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OPTIMALITY CONDITIONS FOR NONSMOOTH INTERVAL-VALUED AND MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING

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Abstract. We consider a nonsmooth semi-infinite interval-valued vector programming problem, where the objectives and constraint functions need not to be locally Lipschitz. Using Abadie's constraint qualification and convexificators, we provide Karush–Kuhn–Tucker necessary optimality conditions by converting the initial problem into a bi-criteria optimization problem. Furthermore, we establish sufficient optimality conditions under the asymptotic convexity assumption.

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1. Introduction

Over the last two decades, optimization problems in which the objective and/or the constraint functions are supposed to take interval values have sparked intense research efforts. Such problems provide a powerful tool to account the effects of uncertainties or any unexpected errors in the final efficient solutions. In a similar fashion to stochastic programming, uncertain variables are represented as intervals of real numbers or functions, by assuming these intervals of variations to be known. The idea of deterministic modeling of uncertainty by considering the coefficients as intervals or sets goes back to [3, 5] in the case of linear programming. This technique was applied later by Ishibuchi and Tanaka [13] to study multiobjective linear programs.

Recently, treating the uncertainty in a nonlinear program with interval-valued quantities was considered by Wu [30]. The problem consisted of a scalar interval-valued function and finitely many real-valued constraint functions. In particular, he proposed two ordering relationships for closed real intervals to introduce two solution concepts, and then to derive Karush–Kuhn–Tucker (KKT) optimality conditions. Later, in [31], after stating the continuity and differentiability concepts for interval-valued functions, necessary optimality conditions were established. These results were extended in [33] for interval-valued objective functions under weakly continuous differentiability and invexity assumptions. In [27], a new concept of generalized differentiability for interval-valued functions was proposed, and later used to establish necessary and sufficient optimality conditions [23,26]. In [19], The KKT optimality conditions for interval-valued semi-infinite programs were derived by considering generalized invexity assumptions.

Keywords. Multiobjective semi-infinite programming, interval-valued functions, Karush–Kuhn–Tucker optimality conditions, Convexificators, Abadie's constraint qualification.

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More recently, considerable interests have centered about studying multiobjective interval-valued optimization problems. In [24,25], necessary and sufficient efficiency conditions were formulated for various concepts of efficient solutions. Using Clarke's subdifferential, extension of the previous results was proposed in [2] for nonsmooth vector optimization problems with locally Lipschitz interval-valued multiobjective functions. In [29], the case of semi-infinite programming with infinite number of real-valued convex constraints was investigated, and KKT optimality conditions were derived for convex interval-valued multiobjective functions.

In the present paper, we seek to develop KKT type optimality conditions for semi-infinite programs where the multiobjective function and constraints are both interval-valued but need not be locally Lipschitz. We introduce Abadie's constraint qualification by making use of upper convexifactors to deal with the non-smoothness in our problem. Convexifactors were introduced by Demyanov [6], and later investigated in many papers [7–9, 14]. It is worth pointing out that this concept can be seen as a convenient extension of some known subdifferentials, like those of Clarke [4], Michel–Penot [21], Mordukhovich [22] and Treiman [28]. Therefore, the advantage of using convexifactors includes obtaining sharper optimality conditions than what we get when using Clarke's, Michel–Penot's or other subdifferentials.

The paper is structured as follows: In Section 2, we set up notation, recall definitions and state two results for later use. Section 3 introduces the interval-valued and multiobjective semi-infinite program studied in this paper, and presents an equivalent reformulation. To characterize the weak efficient solutions, we establish necessary KKT conditions in Section 4 by using Abadie's constraint qualification, and give sufficient optimality conditions under a generalized asymptotic convexity condition in Section 5. We illustrate the obtained results by providing an example in Section 6. Finally, we draw our conclusions in the final section.

2. Preliminaries

For any points x and y in $\mathbb{R}^n$, we write $x < y$ if $x_i < y_i$ for all $i = 1, 2, \dots, n$, and $x \leq y$ if $x_i \leq y_i$ for all $i = 1, 2, \dots, n$ with strict inequality holding for at least one i.

Let S be a nonempty set in $\mathbb{R}^n$. We denote by $co\, S$, $int\, S$, S°, $cl\, S$ and $cl\, co\, S$ the convex hull, interior, polar cone, closure and closed convex hull of S, respectively. At a given point $\overline{x}$ in $cl\, S$, the cone of feasible directions $D_S(\overline{x})$, the tangent cone $T_S(\overline{x})$, and the normal cone $N_S(\overline{x})$ are defined with respect to S by

$$D_S(\overline{x}) := \{d \in \mathbb{R}^n : \exists \delta > 0,\ \forall \lambda \in (0, \delta),\ \overline{x} + \lambda d \in S\},$$
$$T_S(\overline{x}) := \{d \in \mathbb{R}^n : \exists t_n \downarrow 0,\ \exists d_n \to d,\ \overline{x} + t_n d_n \in S\},$$
$$N_S(\overline{x}) := \{\zeta \in \mathbb{R}^n : \langle \zeta, d \rangle \leq 0,\ \forall d \in T_S(\overline{x})\} = T_S(\overline{x})^\circ.$$

Remark 2.1. The cone $D_S(\overline{x})$ is neither closed nor convex necessarily, while $T_S(\overline{x})$ is closed but not necessarily convex. We have, in general, $D_S(\overline{x}) \subset T_S(\overline{x})$.

The convex cone generated by S is the set containing all conic combinations of the elements of S, which can be expressed as follows:

$$cone(S) := \left\{ y \in \mathbb{R}^n : y = \sum_{i=1}^{l} \lambda_i y_i,\ \lambda_i \geq 0,\ y_i \in S,\ i = 1, 2, \dots, l,\ l \geq 0 \right\}.$$

Definition 2.2 ([10])**.** A nonempty set $S \subset \mathbb{R}^n$ is said to be locally star-shaped at $\overline{x} \in S$, if there exists a scalar $\delta \in (0, 1]$ such that $\overline{x} + \lambda(x - \overline{x}) \in S$, for all $\lambda \in (0, \delta)$.

Note that open sets and convex sets are locally star-shaped at each of their elements, and cones are locally star-shaped at the origin. Moreover, if S is closed and locally star-shaped at each $x \in S$, then S is convex (see [18]).

Lemma 2.3 ([16])**.** *Let S be locally star-shaped at $\overline{x} \in S$. Then* $T_S(\overline{x}) = cl\,(D_S(\overline{x}))$.

Now, Let $f:\mathbb{R}^n\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{+\infty\}$. The lower and upper Dini derivatives of f at $x\in dom\, f$ in the direction of a vector $v\in\mathbb{R}^n$ are defined, respectively, by

$$f^-(x,v):=\liminf_{t\downarrow 0}\frac{f(x+tv)-f(x)}{t},$$
$$f^+(x,v):=\limsup_{t\downarrow 0}\frac{f(x+tv)-f(x)}{t}.$$

Next, let us recall some notions of upper convexificators of f given by Jeyakumar and Luc [14].

Definition 2.4. Let $x\in\mathbb{R}^n$. f is said to have an

(i) upper convexificator $\partial^* f(x)$ at x if this set is closed, and for all $v\in\mathbb{R}^n$,

$$f^-(x,v)\le\sup_{\xi\in\partial^* f(x)}\langle\xi,v\rangle;$$

(ii) upper semi-regular convexificator (USRC) $\partial^* f(x)$ at x if this set is closed, and for all $v\in\mathbb{R}^n$,

$$f^+(x,v)\le\sup_{\xi\in\partial^* f(x)}\langle\xi,v\rangle;\tag{2.1}$$

(iii) upper regular convexificator $\partial^* f(x)$ at x if equality holds in (2.1).

Obviously, each upper regular is an upper semi-regular, and each upper semi-regular is an upper convexificator of f at x. However, the converse claim is not necessarily true (see [1], Example 2.2).

We close the list of notation with interval-valued related concepts (for more details see [30]). First, we denote the class of all closed intervals in $\mathbb{R}$ by $\mathcal{I}$. For two elements $A=\left[a^L,a^U\right]$ and $B=\left[b^L,b^U\right]$ in $\mathcal{I}$, we say that $A\le_{LU}B$ if $a^L\le b^L$ and $a^U\le b^U$ with at least one strict inequality. We write $A<_{LU}B$ if $a^L<b^L$ and $a^U<b^U$. On the other hand, $A=(A_1,\ldots,A_p)$ is called an interval-valued vector if $A_k=\left[a_k^L,a_k^U\right]\in\mathcal{I}$ for each $k=1,\ldots,p$. For two interval-valued vectors $A=(A_1,\ldots,A_p)$ and $B=(B_1,\ldots,B_p)$, we write $A\le_{LU}B$ if $A_k\le_{LU}B_k$ for each $k=1,\ldots,p$ except at least one index for which the inequality is strict, and $A<_{LU}B$ if $A_k<_{LU}B_k$ for each $k=1,\ldots,p$.

Functions that take values in $\mathcal{I}$ are said to be interval-valued; *i.e.* when we write $f:\mathbb{R}^n\to\mathcal{I}$, then we mean a function $f(x)=\left[f^L(x),f^U(x)\right]$, where $f^L,f^U:\mathbb{R}^n\to\mathbb{R}$ are such that $f^L(x)\le f^U(x)$ for each $x\in\mathbb{R}^n$. In a similar way, an interval-valued vector function on $\mathbb{R}^n$ will be written as $f=(f_1,\ldots,f_p):\mathbb{R}^n\to\mathcal{I}^p$, where each $f_k(x)=\left[f_k^L(x),f_k^U(x)\right],k=1,\ldots,p$ is an interval-valued function.

3. The problem and its reformulation

In this section we seek to address the following semi-infinite interval-valued vector program

$$\begin{cases}\min & \{F(x)=(F_1(x),\ldots,F_p(x)):x\in\Omega\},\\ \Omega= & \{x\in\mathbb{R}^n:G_t(x)\le_{LU}A_t,\ \forall t\in T\},\end{cases}\tag{3.1}$$

where T is an arbitrary (possibly infinite) index set, $F_k=\left[F_k^L,F_k^U\right]$ and $G_t=\left[G_t^L,G_t^U\right]$ are interval-valued functions defined on $\mathbb{R}^n$ for all $k=1,\ldots,p$ and $t\in T$, and $A_t=\left[A_t^L,A_t^U\right]\in\mathcal{I}$ for all $t\in T$. The terminology *semi-infinite* comes from the fact that the feasible set Ω is included in a finite dimensional space $\mathbb{R}^n$ but the index set T can be infinite.

For the above problem, we use the following concept of solutions introduced by Wu [31].

Definition 3.1 ([31])**.** We say that $\overline{x}\in\Omega$ is a (weak) efficient solution to Problem (3.1) if there exist no $x\in\Omega$ such that $F(x)\le_{LU}(<_{LU})F(\overline{x})$.

Remark 3.2. If F_k, is a scalar function for all $k = 1, \dots, p$, then the above definition is equivalent to the known (weak) minimum definition.

For $x \in \Omega$, we set

$$g_t(x) = \max\left\{G_t^L(x) - A_t^L; G_t^U(x) - A_t^U\right\}, \quad \forall t \in T, \tag{3.2}$$

and

$$T(x) = \{t \in T : g_t(x) = 0\}.$$

Lemma 3.3. *The feasible set of Problem* (3.1) *satisfies*

$$\Omega = \{x \in \mathbb{R}^n : g_t(x) \leq 0, \ \forall t \in T\}.$$

Proof. Let $x \in \mathbb{R}^n$ and $t \in T$. The proof follows immediately from the equivalences:

$$G_t(x) \leq_{LU} A_t \ \Leftrightarrow \ \left(G_t^L(x) \leq A_t^L \text{ and } G_t^U(x) \leq A_t^U\right) \ \Leftrightarrow \ g_t(x) \leq 0.$$

□

Next, let us look at how Problem (3.1) is connected to the following bicriteria optimization problem

$$\begin{cases} \min\{f(x) = (f_1(x), f_2(x)) : x \in \Omega\}, \\ \Omega = \{x \in \mathbb{R}^n : g_t(x) \leq 0, \ \forall t \in T\}, \end{cases} \tag{3.3}$$

where

$$f_1(x) = \max_{k=1}^{p}\left\{F_k^L(x) - F_k^L(\overline{x})\right\} \quad \text{and} \quad f_2(x) = \max_{k=1}^{p}\left\{F_k^U(x) - F_k^U(\overline{x})\right\}. \tag{3.4}$$

Lemma 3.4. *The set of weak efficient solutions of* (3.1) *is equal to that of weak minima of* (3.3)*.*

Proof. Suppose we are given $\overline{x} \in \Omega$ which is not a weak minimum of (3.3). This means there exist $x \in \Omega$ satisfying $f(x) < f(\overline{x})$. Hence

$$f_1(x) < f_1(\overline{x}) = 0 \quad \text{and} \quad f_2(x) < f_2(\overline{x}) = 0.$$

Thus

$$F_k^L(x) < F_k^L(\overline{x}) \quad \text{and} \quad F_k^U(x) < F_k^U(\overline{x}), \quad \text{for all} \quad k = 1, \dots, p.$$

It follows that

$$F_k(x) <_{LU} F_k(\overline{x}) \quad \text{for all} \quad k = 1, \dots, p.$$

Consequently

$$F(x) <_{LU} F(\overline{x}),$$

which contradict the fact that $\overline{x}$ is a weak efficient solution of (3.1). To get the converse, we proceed by the same argument as above. □

4. Necessary condition for weak efficient solutions

In this section we derive first order necessary optimality conditions for the initial problem. To proceed, we need the following assumption which is sometimes called Pshenichyni–Levin–Valadier property [17].

Assumption 4.1. (1) *The functions f_1 and f_2 which are defined in (3.4) have an USRCs at $\overline{x} \in \Omega$, respectively as*

$$\partial^* f_1(\overline{x}) \subset co\left(\bigcup_{k\in I^L(\overline{x})} \partial^* F_k^L(\overline{x})\right) \quad \text{and} \quad \partial^* f_2(\overline{x}) \subset co\left(\bigcup_{k\in I^U(\overline{x})} \partial^* F_k^U(\overline{x})\right),$$

where

$$I^L(\overline{x}) = \left\{k \in \{1, 2, \dots, p\} : f_1(\overline{x}) = F_k^L(\overline{x})\right\},$$

and

$$I^U(\overline{x}) = \left\{k \in \{1, 2, \dots, p\} : f_2(\overline{x}) = F_k^U(\overline{x})\right\}.$$

(2) *For all $t \in T$, the function g_t which is defined in (3.2) has a USRC at $\overline{x} \in \Omega$ such that*

$$\partial^* g_t(\overline{x}) \subset co\left(\partial^* G_t^L(\overline{x}) \cup \partial^* G_t^U(\overline{x})\right).$$

Note that Assumption 4.1 might hold even for discontinuous functions (see [15]). Moreover, if for all $t \in T$, G_t^L and G_t^U are continuous and admit an upper convexificator at $\overline{x}$, then from V. Jeyakumar and D.T. Luc [14] Rule 4.4 one has

$$\partial^* g_t(\overline{x}) = \partial^* G_t^L(\overline{x}) \cup \partial^* G_t^U(\overline{x})$$

is an upper convexificator of g_t at $\overline{x}$.

To develop KKT necessary conditions for weak efficient solution $\overline{x} \in \Omega$ of (3.1), we recall the known Abadie constraint qualification (ACQ) which will be used in the sequal.

Definition 4.2 ([20])**.** Let $\overline{x} \in \Omega$ and $\partial^* g_t(\overline{x})$ be an USRC of g_t for any $t \in T$. We say that the Abadie Constraint Qualification (ACQ) holds at $\overline{x}$ if

$$\Gamma^\circ(\overline{x}) \subseteq T_\Omega(\overline{x}),$$

where

$$\Gamma(\overline{x}) = \bigcup_{t\in T(\overline{x})} \partial^* g_t(\overline{x}).$$

Remark 4.3 ([15])**.** Assume that Ω is locally star-shaped at $\overline{x} \in \Omega$ and ACQ holds at $\overline{x}$. Then

- $\Gamma^\circ(\overline{x}) = T_\Omega(\overline{x})$.
- $N_\Omega(\overline{x}) = cl\, cone\,(\Gamma(\overline{x})) = cl\, cone\left(\bigcup_{t\in T(\overline{x})} co\,(\partial^* g_t(\overline{x}))\right)$.

The following lemma will be used in the proof of Theorem 4.5.

Lemma 4.4 ([11])**.** *Let A and B be two nonempty subsets of $\mathbb{R}^n$. Then*

(i) $co(A + B) = co(A) + co(B)$,
(ii) $cl(cl\, A + cl\, B) = cl(A + cl\, B) = cl(A + B)$,
(iii) $cl\, co(A) = cl\, co(cl\, A) = cl\, co(co\, A)$.

The next theorem gives KKT-type necessary conditions for weak efficiency.

Theorem 4.5. *Let Ω be locally star-shaped at $\overline{x} \in \Omega$, and let F_k^L, F_k^U, G_t^L and G_t^U ($i \in \{1, \dots, p\}$ and $t \in T$), admit respectively USRCs, $\partial^* F_k^L(\overline{x})$, $\partial^* F_k^U(\overline{x})$, $\partial^* G_t^L(\overline{x})$ and $\partial^* G_t^U(\overline{x})$ at $\overline{x}$. Moreover, assume that ACQ holds* *at $\overline{x}$ and Assumption 4.1 is fulfilled. If $\overline{x}$ is a weak efficient solution of* (3.1)*, then there exist an index set $T' \subseteq T(\overline{x})$ with $|T'| \leq n$, $\alpha \in \mathbb{R}_+^{|I^L(\overline{x})|}$, $\beta \in \mathbb{R}_+^{|I^U(\overline{x})|}$, $\mu \in \mathbb{R}_+^{|T'|}$, $\gamma_t^L \in \mathbb{R}_+^{|T'|}$, $\gamma_t^U \in \mathbb{R}_+^{|T'|}$ and $\lambda \in \mathbb{R}_+^2$ with*

$$\lambda_1 + \lambda_2 = \sum_{k \in I^L(\overline{x})} \alpha_k = \sum_{k \in I^U(\overline{x})} \beta_k = \sum_{t \in T'} \gamma_t^L = \sum_{t \in T'} \gamma_t^U = 1,$$

such that

$$0 \in cl \Bigg[\lambda_1 \sum_{k \in I^L(\overline{x})} \alpha_k co \left(\partial^* F_k^L(\overline{x}) \right) + \lambda_2 \sum_{k \in I^U(\overline{x})} \beta_k co \left(\partial^* F_k^U(\overline{x}) \right) + \sum_{t \in T'} \mu_t \gamma_t^L co \left(\partial^* G_t^L(\overline{x}) \right) + \sum_{t \in T'} \mu_t \gamma_t^U co \left(\partial^* G_t^U(\overline{x}) \right) \Bigg].$$

Proof. Let $\overline{x}$ be a weakly efficient solution of (3.1). Setting

$$\mathbb{S} := \left\{ \lambda \in \mathbb{R}_+^2 : \lambda_1 + \lambda_2 = 1 \right\}$$

and

$$\Upsilon(\overline{x}) := cl \bigcup_{\lambda \in \mathbb{S}} \left(\lambda_1 co \left(\partial^* f_1(\overline{x}) \right) + \lambda_2 co \left(\partial^* f_2(\overline{x}) \right) \right), \tag{4.1}$$

where f_1 and f_2 are given by (3.4), one has

$$\sup_{\eta \in \Upsilon(\overline{x})} \langle \eta, d \rangle \geq 0, \quad \forall d \in D_\Omega(\overline{x}). \tag{4.2}$$

Indeed, suppose contrary to our claim, that there exist $d \in D_\Omega(\overline{x})$ such that $\sup_{\eta \in \Upsilon(\overline{x})} \langle \eta, d \rangle < 0$. Then, by using 2.4 and (4.1), we obtain for any $j \in \{1, 2\}$

$$f_j^+(\overline{x}; d) \leq \sup_{\eta \in \partial^* f_j(\overline{x})} \langle \eta, d \rangle \leq \sup_{\eta \in \Upsilon(\overline{x})} \langle \eta, d \rangle < 0.$$

Hence

$$\overline{x} + t_0 d \in \Omega \quad \text{and} \quad f(\overline{x} + t_0 d) < f(\overline{x}),$$

for t_0 small enough. This contradicts the weak efficiency hypothesis.

On the other hand, by taking into account that (4.2) holds also for any vector $d \in cl\, D_\Omega(\overline{x})$ and from $T_\Omega(\overline{x}) = cl\, D_\Omega(\overline{x})$ (see Lem. 2.3), we obtain

$$\sup_{\eta \in \Upsilon(\overline{x})} \langle \eta, d \rangle + I_{T_\Omega(\overline{x})}(d) \geq 0, \quad \forall d \in \mathbb{R}^n,$$

such that $I_{T_\Omega(\overline{x})}$ is the indicator function of $T_\Omega(\overline{x})$.

Moreover, since $T_\Omega(\overline{x})^\circ = N_\Omega(\overline{x})$, it follows from J.B. Hiriart-Urruty and C. Lemarechal [12] Example 2.3.1 that

$$\sigma_{\Upsilon(\overline{x})}(d) + \sigma_{N_\Omega(\overline{x})}(d) \geq 0, \quad \forall d \in \mathbb{R}^n,$$

where $\sigma_{\Upsilon(\overline{x})} + \sigma_{N_\Omega(\overline{x})}$ is the support function of $cl\left(\Upsilon(\overline{x}) + N_\Omega(\overline{x}) \right)$ (see [12], Thm. V.3.3.3(i)). According to J.B. Hiriart-Urruty and C. Lemarechal [12] Theorem V.2.2.2, we have consequently

$$0 \in cl\, co \left(cl \left(\Upsilon(\overline{x}) + N_\Omega(\overline{x}) \right) \right).$$

By Lemma 4.4 we deduce that

$$0 \in cl\left(co\left[\bigcup_{\lambda\in\mathbb{S}}\left(\lambda_1 co\left(\partial^* f_1(\overline{x})\right) + \lambda_2 co\left(\partial^* f_2(\overline{x})\right)\right)\right] + N_\Omega(\overline{x})\right).$$

Therefore there exist two scalars $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ such that

$$0 \in cl\left[\lambda_1 co\left(\partial^* f_1(\overline{x})\right) + \lambda_2 co\left(\partial^* f_2(\overline{x})\right) + N_\Omega(\overline{x})\right].$$

Hence, from Assumption 4.1, we get

$$0 \in cl\left[\lambda_1 co\left(\bigcup_{k\in I^L(\overline{x})} \partial^* F_k^L(\overline{x})\right) + \lambda_2 co\left(\bigcup_{k\in I^U(\overline{x})} \partial^* F_k^U(\overline{x})\right) + N_\Omega(\overline{x})\right].$$

Thus

$$0 \in cl\Big[\lambda_1 co\left(\textstyle\bigcup_{k\in I^L(\overline{x})} co\left(\partial^* F_k^L(\overline{x})\right)\right) + \lambda_2 co\left(\textstyle\bigcup_{k\in I^U(\overline{x})} co\left(\partial^* F_k^U(\overline{x})\right)\right) + N_\Omega(\overline{x})\Big].$$

The convex hull property gives us $\alpha \in \mathbb{R}^{|I^L(\overline{x})|}$ and $\beta \in \mathbb{R}^{|I^U(\overline{x})|}$ with $\sum_{k\in I^L(\overline{x})} \alpha_k = \sum_{k\in I^U(\overline{x})} \beta_k = 1$ such that

$$0 \in cl\left[\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k co\left(\partial^* F_k^L(\overline{x})\right) + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k co\left(\partial^* F_k^U(\overline{x})\right) + N_\Omega(\overline{x})\right].$$

Now, since ACQ holds, then by Proposition (4.3), there exist an index set $T' \subseteq T(\overline{x})$ with $|T'| \leq n$, and a vector $\mu \in \mathbb{R}_+^{|T'|}$ such that

$$0 \in cl\left[\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k co\left(\partial^* F_k^L(\overline{x})\right) + \lambda_2 \sum_{k\in I^L(\overline{x})} \beta_k co\left(\partial^* F_k^U(\overline{x})\right) + \sum_{t\in T'} \mu_t co\left(\partial^* g_t(\overline{x})\right)\right].$$

Consequently, (3.2) yields

$$0 \in cl\left[\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k co\left(\partial^* F_k^L(\overline{x})\right) + \lambda_2 \sum_{k\in I^L(\overline{x})} \beta_k co\left(\partial^* F_k^U(\overline{x})\right) + \sum_{t\in T'} \mu_t co\left(\partial^* G_t^L(\overline{x}) \cup \partial^* G_t^U(\overline{x})\right)\right].$$

Finally, we deduce that there exist $\gamma_t^L \in \mathbb{R}_+^{|T'|}$ and $\gamma_t^U \in \mathbb{R}_+^{|T'|}$ with $\sum_{t\in T'} \gamma_t^L = \sum_{t\in T'} \gamma_t^U = 1$ such that

$$\begin{aligned} 0 \in cl\Bigg[&\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k co\left(\partial^* F_k^L(\overline{x})\right) + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k co\left(\partial^* F_k^U(\overline{x})\right) \\ &+ \sum_{t\in T'} \mu_t \gamma_t^L co\left(\partial^* G_t^L(\overline{x})\right) + \sum_{t\in T'} \mu_t \gamma_t^U co\left(\partial^* G_t^U(\overline{x})\right)\Bigg]. \end{aligned}$$

This completes the proof.

□

5. Sufficient optimality conditions for weak efficient solutions

In order to provide sufficient optimality conditions for the initial problem (3.1), we need the following generalized asymptotic convexity concepts introduced by Yang in [32].

Definition 5.1. Suppose that f has an upper convexificator $\partial^* f(x)$ at every $x \in \mathbb{R}^n$. We say that f is

(i) asymptotic pseudoconvex at $\overline{x} \in \mathbb{R}^n$ if for every $x \in \mathbb{R}^n$,

$$\left(\exists x_n^* \in \mathrm{co}\left(\partial^* f(x)\right) : \lim_{n\to\infty} \langle x_n^*, x - \overline{x}\rangle \geq 0\right) \Rightarrow f(x) \geq f(\overline{x});$$

(ii) asymptotic quasiconvex at $\overline{x} \in \mathbb{R}^n$ if for every $x \in \mathbb{R}^n$,

$$f(x) \leq f(\overline{x}) \Rightarrow \left(\forall x_n^* \in \mathrm{co}\left(\partial^* f(x)\right) : \lim_{n\to\infty} \langle x_n^*, x - \overline{x}\rangle \leq 0\right).$$

In the sequel, we give sufficient conditions for weak efficient solutions.

Theorem 5.2. *Let $\overline{x}$ be a feasible point of (3.1). Assume that the following assertions hold*

(i) *F_k^L, $i \in I^L(\overline{x})$, F_k^U, $i \in I^U(\overline{x})$, G_t^L and G_t^U $(t \in T(\overline{x}))$, admit respectively upper convexificators, $\partial^* F_k^L(\overline{x})$, $\partial^* F_k^U(\overline{x})$, $\partial^* G_t^L(\overline{x})$ and $\partial^* G_t^U(\overline{x})$ at $\overline{x}$, such that one of the upper convexificators $\partial^* F_k^L(\overline{x})$, $i \in I^L(\overline{x})$, and $\partial^* F_k^U(\overline{x})$, $i \in I^U(\overline{x})$, are upper regular at $\overline{x}$.*

(ii) *There exist $\lambda_k > 0$,$k \in \{1, 2\}$, there exist an index set $T' \subseteq T(\overline{x})$ with $|T'| \leq n$, and $\alpha \in \mathbb{R}_+^{|I^L(\overline{x})|}$, $\beta \in \mathbb{R}_+^{|I^U(\overline{x})|}$, $\mu \in \mathbb{R}_+^{|T'|}$, $\gamma_t^L \in \mathbb{R}_+^{|T'|}$, $\gamma_t^U \in \mathbb{R}_+^{|T'|}$, such that*

$$\begin{aligned} 0 \in cl \Bigg[&\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k co\left(\partial^* F_k^L(\overline{x})\right) + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k co\left(\partial^* F_k^U(\overline{x})\right) \\ &+ \sum_{t\in T'} \mu_t \gamma_t^L co\left(\partial^* G_t^L(\overline{x})\right) + \sum_{t\in T'} \mu_t \gamma_t^U co\left(\partial^* G_t^U(\overline{x})\right) \Bigg]. \end{aligned} \tag{5.1}$$

(iii) *The function $\lambda\widetilde{F} := \lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k F_k^L + \lambda_2 \sum_{k\in I^L(\overline{x})} \beta_k F_k^U$ is asymptotic pseudoconvex at $\overline{x} \in \Omega$.*

(iv) *Each function G_t^L and G_t^U, $t \in T'$, is asymptotic quasiconvex at $\overline{x} \in \Omega$.*

Then $\overline{x}$ is a weak efficient solutions of (3.1).

Proof. Assume that (i)–(iv) hold. From (i), there exist $\chi_k^{(n)} \in co\left(\partial^* F_k^L(\overline{x})\right)$, $i \in I^L(\overline{x})$, $\psi_k^{(n)} \in co\left(\partial^* F_k^U(\overline{x})\right)$, $i \in I^U(\overline{x})$, $\xi_t^{(n)} \in co\left(\partial^* G_t^L(\overline{x})\right)$, $\zeta_t^{(n)} \in co\left(\partial^* G_t^U(\overline{x})\right)$, $t \in T'$ such that

$$0 = \lim_{n\to\infty} \left[\lambda_1 \textstyle\sum_{k\in I^L(\overline{x})} \alpha_k \chi_k^{(n)} + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k \psi_k^{(n)} + \sum_{t\in T'} \mu_t \gamma_t^L \xi_t^{(n)} + \sum_{t\in T'} \mu_t \gamma_t^U \zeta_t^{(n)}\right].$$

Then, for all $x \in \Omega$, one has

$$\lim_{n\to\infty} \left\langle \lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k \chi_k^{(n)} + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k \psi_k^{(n)}, x - \overline{x} \right\rangle + \sum_{t\in T'} \mu_t \gamma_t^L \lim_{n\to\infty} \left\langle \xi_k^{(n)} x - \overline{x}\right\rangle + \sum_{t\in T'} \mu_t \gamma_t^U \lim_{n\to\infty} \left\langle \zeta^{(n)}, x - \overline{x}\right\rangle = 0. \tag{5.2}$$

Observing that $G_t^L(x) \leq G_t^L(\overline{x})$ and $G_t^U(x) \leq G_t^U(\overline{x})$ for any $x \in \Omega$ and $t \in T'$, and taking into account the asymptotic quasiconvexity of G_t^L and G_t^U at $\overline{x}$, we deduce

$$\lim_{n\to\infty} \langle \xi_k^{(n)} x - \overline{x}\rangle \leq 0 \quad \text{and} \quad \lim_{n\to\infty} \langle \zeta_k^{(n)} x - \overline{x}\rangle \leq 0. \tag{5.3}$$

On the other hand, since one of the upper convexificators $\partial^* F_k^L(\overline{x})$, $i \in I^L(\overline{x})$, $\partial^* F_k^U(\overline{x})$, $i \in I^U(\overline{x})$ is upper regular, by V. Jeyakumar and D.T. Luc [14] Rule 4.2, $\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k \partial^* F_k^L(\overline{x}) + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k \partial^* F_k^U(\overline{x})$ is an upper convexificator for the function $\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k F_k^L + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k F_k^U$ at $\overline{x}$. Combining (5.2) and (5.3) we get,

$$\lim_{n\to\infty} \left\langle \lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k \chi_k^{(n)} + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k \psi_k^{(n)}, x - \overline{x} \right\rangle \geq 0, \quad \forall x \in \Omega.$$

Now, using the asymptotic pseudoconvexity of $\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k F_k^L + \lambda_2 \sum_{k\in I^U(\overline{x})} \beta_k F_k^U$ at $\overline{x}$, we obtain

$$\begin{aligned}\lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k F_k^L(x) + \lambda_2 \textstyle\sum_{k\in I^U(\overline{x})} \beta_k F_k^U(x) \geq \lambda_1 \sum_{k\in I^L(\overline{x})} \alpha_k F_k^L(\overline{x}) \\ + \lambda_2 \textstyle\sum_{k\in I^U(\overline{x})} \beta_k F_k^U(\overline{x}), \quad \forall x \in \Omega,\end{aligned}$$

which means that $\lambda \widetilde{F}(x) \geq \lambda \widetilde{F}(\overline{x})$. Since $\lambda_1 > 0, \lambda_2 > 0$, $\alpha \in \mathbb{R}_+^{|I^L(\overline{x})|}$ and $\beta \in \mathbb{R}_+^{|I^U(\overline{x})|}$, we deduce that there is no $x \in \Omega$ that satisfies one of the following inequalities

$$F_k^L(x) < F_k^L(\overline{x}),\ F_k^U(x) < F_k^U(\overline{x}),$$

or

$$F_k^L(x) < F_k^L(\overline{x}),\ F_k^U(x) \leq F_k^U(\overline{x}),$$

or

$$F_k^L(x) \leq F_k^L(\overline{x}),\ F_k^U(x) < F_k^U(\overline{x}),$$

which implies that there is no $x \in \Omega$ such that

$$F_k^L(x) < F_k^L(\overline{x}),\ F_k^U(x) < F_k^U(\overline{x}).$$

We conclude that $\overline{x}$ is a weak efficient solution of (3.1). □

6. Example

As an illustration of the main result of this paper (Thm. 4.5), we consider the following example of semi-infinite interval-valued vector program with interval-valued constraints

$$\begin{cases} \min & \left\{F(x) = \left(\left[F_1^L(x), F_1^U(x)\right], \left[F_2^L(x), F_2^U(x)\right]\right) : x \in \Omega\right\}, \\ \Omega = & \left\{x \in \mathbb{R}^2 : G_t(x) = \left[G_t^L(x), G_t^U(x)\right] \leq_{LU} [0,1],\ \forall t \in T = [-1,1]\right\}. \end{cases} \tag{6.1}$$

The objective and constraint functions are given by

$$\begin{aligned} &F_1^L(x_1,x_2) = \begin{cases} 0, & x_2 \leq 0, \\ 1, & x_2 > 0, \end{cases} \quad F_1^U(x_1,x_2) = \begin{cases} 0, & x_1 \leq 0 \quad \text{and} \quad x_2 \leq 0, \\ 1 + \sqrt{|x_1|}, & \text{otherwise}, \end{cases} \\ &F_2^L(x_1,x_2) = x_2, \quad F_2^U(x_1,x_2) = \max\{x_1,x_2\}, \\ &G_t^L(x_1,x_2) = tx_1 - x_2 \quad \text{and} \quad G_t^U(x_1,x_2) = tx_1 - x_2 + 1, \quad \text{for} \quad t \in T. \end{aligned}$$

Note that the set of efficient solutions is $\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \geq |x_1|\}$.

Observe that $\overline{x} = (0,0)$ is a weak efficient solution of Problem (6.1). It can be seen that $T(\overline{x}) = [-1,1]$, $\Gamma(\overline{x}) = \{(t,-1) : t \in [-1,1]\}$ and $\Gamma^\circ(\overline{x}) = T_\Omega(\overline{x})$. Then ACQ holds at $\overline{x}$.

On the other hand, $\partial^* F_1^L(\overline{x}) = \{0\} \times [0,\infty)$, $\partial^* F_1^U(\overline{x}) = [0,\infty) \times [0,\infty)$, $\partial^* F_2^L(\overline{x}) = \{(0,1)\}$ and $\partial^* F_2^U(\overline{x}) = \{(0,1),(1,0)\}$, are respectively USRCs of F_1^L, F_1^U, F_2^L and F_2^U at $\overline{x}$.

The set Ω is locally star-shaped at $\overline{x}$ because it is convex. Also, we have $I^L(\overline{x}) = I^U(\overline{x}) = \{1,2\}$ and $\partial^* G_t^L(\overline{x}) = \partial^* G_t^U(\overline{x}) = \{(t,-1)\}$ are USRCs of G_t^L and G_t^U, $t \in T$, at $\overline{x}$.

Setting $T' = \{1\}$ and taking $\lambda_1 = \lambda_2 = \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \frac{1}{2}$, $\mu_t = 1$, $\gamma_t^L = 1$ and $\gamma_t^U = 1$, one can easily verify that

$$(0,0) \in cl\Bigg[\lambda_1\alpha_1 co\left(\partial^* F_1^L(\overline{x})\right) + \lambda_2\beta_1 co\left(\partial^* F_1^U(\overline{x})\right) + \lambda_1\alpha_2 co\left(\partial^* F_2^L(\overline{x})\right) + \lambda_2\beta_2 co\left(\partial^* F_2^U(\overline{x})\right) + \sum_{t\in T'} \mu_t\gamma_t^L co\left(\partial^* G_t^L(\overline{x})\right) + \sum_{t\in T'} \mu_t\gamma_t^U co\left(\partial^* G_t^U(\overline{x})\right)\Bigg].$$

Consequently, Theorem 4.5 is verified.

7. Conclusions

In this work, we have studied a nonsmooth semi-infinite programming problem where both the multiobjective and constraint functions are interval-valued. Using an intermediate bicriteria optimization problem, we have derived necessary optimality conditions in terms of convexificators. Moreover, under assumptions on the asymptotic pseudoconvexity of the multiobjective function and the asymptotic quasiconvexity of inequality constraints, we have shown that the Karush–Kuhn–Tucker necessary conditions become also sufficient. Our results have been obtained without assuming neither convexity nor locally Lipshitz assumptions of the involved functions.

References

[1] A. Ansari Ardali, N. Movahedian and S. Nobakhtian, Convexificators and boundedness of the Kuhn–Tucker multipliers set. *Optimization* **66** (2017) 1445–1463.

[2] T. Antczak, Optimality conditions and duality results for nonsmooth vector optimization problems with the multiple interval-valued objective function. *Acta Math. Sci.* **37** (2017) 1133–1150.

[3] A. Ben-Israel and P.D. Robers, A decomposition method for interval linear programming. *Manage. Sci.* **16** (1970) 374–387.

[4] F.H. Clarke, Optimization and Nonsmooth Analysis. SIAM **5** (1990).

[5] G.B. Dantzig, Linear Programming and Extensions. In: Vol. 48 of *Princeton Landmarks in Mathematics and Physics*. Princeton University Press, Princeton, NJ (1998).

[6] V.F. Demyanov, Convexification and concavification of a positively homogeneous function by the same family of linear functions. Universia di Pisa, Report, 3, 208, 802 (1994).

[7] V.F. Demyanov and V. Jeyakumar, Hunting for a smaller convex subdifferential. *J. Global Optim.* **10** (1997) 305–326.

[8] J. Dutta and S. Chandra, Convexifactors, generalized convexity and optimality conditions. *J. Optim. Theory App.* **113** (2002) 41–65.

[9] J. Dutta and S. Chandra, Convexifactors, generalized convexity and vector optimization. *Optimization* **53** (2004) 77–94.

[10] G.M. Ewing, Sufficient conditions for global minima of suitably convex functionals from variational and control theory. *SIAM Rev.* **19** (1977) 202–220.

[11] J.B. Hiriart-Urruty and C. Lemaréchal, Fundamentals of Convex Analysis. Springer, New York, NY (2012).

[12] J.B. Hiriart-Urruty and C. Lemaréchal, Convex Analysis and Minimization Algorithms I: Fundamentals. In: Vol. 305 of *Series of Comprehensive Studies in Mathematics*. Springer, New York, NY (2013).

[13] H. Ishibuchi and H. Tanaka, Multiobjective programming in optimization of the interval objective function. *Eur. J. Oper. Res.* **48** (1990) 219–225.

[14] V. Jeyakumar and D.T. Luc, Nonsmooth calculus, minimality, and monotonicity of convexificators. *J. Opt. Theory App.* **101** (1999) 599–621.

[15] A. Kabgani and M. Soleimani-Damaneh, Characterization of (weakly/properly/robust) efficient solutions in nonsmooth semi-infinite multiobjective optimization using convexificators. *Optimization* **67** (2017) 217–235.

[16] A. Kabgani and M. Soleimani-Damaneh, Constraint qualifications and optimality conditions in nonsmooth locally star-shaped optimization using convexificators. *Pac. J. Opt.* **15** (2019) 399–413.

[17] N. Kanzi, Necessary optimality conditions for nonsmooth semi-infinite programming problems. *J. Global Optim.* **49** (2011) 713–725.

[18] R.N. Kaul and S. Kaur, Generalized convex functions: properties, optimality and duality (No. Systems Optimization Lab. SOL-84-4). Stanford University, Stanford, CA (1984).

[19] P. Kumar, B. Sharma and J. Dagar, Interval-valued programming problem with infinite constraints. *J. Oper. Res. Soc. Chin.* **6** (2018) 611–626.

[20] X.F. Li and J.Z. Zhang, Necessary optimality conditions in terms of convexificators in Lipschitz optimization. *J. Optim. Theory App.* **131** (2006) 429–452.

[21] P. Michel and J.P. Penot, A generalized derivative for calm and stable functions. *Differ. Integr. Equ.* **5** (1992) 433–454.

[22] B.S. Mordukhovich and Y. Shao, On nonconvex subdifferential calculus in Banach spaces. *J. Conv. Anal.* **2** (1995) 211–228.

[23] R. Osuna-Gómez, Y. Chalco-Cano, B. Hernández-Jiménez and G. Ruiz-Garzón, Optimality conditions for generalized differentiable interval-valued functions. *Inf. Sci.* **321** (2015) 136–146.

[24] R. Osuna-Gómez, B. Hernádez-Jiménez, Y. Chalco-Cano and G. Ruiz-Gazón, New efficiency conditions for multiobjective interval-valued programming problems. *Inf. Sci.* **420** (2017) 235–248.

[25] D. Singh, B.A. Dar and D.S. Kim, KKT optimality conditions in interval-valued multiobjective programming with generalized differentiable functions. *Eur. J. Oper. Res.* **254** (2016) 29–39.

[26] L. Stefanini and M. Arana-Jiménez, Karush–Kuhn–Tucker conditions for interval and fuzzy optimization in several variables under total and directional generalized differentiability. *Fuzzy Sets Syst.* **362** (2019) 1–34.

[27] L. Stefanini and B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Anal. Theory Methods App.* **71** (2009) 1311–1328.

[28] J.S. Treiman, The linear nonconvex generalized gradient and Lagrange multipliers. *SIAM J. Optim.* **5** (1995) 670–680.

[29] L.T. Tung, Karush–Kuhn–Tucker optimality conditions and duality for convex semi-infinite programming with multiple interval-valued objective functions. *J. Appl. Math. Comput.* (2019) 1–25.

[30] H.C. Wu, The Karush–Kuhn–Tucker optimality conditions in an optimization problem with interval-valued objective function. *Eur. J. Oper. Res.* **176** (2007) 46–59.

[31] H.C. Wu, The Karush–Kuhn–Tucker optimality conditions in multiobjective programming problems with interval-valued objective function. *Eur. J. Oper. Res.* **196** (2009) 49–60.

[32] X.Q. Yang, Continuous generalized convex functions and their characterizations. *Optimization* **54** (2005) 495–506.

[33] J. Zhang, S. Liu, L. Li and Q. Feng, The KKT optimality conditions in a class of generalized convex optimization problems with an interval-valued objective function. *Optim. Lett.* **8** (2014) 607–631.