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COMMENTS ON "OPTIMALITY CONDITIONS FOR NONSMOOTH INTERVAL-VALUED AND MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING"

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Abstract. Necessary optimality conditions for a nonsmooth semi-infinite interval-valued vector programming problem are given in the paper by Jennane *et al.* (*RAIRO:OR* (2020). DOI: 10.1051/ro/2020066). Having noticed inconsistencies in their paper, Gadhi and Ichatouhane (*RAIRO:OR* (2020). DOI: 10.1051/ro/2020107) made the necessary corrections and proposed what they considered a more pertinent formulation of their main Theorem. Recently, Jennane *et al.* (*RAIRO:OR* (2020). DOI: 10.1051/ro/2020134) have criticised our work. This note is a critical response to this criticism.

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1. Introduction

In the paper [2], Jennane *et al.* investigated the following semi-infinite interval-valued vector program

$$(2):\begin{cases}\text{Min } \{F(x)=(F_1(x),\ \dots,\ F_p(x)) : x\in\Omega\},\\ \Omega=\{x\in\mathbb{R}^n : G_t(x)\leq_{LU} A_t,\ \forall t\in T\}\end{cases}$$

where T is an arbitrary (possibly infinite) index set, $A_t=\left[A_t^L,\ A_t^U\right]\subseteq\mathbb{R}$ is a closed interval for all $t\in T,\ F_k=\left[F_k^L,\ F_k^U\right]$ and $G_t=\left[G_t^L,\ G_t^U\right]$ are interval-valued functions defined on $\mathbb{R}^n$ for all $k\in I:=\{1,\ \dots,\ p\}$ and $t\in T$.

For $A=\left[a^L,a^U\right]$ and $B=\left[b^L,b^U\right]$, they considered the partial ordering for intervals defined by:

$$A\leq_{LU} B \text{ iff } a^L\leq b^L \text{ and } a^U\leq b^U,\ \text{with at least one strict inequality.} \tag{1}$$

Under a nonsmooth constraint qualification (ACQ) given in terms of convexifactors, Jennane *et al.* [2] established necessary optimality condition for Problem (2). However, their examples and results did not take into consideration the requirement "at least one strict inequality".

Later on, Gadhi and Ichatouhane [1] pointed out these inconsistencies in the examples of Jennane *et al.* [2] and used an extra assumption (Assumption 3) for the results to hold if (1) is used for $\leq_{LU}$. They also specified

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that the closedness of the set $cone\left(\bigcup_{t\in T(\overline{x})}\partial^{us}g_t(\overline{x})\right)$ is necessary for the second part of Remark 4.3 from [2] to hold; and demonstrated its necessity by Example 10 of [1].

Recently, Jennane *et al.* published an erratum [3] where they removed "at least one strict inequality" from (1) and used another partial ordering for intervals defined by

$$A \leq_{LU} B \text{ iff } a^L \leq b^L \text{ and } a^U \leq b^U. \tag{2}$$

With this change, they resolved the issues with the examples and lemmas ([2], Lem. 3.3 and [2], Lem. 3.4). The authors have also corrected ([2], Thm. 4.5) by adding the closedness condition on the set cone. However, in an attempt to demonstrate that Example 3.6 of [1] is wrong, Jennane *et al.* [3] made further statements, which are incorrect (see Rems. 2.4 and 2.5). In fact, all our arguments in [1] are correct and this note provides the factual evidence.

The rest of the paper is organized in this way: Section 2 contains our comments. A conclusion is given in Section 3.

2. Comments

Below, we show that our comments and results are relevant and necessary.

Remark 2.1. When definition (1) is used our results Examples 3.1, 3.2, 3.6, Remarks 3.3, 3.4, Assumption 3, and Lemmas 4.2, 4.3 from [1] are not superfluous. It is clear that the definition used for the partial ordering has a great influence on the correctness of each result.

Remark 2.2. Assumption 3 of [1] is not a strong assumption (see [1], Example 4.1). It is useful and verifiable if we use the initial partial ordering (1) for intervals.

Remark 2.3. Our comments in Remark 3.4 of [1] are correct. Notice that Jennane *et al.* changed the meaning of Lemma 3.4 from [2] in [3] (see p. 2, lines 22 and 23).

In the following remark, we highlight a reasoning error that illustrates the inconsistency of the arguments of the authors in both [2,3].

Remark 2.4. Remark 3.5 of [1] is correct and the closedness of the set $cone\left(\bigcup_{t\in T(\overline{x})}\partial^{us}g_t(\overline{x})\right)$ is necessary to get

$$N_\Omega(\overline{x}) = cl\ cone\,(\Gamma(\overline{x})) = cl\ cone\left(\bigcup_{t\in T(\overline{x})} co\,(\partial^{us}g_t(\overline{x}))\right).$$

According to [4], we have the following.

- If DCQ holds at $\overline{x}$, then ACQ holds at $\overline{x}$ (see [4], Thm. 3.3(i)). However, the converse implication is not necessarily true.
- If ACQ holds at $\overline{x}$ and if $cone\,(\Gamma(\overline{x}))$ is closed, then

$$N_\Omega(\overline{x}) \subseteq cl\ cone\left(\bigcup_{t\in T(\overline{x})} co\,(\partial^{us}g_t(\overline{x}))\right)$$

  where $cone\,(\Gamma(\overline{x})) = pos\,(\Gamma(\overline{x}))$ is the convex cone generated by the set $\Gamma(\overline{x})$. For more details, see Summary of the Section H(iii) of [4].

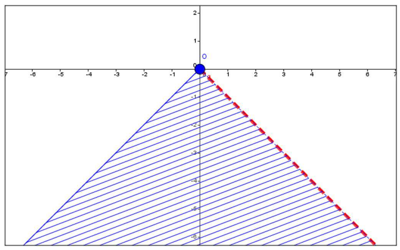
Remark 2.5. Example 3.6 of [1] supports the claim that the set cone is not necessarily closed even if the Abadie constraint qualification is satisfied. Indeed, since

$$cone\left(\bigcup_{t\in T(\overline{x})} co\left(\partial^{us} g_t\left(\overline{x}\right)\right)\right) = cone\left(T\times\{-1\}\right) = cone\left([-1,1)\times\{-1\}\right)$$

we have

$$cone\left(\bigcup_{t\in T(\overline{x})} co\left(\partial^{us} g_t\left(\overline{x}\right)\right)\right) = \left\{(x_1,x_2)\in\mathbb{R}^2 : x_2 < -\left|x_1\right|\right\} \cup \left\{(x_1,x_2)\in\mathbb{R}^2 : x_2 = x_1,\ x_2\leq 0\right\}.$$

Notice that $cone\left(\bigcup_{t\in T(\overline{x})} co\left(\partial^{us} g_t\left(\overline{x}\right)\right)\right)$ is not closed (blue hatched part of the graph below).



For more details, see Example 3.6 of [1]. Notice that Example 3.6 of [1] is taken from [4].

3. Conclusion

In the paper [1], we corrected some results established by Jennane *et al.* [2]. Recently, our approach was criticized in [3]. This note is a corroboration of the validity of our critique.

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