *i.e.*, the solutions are basic feasible solutions, since the number of non-trivial solutions is $(m+n+l-2)$. From Tables 3 to 5, it is clear that more expected demand influences the decision-makers to transport more quantity of items. Consequently, more amount transportation increases the total cost of the system as well as profit of the system. Here, it is also seen that most of the Lagrangian multipliers are positive, which indicate for the global optimal solution of the model.

TABLE 5. Resulting solutions of Sub-Model-3 for different d_j.

Demand parameters	Conveyance K	1		2		Lagrange	Multipliers	Z
	i/j	1	2	1	2			
$d_1 = (57, 0.3)$, $d_2 = (51, 0.2)$	1	37.34	0	0	0	$u_1 = 4.11$	$v_1 = 2.44$	5600.84
$d_1 = (58, 0.4)$, $d_2 = (52, 0.5)$	2	0	19	29.23	0	$u_2 = 6.11$	$v_2 = 3.10$	
$d_1 = (59, 0.3)$, $d_2 = (53, 0.3)$	3	0	21.56	0	31.14			
$d_1 = (61, 0.3)$, $d_2 = (54, 0.2)$	1	29.25	0	20.2	0	$u_1 = 4.11$	$v_1 = 2.44$	6189.45
$d_1 = (62, 0.4)$, $d_2 = (55, 0.5)$	2	0	46	0	0	$u_2 = 6.11$	$v_2 = 3.10$	
$d_1 = (63, 0.3)$, $d_2 = (56, 0.3)$	3	23.5	0	0	31.6			
$d_1 = (59, 0.3)$, $d_2 = (59, 0.2)$	1	0	49.75	0	0	$u_1 = 4.11$	$v_1 = 2.44$	6554.3
$d_1 = (60, 0.4)$, $d_2 = (60, 0.5)$	2	38.1	0	12	0	$u_2 = 6.11$	$v_2 = 3.10$	
$d_1 = (61, 0.3)$, $d_2 = (61, 0.3)$	3	0	0	24.2	37			

6. Practical implementation

In developing countries, like India, Bangladesh, Nepal, China, etc., due to several reasons, the parameters especially the demand of an item is uncertain in nature. For these reasons, if we collect the previous data from any management system belong to these countries and followed by statistical regularity criteria, its probability distribution can be obtained for future correspondence. For this type of random demand with exponential and logistic distribution, has a wide range of applications (*cf.* [22]) has been considered here. Not only that, the more realistic discrete demand distribution is taken here. Such type of transportation system is found for the seasons products, like-winter garments, raincoats, summer creams, seasonal fruit juice, etc., or the attractive items, like bikes, cars, and mobile phones, etc. Here as per reality, profit maximization criteria are introduced instead of cost minimization of the system. So the above transportation model has a wide practical area of the above mentioned transportation management.

7. Discussion and conclusion

In today's highly comparative market, the decision-makers deliver the product to the customer in effective ways, although the system has a heavy uncertainty. The literature focused on a solid transportation problem (STP) where the demand of the destinations are not fixed quantities, but uncertain in nature. The presented numerical result is a choice of such decision making problem for a particular illustration. Here such a STP is taken into consideration with a more realistic profit maximization form. The formulated STP is simplified with different probability distributions and their comparative results are illustrated.

The proposed model is one of the realistic models, so it can be developed with other different types of environments, like rough, fuzzy rough, fuzzy stochastic, etc. The model can be extended with a multi-item solid transportation problem, model with fixed charge, model with more vehicle costs, etc. The proposed SSTP can be solved with some other programming problems, like genetic algorithms, simulation methods, etc. All these can be taken as future research.

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