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ON k-ORTHOGONAL FACTORIZATIONS IN NETWORKS

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Abstract. Let m, n, k, r and k_i $(1 \le i \le m)$ are positive integers such that $1 \le n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m \ge (r+1)k$. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and $H_1, H_2, \cdots, H_r$ be r vertex-disjoint nk-subgraphs of G. In this article, we demonstrate that a graph G with maximum degree at most $\sum_{i=1}^{m} k_i - (n-1)k$ has a set $\mathcal{F} = \{F_1, \cdots, F_n\}$ of n pairwise edge-disjoint factors of G such that F_i has maximum degree at most k_i for $1 \le i \le n$ and $\mathcal{F}$ is k-orthogonal to every H_j for $1 \le j \le r$.

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1. INTRODUCTION

Many real-world networks can be modelled by graphs or networks. The vertices of the graph stand for the nodes of the network, and the edges of the graph act for the links between the nodes in the network. Next, we show an example: an online social network with nodes representing persons and links corresponding to personal contacts of each user. Other examples include an aviation network with nodes modelling aviation stations and links representing air lines between two stations, or the World Wide Web with nodes corresponding to web pages and links modelling hyperlinks between web pages, or a communication network with nodes acting for cities and links standing for communication channels. Henceforth, we employ the term "graph" instead of "network". Network Science (a.k.a. Complex Network Analysis) is an emerging area of interest in the big data paradigm and corresponds to analyzing complex real-world networks and theoretical model-based networks from a graph theory point of view. Many real-life problems on network design, combinatorial design, circuit layout, and so on are related to the factors, factorizations and orthogonal factorizations in networks [1], and attract a great deal of attention [2, 4–6, 8, 14–17, 19–29, 31–35].

We deal with finite undirected simple graphs. Let G be a graph. We denote by $V(G)$ the vertex set of G, and by $E(G)$ the edge set of G. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G. A vertex of G is called an isolated vertex if its degree in G is 0. Let $g, f : V(G) \to \mathbb{Z}$ be two functions with $0 \le g(x) \le f(x)$ for every $x \in V(G)$. A spanning subgraph F of G is called a (g, f)-factor of G if $g(x) \le d_F(x) \le f(x)$ holds for every $x \in V(G)$. We call G a (g, f)-graph if G itself is a (g, f)-factor. Especially, a (g, f)-factor is called an $[a, b]$-factor and a (g, f)-graph is called an $[a, b]$-graph if $g(x) = a$ and $f(x) = b$ for every $x \in V(G)$. Let $k_1, k_2, \cdots, k_m$ be

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positive integers. If the edges of G can be decomposed into edge-disjoint $[0,k_1]$-factor F_1, $[0,k_2]$-factor $F_2, \cdots,$ $[0,k_m]$-factor F_m, then we call $\mathcal{F}=\{F_1,F_2,\cdots,F_m\}$ is a $[0,k_i]_{i=1}^m$-factorization of G.

A subgraph with m edges is said to be an m-subgraph. Let H be an mk-subgraph of G and $\mathcal{F}=\{F_1,F_2,\cdots,F_m\}$ be a $[0,k_i]_{i=1}^m$-factorization of G. If $|E(H)\cap E(F_i)|=k$ for any $i\in[1,m]$, then we say that $\mathcal{F}$ is k-orthogonal to H, namely, $\mathcal{F}$ is a k-orthogonal $[0,k_i]_{i=1}^m$-factorization of G. Note that 1-orthogonal is also called orthogonal.

Alspach *et al.* [1] presented the following problem: Given a subgraph H of G, does there exist a factorization $\mathcal{F}$ of G with a given property orthogonal to H? Liu [11] proved that every $(mg+m-1,mf-m+1)$-graph has a (g,f)-factorization orthogonal to any given m-matching. Li and Liu [9] verified that every $(mg+m-1,mf-m+1)$-graph admits a (g,f)-factorization orthogonal to any given m-subgraph, which is an improvement of Liu's previous result [11]. Lam *et al.* [7] justified that every $(mg+m-1,mf-m+1)$-graph admits a (g,f)-factorization orthogonal to k vertex-disjoint m-subgraphs if $k\le g(x)\le f(x)$ for any $x\in V(G)$, which is a generalization of Li and Liu's previous result [9]. Li *et al.* [10] investigated the existence of a subgraph with orthogonal factorization in an $(mg+k,mf-k)$-graph. Feng and Liu [3] showed the existence of orthogonal factorizations of $[0,k_1+k_2+\cdots+k_m-m+1]$-graphs. Wang [18] discussed the existence of a subgraph with orthogonal factorization in a $[0,k_1+k_2+\cdots+k_m-n+1]$-graph. The k-orthogonal factorizations of some graphs were studied in [12, 13, 30].

In the present article, we deal with the following problem: Given r vertex-disjoint subgraphs $H_1,H_2,\cdots,H_r$ of G, does there exist a subgraph R of G such that R possesses a factorization k-orthogonal to every H_i for $1\le i\le r$?

We now present the main result of this article, which answers the above question.

Theorem 1.1. *Let G be a graph with maximum degree at most $\sum_{i=1}^m k_i-(n-1)k$, and let $H_1,H_2,\cdots,H_r$ be r pairwise vertex-disjoint nk-subgraphs of G, where m, n, k, r and k_i $(1\le i\le m)$ are positive integers such that $1\le n\le m$ and $k_1\ge k_2\ge\cdots\ge k_m\ge(r+1)k$. Then G has a set $\mathcal{F}=\{F_1,\cdots,F_n\}$ of n pairwise edge-disjoint factors of G such that F_i has maximum degree at most k_i for $1\le i\le n$ and $\mathcal{F}$ is k-orthogonal to every H_j for $1\le j\le r$.*

2. Lemmas

Let G be a graph. For $S\subseteq V(G)$ and $A\subseteq E(G)$, we denote by $G-S$ the subgraph induced by $V(G)-S$, and by $G-A$ the subgraph induced by $E(G)-A$. For any function φ defined on $V(G)$ and $S\subseteq V(G)$, we write $\varphi(S)=\sum_{x\in S}\varphi(x)$ and $\varphi(\emptyset)=0$. Let S and T be two subsets of $V(G)$, and $S\cap T=\emptyset$. We denote by $E_G(S,T)$ the set of edges in G with one end in S and the other in T, and write $e_G(S,T)=|E_G(S,T)|$. Set

$$U=V(G)-(S\cup T),\quad E(S)=\{xy:xy\in E(G),x,y\in S\}$$

and

$$E(T)=\{xy:xy\in E(G),x,y\in T\}.$$

Let E_1 and E_2 be two subsets of $E(G)$, and $E_1\cap E_2=\emptyset$. Put

$$E_1'=E_1\cap E(S),\quad E_1''=E_1\cap E_G(S,U),$$
$$E_2'=E_2\cap E(T),\quad E_2''=E_2\cap E_G(T,U),$$
$$\alpha_G(S,T;E_1)=2|E_1'|+|E_1''|$$

and

$$\beta_G(S,T;E_2)=2|E_2'|+|E_2''|.$$

If it causes no ambiguity, we write α for $\alpha_G(S,T;E_1)$, and β for $\beta_G(S,T;E_2)$. We easily see that $\alpha\le d_{G-T}(S)$ and $\beta\le d_{G-S}(T)$.

The following lemma, whose proof can be shown by Lam *et al.* [7], Li *et al.* [10] and Liu *et al.* [9], is useful for verifying our main theorem.

Lemma 2.1 ([7,9,10])**.** *Let G be a graph, and let $g, f : V(G) \to Z$ be two functions such that $0 \le g(x) < f(x) \le d_G(x)$ for all $x \in V(G)$. Let E_1 and E_2 be two subsets of $E(G)$, and $E_1 \cap E_2 = \emptyset$. Then G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if*

$$\delta_G(S, T; g, f) = d_{G-S}(T) - g(T) + f(S) \ge \alpha_G(S, T; E_1) + \beta_G(S, T; E_2)$$

for any two disjoint subsets S and T of $V(G)$.

The following lemma, which was obtained by Wang [18], will be used in the proof of our main theorem.

Lemma 2.2 (Wang [18])**.** *Let G be a graph with maximum degree $\sum_{i=1}^{m} k_i - n + 1$, where m, n and k_i $(1 \le i \le m)$ are positive integers with $n \le m$ and $k_1 \ge k_2 \ge \cdots \ge k_m$. Let H be an arbitrary n-subgraph of G. Then G has a set $\mathcal{F} = \{F_1, \cdots, F_n\}$ of n pairwise edge-disjoint factors of G such that F_i has maximum degree at most k_i for $1 \le i \le n$ and $\mathcal{F}$ is orthogonal to H.*

Now, we shall prove the following lemma, which will be used in the proof of our main theorem.

Lemma 2.3. *Let G be a $[0, k_1 + k_2 + \cdots + k_m]$-graph, and let $H_1, H_2, \cdots, H_r$ be r vertex-disjoint k-subgraphs of G, where m, r, k and k_i $(1 \le i \le m)$ are positive integers such that $k_1 \ge k_2 \ge \cdots \ge k_m \ge (r + 1)k$. Then G has a $[0, k_1]$-factor F_1 satisfying $E(H_i) \subseteq E(F_1)$ for $1 \le i \le r$.*

Proof. Let $E_1 = \bigcup_{i=1}^{r} E(H_i)$ and $E_2 = \emptyset$. We write $E(H_i) = \{e_{i1}, \cdots, e_{ik}\}$, where $e_{ij} = x_{ij}y_{ij}$ for $1 \le j \le k$, and write $E(H_i) \cap E(S) = \{x_{i1}y_{i1}, \cdots, x_{id_i}y_{id_i}\}$ and $E(H_i) \cap E_G(S, U) = \{x_{id_i+1}y_{id_i+1}, \cdots, x_{it_i}y_{it_i}\}$, where $t_i \le k$ is a nonnegative integer. Thus, we admit

$$\begin{aligned}
2|E_1 \cap E(S)| + |E_1 \cap E_G(S, U)| &= 2\left|\left(\bigcup_{i=1}^{r} E(H_i)\right) \cap E(S)\right| + \left|\left(\bigcup_{i=1}^{r} E(H_i)\right) \cap E_G(S, U)\right| \\
&= 2\left|\bigcup_{i=1}^{r}(E(H_i) \cap E(S))\right| + \left|\bigcup_{i=1}^{r}(E(H_i) \cap E_G(S, U))\right| \\
&= 2\sum_{i=1}^{r} |E(H_i) \cap E(S)| + \sum_{i=1}^{r} |E(H_i) \cap E_G(S, U)| \\
&= 2\sum_{i=1}^{r} d_i + \sum_{i=1}^{r}(t_i - d_i) \\
&= \sum_{i=1}^{r}(d_i + t_i)
\end{aligned}$$

and

$$k|S| \ge k\sum_{i=1}^{r}\left(\frac{2d_i + (t_i - d_i)}{k}\right) = \sum_{i=1}^{r}(d_i + t_i).$$

Hence, we derive

$$2|E_1 \cap E(S)| + |E_1 \cap E_G(S, U)| \le k|S|.$$

We define α and β as before for two disjoint vertex subsets S and T of G. By the definitions of α and β, we obtain

$$\alpha = 2|E_1 \cap E(S)| + |E_1 \cap E_G(S, U)| \le \min\{2kr, k|S|\}$$

and

$$\beta = 2|E_2 \cap E(T)| + |E_2 \cap E_G(T, U)| = 0.$$

Then it follows from $k_1 \geq (r+1)k$ that

$$\delta_G(S,T;0,k_1) = d_{G-S}(T) - 0 \cdot |T| + k_1|S| \geq k_1|S| \geq (r+1)k|S| \geq k|S| \geq \alpha = \alpha + \beta.$$

In terms of Lemma 2.1, G has a $[0,k_1]$-factor F_1 such that $E(H_i) \subseteq E(F_1)$ for $1 \leq i \leq r$. Lemma 2.3 is demonstrated. □

3. Proof of Theorem 1.1

Let m,n,k,r and k_i $(1 \leq i \leq m)$ are positive integers such that $1 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq (r+1)k$. In the following, we assume that G is a $[0, k_1 + k_2 + \cdots + k_m - (n-1)k]$-graph. For every $[0,k_i]$-factor F_i and every isolated vertex $x \in V(G)$, x is also isolated in F_i, which automatically satisfies the required condition. Therefore, we may assume that G does not admit isolated vertices. Define

$$p(x) = \max\{0, d_G(x) - (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k)\}$$

and

$$q(x) = \min\{k_m, d_G(x)\}.$$

By the definitions of $p(x)$ and $q(x)$, we obtain $0 \leq p(x) < q(x) = \min\{k_m, d_G(x)\}$.

Proof of Theorem 1.1. If $kr = 1$, namely, $k = 1$ and $r = 1$, then the theorem holds by Lemma 2.2. Next, we may assume $kr \geq 2$.

We proceed by induction on m and n. The theorem obviously holds for $n = 1$ by Lemma 2.3. So we may assume $n \geq 2$ in the following. For the inductive step, we assume that Theorem 1.1 holds for any graph G' with maximum degree at most $\sum_{i=1}^{m'} k_i - (n'-1)k$ ($m' < m$, $n' < n$ and $1 \leq n' \leq m'$), and any r pairwise vertex-disjoint $n'k$-subgraphs $H'_1, H'_2, \cdots, H'_r$ of G'. We now discuss a graph G with maximum degree at most $\sum_{i=1}^{m} k_i - (n-1)k$ and any r pairwise vertex-disjoint nk-subgraphs $H_1, H_2, \cdots, H_r$ of G.

We define $p(x)$ and $q(x)$ the same as before, and choose any $A_j \subseteq E(H_j)$ with $|A_j| = k$, $1 \leq j \leq r$. Let $E_1 = \bigcup_{j=1}^{r} A_j$ and $E_2 = \left(\bigcup_{j=1}^{r} E(H_j)\right) \setminus E_1$. Then $|E_1| = kr$ and $|E_2| = (n-1)kr$. For two disjoint subsets S and T of $V(G)$, we define E'_1, E''_1, E'_2, E''_2, α and β as before. According to the definitions of α and β, we obtain

$$\alpha \leq \min\{2kr, k|S|\}$$

and

$$\beta \leq \min\{2(n-1)kr, (n-1)k|T|\}.$$

In what follows, we choose two disjoint subsets S and T of $V(G)$ such that

(a) $\delta_G(S,T;p,q) - \alpha_G(S,T;E_1) - \beta_G(S,T;E_2)$ is as small as possible;
(b) $|S|$ is minimum subject to (a).

Claim 1. If $S \neq \emptyset$, then $q(x) \leq d_G(x) - 1$ for all $x \in S$. Thus, $q(x) = k_m$ for all $x \in S$.

Proof. Suppose that $S_1 = \{x : x \in S, q(x) \geq d_G(x)\} \neq \emptyset$. Write $S_0 = S \setminus S_1$. Thus,

$$\begin{aligned}
\delta_G(S,T;p,q) &= d_{G-S}(T) - p(T) + q(S) \\
&= d_G(T) - e_G(S_0,T) - e_G(S_1,T) - p(T) + q(S_0) + q(S_1) \\
&= d_{G-S_0}(T) - p(T) + q(S_0) + q(S_1) - e_G(S_1,T) \\
&= \delta_G(S_0,T;p,q) + q(S_1) - e_G(S_1,T) \\
&\geq \delta_G(S_0,T;p,q) + d_G(S_1) - e_G(S_1,T) \\
&= \delta_G(S_0,T;p,q) + d_{G-T}(S_1).
\end{aligned}$$

Note that

$$\alpha_G(S,T;E_1)+\beta_G(S,T;E_2)\le\alpha_G(S_0,T;E_1)+\beta_G(S_0,T;E_2)+\alpha_G(S_1,T;E_1)$$

and

$$d_{G-T}(S_1)\ge\alpha_G(S_1,T;E_1).$$

Thus,

$$\begin{aligned}\delta_G(S,T;p,q)-\alpha_G(S,T;E_1)-\beta_G(S,T;E_2)&\ge\delta_G(S_0,T;p,q)+d_{G-T}(S_1)-\alpha_G(S_0,T;E_1)\\&-\beta_G(S_0,T;E_2)-\alpha_G(S_1,T;E_1)\ge\delta_G(S_0,T;p,q)-\alpha_G(S_0,T;E_1)-\beta_G(S_0,T;E_2),\end{aligned}$$

which conflicts the choice of S. Therefore, $S_1=\emptyset$. And so, if $S\neq\emptyset$, then $q(x)\le d_G(x)-1$ for all $x\in S$. Combining this with the definition of $q(x)$, we have $q(x)=k_m$ for all $x\in S$. Claim 1 is proved. □

The remaining of the proof is dedicated to demonstrating that G has a (p,q)-factor F_n (which is also a $[0,k_n]$-factor) such that $E_1\subseteq E(F_n)$ and $E_2\cap E(F_n)=\emptyset$. In light of Lemma 2.1, and the choice of S and T, it suffices to verify that $\delta_G(S,T;p,q)\ge\alpha+\beta$.

In what follows, we let $T_1=\{x: d_G(x)-(k_1+k_2+\cdots+k_{m-1}-(n-2)k)\ge 1, x\in T\}$, and $T_0=T\setminus T_1$. We easily see that $p(x)=0$ for all $x\in T_0$, and $p(x)=d_G(x)-(k_1+k_2+\cdots+k_{m-1}-(n-2)k)$ for all $x\in T_1$. Thus,

$$p(T_0)=0 \tag{3.1}$$

and

$$p(T_1)=d_G(T_1)-(k_1+k_2+\cdots+k_{m-1}-(n-2)k)|T_1|. \tag{3.2}$$

By the definition of $\beta_G(S,T;E_2)$, we easily see that

$$\beta_G(S,T_0;E_2)+\beta_G(S,T_1;E_2)=\beta_G(S,T;E_2). \tag{3.3}$$

Note that $\alpha\le\min\{2kr,k|S|\}\le k|S|$, and $\beta\le d_{G-S}(T)$. If $T_1=\emptyset$, then it follows from (3.1), Claim 1 and $k_m\ge(r+1)k$ that

$$\begin{aligned}\delta_G(S,T;p,q)&=d_{G-S}(T)-p(T)+q(S)\\&=d_{G-S}(T)-p(T_0)-p(T_1)+q(S)\\&=d_{G-S}(T)+q(S)\\&=d_{G-S}(T)+k_m|S|\\&\ge d_{G-S}(T)+(r+1)k|S|\\&\ge d_{G-S}(T)+k|S|\\&\ge\alpha+\beta.\end{aligned}$$

If $S=\emptyset$, then $\alpha=0$. Using (3.1)–(3.3), $2\le n\le m$, and $k_1\ge k_2\ge\cdots\ge k_m\ge(r+1)k$, we get

$$\begin{aligned}\delta_G(S,T;p,q)&=d_{G-S}(T)-p(T)+q(S)\\&=d_G(T)-p(T)\\&=d_G(T_0)+d_G(T_1)-p(T_0)-p(T_1)\\&=d_G(T_0)+d_G(T_1)-(d_G(T_1)-(k_1+k_2+\cdots+k_{m-1}-(n-2)k)|T_1|)\\&=d_G(T_0)+(k_1+k_2+\cdots+k_{m-1}-(n-2)k)|T_1|\\&\ge d_G(T_0)+((m-1)k_m-(n-2)k)|T_1|\\&\ge d_G(T_0)+((n-1)(r+1)k-(n-2)k)|T_1|\\&\ge d_G(T_0)+(n-1)k|T_1|\\&\ge\beta_G(\emptyset,T_0;E_2)+\beta_G(\emptyset,T_1;E_2)\\&=\beta_G(\emptyset,T;E_2)=\beta=\alpha+\beta.\end{aligned}$$

Next, we always assume that $S \neq \emptyset$ and $T_1 \neq \emptyset$, and consider two cases.

Case 1. $|S| \geq |T_1|$.
Note that G is a graph with maximum degree at most $\sum_{i=1}^{m} k_i - (n-1)k$. Thus, $d_G(T_1) \leq (k_1 + k_2 + \cdots + k_m - (n-1)k)|T_1|$. Combining this with (3.1), (3.2), Claim 1 and $k_m \geq (r+1)k$, we get

$$\begin{aligned}
\delta_G(S,T;p,q) &= d_{G-S}(T) - p(T) + q(S)\\
&= d_{G-S}(T) - p(T_0) - p(T_1) + q(S)\\
&= d_{G-S}(T) - p(T_1) + k_m|S|\\
&= d_{G-S}(T) - (d_G(T_1) - (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k)|T_1|) + k_m|S|\\
&= d_{G-S}(T) + ((k_1 + k_2 + \cdots + k_{m-1} + k_m - (n-1)k)|T_1| - d_G(T_1))\\
&\quad + k|T_1| + k_m(|S| - |T_1|)\\
&\geq d_{G-S}(T) + k|T_1| + k_m(|S| - |T_1|)\\
&= d_{G-S}(T) + k|S| + (k_m - k)(|S| - |T_1|)\\
&\geq d_{G-S}(T) + k|S|\\
&\geq \alpha + \beta.
\end{aligned}$$

Case 2. $|S| \leq |T_1| - 1$.
It follows from (3.1), (3.2), Claim 1, $2 \leq n \leq m$ and $k_1 \geq k_2 \geq \cdots \geq k_m \geq (r+1)k$ that

$$\begin{aligned}
\delta_G(S,T;p,q) &= d_{G-S}(T) - p(T) + q(S)\\
&= d_{G-S}(T_0) + d_{G-S}(T_1) - p(T_0) - p(T_1) + q(S)\\
&= d_{G-S}(T_0) + d_G(T_1) - e_G(S,T_1)\\
&\quad - (d_G(T_1) - (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k)|T_1|) + q(S)\\
&= d_{G-S}(T_0) - e_G(S,T_1) + (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k)|T_1| + k_m|S|\\
&\geq d_{G-S}(T_0) - |S||T_1| + ((m-1)k_m - (n-2)k)|T_1| + k_m|S|\\
&\geq d_{G-S}(T_0) - |S||T_1| + ((n-1)(r+1)k - (n-2)k)|T_1| + k_m|S|\\
&\geq d_{G-S}(T_0) - |S||T_1| + (n-1)rk|T_1| + k_m|S|,
\end{aligned}$$

that is,

$$\delta_G(S,T;p,q) \geq d_{G-S}(T_0) - |S||T_1| + (n-1)rk|T_1| + k_m|S|. \tag{3.4}$$

Subcase 2.1. $|T_1| \leq k_m - k$.
By (3.3) and (3.4), we obtain

$$\begin{aligned}
\delta_G(S,T;p,q) &\geq d_{G-S}(T_0) - |S||T_1| + (n-1)rk|T_1| + k_m|S|\\
&\geq d_{G-S}(T_0) - (k_m - k)|S| + (n-1)k|T_1| + k_m|S|\\
&= d_{G-S}(T_0) + k|S| + (n-1)k|T_1|\\
&\geq \alpha + \beta_G(S,T_0;E_2) + \beta_G(S,T_1;E_2)\\
&= \alpha + \beta_G(S,T;E_2)\\
&= \alpha + \beta.
\end{aligned}$$

Subcase 2.2. $|T_1| \geq k_m - k + 1$.
Subcase 2.2.1. $|S| \leq (n-1)rk$.

According to (3.4), $k_m \geq (r+1)k$ and $rk \geq 2$, we obtain

$$\begin{aligned}\delta_G(S,T;p,q) &\geq d_{G-S}(T_0) - |S||T_1| + (n-1)rk|T_1| + k_m|S| \\ &\geq k_m|S| - |S||T_1| + (n-1)rk|T_1| \\ &= k_m|S| + ((n-1)rk - |S|)|T_1| \\ &\geq k_m|S| + ((n-1)rk - |S|)(k_m - k + 1) \\ &= k|S| - |S| + (k_m - k + 1)(n-1)rk \\ &\geq k|S| - (n-1)rk + (k_m - k + 1)(n-1)rk \\ &= k|S| + (k_m - k)(n-1)rk \\ &\geq k|S| + ((r+1)k - k)(n-1)rk \\ &= k|S| + (n-1)(rk)^2 \\ &\geq k|S| + 2(n-1)kr \\ &\geq \alpha + \beta.\end{aligned}$$

Subcase 2.2.2. $|S| \geq (n-1)rk + 1$.

Note that G is a graph with maximum degree at most $\sum_{i=1}^{m} k_i - (n-1)k$. Thus, $d_G(S) \leq (k_1 + k_2 + \cdots + k_m - (n-1)k)|S|$. Combining this with (3.1), (3.2), Claim 1, $k_1 \geq k_2 \geq \cdots \geq k_m \geq (r+1)k$ and $2 \leq n \leq m$, we have

$$\begin{aligned}\delta_G(S,T;p,q) &= d_{G-S}(T) - p(T) + q(S) \\ &= d_{G-S}(T) - p(T_0) - p(T_1) + q(S) \\ &= d_G(T) - e_G(S,T) - p(T_1) + k_m|S| \\ &= d_G(T) - e_G(S,T) - (d_G(T_1) - (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k)|T_1|) + k_m|S| \\ &\geq k_m|S| + (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k)|T_1| - e_G(S,T) \\ &= (k_1 + k_2 + \cdots + k_{m-1} + k_m - (n-2)k)|S| \\ &\quad + (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k)(|T_1| - |S|) - e_G(S,T) \\ &\geq d_G(S) + k|S| + (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k) - e_G(S,T) \\ &\geq d_G(S) + k|S| + (m-1)k_m - (n-2)k - e_G(S,T) \\ &\geq d_G(S) + k((n-1)rk + 1) + (n-1)(r+1)k - (n-2)k - e_G(S,T) \\ &= d_{G-T}(S) + (n-1)rk^2 + (n-1)rk + 2k \\ &> d_{G-T}(S) + 2(n-1)kr \\ &\geq \alpha + \beta.\end{aligned}$$

Thus, we have $\delta_G(S,T;p,q) \geq \alpha_G(S,T;E_1) + \beta_G(S,T;E_2)$. By the choice of S and T, we obtain $\delta_G(S',T';p,q) \geq \alpha_G(S',T';E_1) + \beta_G(S',T';E_2)$ for arbitrary disjoint subsets S' and T' of $V(G)$. In light of Lemma 2.1, G has a (p,q)-factor F_n such that $E_1 \subseteq E(F_n)$ and $E_2 \cap E(F_n) = \emptyset$, and F_n is also a $[0,k_n]$-factor of G. It follows from $p(x)$ and $q(x)$ that

$$d_{G-F_n}(x) = d_G(x) - d_{F_n}(x) \geq d_G(x) - q(x) \geq 0$$

and

$$\begin{aligned}d_{G-F_n}(x) &= d_G(x) - d_{F_n}(x) \leq d_G(x) - p(x) \\ &\leq d_G(x) - (d_G(x) - (k_1 + k_2 + \cdots + k_{m-1} - (n-2)k)) \\ &= k_1 + k_2 + \cdots + k_{m-1} - ((n-1)-1)k\end{aligned}$$

for each $x \in V(G)$. Therefore, $G - F_n$ is a graph with maximum degree at most $\sum_{i=1}^{m-1} k_i - ((n-1)-1)k$. Write $H'_j = H_j - A_j$ for $1 \le j \le r$. It is obvious that $H'_1, H'_2, \cdots, H'_r$ are r pairwise vertex-disjoint $(n-1)k$-subgraphs of $G - F_n$. By the induction hypothesis, $G - F_n$ has a set $\mathcal{F}' = \{F_1, \cdots, F_{n-1}\}$ of $(n-1)$ pairwise edge-disjoint factors of $G - F_n$ such that F_i has maximum degree at most k_i for $1 \le i \le n-1$ and $\mathcal{F}'$ is k-orthogonal to every H'_j for $1 \le j \le r$. Consequently, G has a set $\mathcal{F} = \{F_1, \cdots, F_n\}$ of n pairwise edge-disjoint factors of G such that F_i has maximum degree at most k_i for $1 \le i \le n$ and $\mathcal{F}$ is k-orthogonal to every H_j for $1 \le j \le r$. We prove Theorem 1.1.

□

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