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A LOWER BOUND ON THE GLOBAL POWERFUL ALLIANCE NUMBER IN TREES

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Abstract. For a graph $G = (V, E)$, a set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is either in D or has a neighbor in D. A dominating set D is a global offensive alliance (resp. a global defensive alliance) if for each vertex v in $V - D$ (resp. v in D) at least half the vertices from the closed neighborhood of v are in D. A global powerful alliance is both global defensive and global offensive. The global powerful alliance number $\gamma_{pa}(G)$ is the minimum cardinality of a global powerful alliance of G. We show that if T is a tree of order n with l leaves and s support vertices, then $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5}$. Moreover, we provide a constructive characterization of all extremal trees attaining this bound.

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1. Introduction

Let $G = (V, E)$ be a finite and simple graph of order n. The *open neighborhood* of a vertex v is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $d_G(v)$, is the size of its open neighborhood. For a vertex v, the *eccentricity* of v is the maximum of the distance to any vertex in the graph. The *diameter* of G noted $diam(G)$ is the maximum of the eccentricity of any vertex in the graph. A vertex of degree one is called a *pendant vertex* or a *leaf* and its neighbor is called a *support vertex*. A support vertex with exactly one non-leaf neighbor is called a *pendant support vertex*. If v is a support vertex, then L_v will denote the set of the leaves attached at v. A support vertex v is said to be *strong* if $|L_v| \geq 2$ and *weak* otherwise. We denote the set of leaves of a graph G by $L(G)$ and the set of support vertices by $S(G)$, and let $|L(G)| = l(G)$, $|S(G)| = s(G)$ (we use l, s if there is no ambiguity). We write P_n for the path of order n. If a tree $T = P_2$, we consider without loss of generality that P_2 has one support vertex and one leaf, so $l(P_2) = s(P_2) = 1$. A *star* S_p is the complete bipartite graph $K_{1,p}$. A tree containing exactly two non-pendant vertices is called a *double star*. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. Denote by T_x the subtree induced by a vertex x and its descendants in a rooted tree T.

In [8], Hedetniemi *et al.* introduced several types of alliances in graphs including the offensive and the defensive alliances and the powerful alliances we consider here. A dominating set D of G is called a *global offensive alliance* (resp. a *global defensive alliance*) if for each vertex v in $V - D$ (resp. v in D) at least half the vertices from

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the closed neighborhood of v are in $D : |N[v] \cap D| \geq |N[v] - D|$. Alliances in graph were studied in [1–9]. A *global powerful alliance* is both global defensive and global offensive which is equivalent to saying that for every vertex v in $V, |N[v] \cap D| \geq |N[v] - D|$. The *global powerful alliance number* $\gamma_{pa}(G)$ is the minimum cardinality of a global powerful alliance. The entire vertex set is both a global offensive alliance and a global defensive alliance for any graph G, so every graph G has a global powerful alliance number. Note that this parameter has been little studied in literature since its introduction. We abbreviate global powerful alliance as *gpa*. A gpa with minimum cardinality $\gamma_{pa}(G)$ is called a $\gamma_{pa}(G)$-set. A lower bound for a global powerful alliance number for any tree with equal powerful alliance and global powerful alliance numbers was given in [2] in term of the maximum degree $\triangle(G)$. They show that $\gamma_{pa}(T) \geq \lceil \frac{\triangle(T)+1}{2} \rceil$ and they characterize all such trees achieving this bound. In [3], Cami *et al.* have proved that finding an optimal global powerful (offensive, defensive) alliance is an NP-complete problem. However, a linear time algorithm that finds the smallest global powerful alliance of any weighted tree was given in [7].

2. Lower bound

We begin by giving the global powerful alliance number of a tree T, where T is either a star or a double star.

Observation 2.1. If T is a star S_p, then $\gamma_{pa}(S_p) = \lceil \frac{p+1}{2} \rceil$.

Observation 2.2. If T is a double star $S_{p,q}$, then $\gamma_{pa}(S_{p,q}) = \lfloor \frac{p+1}{2} \rfloor + \lfloor \frac{q+1}{2} \rfloor$.

The following observation about a support in a $\gamma_{pa}(G)$-set will be useful.

Observation 2.3. If G is a connected graph of order at least three, then there exists a $\gamma_{pa}(G)$-set that contains all support vertices.

Proof. If a $\gamma_{pa}(G)$-set, D does not contain a support vertex u, then D contains all leaves of u. So, we can replace any leaf of u by u in D. □

We now present our main result of this section.

Theorem 2.4. *Let T be a tree of order n with l leaves and s support vertices. Then $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5}$.*

Proof. We proceed by induction on the order of T. Clearly, the result holds for $1 \leq n \leq 3$ where T is P_n and thus $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5}$. Let $n \geq 4$ and assume that every tree T' of order $n', 4 \leq n' < n$ with l' leaves and s' support vertices verifies $\gamma_{pa}(T') \geq \frac{3n'-2l'-s'+2}{5}$. Let T be a tree with l leaves and s support vertices. If $diam(T) = 2$, then T is a star S_p and from Observation 2.1, $\gamma_{pa}(S_p) = \lceil \frac{p+1}{2} \rceil$. Since, $n = p+1, s = 1$ and $l = p$, we get $\gamma_{pa}(T) > \frac{3n-2l-s+2}{5} = \frac{p+4}{5}$. If $diam(T) = 3$, then $T = S_{p,q}$ and from Observation 2.2, $\gamma_{pa}(S_{p,q}) = \lfloor \frac{p+1}{2} \rfloor + \lfloor \frac{q+1}{2} \rfloor$. Since $n = p+q+2, l = p+q$ and $s = 2$, we obtain $\gamma_{pa}(S_{p,q}) \geq \frac{p+q+6}{5} = \frac{3n-2l-s+2}{5}$ and hence the result is valid. Assume that $diam(T) = t \geq 4$. Let T be rooted at a leaf u_t of a maximum eccentricity, that is $ecc(u_t) = diam(T)$ and let u_1 be a support vertex at distance $diam(T) - 1$ from u_t. Let $u_{i+1}, 0 \leq i \leq t-1$ be the parent of u_i in the rooted tree. Let $P : u_0, u_1, \ldots, u_t$ be then the resultant diametral path. It's clear that u_1 is a pendant support vertex. Let D be a $\gamma_{pa}(T)$-set with a fewest possible number of leaves. Consider the following cases.

Case 1. $|L_{u_1}| = k \geq 3$. By the choice of D, both vertices u_1 and u_2 are in D and D contains $\lfloor \frac{k-1}{2} \rfloor$ leaves of u_1. Let u' be any leaf of u_1 not in D. Let $T' = T - (L_{u_1} - u')$ (see Fig. 1), then $n' = n-k+1, l' = l-k+1$ and $s' = s$. Since, $diam(T) \geq 4$ then $n' \geq 5$. Clearly $D \cap V(T')$ is a gpa of T' and so $\gamma_{pa}(T') \leq \gamma_{pa}(T) - \lfloor \frac{k-1}{2} \rfloor$. Using the inductive hypothesis on T', we get $\gamma_{pa}(T) \geq \frac{3n'-2l'-s'+2}{5} + \lfloor \frac{k-1}{2} \rfloor$. Since $n' = n-k+1, l' = l-k+1$ and $s' = s$ then $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5} + \frac{1-k}{5} + \lfloor \frac{k-1}{2} \rfloor > \frac{3n-2l-s+2}{5}$.

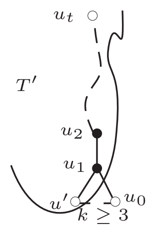


FIGURE 1. $T' = T - (L_{u_1} - u')$.

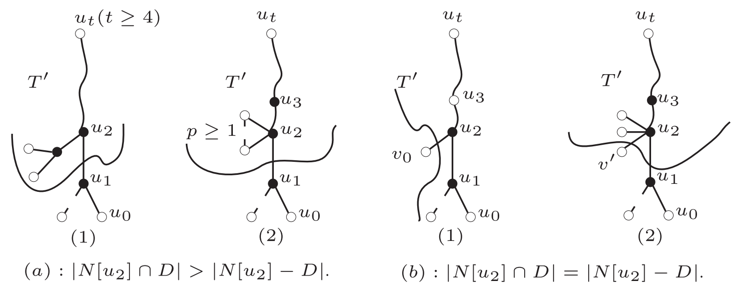


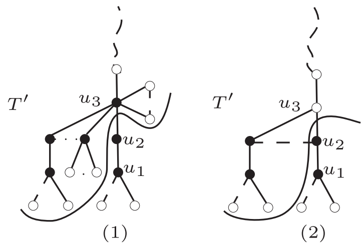
FIGURE 2. The black vertices are in D. In (a), $T' = T - T_{u_1}$. In (b), $T' = T - \{v_0\}$ or $T' = T - (T_{u_1} \cup \{v'\})$.

Case 2. $|L_{u_1}| = k \leq 2$ and $D \cap L(u_1) = \emptyset$. Any child of u_2 is either a leaf or a support vertex. By Observation 2.3 and the minimality of D, both u_1 and u_2 are in D.

Subcase 2.1 $d_T(u_2) \geq 3$. Assume u_2 is a support vertex with $|L_{u_2}| = p \geq 2$ or u_2 has a child which is a support vertex. Then, if $|N[u_2] \cap D| > |N[u_2] - D|$, let us consider $T' = T - T_{u_1}$ (see (a) in Fig. 2). As $diam(T) \geq 4$, then $n' \geq 4$. It is obvious that $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 1$. Hence, $\gamma_{pa}(T) \geq \gamma_{pa}(T') + 1$. Using the inductive hypothesis on T', we obtain $\gamma_{pa}(T) \geq \frac{3n'-2l'-s'+2}{5} + 1$. Since $n' = n - 1 - k, l' = l - k$ and $s' = s - 1$, we get $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5} + \frac{3-k}{5} > \frac{3n-2l-s+2}{5}$. Suppose now that u_2 is a support vertex with $|N[u_2] \cap D| = |N[u_2] - D|$. Let v' be a leaf-neighbor of u_2 not in D and let $T' = T - (T_{u_1} \cup \{v'\})$ (see (b.2) in Fig. 2). Clearly, $D \cap V(T')$ is a gpa of T' which implies that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 1$. Using the inductive hypothesis on T', we get $\gamma_{pa}(T) \geq \frac{3n'-2l'-s'+2}{5} + 1$. Since $n' = n - 2 - k, l' = l - k - 1$ and $s' = s - 1$, we obtain $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5} + \frac{2-k}{5} \geq \frac{3n-2l-s+2}{5}$.

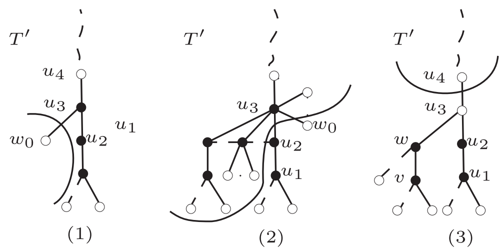
Suppose now that u_2 is a support vertex with $|L_{u_2}| = 1$ and $d_T(u_2) = 3$ (see (a.2) in Fig. 2). If $u_3 \in D$, then we consider $T' = T - T_{u_1}$. As before, it can be seen that $\gamma_{pa}(T) > \frac{3n-2l-s+2}{5}$. Thus, assume that $u_3 \notin D$. Hence, u_3 is not a support vertex according to the Observation 2.3. Let v_0 be the unique leaf of u_2 in T and let us consider $T' = T - \{v_0\}$ (see (b.1) in Fig. 2). We have $diam(T') \geq 4$ and so $n' \geq 5$. The set $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T)$. Using the inductive hypothesis on T' and since $n' = n - 1, l' = l - 1$ and $s' = s - 1$, we obtain $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5}$.

Subcase 2.2 $d_T(u_2) = 2$. Suppose $u_3 \in D$. Any child of u_3 is either a leaf or a support vertex or a non-leaf neighbor of a support vertex. According to Observation 2.3, and the minimality of D, any non-leaf neighbor of u_3 is in D. Assume that $|N[u_3] \cap D| > |N[u_3] - D|$ (see (a.1) in Fig. 3). Let us consider $T' = T - T_{u_2}$. It is easy to check that if $T' = K_2$ then $n = k + 4, l = k + 1$ and $s = 2$ and so $\gamma_{pa}(T) = 3 > \frac{3n-2l-s+2}{5} = \frac{k+10}{5}$ and the result holds. Further, $T' \neq S_2$ otherwise $|N[u_3] \cap D| = |N[u_3] \cap (V - D)|$. Thus $n' \geq 4$ and $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 2$. Using the inductive hypothesis on T' and since $n' = n - 2 - k, l' \leq l - k + 1$



$(a) : |N[u_3] \cap D| > |N[u_3] - D|$.

FIGURE 3. In (a): either u_3 is in D or not, we consider $T' = T - T_{u_2}$.



$(b) : |N[u_3] \cap D| = |N[u_3] - D|$.

FIGURE 4. In (b), if $u_3 \in D, T' = T - \{w_0\}$ if $d_T(u_3) = 3$, otherwise, $T' = T - (T_{u_2} \cup \{w_0\})$. If $u_3 \notin D, T' = T - T_{u_3}$.

and $s' \leq s$, we obtain $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5} + \frac{2-k}{5} \geq \frac{3n-2l-s+2}{5}$. Let us remark that the case when $d_T(u_3) = 2$ is included in this case.

Suppose now that $|N[u_3] \cap D| = |N[u_3] - D|$. If $d_T(u_3) = 3$, then $|L_{u_3}| = 1$ and u_4 is not in D (see (b.1) in Fig. 4). Let w_0 be the unique leaf of u_3 and let us consider $T' = T - \{w_0\}$. As $diam(T) \geq 4$ then $n' \geq 5$. Clearly, $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T)$. Using the inductive hypothesis on T' and since $n' = n - 1, l' = l - 1$ and $s' = s - 1$, we obtain $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5}$.

Suppose now that $d_T(u_3) > 3$. So, $|L_{u_3}| \geq 2$ and u_3 has necessarily a leaf say $w_0 \notin D$. Let us consider $T' = T - (T_{u_2} \cup \{w_0\})$ (see (b.2) in Fig. 4). If $u_4 \in D$ then $|L_{u_3}| \geq 3$ and so $n' \geq 4$. Otherwise, $u_4 \notin D, |L_{u_3}| \geq 2$ and $L_{u_3} \cap D = \emptyset$. Thus, u_3 has a non-leaf child in $D \cap V(T')$ and then $n' \geq 4$. Clearly, $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 2$. Using the inductive hypothesis on T' and since $n' = n - 3 - k, l' = l - 1 - k$ and $s' = s - 1$, we obtain $\gamma_{pa}(T) \geq \frac{3n'-2l'-s'+2}{5} + 2 \geq \frac{3n-2l-s+2}{5} + \frac{4-k}{5}$. Since $k \leq 2$ then we obtain $\gamma_{pa}(T) > \frac{3n-2l-s+2}{5}$. Suppose now that $u_3 \notin D$. Then u_3 is different from a support vertex according to Observation 2.3. Further, u_3 cannot be the unique non-leaf neighbor of a support vertex as it can replace any leaf in D of its support neighbor which is a contradiction. So, any neighbor of u_3 is in D. If $|N[u_3] \cap D| > |N[u_3] - D|$ (see (a.2) in Fig. 3). Let us consider $T' = T - T_{u_2}$ and we proceed with the same manner as previously (the case $d_T(u_3) = 2$ is included here). Assume now that $|N[u_3] \cap D| = |N[u_3] - D|$. So, $d_T(u_3) = 3$ and u_4 is not in D (see (b.3) in Fig. 4). Let w, v be the children of u_3 and w, respectively in T_{u_3} and let us consider $T' = T - T_{u_3}$. If w is a support vertex then $|L_w| = 1$ otherwise, u_3 may replace a leaf of w in D which is a contradiction. Set $|L_v| = p$. By analogy to $L_{u_1}, p \leq 2$. As $u_4, u_3 \notin D$, then u_4 has at least two neighbors in $D \cap V(T')$. The vertex u_t cannot be a parent of u_4 otherwise u_4 is a support vertex and then u_4 will be in D which is a contradiction. So, $n' \geq 4$. It is obvious that $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 4$. Using inductive hypothesis on T' and since $n' \geq n - 6 - p - k, l' \leq l - p - k$ and $s' \leq s - 2$, we obtain $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5} + \frac{4-(p+k)}{5}$. As $p + k \leq 4$ then we get $\gamma_{pa}(T) \geq \frac{3n-2l-s+2}{5}$ and the result holds.

□

3. Trees with $\gamma_{pa}(T) = \frac{3n-2l-s+2}{5}$

In this section, we characterize the extremal trees attaining the bound given in the Theorem 2.4. For the purpose, we define a family $\mathfrak{F}$ of all trees T that can be obtained from the sequence $T_1, T_2, \ldots, T_k (k \geq 1)$ of trees, where T_1 is the path $P_2, T = T_k$, and if $k \geq 2, T_{i+1}$ is obtained recursively from T_i by the operation listed below.

– Operation O: Assume u is a support vertex of T_i. Then the tree T_{i+1} is obtained from T_i by adding one leaf by attachment edge to u and by adding a path $P_3 : xyz$ by joining u to y.

3.1. Preliminary results

From the way in which a tree $T \in \mathfrak{F}$ is constructed, we make the following observation.

Observation 3.1. Let T be a tree of $\mathfrak{F}$ different from P_2. Then every non-leaf vertex of T is a strong support vertex.

Theorem 3.2. *Let T be a nontrivial tree with l leaves and s support vertices. If $T \in \mathfrak{F}$ then $\gamma_{pa}(T) = \frac{3n-2l-s+2}{5}$.*

Proof. Let T be a tree of $\mathfrak{F}$. We use induction on the number of operations O performed to construct T. Clearly, if $T = T_1 = P_2$ then $\gamma_{pa}(T) = 1 = \frac{3n-2l-s+2}{5}$. Assume that the property is true for all trees of $\mathfrak{F}$ constructed with $k-1 \geq 0$ operations, and let T be a tree of $\mathfrak{F}$ constructed with k operations. Thus T is obtained by performing the operation O on a tree $T' = T_{k-1} \in \mathfrak{F}$ of order n' with l' leaves and s' support vertices. By the induction hypothesis, we have $\gamma_{pa}(T') = \frac{3n'-2l'-s'+2}{5}$. By Observation 2.3, there exist a $\gamma_{pa}(T')$-set that contains the support vertex u. Such a set can be extend to a gpa of T by adding y. Then, $\gamma_{pa}(T) \leq \gamma_{pa}(T') + 1 = \frac{3n'-2l'-s'+2}{5} + 1$. On the other hand, let D be a $\gamma_{pa}(T)$-set with a fewest possible number of leaves. So, by the minimality of D and Observation 2.3, both u and y are in D. Let u' be the leaf of u in $T - T'$. If $u' \notin D$ then $D \cap V(T')$ is a gpa of T'. Otherwise, we replace $u' \in D$ by any leaf of u in T' not in D. It follows then that $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 1$. We deduce then that $\gamma_{pa}(T) = \gamma_{pa}(T') + 1 = \frac{3n'-2l'-s'+2}{5} + 1$. Since $n' = n-4, l' = l-3$ and $s' = s-1$, we get $\gamma_{pa}(T) = \frac{3n-2l-s+2}{5}$. □

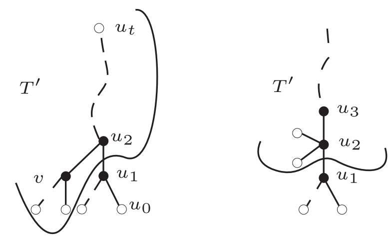
3.2. The main result

Theorem 3.3. *Let T be a nontrivial tree with l leaves and s support vertices. Then $\gamma_{pa}(T) = \frac{3n-2l-s+2}{5}$ if and only if $T \in \mathfrak{F}$.*

Proof. The sufficiency follows from Theorem 3.2. To prove the necessity, we proceed by induction on the order n of a tree T verifying $\gamma_{pa}(T) = \frac{3n-2l-s+2}{5}$. We shall prove that $T \in \mathfrak{F}$. If $diam(T) = 1$, then $T = K_2 \in \mathfrak{F}$. If $diam(T) = 2$ then T is a star S_p with $p \geq 2$ and from Observation 2.1, $\gamma_{pa}(T) = \lceil \frac{p+1}{2} \rceil > \frac{p+4}{5} = \frac{3n-2l-s+2}{5}$. If $diam(T) = 3$ then T is a double star $S_{p,q}$ and by Observation 2.2, $\gamma_{pa}(T) = \lfloor \frac{p+1}{2} \rfloor + \lfloor \frac{q+1}{2} \rfloor = \frac{p+q+6}{5} = \frac{3n-2l-s+2}{5}$ if and only if $p = q = 2$ and $S_{2,2} \in \mathfrak{F}$. So, $diam(T) \geq 4$. Suppose now that every tree T' of order $n', 5 \leq n' < n$ with l' leaves and s' support vertices such that $\gamma_{pa}(T') = \frac{3n'-2l'-s'+2}{5}$ is in $\mathfrak{F}$.

If any non-pendant support vertex, say t of T is weak, then let T' be the tree obtained from T by removing a leaf, say t' adjacent to t. So, any $\gamma_{pa}(T)$-set with a fewest possible number of leaves is a gpa of T' implying that $\gamma_{pa}(T') \leq \frac{3n-2l-s+2}{3}$. Since $n = n'+1, l = l'+1$ and $s = s'+1$ we get $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{3}$. The equality holds by Theorem 2.4 and consequently by using the inductive hypothesis, we deduce that $T' \in \mathfrak{F}$. By Observation 3.1, we deduce that t is a strong support vertex of T' as it is different from a leaf which is a contradiction. Henceforth, we can assume that any non-pendant support vertex of T is strong.

Let $diam(T) = t$ and let $P : u_0, u_1, \ldots, u_t, (t \geq 4)$ be a diametral path and root T at u_t. Clearly u_1 is a pendant support vertex. Let D be any $\gamma_{pa}(T)$-set that contains the fewest possible number of leaves.



(a) : u_2 is not a support. (b) : $|L_{u_2}| = 2$.

FIGURE 5. In both cases (a) and (b), $T' = T - T_{u_1}$.

Claim 3.4. $|L_{u_1}| \le 2$.

Proof. Suppose $|L_{u_1}| = k \ge 3$. By the choice of D, both vertices u_1 and u_2 are in D and D contains $\lfloor \frac{k-1}{2} \rfloor$ leaves of u_1. Let u' be any leaf of u_1 not in D. Let $T' = T - (L_{u_1} - \{u'\})$ (see Fig. 1). As $diam(T) \ge 4$ and $k - 1 \ge 2$ then $n' \ge 5$. Clearly, $D \cap V(T')$ is a gpa of T' and so $\gamma_{pa}(T') \le \gamma_{pa}(T) - \lfloor \frac{k-1}{2} \rfloor = \frac{3n-2l-s+2}{5} - \lfloor \frac{k-1}{2} \rfloor$. Since $n = n' + k - 1, l = l' + k - 1$ and $s = s'$, we obtain $\gamma_{pa}(T') \le \frac{3n'-2l'-s'+2}{5} + \frac{k-1}{5} - \lfloor \frac{k-1}{2} \rfloor < \frac{3n'-2l'-s'+2}{5}$ which contradicts Theorem 2.4. □

By Claim 3.4 and the choice of D, both u_1 and u_2 are in D and $D \cap L(u_1) = \emptyset$. Consider the following cases.
Case 1. $d_T(u_2) \ge 3$.

Claim 3.5. u_2 is a support vertex.

Proof. Any child of u_2 is either a leaf or a support vertex. Suppose u_2 has a neighbor say v which is a support vertex (see (a) in Fig. 5). By Claim 3.4, $|L_v| \le 2$ and so $\{u_1, u_2, v\} \subseteq D$. Let us consider $T' = T - T_{u_1}$. So $diam(T') \ge 4$ and then $n' \ge 5$. Obviously, $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \le \gamma_{pa}(T) - 1 = \frac{3n-2l-s+2}{5} - 1$. Since $n = n' + 1 + |L_{u_1}|, l = l' + |L_{u_1}|$ and $s = s' + 1$, we obtain $\gamma_{pa}(T') \le \frac{3n'-2l'-s'+2}{5} + \frac{|L_{u_1}|-3}{5} \le \frac{3n'-2l'-s'+2}{5} - \frac{1}{5} < \frac{3n'-2l'-s'+2}{5}$ which contradicts Theorem 2.4. □

Claim 3.6. $|L_{u_2}| \ge 3$.

Proof. By Claim 3.5, u_2 is a non-pendant support vertex, then by the remark given above, u_2 is a strong support vertex. Assume that $|L_{u_2}| = 2$. By the minimality of D and Observation 2.3, $\{u_1, u_2, u_3\} \subseteq D$. Let us consider $T' = T - T_{u_1}$ (see (b) in Fig. 5). As $diam(T) \ge 4$ and $|L_{u_2}| = 2$ then $n' \ge 5$. We proceed with the same manner as in the proof of Claim 3.5 and we get a contradiction. □

Claim 3.7. $|L_{u_1}| = 2$.

Proof. By Claim 3.4, $|L_{u_1}| \le 2$. Suppose $|L_{u_1}| = 1$. By the minimality of D, Observation 2.3 and Claim 3.6, D contains $\{u_1, u_2, u_3\}$ and $\lfloor \frac{|L_{u_2}|-2}{2} \rfloor$ leaves of u_2. Since $u_2 \in D$ and D is a defensive alliance, then $|N[u_2] \cap D| \ge |N[u_2] - D|$ (see Fig. 6). Suppose this inequality is strict and let us consider $T' = T - T_{u_1}$ (see (a) in Fig. 6). We get a contradiction with the same manner as in the proof of the Claim 3.5.

Thus, $|N[u_2] \cap D| = |N[u_2] - D|$. Let v be a leaf of u_2 not in D. Let us consider then $T' = T - T_{u_1} - \{v\}$ (see (b) in Fig. 6). As $diam(T) \ge 4$ and $|L_{u_2}| \ge 3$, then $n' \ge 5$. Clearly, $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \le \gamma_{pa}(T) - 1 = \frac{3n-2l-s+2}{5} - 1$. Since $n = n' + 2 + |L_{u_1}|, l = l' + 1 + |L_{u_1}|$ and $s = s' + 1$, we deduce that $\gamma_{pa}(T') \le \frac{3n'-2l'-s'+2}{5} + \frac{|L_{u_1}|-2}{5}$. Since $|L_{u_1}| = 1$, then $\gamma_{pa}(T') < \frac{3n'-2l'-s'+2}{5}$ which contradicts Theorem 2.4. □

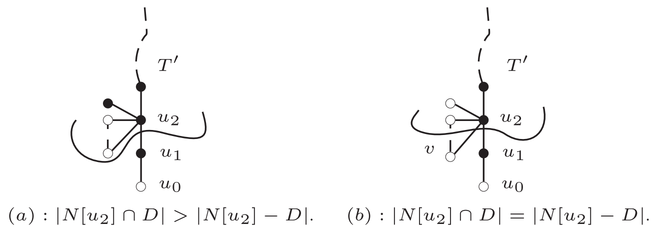


FIGURE 6. Case $|L_{u_1}| = 1$. In (a): $T' = T - T_{u_1}$, in (b): $T' = T - T_{u_1} - \{v\}$.

So, by Claim 3.7 and its proof, $|L_{u_1}| = 2$ and $|N[u_2] \cap D| = |N[u_2] - D|$. Let then v be a leaf of u_2 not in D and let us consider then $T' = T - T_{u_1} - \{v\}$. From the proof of Claim 3.7, we get $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5}$. Thus, the equality holds by Theorem 2.4. Using the inductive hypothesis, we deduce that $T' \in \mathfrak{F}$. Thus $T \in \mathfrak{F}$ since it is obtained from T' by applying operation O.

Case 2. $d_T(u_2) = 2$.

Claim 3.8. $d_T(u_3) \geq 3$.

Proof. Suppose $d_T(u_3) = 2$. Either u_3 or u_4 is in D. Without loss of generality, we suppose that $u_4 \in D$ as we may replace u_3 in D by u_4. Let us consider $T' = T - T_{u_2}$ (see (a) in Fig. 7). It is easy to check that if $T' = S_p, 1 \leq p \leq 3$ or $T' = P_4$ then $\gamma_{pa}(T) \in \{3, 4\} > \frac{3n-2l-s+2}{5}$. So, $n' \geq 5$. Clearly, $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 2 = \frac{3n-2l-s+2}{5} - 2$. We have $n = n' + 2 + |L_{u_1}|$. If u_4 is a support vertex then $s = s' + 1$ and $l = l' - 1 + |L_{u_1}|$. Thus, $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5} + \frac{|L_{u_1}|-3}{5}$. By Claim 3.4, $|L_{u_1}| \leq 2$ then $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5} - \frac{1}{5} < \frac{3n'-2l'-s'+2}{5}$ which contradicts Theorem 2.4. Suppose now that u_4 is not a support vertex, so $s = s'$ and $l = l' - 1 + |L_{u_1}|$. It follows that $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5} + \frac{|L_{u_1}|-2}{5}$. Since $|L_{u_1}| \leq 2$, if $|L_{u_1}| = 1$, we get $\gamma_{pa}(T') < \frac{3n'-2l'-s'+2}{5}$ which contradicts Theorem 2.4. So $|L_{u_1}| = 2$ and then $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5}$. We get the equality by Theorem 2.4 and using the induction hypothesis, we deduce that $T' \in \mathfrak{F}$. So, by Observation 3.1, every non-leaf vertex of T' is a strong support vertex. Thus, u_4 is a support vertex in T which contradicts our assumption. □

Claim 3.9. $|N[u_3] \cap D| = |N[u_3] - D|$.

Proof. By Claim 3.8, $d_T(u_3) \geq 3$. Each child of u_3 is either a leaf or a support vertex or a non-leaf neighbor of a support vertex. Assume that $|N[u_3] \cap D| > |N[u_3] - D|$. Let us consider $T' = T - T_{u_2}$ (see (b) in Fig. 7). We can easily check that $T' \neq S_p, 2 \leq p \leq 3$ and $T' \neq P_4$ otherwise $\gamma_{pa}(T) > \frac{3n-2l-s+2}{5}$ which contradicts the assumption on T. So $n' \geq 5$. Clearly, $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 2 = \frac{3n-2l-s+2}{5} - 2$. Since $n = n' + 2 + |L_{u_1}|, l = l' + |L_{u_1}|$ and $s = s' + 1$, we obtain $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5} + \frac{|L_{u_1}|-5}{5}$. By Claim 3.4, $|L_{u_1}| \leq 2$ then $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5} - \frac{3}{5} < \frac{3n'-2l'-s'+2}{5}$ which contradicts Theorem 2.4. □

Claim 3.10. u_3 is a support vertex.

Proof. Assume that u_3 is not a support vertex. So, by the minimality of D and Observation 2.3, every child of u_3 is in D as it is either a support vertex or a non-leaf neighbor of a support vertex. Consequently, if $u_3 \in D$ then $|N[u_3] \cap D| > |N[u_3] - D|$ (see (b) in Fig. 7) which contradicts Claim 3.9. So, $u_3 \notin D$ and according to Claim 3.9, $d_T(u_3) = 3$ and $u_4 \notin D$ (see (a) in Fig. 8). Then u_4 is different from a support vertex, otherwise, by the minimality of D and Observation 2.3, u_4 will be in D. Let v, u be the children of u_3 and v, respectively in T_{u_3}. If v is a support vertex then $|L_v| = 1$ otherwise u_3 will be in D as it may replace any leaf of v in D. But the

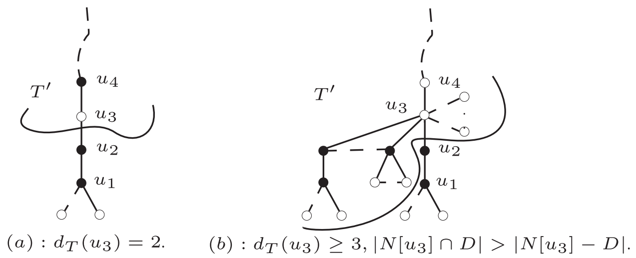


FIGURE 7. In $(b), u_3$ may be a support. In both cases (a) and (b), $T' = T - T_{u_2}$.

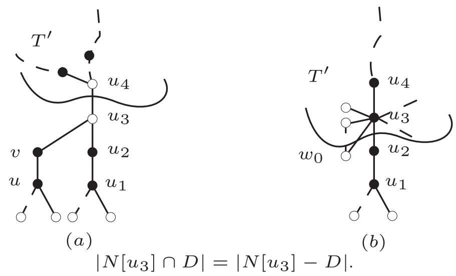


FIGURE 8. In (a): $u_3, u_4 \notin D, T' = T - T_{u_3}$. In (b), u_3 is a strong support vertex and $T' = T - T_{u_2} - \{w_0\}$.

weakness of v contradicts our remark given above about the non-pendant support vertices which must be strong. So, v is not a support vertex. Let us consider $T' = T - T_{u_3}$. Since $u_4 \notin D$, then u_4 has at least two neighbors in $D \cap V(T')$. The vertex u_t cannot be the parent of u_4, otherwise $u_4 \in D$ as it may replace u_t in D. Further, u_4 cannot be adjacent to a leaf or a support vertex, otherwise u_4 is in D as it is a support or it can replace any leaf in D of its support neighbor. It follows that $diam(T') \geq 4$ and $n' \geq 5$. The set $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 4 = \frac{3n-2l-s+2}{5} - 4$. Since $n = n' + 5 + |L_{u_1}| + |L_u|, l = l' + |L_{u_1}| + |L_u|$ and $s = s' + 2$, we obtain $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5} + \frac{|L_{u_1}|+|L_u|-7}{5}$. By Claim 3.4, $|L_{u_1}| + |L_u| \leq 4$, we get $\gamma_{pa}(T') < \frac{3n'-2l'-s'+2}{5}$ which contradicts Theorem 2.4. □

It follows from the previous Claims and the remark given above that u_3 is a strong support vertex verifying $|N[u_3] \cap D| = |N[u_3] - D|$. Let w_0 be a leaf of u_3 not in D. Let us consider $T' = T - (T_{u_2} \cup \{w_0\})$ (see (b) in Fig. 8). It is easy to check that u_t cannot be the parent of u_3 otherwise $T' = S_3$ and then $\gamma_{pa}(T) = 4$. So $\gamma_{pa}(T) > \frac{3n-2l-s+2}{5}$ which is a contradiction. So, $n' \geq 5$. Clearly, $D \cap V(T')$ is a gpa of T' implying that $\gamma_{pa}(T') \leq \gamma_{pa}(T) - 2 = \frac{3n-2l-s+2}{5} - 2$. Since $n = n' + 3 + |L_{u_1}|, l = l' + 1 + |L_{u_1}|$ and $s = s' + 1$ we get $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5} + \frac{|L_{u_1}|-4}{5}$. By Claim 3.4, $|L_{u_1}| \leq 2$ and then $\gamma_{pa}(T') \leq \frac{3n'-2l'-s'+2}{5} - \frac{2}{5} < \frac{3n'-2l'-s'+2}{5}$ which contradicts Theorem 2.4 and the proof is complete. □

4. CONCLUSION

We give in this paper a lower bound on the global powerful alliance number of any tree in terms of its order and its numbers of leaves and support vertices. Moreover, we characterize all extremal trees attaining this bound. Bouzefrane [1] shows that any tree T different from a star S_p with order $n \geq 4, l$ leaves and s support vertices verifies $\gamma_{pa}(T) \leq \frac{4n-l+s}{6}$. The first author of this paper characterizes all extremal trees achieving this bound in [9]. Thus, we obtain a framing of the global powerful alliance number in the class of trees. Among the open problems raised by our results, the following are of particular interest.

– Explore the bounds on the global powerful alliance number in particular classes of graphs like the unicycle graphs, bipartite ones and the cactus.
– Characterize trees with a unique minimum global powerful alliance.

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