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ADVANCE SUSTAINABLE INVENTORY MANAGEMENT THROUGH ADVERTISEMENT AND TRADE-CREDIT POLICY

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Abstract. The concept of advanced sustainable inventory management, where demand pattern stock level and advertising dependent under trade-credit policy is taking account in this present study. Optimal credit period and cycle time are the main objective of this advanced system. A developed solution methodology is derived to show the existence of global optimality under optimum credit period and cycle time. The main concern of this advanced system is to maximize the annual total system profit of retailer with finite replenishment rate. Numerical illustration are carry forward for different cases to prove the stainability along with real impact of this model. Sensitive analysis for the key parameters is discussed in sensitivity analysis section along with some real managerial insights.

Mathematics Subject Classification. 90B05, 90B06.

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1. Introduction

In today's competitive business world each and every industry wants more profits, as advertising gives a chance to sell more product, thus every industry invests more in advertising to optimum their profit. Thus, every industry maintains a good advertising team to optimum their profit. This is a strategy to sell maximum items at minimum duration. Due to increases of demand, selling of the items also increased. This advertisement helps to identify products with different facilities, and which one is more reliable among all other products in the market. The advertising increases the probability of successful marketing targets with the fulfill demand of the customers. As a result, maximum investment for advertising gives more profits to any company. In this competitive marketing system, every supplier or manufacturer wants to sells more items to earn more revenues, reminding this most of the suppliers/manufacturer offer a certain delay period to convince the retailer/customer to buy more items. Retailer can sell his/her product before the end of the delay period, and interest earns

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with revenue. So, the delay-in-payments strategy is one kind of price-discount policy which encourages the retailers/customer to increase their order quantity.

Demand for any inventory system one of the most important issues in today's business industry. For simplification of the model, demand will be considered constant as a basic assumption in many existing literature [3,6,28]. But in our daily life it is near about impossible that demand of any product always constant in general. Some times looks of any product attract the customers, so demand of the products can be vary with displayed stock level (DSL) and some times this demand may vary with the time, that is time dependent demand rate, as a result the demand is not constant. To increases the profit, any retailer must have to stocks as much as he can, other wise due to short stock customer can go away from his/her shop. Thus, in any industry DSL have a great impact. Display of inventory is proportional to the sales in retail shop was calculated by Silver and Peterson [49]. In the same direction Mashud *et al.* [21], Shaikh *et al.* [48] and Khan *et al.* [13] developed different inventory model where demand was depend on stock-level. The developed the model for perishable items. More recent, Sana and Chaudhuri [29] developed a volume-flexible stock-dependent demand model. A generalised model for stock dependent was formulated by Shaikh *et al.* [47] in this model they also considered full backlogging. All research discussed about stock dependent demand model but Sarkar *et al.* [35] developed a different strategy to find out the optimality for the supplier.

There were different inventory models were developed by different researchers. An advanced stock-dependent inventory model was discussed by Sarkar and Sarkar [33] where deterioration depend on time. This model was extended by Sarkar [31] to a production-inventory model and by considering probabilistic deterioration. In the same year Sarkar *et al.* [36] developed an inventory model by considering trade-credit policy and price-discount offer. A distribution free continuous-review model with service level constraint was developed by Moon *et al.* [26]. An integrated inventory model with unequal lot size and variable setup cost was developed by Sarkar *et al.* [38]. A continuous-review inventory model with controllable lead time and service level constraint was presented by Shin *et al.* [39]. All researcher developed different advanced inventory model by considering lead time, deterioration, tread-credit etc., but no one concerned about increasing the demand of the product, which directly optimised the profit of the whole system. To increase the demand of a particular product in this research the concept of advertisement policy is adopted along with trade-credit policy, which directly increases the total profit of the system.

Researchers are always busy to developed different type of model of different marketing sector. In today's active social sites and presence of media advertisement for any product has a great impact for any industry. So advertising demand pattern for any industry is one of the great finding for any researcher. It is quite natural that any industry can use the best advertise policy for their product and for that they formulate their best marketing team. Normally the aim of advertising is to awareness of the common people. It inform the general public about the product and influence to the customer to buy their product instead of other, so it compare the difference of product. Any advertisement base on the message which it try to give the general people and what type of media are used. There are different types of media are available for any type of advertisement like as electronic medias, magazines, cable networks, door-to-door sales, newspapers, etc. which implies more demands of the products. Therefore, investing on advertisement definitely incress the demand of any product thus, industries are now invest more on advertisement section to optimum their profit by selling more products. Considering this, Sethi [46] survey a dynamic optimal control model where advertising takes place. Goodwill for any company is one of the key parameter to sell their product, reminding this Tapiero [52] developed the inventory model in which both advertising and goodwill take places in the presence of uncertainty. An advertising model for non-instantaneous deterioration along with trade-credit and preservation technology was developed by Mashud *et al.* [22,23]. An interesting cooperative advertising collaboration policy for supply chain model was introduced by Sarkar *et al.* [43]. Recently, an inventory model for stock and advertising demand model was presented by Sana and Chaudhuri [30], where they approach a concentrate investment for advertising as well as increasing the sales effort to maximize the profit. Recently, advertisement dependent demand for two-echelon supply chain model was proposed by Noh *et al.* [27]. They used Stackelberg game policy to solve the model. There different model for stock dependent demand and advertising dependent demand were separately developed by several researchers, but a sustainable inventory model where demand depend both on stock-level and advertisement still not considered by any researcher. In reality, advertisement is one of the best strategy for increase any business industry. Thus, advertisement and stock-level dependent demand is always helpful for any industry to increase their profit, this pioneer attempt is taken in this model.

In this huge competitive market, to increase the sales of any particular product, the supplier made a strategy in which he/she offers a certain fixed time period in that time the retailer pay the amount for the purchased item. This is the trade-credit period, which offered to the retailer by the supplier. There is a chance to the retailers to concentrate their revenue and collect interest with the help of the products during the delay period, whereas a larger interest amount has to be charged if the deposit of amount is not done at the end of the credit period. There is no penalty is charge if the remaining money is paid within the given permissible delay period. In this direction, in response to temporary price/credit incentives Tiwari *et al.* [54] discussed about retailers pricing credit and some inventory policies for deteriorating items. In the same direction a model with exponential deterioration and partial trade-credit was introduced by Sarkar and Saren [32]. An optimal economic ordering policy is proposed by Sarkar *et al.* [44] for deteriorating items. Where, they assumed different cases of trade credit period for finite horizon. An EOQ (economic order quantity) model with deteriorating items was deduced by Chang *et al.* [4], in which the supplier provides a permissible time period to the customer, provided the quantity that he/she will ordered must be larger than or equal to the quantity which he/she predetermined to take. They characterized the optimal solution. More over they developed an easy-to-use algorithm which helps to obtain the optimal order quantity with replenishment time. A permissible delay-in-payments for EOQ models, which was depends on some conditions, is introduced by Chen *et al.* [5]. A partial tread-credit policy for retailer was proposed by Chung [7]. In this model, he minimised the total cost of the system with optimum order quantity. The concept of two-levels trade-credit was used by Mashud *et al.* [24]. They consider price-dependent demand for an inventory model along with shortages. A lead time depended ordering cost for multi-stage production model was developed by Kim and Sarkar [20]. Different study about supply chain or integrated inventory were done by different researchers where warranty policy takes a major role in the optimization of total cost or profit [11, 17]. Taleizadeh *et al.* [51] developed a production model for multi-product, where delayed payment takes places. In this model they also considered the repair failure and backlogging strategies. To increase the life-time of the deteriorating items preservation technology can helps allot, proved by Iqbal and Sarkar [12]. In long run process a machine definitely produced defective items in *out-of-control* state. To detect the imperfect items and remove them an inspection is needed Sarkar *et al.* [37]. Some time inspection error had a great impact on any inventory system, proposed by Khanna *et al.* [18]. More recent, Dey *et al.* [8] derived an integrated model where demand is selling price dependent. An integrated model with imperfect production was developed by Sarkar *et al.* [40]. In the presence of two-level trade-credit policy Soni and Patel [50] derived an integrated inventory model where production rate is variable and demand is price-sensitive. An inventory model with expiration date and pricing decisions was presented by Khan *et al.* [14] Considering advanced payment policy and trade-credit, different inventory models were formulated by different researchers [15,23,24]. All researcher developed different model for trade-credit or for advertisement policy, but the benefit for advertisement and stock-level dependent demand in the presence of trade-credit for inventory model still not considered. Thus, this model try to fulfill this gap. An author(s) contribution table (Tab. 1), is provided to show the research gaps in this direction.

This model is structured as follows, the next section contains the problem definition along with notation and assumptions. In Section 3, the solution procedure of this model is discussed followed by the mathematical model. Next section consists some numerical examples to elaborate the model. Sensitivity analysis of key parameters and some managerial insights are discussed in Sections 5 and 6. Finally, some concluding remarks with future extension are discussed in conclusion section.

2. Problem definition, notation, and assumptions

In this section, one can find the definition of the problem, that is formulated in this current study along with Notation, and assumptions which are considered to developed this model.

Table 1. Research gaps and contributions of previous Author(s).

Author(s)	Demand	Model	Strategy	Deterioration rate
Shaikh *et al.* [48]	Stock dependent	Inventory	Price discount	Yes
Sarkar *et al.* [35]	Stock dependent	Inventory	NA	NA
Mashud *et al.* [23]	Inventory-level	Inventory	Backlogging	Yes
Chang *et al.* [4]	Stock dependent	EOQ	Backlogging	Yes
Dey *et al.* [8]	Selling-price	Integrated inventory	Cost reduction	NA
Khan *et al.* [15]	Constant	Inventory	Advance payment	Yes
Soni and Patel [50]	Constant	Integrated inventory	Trade-credit	NA
Khanra *et al.* [19]	Time dependent	Inventory	Trade-credit	NA
This paper	Advertisement & stock-level dependent	Inventory	Tread-credit	Constant function of on-hand inventory

Notes. NA-Not applicable.

2.1. Problem definition

Here, an advanced inventory model for single type of item, which has a constant deterioration rate is derived. To illustrate the practical situation, demand is considered as stock-level and advertising dependent. To make the model more sustainable, deterioration rate is considered as the constant function of on-hand inventory. No outstanding payment to the supplier during placing an order along with no interest is to be charged after commencement. Due to large production, shortages are not allowed. A partial trade-credit policy is implied to optimised the total system profit. Finally, the total profit of the system was optimised based on the optimum value of delay period and total cycle time.

2.2. Notation

The following notation are use to develop the model.

Decision variables	
T	Length of the cycle (year)
T_1	Length of time in which inventory reach the zero level (year)
Parameters	
$d(I(t), t, v)$	Demand rate function
$I(t)$	Initial inventory at time $t \geq 0$
θ	Constant deterioration rate of on-hand inventory, $0 < \theta < 1$
m	permissible delay time in staling the account (year)
I_r	Chargeable interest per rupee investment in stock (\$/year)
I_e	Earned interest per rupee investment in a year, $I_e \leq I_r$ (\$/year)
c_h	Inventory holding cost per unit at time (\$/unit/year)
c_p	Production cost per unit (\$/unit)
c_0	Setup cost per cycle (\$/setup)
p	Selling price per unit (\$/unit)
v	Advertisement cost per cycle (\$/cycle)

2.3. Assumptions

The following presumptions are constructed to formulate this model.

(1) The inventory system covers single type of item and the demand rate is a function of stock on display, time and advertisement cost.
(2) No outstanding payment to the supplier during placing an order that is $m < T$ as well as no interest is to be changed after commencement.
(3) The rate of deterioration per unit time is a constant function of on-hand inventory. *i.e.*, Deterioration depends on the total on-hand inventory. If inventory is large, then deterioration rate also increased.
(4) Lead time is the time gap between placing an order and receiving the order, which is considered as negligible in this model, *i.e.*, replenishment occurs instantly just after placing the order.
(5) Production rate always greater then demand so, shortage are not allowed.
(6) The planning period is infinite length and the ending inventory levels are zero.

3. Model formulation

The demand pattern for the model depend on stock level and advertisement and that is $D(I(t), t, v) = \alpha.I(t) + \beta(at + b) + xv^y$ where x and y are empirically determined constants, which indicate the effectiveness of advertising $0 \leq y \leq 1$. α, β, a, b are the scaling parameters. $x = 0$ indicates that demand is independent of advertising expenditure. For any $y > 0$, the increase value of y, is more effective in the advertising. Thus, the basic differential equation is

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\alpha\theta I(t) - \beta(at + b) - xv^y, \quad 0 \leq t \leq T \tag{3.1}$$

with $I(0) = Q_0$ and $I(T) = 0$.

The solution of equation (3.1) in $0 \leq t \leq T$ is

$$\begin{aligned} I(t) = & -\beta\left(\frac{at}{\alpha\theta} - \frac{a}{\alpha^2\theta^2} + \frac{b}{\alpha\theta}\right) - \frac{xv^y}{\alpha\theta} \\ & + \left\{\left(\beta\left(\frac{aT}{\alpha\theta} - \frac{a}{\alpha^2\theta^2} + \frac{b}{\alpha\theta}\right) + \frac{xv^y}{\alpha\theta}\right) e^{\alpha\theta(T-t)}\right\}. \end{aligned} \tag{3.2}$$

As the delay period has a great impact on demand, so, distributor or whole-seller allowed a delay period m for the retailer or customers. So, depend on the value of m, T_1, and T, three different models developed in this paper.

Case 1. Let $m \leq T_1 \leq T$.

In Figure 1, inventory *versus* with the time for model I, $m \leq T_1 < T$, that is the length of the total planing horizon is grater then the credit period. In this situation buyer can use the revenue of sales to assembled the delay charge with an annual rate I_e in $[0, m]$. The interest earned E_1 is given by

$$\begin{aligned} E_1 &= pI_e \int_0^m tD(t)\mathrm{d}t \\ &= pI_e \int_0^m t \cdot \{\alpha \cdot I(t) + \beta(at + b) + xv^y\}\mathrm{d}t \\ &= pI_e\left\{-\beta\left(\frac{am^3}{3\theta} - \frac{am^2}{2\alpha\theta^2} + \frac{bm^2}{2\theta}\right) + \frac{xv^y}{2}m\left(1 - \frac{1}{\theta}\right) + \frac{a\beta m^3}{3} + \frac{b\beta}{2}m^2\right\} \\ &\quad + pI_e\left\{\beta\left(\frac{aTm^2}{2\theta} - \frac{am^2}{2\alpha\theta^2} + \frac{bm^2}{\theta}\right) + \frac{xv^y}{\theta}\right\}\left\{e^{\theta T} - (1 + m)e^{\theta(T-m)}\right\}. \end{aligned}$$

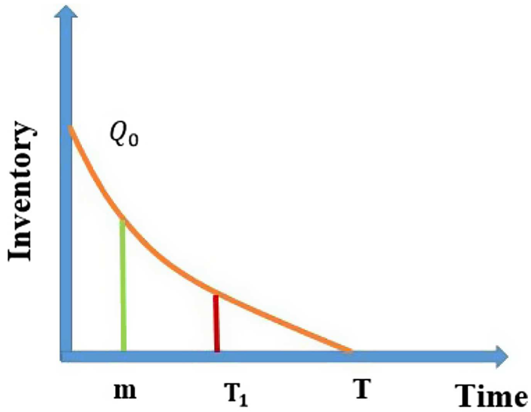

FIGURE 1. Inventory level *versus* time for Model I $m \le T_1 < T$.

After credit-period, the stock which was not sold is to be economics with an annual rate I_r so the payable interest I is as follows

$$
\begin{aligned}
I &= pI_r \int_m^{T_1} I(t)\mathrm{d}t \\
&= -pI_r \frac{\beta}{2\alpha\theta}(T_1^2 - m^2) + pI_r \left\{\beta\left(\frac{a}{\alpha^2\theta^2} - \frac{b}{\alpha\theta}\right)(T_1 - m) - \frac{xv^y}{\alpha\theta}(T_1 - m)\right\} \\
&\quad + pI_r \left\{\beta\left(\frac{aT}{\alpha\theta} - \frac{a}{\alpha^2\theta^2} + \frac{b}{\alpha\theta}\right) + \frac{xv^y}{\alpha\theta}\right\}\left\{e^{\alpha\theta(T-m)} - e^{\alpha\theta(T-T_1)}\right\}.
\end{aligned}
$$

Therefore, the average profit is

$$
\begin{aligned}
Z_1 &= \frac{1}{T}\left[(p - c_p)\int_0^T D(I(t), t, v)\mathrm{d}t - c_0 - v - c_h \int_0^T I(t)\mathrm{d}t - \theta c_p \int_{T_1}^T I(t)\mathrm{d}t - I + E_1\right] \\
&= \frac{1}{T}\left[(p - c_p)\int_0^T \{\beta(at + b) + xv^y\}\mathrm{d}t + E_1 - c_0 - v - I\right. \\
&\quad \left. + (p - c_p - c_h)(1 + \alpha)\int_0^T I(t)\mathrm{d}t - \theta c_p \int_{T_1}^T I(t)\mathrm{d}t - E_1\right] \\
&= \frac{1}{T}\left[(p - c_p)\left\{\beta\left(\frac{aT^2}{2} + bT\right) + xV^yT\right\} - (c_0 + v) + pI_e\left\{-\alpha\beta\left(\frac{am^3}{3\theta} - \frac{m^2}{2\theta} + \frac{bm^2}{2\theta}\right)\right.\right. \\
&\quad \left. + \frac{xv^y}{2}m\left(\frac{1}{\theta} + 1\right) + \frac{a\beta m^2}{3} + \frac{b\beta}{2}m^2\right\} + pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{e^{\theta T} - (1 + m)e^{\theta(T-m)}\right\} \\
&\quad + (p - c_p - c_h)(1 - \alpha)\left\{-\beta\left(\frac{aT^2}{2\theta} - \frac{T^2}{\theta^2} + \frac{bT}{\theta}\right) + \frac{xv^y}{\theta}T + \beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta T} \\
&\quad + pI_r \frac{\alpha\beta}{2\theta}\left(T_1^2 - m^2\right) - pI_r\left\{\beta\left(\frac{1}{\theta^2} - \frac{b}{\theta}\right) + \frac{xv^y}{\theta}\right\}(T_1 - m) \\
&\quad - pI_r\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{e^{\theta(T-m)} - e^{\theta(T-T_1)}\right\} + \theta c_p \frac{a\beta}{2\theta}\left(T^2 - T_1^2\right)
\end{aligned}
$$

$$-\theta c_p\left\{\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}(T-T_1)-\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}\Bigg]. \tag{3.3}$$

The purpose is to achieve the maximum average profit per unit time.

The following theorem was used from Khanra *et al.* [19].

Theorem 3.1. *If a function $U(T_1,T)=\frac{1}{T}W(T_1,T)$ where $W(T_1,T)$ is a continuously partial differentiable function of T_1 and T of second order then $U(T_1,T)$ is maximum at $T_1=T_1^*$, $T=T^*$ if $d^2W(T_1,T)$ is positive definite i.e., if*

$$\begin{vmatrix}\frac{\partial^2 W(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W(T_1,T)}{\partial T_1\partial T}\\ \frac{\partial^2 W(T_1,T)}{\partial T_1\partial T} & \frac{\partial^2 W(T_1,T)}{\partial T^2}\end{vmatrix}>0 \quad\text{and}\quad \frac{\partial^2 W(T_1,T)}{\partial T_1^2}<0 \quad\text{or}\quad \frac{\partial^2 W(T_1,T)}{\partial T^2}<0.$$

[See Appendix A for proof].

Lemma 3.2. *$Z_1(T_1,T)$ has the maximum value at the optimised values of T and T_1, that satisfy following equations*

$$\begin{aligned}
\text{(i)}\quad & pI_r\left\{\frac{\alpha\beta}{\theta}T_1-\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}-\frac{xv^y}{\theta}\right)\right\}-\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}\\
&-\theta c_p\left\{\frac{\alpha\beta}{\theta}T_1-\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\\
&+\theta c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}=0\\
\text{(ii)}\quad & (p-c_p)\left\{\beta(aT+b)+\frac{xv^y}{\theta}\right\}+pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\\
&\times\left\{e^{\theta T}-(1+m)e^{\theta(T-m)}\right\}+\frac{pI_e\alpha\beta}{\theta}\left\{e^{\theta T}-(1+m)e^{\theta(T-m)}\right\}\\
&+(p-c_p-c_h)(1+\alpha)\left\{-\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}\\
&+(p-c_p-c_h)(1+\alpha)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta T}\\
&+(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}e^{\theta T}-p\theta I_r\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta(T-m)}-e^{\theta(T-T_1)}\right\}\\
&-p\alpha\beta I_r\left\{e^{\theta(T-m)}-e^{\theta(T-T_1)}\right\}+\alpha\beta c_pT-\theta c_p\left\{\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}\\
&+a\beta c_pe^{\theta(T-T_1)}+\theta^2c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}
\end{aligned}$$

provided the following sufficient conditions are satisfied

$$\begin{aligned}
\text{(i)}\quad & \frac{pI_r\alpha\beta}{\theta}-pI_r\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}\\
&>\alpha\beta c_p-pI_r\theta c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}\\
\text{(ii)}\quad & (p-c_p)\alpha\beta+pI_e\theta\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+m)e^{\theta(T-m)}\right\}
\end{aligned}$$

$$+ (p - c_p - c_h)(1+\alpha)\left(\frac{1+\theta}{\theta}\right)\alpha\beta e^{\theta T}$$
$$+ \theta(p - c_p - c_h)(1+\alpha)\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\} e^{\theta T} + a\beta c_p$$
$$> (p - c_p - c_h)(1+\alpha)\frac{\alpha\beta}{\theta} + pa\beta I_r \frac{1+\theta}{\theta}\left\{e^{\theta(T-m)} - e^{\theta(T-T_1)}\right\}$$
$$+ p\theta^2 I_r \left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{e^{\theta(T-m)} - e^{\theta(T-T_1)}\right\}$$

(iii) $$\theta c_p \left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\} e^{\theta(T-T_1)} + a\beta c_p e^{\theta(T-T_1)}$$
$$> pI_r\theta \left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\} e^{\theta(T-T_1)} + \frac{a\beta p I_r}{\theta} e^{\theta(T-T_1)}.$$

[See Appendix B for proof].

Case 2. $T_1 < m < T$.

Figure 2 shows that no interest should be paid by the customer, but earns some interest at an annual rate I_e, in the time period $[0, m]$, that earned interest is E_2 (see [19]) given by

$$\begin{aligned} E_2 &= pI_e \int_0^{T_1} tD(t)\mathrm{d}t + pI_e(m - T_1)\int_0^{T_1} D(t)\mathrm{d}t \\ &= pI_e\left\{-\alpha\beta\left(\frac{aT_1^3}{3\theta} - \frac{T_1^2}{2\theta^2} + \frac{bT_1^2}{2\theta}\right) + \frac{xv^y}{2}T_1^2\left(\frac{1}{\theta}+1\right)\right. \\ &\quad \left. + \frac{a\beta T_1^2}{3} + \frac{b\beta}{2}T_1^2\right\} + pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{e^{\theta T} - (1+T_1)e^{\theta(T-T_1)}\right\} \\ &\quad + pI_e(m - T_1)\left\{-\alpha\beta\left(\frac{aT_1^3}{3\theta} - \frac{T_1}{2\theta^2} + \frac{bT_1^2}{2\theta}\right) + \frac{xv^y}{2}T_1\left(\frac{1}{\theta}+1\right) + \frac{a\beta T_1^2}{3} + \frac{b\beta}{2}T_1\right\} \\ &\quad + pi_e(m - T_1)\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{e^{\theta T} - e^{\theta(T-T_1)}\right\}. \end{aligned} \tag{3.4}$$

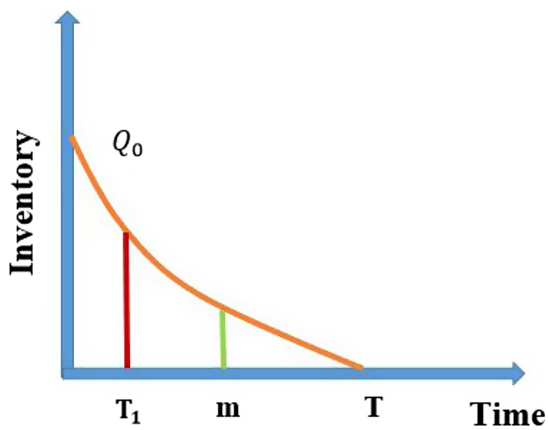

FIGURE 2. Inventory level *versus* time for Model II $T_1 < m < T$.

As all items are sold in this case so there is no payable interest for this case. Therefore, the total average profit in this case is as follows

$$
\begin{aligned}
Z_2 &= \frac{1}{T}\left[(P-c_P)\int_0^T D(I(t),t,v)\mathrm{d}t - c_0 - v - c_h\int_0^T I(t)\mathrm{d}t - \theta c_p\int_{T_1}^T I(t)\mathrm{d}t + E_2\right]\\
&= \frac{1}{T}\left[(p-c_p)\left\{\beta\left(\frac{aT^2}{2}+bT\right)+xv^yT\right\}-(c_0+V)\right.\\
&\quad + pI_e\left\{-\alpha\beta\left(\frac{aT_1^3}{3\theta}-\frac{T_1^2}{2\theta^2}+\frac{bT_1^2}{2\theta}\right)+\frac{xv^y}{2}T_1^2\left(\frac{1}{\theta}+1\right)+\frac{a\beta T_1^2}{3}+\frac{b\beta}{2}T_1^2\right\}\\
&\quad + pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}\\
&\quad + pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1^2}{3\theta}-\frac{T_1}{2\theta^2}+\frac{bT_1}{2\theta}\right)+\frac{xv^y}{2}T_1\left(\frac{1}{\theta}+1\right)+\frac{a\beta T_1}{3}+\frac{b\beta}{2}T_1\right\}\\
&\quad + pI_e(m-T_1)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-e^{\theta(T-T_1)}\right\}\\
&\quad + (p-c_p-c_h)(1+\alpha)\left\{-\beta\left(\frac{aT^2}{\theta}-\frac{T}{\theta^2}+\frac{bT}{\theta}\right)+\frac{xv^y}{\theta}T\right\}\\
&\quad + (p-c_p-c_h)(1+\alpha)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta T}+\theta c_p\frac{a\beta}{2\theta}(T^2-T_1^2)\\
&\quad - \theta c_p\left\{\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}(T-T_1)\\
&\quad \left.- \theta c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}\right].
\end{aligned}
\tag{3.5}
$$

Lemma 3.3. *$Z_2(T_1,T)$ attaints maximum value for the optimised value of T and T_1 that also satisfy the following equations*

(i)
$$
\begin{aligned}
&pI_e\left\{-\alpha\beta\left(\frac{aT_1^2}{\theta}-\frac{T_1}{\theta^2}+\frac{bT_1}{\theta}\right)+\frac{xv^y}{2}T_1\left(\frac{1}{\theta}+1\right)\right.\\
&\quad\left.+a\beta T_1^2+b\beta T_1\right\}-p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{(1+T_1)e^{\theta(T-T_1)}\right\}\\
&\quad - pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}-pI_e\left\{-\alpha\beta\left(\frac{aT_1^2}{\theta}-\frac{T_1}{\theta^2}+\frac{bT_1}{\theta}\right)+\frac{xv^y}{\theta}T_1\right\}\\
&\quad + pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}\\
&\quad - pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-e^{\theta(T-T_1)}\right\}\\
&\quad - p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}-a\beta c_pT_1+\theta c_p\left\{\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}\\
&\quad + \theta^2c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}=0
\end{aligned}
$$

$$\begin{aligned}
\text{(ii)}\qquad & (p-c_p)\left\{\beta(aT+b)+xv^y\right\}+p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\\
&\times\left\{e^{\theta T}-(1-T_1)e^{\theta(T-T_1)}\right\}+\frac{a\beta pI_e}{\theta}\left\{e^{\theta T}-(1-T_1)e^{\theta(T-T_1)}\right\}\\
&+(m-T_1)p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1-T_1)e^{\theta(T-T_1)}\right\}\\
&+(m-T_1)\frac{a\beta pI_e}{\theta}\left\{e^{\theta T}-(1-T_1)e^{\theta(T-T_1)}\right\}\\
&+(p-c_p-c_h)(1+\alpha)\left\{-\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}\\
&+\theta(p-c_p-c_h)(1+\alpha)\left\{-\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}e^{\theta T}\\
&+(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}+a\beta c_pT-\theta c_p\left\{\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}\\
&-\theta^2c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}-a\beta c_pe^{\theta(T-T_1)}=0
\end{aligned}$$

provided the following sufficient conditions are satisfied

$$\begin{aligned}
\text{(i)}\qquad & pI_e\left\{-\alpha\beta\left(\frac{2aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{2}\left(\frac{1}{\theta}+1\right)+2a\beta T_1+b\beta\right\}\\
&-pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}\\
&+p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}(1+T_1)e^{\theta(T-T_1)}\\
&-2pI_e\left\{-\alpha\beta\left(\frac{2aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}-\frac{a\alpha\beta pI_e}{\theta}(m-T_1)\\
&-pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}-p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)\right.\\
&\left.-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}-pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}\\
&-p\theta^2I_e(m-T_1)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}a\beta c_p\\
&-\theta^3c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}=0\\
\text{(ii)}\qquad & (p-c_p)a\beta+p\theta^2I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}\\
&+\frac{a\beta pI_e}{\theta}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}+(m-T_1)p\theta^2I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)\right.\\
&\left.-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}+2(m-T_1)a\beta pI_e\left\{e^{\theta T}-e^{\theta(T-T_1)}\right\}\\
&+\theta^2(p-c_p-c_h)(1+\alpha)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}e^{\theta T}\\
&+\theta(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}e^{\theta T}+\theta(p-c_p-c_h)(1+\alpha)a\beta e^{\theta T}+a\beta c_p
\end{aligned}$$

(iii)
$$> \theta(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}+\theta^3 c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$-2a\beta\theta c_p e^{\theta(T-T_1)}(1+m+\theta c_P)pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$+(\theta c_p-1)\frac{a\beta pI_e}{\theta}e^{\theta(T-T_1)} > pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta T}.$$

[See Appendix C for proof].

Case 3. Let $m \geq T$.

In this case, from Figure 3, some interest was earned by the customer from the sales revenue and this should be earned upto the permissible delay-period. For this time interval $[0, T]$ no interest should be payable.

$$pI_e\int_0^{T_1} tD(t) = pI_e\left\{-\alpha\beta\left(\frac{aT_1^3}{3\theta}-\frac{T_1^2}{2\theta}+\frac{bT_1^2}{2\theta}\right)+\frac{xv^y}{2}T_1^2\left(\frac{1}{\theta}+1\right)+\frac{a\beta T_1^2}{3}+\frac{b\beta}{2}T_1^2\right\}$$
$$+pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}.$$

Interest earned for the permissible delay $[T_1, m]$ is

$$pI_e\int_0^{T_1}(m-T_1)D(t)\mathrm{d}t = pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1^2}{3\theta}-\frac{T_1}{2\theta}+\frac{bT_1}{2\theta}\right)+\frac{xv^y}{2}T_1\left(\frac{1}{\theta}+1\right)\right.$$
$$\left.+\frac{a\beta T_1}{3}+\frac{b\beta}{2}T_1\right\}+pI_e(m-T_1)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-e^{\theta(T-T_1)}\right\}.$$

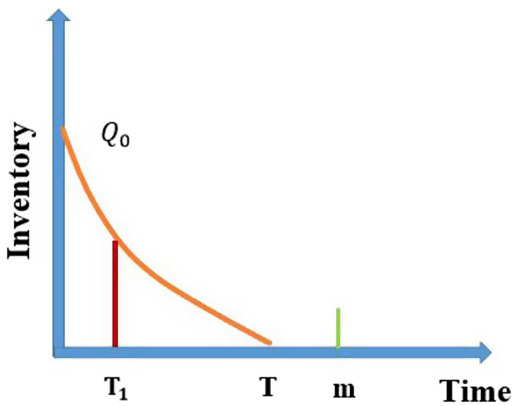

FIGURE 3. Inventory level *versus* time for Model III $m \geq T$.

Hence, the total earned during the cycle is

$$E_3 = pI_e\left\{-\alpha\beta\left(\frac{aT_1^3}{3\theta}-\frac{T_1^2}{2\theta}+\frac{bT_1^2}{2\theta}\right)+\frac{xv^y}{2}T_1^2\left(\frac{1}{\theta}+1\right)\right.$$
$$\left.+\frac{a\beta T_1^2}{3}+\frac{b\beta}{2}T_1^2\right\}+pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}$$
$$+pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1^2}{3\theta}-\frac{T_1}{2\theta}+\frac{bT_1}{2\theta}\right)+\frac{xv^y}{2}T_1\left(\frac{1}{\theta}+1\right)+\frac{a\beta T_1}{3}+\frac{b\beta}{2}T_1\right\}$$

$$+ pI_e(m - T_1)\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{e^{\theta T} - e^{\theta(T-T_1)}\right\}. \tag{3.6}$$

There is no interest should be pay due to absence of unsold items. Therefore, the total average profit is

$$\begin{aligned}
Z_3 &= \frac{1}{T}\left[(P - c_P)\int_0^T D(I(t), t, v)\mathrm{d}t - c_0 - v - c_h\int_0^T I(t)\mathrm{d}t - \theta c_p\int_{T_1}^T I(t)\mathrm{d}t + E_3\right] \\
&= \frac{1}{T}\Bigg[(p - c_p)\left\{\beta\left(\frac{aT^2}{2} + bT\right) + xv^yT\right\} - (c_0 + v) + pI_e\Bigg\{-\alpha\beta\left(\frac{aT_1^3}{3\theta} - \frac{T_1^2}{2\theta^2} + \frac{bT_1^2}{2\theta}\right) \\
&\quad + \frac{xv^y}{2}T_1^2\left(\frac{1}{\theta} + 1\right) + \frac{a\beta T_1^2}{3} + \frac{b\beta}{2}T_1^2\Bigg\} + pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\} \\
&\quad \times \left\{e^{\theta T} - (1 + T_1)e^{\theta(T-T_1)}\right\} + pI_e(m - T_1)\Bigg\{-\alpha\left(\frac{aT_1^2}{3\theta} - \frac{T_1}{2\theta^2} + \frac{bT_1}{2\theta}\right) \\
&\quad + \frac{xv^y}{2}T_1\left(\frac{1}{\theta} + 1\right) + \frac{a\beta T_1}{3} + \frac{b\beta}{2}T_1\Bigg\} + pI_e(m - T_1)\Bigg\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) \\
&\quad - \frac{xv^y}{\theta}\Bigg\}\left\{e^{\theta T} - e^{\theta(T-T_1)}\right\} + (p - c_p - c_h)(1 + \alpha)\left\{-\beta\left(\frac{aT^2}{2\theta} - \frac{T}{\theta^2} + \frac{bT}{\theta}\right) + \frac{xv^y}{\theta}T\right\} \\
&\quad + (p - c_p - c_h)(1 + \alpha)\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta T} + \theta c_p\frac{a\beta}{2\theta}(T^2 - T_1^2) \\
&\quad - \theta c_p\left\{\beta\left(\frac{1}{\theta^2} - \frac{b}{\theta}\right) + \frac{xv^y}{\theta}\right\}(T - T_1) - \theta c_p\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}\Bigg].
\end{aligned} \tag{3.7}$$

$Z_3(T_1, T)$ is now maximized.

Lemma 3.4. *The maximum value for $Z_3 = (T_1, T)$ is obtain for the optimised values of T and T_1, which satisfy the following equations*

(i)
$$\begin{aligned}
&pI_e\left\{-\alpha\beta\left(\frac{aT_1^2}{\theta} - \frac{T_1}{\theta^2} + \frac{bT_1}{\theta}\right) + \frac{xv^y}{2}T_1\left(\frac{1}{\theta} + 1\right) + a\beta T_1^2 + b\beta T_1\right\} \\
&- p\theta I_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{(1 + T_1)e^{\theta(T-T_1)}\right\} \\
&- pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} - pI_e\Bigg\{-\alpha\beta\left(\frac{aT_1^2}{\theta} - \frac{T_1}{\theta^2} + \frac{bT_1}{\theta}\right) \\
&+ \frac{xv^y}{\theta}T_1\Bigg\} + pI_e(m - T_1)\left\{-\alpha\beta\left(\frac{aT_1}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) + \frac{xv^y}{\theta}\right\} \\
&- pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{e^{\theta T} - e^{\theta(T-T_1)}\right\} \\
&- p\theta I_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} - a\beta c_pT_1 + \theta c_p\Bigg\{\beta\left(\frac{1}{\theta^2} - \frac{b}{\theta}\right) \\
&+ \frac{xv^y}{\theta}\Bigg\} + \theta^2c_p\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} = 0
\end{aligned}$$

(ii)
$$\begin{aligned}
&(p - c_p)\left\{\beta(aT + b) + xv^y\right\} + p\theta I_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\} \\
&\times \left\{e^{\theta T} - (1 - T_1)e^{\theta(T-T_1)}\right\} + \frac{a\beta pI_e}{\theta}\left\{e^{\theta T} - (1 - T_1)e^{\theta(T-T_1)}\right\}
\end{aligned}$$

$$+ (m-T_1)p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1-T_1)e^{\theta(T-T_1)}\right\}$$
$$+ (m-T_1)\frac{a\beta pI_e}{\theta}\left\{e^{\theta T}-(1-T_1)e^{\theta(T-T_1)}\right\}+(p-c_p-c_h)(1+\alpha)$$
$$\times\left\{-\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}+\theta(p-c_p-c_h)(1+\alpha)\left\{-\beta\left(\frac{aT}{\theta}\right.\right.$$
$$\left.\left.-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}e^{\theta T}+(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}+a\beta c_pT$$
$$-\theta c_p\left\{\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}-\theta^2c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$-a\beta c_pe^{\theta(T-T_1)}=0$$

provided the following sufficient conditions are satisfied

(i)
$$pI_e\left\{-\alpha\beta\left(\frac{2aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{2}\left(\frac{1}{\theta}+1\right)+2a\beta T_1+b\beta\right\}$$
$$-pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}+p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)\right.$$
$$\left.-\frac{xv^y}{\theta}\right\}(1+T_1)e^{\theta(T-T_1)}-2pI_e\left\{-\alpha\beta\left(\frac{2aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}$$
$$-\frac{a\alpha\beta pI_e}{\theta}(m-T_1)-pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)\right.$$
$$\left.+\frac{xv^y}{\theta}\right\}-p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$-pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}-p\theta^2I_e(m-T_1)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)\right.$$
$$\left.-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}a\beta c_p-\theta^3c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}=0$$

(ii)
$$(p-c_p)a\beta+p\theta^2I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}$$
$$+\frac{a\beta pI_e}{\theta}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}+(m-T_1)p\theta^2I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)\right.$$
$$\left.-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}+2(m-T_1)a\beta pI_e\left\{e^{\theta T}-e^{\theta(T-T_1)}\right\}$$
$$+\theta^2(p-c_p-c_h)(1+\alpha)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}e^{\theta T}$$
$$+\theta(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}e^{\theta T}+\theta(p-c_p-c_h)(1+\alpha)a\beta e^{\theta T}+a\beta c_p$$
$$>\theta(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}+\theta^3c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$-2a\beta\theta c_pe^{\theta(T-T_1)}$$

(iii)
$$(1+m+\theta c_P)pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$+(\theta c_p-1)\frac{a\beta pI_e}{\theta}e^{\theta(T-T_1)}>pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta T}.$$

[See Appendix D for proof].

One has to find maximum value of

$$Z(T_1, T) = \begin{Bmatrix} Z_1(T_1, T) & m \leq T_1 < T \\ Z_2(T_1, T) & T_1 < m < T \\ Z_3(T_1, T) & m \geq T \end{Bmatrix}. \tag{3.8}$$

4. Numerical examples

To prove the applicability of this model a numerical example is discussed here for the three cases (which are discussed in the Model formulation section). For the numerical result, the parametric values for inventory system are considered as on Khanra *et al.* [19], which are as follows: $a = 200$ units per month, $b = 1.8$ units per month, $\alpha = 0.002$ units per month, $v = \$2$ per unit, $\beta = 0.6$ unit per month, $x = 0.7$ unit per month, $y = 0.7$ unit per month, $c_0 = \$90$ per order, $c_p = \$1.8$ per order, $I_r = \$0.12$ per month, $I_e = \$0.10$ per month, $p = \$7$ per unit, $c_h = \$1.2$ per month, $p = \$2$ per unit and $m = 0.3$ year. Then by using the equations (3.3), (3.5), and (3.7), the optimum results for different cases are shown in the following Table 2. From the optimum result Table 2, one can easily find that the total profit is \$1488.85 when permissible delay period is less then length of time in which inventory reach the zero level and profit is \$1591.56, when delay period is grater than total cycle time but the optimum profit is obtain when delay period is in between cycle time and time length when inventory reach to the zero level. The bold values shows the optimal result of this model.

Table 2. Optimal result for different cases.

	Cycle Length T^* (Year)	Time length inventory reach zero level T_1^* (Year)	Total profit $Z_i^*(T^*, T_1^*)$ (\$)
Case I: $m \leq T_1 \leq T$	1.65	0.42	\$1488.85
Case II: $T_1 < m < T$	**0.92**	**0.53**	**\$1840.41**
Case III: $m \geq T$	1.13	0.74	\$1591.56

Hence, the maximum average profit in this case is $Z_2^*(T_1, T) = \$1840.41$ where $T_1^* = 0.53$ year, $T^* = 0.92$ year.

5. Sensitivity analysis

In this section contained the effect of change of the parametric values on total profit by increasing and decreasing 50% and 20% of the parameter a, b, α, β, x, y, c_p, v, I_e, h, and C_0 respectively.

The effect of change for total profit is shown in Table 3 and a graphical representation (Fig. 4) also provide to show the effect of parametric value over the total profit:

From the sensitivity analysis Table 3, one can easily find that the effect of different parameters as follows:

(i) The effect of scaling parameter a is very much sensitive, small change in a, there is a huge impact on total system profit.
(ii) The parameter β is much more effective in total system profit. Increase or decrease in β is very sensitive for total profit.
(iii) The production cost c_p has a great impact in the total system profit.
(iv) The holding cost is more sensitive as usual. Increase of holding cost is always harmful for any production industry.

(v) The initial setup cost is also very crucial factor for any industry, that was clearly shown in the sensitivity analysis table.
(vi) The earned interest I_e is quite sensitive for any economic order quantity model that was clear from the sensitivity analysis table.
(vii) The shape parameter y for advertising parameter is little bit sensitive in the calculation of profit of the system.
(viii) The other parameters b, α, v, and x are little bit sensitive in the profit calculation.

TABLE 3. Sensitivity analysis table.

Parameters	Change(in %) (in %)	Optimal profit $Z(T_1^*, T^*)$	Change $Z(T_1^*, T^*)$	Parameters	Change (in %)	Optimal profit $Z(T_1^*, T^*)$	Change $Z(T_1^*, T^*)$
	50%	2809.15	+52.64		50%	2808.64	+52.61
	25%	2324.78	+26.32		25%	2324.52	+26.30
a	−25%	1356.03	−26.32	β	−25%	1356.29	−26.30
	−50%	871.66	−52.64		−50%	872.17	−52.61
	50%	1844.50	+0.22		50%	1842.37	+00.11
	25%	1842.45	+00.11		25%	1841.39	+0.05
b	−25%	1838.36	−00.11	x	−25%	1839.42	−0.05
	−50%	1836.32	−00.22		−50%	1838.44	−00.11
	50%	1842.56	+0.12		50%	1841.48	+0.059
	25%	1841.49	+0.06		25%	1840.91	+0.028
α	−25%	1839.33	−0.06	y	−25%	1839.96	−0.024
	−50%	1838.25	−0.12		−50%	1839.56	−0.046
	50%	1840.61	+0.011		50%	1498.97	−18.55
	25%	1840.53	+0.006		25%	1513.50	−17.76
v	−25%	1840.23	−0.009	c_p	−25%	2167.31	+17.76
	−50%	1839.98	−0.023		−50%	2494.22	+35.52
	50%	1683.64	−8.52		50%	1513.55	−17.76
	25%	1762.02	−4.26		25%	1676.98	−8.88
I_e	−25%	1918.79	+4.26	h	−25%	2003.84	+8.88
	−50%	1997.17	+8.52		−50%	2167.26	+17.76
	50%	1791.49	−2.66				
	25%	1815.95	−1.33				
C_0	−20%	1864.86	+1.33				
	−50%	1889.32	+2.66				

6. Managerial insights

This model deals with an inventory model for items with stock level and advertising dependent demand pattern by considering shortages under trade-credit policy. The managerial insights of this chapter are given as follows.

(i) The main aim of any inventory model to optimise profit. DSL has a great impact on optimization of profit. The vendor always want to sell his own product more, compare to other product. The common policy for any vendor that no customer will go away from his/her shop with out the required product, and can easily see the stock of product of his/her shop. Normally the demand will increase if the stored product is higher than the demand. Casually, profit will optimised if vendor can sell more product. In this model one can easily find out that, when demand depend on the on-hand inventory, the profit was optimised.

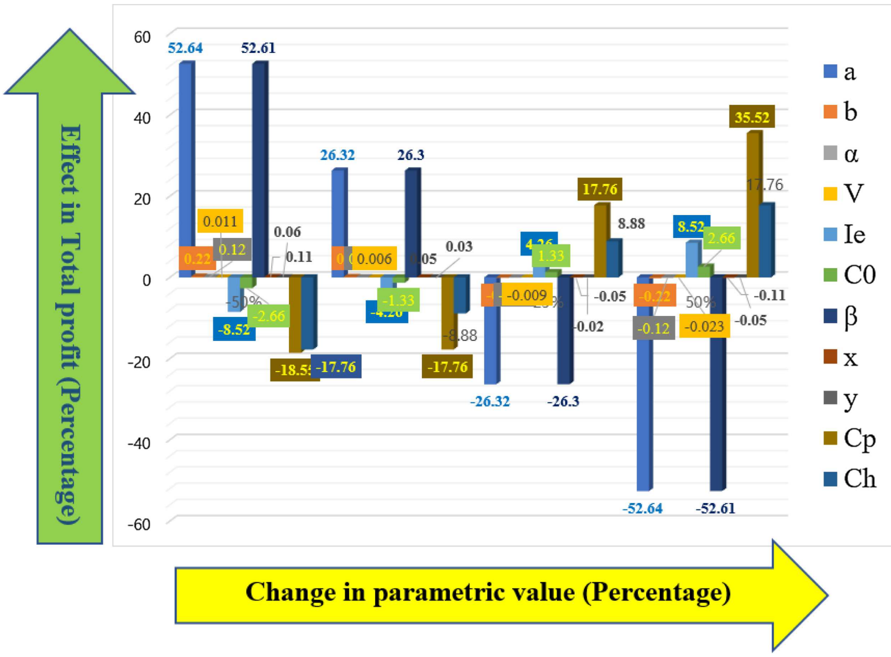

FIGURE 4. Effect of change in parametric value in total profit.

(ii) Any company wants to sell more items by using different type of sales advertisement policy. The main aim of advertising is to aware or inform the general public about the product, it can compared the difference of the other products and influence the common public to buy a particular product. Thus the sell of a particular product's automatically increase by a better advertisement.
Thus advertising for any product is one of the most crucial factor in any industry to increase the profit in modern competitive marketing environment.

(iii) To increase the sell, retailer/supplier offer a certain credit period to their customers without any penalty during the said delay time period, it motivates customer to order more quantities. In this regard the trade credit policy is considered in this model. Thus, sell of a particular product increases by considering trade-credit policy, which directly optimize the total system profit. Thus, from the managerial point of view, this strategy is valid and useful to this competitive business market.

7. Conclusions

The proposed model extends the existing literature by considering finite replenishment rate, stock-dependent and advertising demand and two progressive periods for delay-in-payments. The analytic result proves the global optimality of the solutions. Some theorems also proved to show that the optimal solution is global. The numerical results proved that the model obtained the global optimum profit at the optimal solutions. This model can be extended by considering shortages of the product, deterioration which is time-varying, for multi-item perishable products (see [33]). This model can also be extended by considering environmental effect or by considering the effect of carbon emission [1, 25, 41], which is one of the most important issue in today's business environment. One can develop this model to a supply chain model along with the consideration of variable production rate and variable setup cost [9, 34]. To identify the deteriorate items one can use the autonomation policy [10] along with the latest Radio Frequency Identify Device (RFID) technology [55]. This will be another good research if one considered energy consumption for carbon emission [2] along with increased system reliability [16]. The fully deteriorated items are treated as waste, those waste items management is one another interesting research in this direction [42, 45, 53].

Appendix A.

Proof. One can have $U(T_1,T) = \frac{1}{T}W(T_1,T)$. For maximum value of $U(T_1,T)$ at $T_1 = T_1^*, T = T^*$, necessary condition are $\frac{\partial U(T_1,T)}{\partial T_1} = 0$ and $\frac{\partial U(T_1,T)}{\partial T} = 0$.

Now

$$\frac{\partial U(T_1,T)}{\partial T} = \frac{T\frac{\partial W(T_1,T)}{\partial T} - W(T_1,T)}{T^2} \quad \text{and} \quad \frac{\partial U(T_1,T)}{\partial T_1} = \frac{1}{T}\frac{\partial W(T_1,T)}{\partial T_1}.$$

Therefore, $\frac{\partial U(T_1,T)}{\partial T} = 0$ gives $T\frac{\partial W(T_1,T)}{\partial T} - W(T_1,T) = 0$ and $\frac{\partial U(T_1,T)}{\partial T_1} = 0$ gives $\frac{\partial W(T_1,T)}{\partial T_1} = 0$. This equations are satisfied if one considered $T = T^*$ and $T_1 = T_1^*$. Again,

$$\frac{\partial^2 U(T_1,T)}{\partial T^2} = \frac{T\left(T\frac{\partial^2 W(T_1,T)}{\partial T^2} + \frac{\partial W(T_1,T)}{\partial T} - \frac{\partial W(T_1,T)}{\partial T}\right) - 2T\left(T\frac{\partial W(T_1,T)}{\partial T} - W(T_1,T)\right)}{T^4}.$$

Thus, for $T = T^*$ and $T_1 = T_1^*$, one can write $\frac{\partial^2 U(T_1,T)}{\partial T^2} = \frac{1}{T}\frac{\partial^2 W(T_1,T)}{\partial T^2}$.

Similarly, at $T = T^*$ and $T_1 = T_1^*$, one can obtain $\frac{\partial^2 U(T_1,T)}{\partial T_1^2} = \frac{1}{T}\frac{\partial^2 W(T_1,T)}{\partial T_1^2}$ and $\frac{\partial^2 U(T_1,T)}{\partial T\partial T_1} = \frac{1}{T}\frac{\partial^2 W(T_1,T)}{\partial T\partial T_1}$. Hence,

$$\begin{vmatrix} \frac{\partial^2 U(T_1,T)}{\partial T_1^2} & \frac{\partial^2 U(T_1,T)}{\partial T_1\partial T} \\ \frac{\partial^2 U(T_1,T)}{\partial T_1\partial T} & \frac{\partial^2 U(T_1,T)}{\partial T^2} \end{vmatrix} = \begin{vmatrix} \frac{1}{T}\frac{\partial^2 W(T_1,T)}{\partial T_1^2} & \frac{1}{T}\frac{\partial^2 W(T_1,T)}{\partial T_1\partial T} \\ \frac{1}{T}\frac{\partial^2 W(T_1,T)}{\partial T_1\partial T} & \frac{1}{T}\frac{\partial^2 W(T_1,T)}{\partial T^2} \end{vmatrix} = \frac{1}{T^2}\begin{vmatrix} \frac{\partial^2 W(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W(T_1,T)}{\partial T_1\partial T} \\ \frac{\partial^2 W(T_1,T)}{\partial T_1\partial T} & \frac{\partial^2 W(T_1,T)}{\partial T^2} \end{vmatrix}.$$

Thus,

$$\text{if} \begin{vmatrix} \frac{\partial^2 W(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W(T_1,T)}{\partial T_1\partial T} \\ \frac{\partial^2 W(T_1,T)}{\partial T_1\partial T} & \frac{\partial^2 W(T_1,T)}{\partial T^2} \end{vmatrix} > 0 \quad \text{and} \quad \frac{\partial^2 W(T_1,T)}{\partial T_1^2} < 0$$
$$\text{then} \begin{vmatrix} \frac{\partial^2 U(T_1,T)}{\partial T_1^2} & \frac{\partial^2 U(T_1,T)}{\partial T_1\partial T} \\ \frac{\partial^2 U(T_1,T)}{\partial T_1\partial T} & \frac{\partial^2 U(T_1,T)}{\partial T^2} \end{vmatrix} > 0 \quad \text{and} \quad \frac{\partial^2 U(T_1,T)}{\partial T_1^2} < 0$$

which indicates $U(T_1,T)$ is maximum. □

Appendix B.

Proof. To maximize $Z_1(T_1,T) = \frac{1}{T}W_1(T_1,T)$, the necessary conditions are as $\frac{\partial Z_1(T_1,T)}{\partial T_1} = 0$, $\frac{\partial Z_1(T_1,T)}{\partial T} = 0$ and the above theorem indicate the sufficiency of the condition are

$$\frac{\partial^2 W_1(T_1,T)}{\partial^2 T_1} < 0, \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 W_1(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W_1(T_1,T)}{\partial T_1\partial T} \\ \frac{\partial^2 W_1(T_1,T)}{\partial T_1\partial T} & \frac{\partial^2 W_1(T_1,T)}{\partial T^2} \end{vmatrix} > 0.$$

Now, $\frac{\partial Z_1(T_1,T)}{\partial T_1} = 0$ gives

$$\text{(i)} \qquad pI_r\left\{\frac{\alpha\beta}{\theta}T_1 - \beta\left(\frac{1}{\theta^2} - \frac{b}{\theta} - \frac{xv^y}{\theta}\right)\right\} - \left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$- \theta c_p\left\{\frac{\alpha\beta}{\theta}T_1 - \beta\left(\frac{1}{\theta^2} - \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\} + \theta c_p\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$= 0.$$

Also, $\frac{\partial Z_1(T_1,T)}{\partial T} = 0$ provides

$$\text{(ii)} \quad (p-c_p)\left\{\beta(aT+b)+\frac{xv^y}{\theta}\right\}+pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+m)e^{\theta(T-m)}\right\}$$
$$+\frac{pI_e\alpha\beta}{\theta}\left\{e^{\theta T}-(1+m)e^{\theta(T-m)}\right\}+(p-c_p-c_h)(1+\alpha)\left\{-\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}$$
$$+(p-c_p-c_h)(1+\alpha)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta T}+(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}e^{\theta T}$$
$$-p\theta I_r\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta(T-m)}-e^{\theta(T-T_1)}\right\}-p\alpha\beta I_r\left\{e^{\theta(T-m)}-e^{\theta(T-T_1)}\right\}$$
$$+\alpha\beta c_pT-\theta c_p\left\{\beta\left(\frac{1}{\theta^2}-\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}+a\beta c_pe^{\theta(T-T_1)}+\theta^2c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$=0.$$

By solving the equations, one can obtain optimal values of T_1 and T. Using the above theorem,

$$\frac{\partial^2 W_1(T_1,T)}{\partial^2 T_1}<0, \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 W_1(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W_1(T_1,T)}{\partial T_1\partial T} \\ \frac{\partial^2 W_1(T_1,T)}{\partial T_1\partial T} & \frac{\partial^2 W_1(T_1,T)}{\partial T^2} \end{vmatrix}>0.$$

The following results are obtained

$$\text{(i)} \quad \frac{pI_r\alpha\beta}{\theta}-pI_r\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$
$$>\alpha\beta c_p-pI_r\theta c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}$$

and

$$\text{(ii)} \quad (p-c_p)\alpha\beta+pI_e\theta\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+m)e^{\theta(T-m)}\right\}$$
$$+(p-c_p-c_h)(1+\alpha)\left(\frac{1+\theta}{\theta}\right)a\beta e^{\theta T}$$
$$+\theta(p-c_p-c_h)(1+\alpha)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta T}+a\beta c_p$$
$$>(p-c_p-c_h)(1+\alpha)\frac{\alpha\beta}{\theta}+pa\beta I_r\frac{1+\theta}{\theta}\left\{e^{\theta(T-m)}-e^{\theta(T-T_1)}\right\}$$
$$+p\theta^2I_r\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta(T-m)}-e^{\theta(T-T_1)}\right\}$$

$$\text{(iii)} \quad \theta c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}+a\beta c_pe^{\theta(T-T_1)}$$
$$>pI_r\theta\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}+\frac{a\beta pI_r}{\theta}e^{\theta(T-T_1)}.$$

□

Appendix C.

Proof. To maximize $Z_2(T_1, T) = \frac{1}{T} W_2(T_1, T)$, the necessary conditions are as $\frac{\partial Z_2(T_1,T)}{\partial T_1} = 0$, $\frac{\partial Z_2(T_1,T)}{\partial T} = 0$ and the sufficient condition are $\frac{\partial^2 W_2(T_1,T)}{\partial^2 T_1} < 0$, and

$$\begin{vmatrix} \frac{\partial^2 W_2(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W_2(T_1,T)}{\partial T_1 \partial T} \\ \frac{\partial^2 W_2(T_1,T)}{\partial T_1 \partial T} & \frac{\partial^2 W_2(T_1,T)}{\partial T^2} \end{vmatrix} > 0.$$

Now, $\frac{\partial Z_2(T_1,T)}{\partial T_1} = 0$ gives

$$\begin{aligned}
\text{(i)} \qquad & pI_e \left\{ -\alpha\beta \left(\frac{aT_1^2}{\theta} - \frac{T_1}{\theta^2} + \frac{bT_1}{\theta} \right) + \frac{xv^y}{2} T_1 \left(\frac{1}{\theta} + 1 \right) + a\beta T_1^2 + b\beta T_1 \right\} \\
& - p\theta I_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} \left\{ (1 + T_1) e^{\theta(T - T_1)} \right\} \\
& - pI_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} e^{\theta(T - T_1)} \\
& - pI_e \left\{ -\alpha\beta \left(\frac{aT_1^2}{\theta} - \frac{T_1}{\theta^2} + \frac{bT_1}{\theta} \right) + \frac{xv^y}{\theta} T_1 \right\} \\
& + pI_e (m - T_1) \left\{ -\alpha\beta \left(\frac{aT_1}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) + \frac{xv^y}{\theta} \right\} \\
& - pI_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} \left\{ e^{\theta T} - e^{\theta(T - T_1)} \right\} \\
& - p\theta I_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} e^{\theta(T - T_1)} - a\beta c_p T_1 + \theta c_p \left\{ \beta \left(\frac{1}{\theta^2} - \frac{b}{\theta} \right) + \frac{xv^y}{\theta} \right\} \\
& + \theta^2 c_p \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} e^{\theta(T - T_1)} = 0.
\end{aligned}$$

Also, $\frac{\partial Z_2(T_1,T)}{\partial T} = 0$ provides

$$\begin{aligned}
\text{(ii)} \qquad & (p - c_p) \left\{ \beta(aT + b) + xv^y \right\} + p\theta I_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) \right. \\
& \left. - \frac{xv^y}{\theta} \right\} \left\{ e^{\theta T} - (1 - T_1) e^{\theta(T - T_1)} \right\} + \frac{a\beta p I_e}{\theta} \left\{ e^{\theta T} - (1 - T_1) e^{\theta(T - T_1)} \right\} \\
& + (m - T_1) p\theta I_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} \left\{ e^{\theta T} - (1 - T_1) e^{\theta(T - T_1)} \right\} \\
& + (m - T_1) \frac{a\beta p I_e}{\theta} \left\{ e^{\theta T} - (1 - T_1) e^{\theta(T - T_1)} \right\} \\
& + (p - c_p - c_h)(1 + \alpha) \left\{ -\beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) + \frac{xv^y}{\theta} \right\} \\
& + \theta (p - c_p - c_h)(1 + \alpha) \left\{ -\beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) + \frac{xv^y}{\theta} \right\} e^{\theta T} \\
& + (p - c_p - c_h)(1 + \alpha) \frac{a\beta}{\theta} + a\beta c_p T - \theta c_p \left\{ \beta \left(\frac{1}{\theta^2} - \frac{b}{\theta} \right) + \frac{xv^y}{\theta} \right\} \\
& - \theta^2 c_p \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} e^{\theta(T - T_1)} - a\beta c_p e^{\theta(T - T_1)} = 0.
\end{aligned}$$

By solving the equations, one can obtain optimal values of T_1 and T. Using the above theorem,

$$\frac{\partial^2 W_2(T_1,T)}{\partial^2 T_1} < 0, \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 W_2(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W_2(T_1,T)}{\partial T_1 \partial T} \\ \frac{\partial^2 W_2(T_1,T)}{\partial T_1 \partial T} & \frac{\partial^2 W_2(T_1,T)}{\partial T^2} \end{vmatrix} > 0$$

one can obtain the following results

$$\begin{aligned}
\text{(i)} \qquad & pI_e\left\{-\alpha\beta\left(\frac{2aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{2}\left(\frac{1}{\theta}+1\right)+2a\beta T_1+b\beta\right\} \\
& -pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} \\
& +p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}(1+T_1)e^{\theta(T-T_1)} \\
& -2pI_e\left\{-\alpha\beta\left(\frac{2aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}-\frac{a\alpha\beta pI_e}{\theta}(m-T_1) \\
& -pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}-p\theta I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)\right. \\
& \left.-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}-pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} \\
& -p\theta^2 I_e(m-T_1)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}a\beta c_p \\
& -\theta^3 c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}=0 \\
\text{(ii)} \qquad & (p-c_p)a\beta+p\theta^2 I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\} \\
& +\frac{a\beta pI_e}{\theta}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}+(m-T_1)p\theta^2 I_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)\right. \\
& \left.-\frac{xv^y}{\theta}\right\}\left\{e^{\theta T}-(1+T_1)e^{\theta(T-T_1)}\right\}+2(m-T_1)a\beta pI_e\left\{e^{\theta T}-e^{\theta(T-T_1)}\right\} \\
& +\theta^2(p-c_p-c_h)(1+\alpha)\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)+\frac{xv^y}{\theta}\right\}e^{\theta T} \\
& +\theta(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}e^{\theta T}+\theta(p-c_p-c_h)(1+\alpha)a\beta e^{\theta T}+a\beta c_p \\
& >\theta(p-c_p-c_h)(1+\alpha)\frac{a\beta}{\theta}+\theta^3 c_p\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} \\
\text{(iii)} \qquad & -2a\beta\theta c_p e^{\theta(T-T_1)}(1+m+\theta c_P)pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} \\
& +(\theta c_p-1)\frac{a\beta pI_e}{\theta}e^{\theta(T-T_1)}>pI_e\left\{\beta\left(\frac{aT}{\theta}-\frac{1}{\theta^2}+\frac{b}{\theta}\right)-\frac{xv^y}{\theta}\right\}e^{\theta T}.
\end{aligned}$$

□

Appendix D.

Proof. To maximize $Z_3(T_1,T) = \frac{1}{T}W_3(T_1,T)$, necessary conditions are as $\frac{\partial Z_3(T_1,T)}{\partial T_1} = 0$, $\frac{\partial Z_3(T_1,T)}{\partial T} = 0$ and the sufficient condition are

$$\frac{\partial^2 W_3(T_1,T)}{\partial^2 T_1} < 0, \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 W_3(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W_3(T_1,T)}{\partial T_1 \partial T} \\ \frac{\partial^2 W_3(T_1,T)}{\partial T_1 \partial T} & \frac{\partial^2 W_3(T_1,T)}{\partial T^2} \end{vmatrix} > 0.$$

Now, $\frac{\partial Z_3(T_1,T)}{\partial T_1} = 0$ gives

$$\begin{aligned}
\text{(i)} \qquad & pI_e\left\{-\alpha\beta\left(\frac{aT_1^2}{\theta} - \frac{T_1}{\theta^2} + \frac{bT_1}{\theta}\right) + \frac{xv^y}{2}T_1\left(\frac{1}{\theta}+1\right)\right. \\
& \left.+ a\beta T_1^2 + b\beta T_1\right\} - p\theta I_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{(1+T_1)e^{\theta(T-T_1)}\right\} \\
& - pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} - pI_e\left\{-\alpha\beta\left(\frac{aT_1^2}{\theta} - \frac{T_1}{\theta^2} + \frac{bT_1}{\theta}\right)\right. \\
& \left.+ \frac{xv^y}{\theta}T_1\right\} + pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) + \frac{xv^y}{\theta}\right\} \\
& - pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}\left\{e^{\theta T} - e^{\theta(T-T_1)}\right\} \\
& - p\theta I_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} - a\beta c_p T_1 + \theta c_p\left\{\beta\left(\frac{1}{\theta^2} - \frac{b}{\theta}\right) + \frac{xv^y}{\theta}\right\} \\
& + \theta^2 c_p\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} = 0.
\end{aligned}$$

By solving the equations, one can obtain optimal values of T_1 and T. Using the above theorem,

$$\frac{\partial^2 W_3(T_1,T)}{\partial^2 T_1} < 0, \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 W_3(T_1,T)}{\partial T_1^2} & \frac{\partial^2 W_3(T_1,T)}{\partial T_1 \partial T} \\ \frac{\partial^2 W_3(T_1,T)}{\partial T_1 \partial T} & \frac{\partial^2 W_3(T_1,T)}{\partial T^2} \end{vmatrix} > 0.$$

The following results are obtained

$$\begin{aligned}
\text{(i)} \qquad & pI_e\left\{-\alpha\beta\left(\frac{2aT_1}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) + \frac{xv^y}{2}\left(\frac{1}{\theta}+1\right) + 2a\beta T_1 + b\beta\right\} \\
& - pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} \\
& + p\theta I_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}(1+T_1)e^{\theta(T-T_1)} \\
& - 2pI_e\left\{-\alpha\beta\left(\frac{2aT_1}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) + \frac{xv^y}{\theta}\right\} - \frac{a\alpha\beta pI_e}{\theta}(m-T_1) \\
& - pI_e(m-T_1)\left\{-\alpha\beta\left(\frac{aT_1}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) + \frac{xv^y}{\theta}\right\} - p\theta I_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right)\right. \\
& \left.- \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} - pI_e\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)} \\
& - p\theta^2 I_e(m-T_1)\left\{\beta\left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta}\right) - \frac{xv^y}{\theta}\right\}e^{\theta(T-T_1)}a\beta c_p
\end{aligned}$$

$$- \theta^3 c_p \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} e^{\theta(T-T_1)} = 0$$

(ii)
$$(p - c_p)a\beta + p\theta^2 I_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} \left\{ e^{\theta T} - (1+T_1)e^{\theta(T-T_1)} \right\}$$
$$+ \frac{a\beta p I_e}{\theta} \left\{ e^{\theta T} - (1+T_1)e^{\theta(T-T_1)} \right\} + (m - T_1)p\theta^2 I_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) \right.$$
$$\left. - \frac{xv^y}{\theta} \right\} \left\{ e^{\theta T} - (1+T_1)e^{\theta(T-T_1)} \right\} + 2(m - T_1)a\beta p I_e \left\{ e^{\theta T} - e^{\theta(T-T_1)} \right\}$$
$$+ \theta^2 (p - c_p - c_h)(1+\alpha) \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) + \frac{xv^y}{\theta} \right\} e^{\theta T}$$
$$+ \theta(p - c_p - c_h)(1+\alpha)\frac{a\beta}{\theta} e^{\theta T} + \theta(p - c_p - c_h)(1+\alpha)a\beta e^{\theta T} + a\beta c_p$$
$$> \theta(p - c_p - c_h)(1+\alpha)\frac{a\beta}{\theta} + \theta^3 c_p \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} e^{\theta(T-T_1)}$$
$$- 2a\beta\theta c_p e^{\theta(T-T_1)}$$

(iii)
$$(1 + m + \theta c_P)pI_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} e^{\theta(T-T_1)}$$
$$+ (\theta c_p - 1)\frac{a\beta p I_e}{\theta} e^{\theta(T-T_1)} > pI_e \left\{ \beta \left(\frac{aT}{\theta} - \frac{1}{\theta^2} + \frac{b}{\theta} \right) - \frac{xv^y}{\theta} \right\} e^{\theta T}.$$

□

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