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A MULTISTAGE STOCHASTIC LOT-SIZING PROBLEM WITH CANCELLATION AND POSTPONEMENT UNDER UNCERTAIN DEMANDS

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Abstract. A multistage stochastic capacitated discrete procurement problem with lead times, cancellation and postponement is addressed. The problem determines the procurement of a product under uncertain demand at minimal expected cost during a time horizon. The supply of the product is made through the purchase of optional distinguishable orders of fixed size with delivery time. Due to the unveiling of uncertainty over time it is possible to make cancellation and postponement corrective decisions on order procurement. These decisions involve costs and times of implementation. A model of the problem is formulated as an extension of a discrete capacitated lot-sizing problem under uncertain demand and lead times through a multi-stage stochastic mixed-integer linear optimization approach. Valid inequalities are generated, based on a conventional inequalities approach, to tighten the model formulation. Experiments are performed for several problem instances with different uncertainty information structure. Their results allow to conclude that the incorporation of a subset of the generated inequalities favor the model solution.

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1. Introduction

This work deals with the resolution of a capacitated discrete procurement problem with lead times, cancellation and postponement. The problem determines the expected cost minimization of meeting the uncertain demand of a product during a discrete time planning horizon. Optional distinguishable orders with an indivisible amount of the product are available to accomplish the product demand. Each order, with an associated cost and a delivery time, can be acquired only once on the planning horizon. After covering the product demand in a period, the remaining acquired quantity is stored, up to a certain capacity, to satisfy demand in the following periods. The orders have significant delivery times within the planning horizon; so that a considerable amount of time elapses between the purchase decision and the moment when the product is received. As the product uncertain demand is revealed over time, it may happen that an acquired order not yet received is no longer

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necessary. In such case, it is possible to take corrective decisions: to cancel the order acquisition or postpone the order delivery. Both corrective decisions have associated costs and minimum execution times with regard to the order delivery time.

The motivation of the present work is a real problem arising in an oil company that deals with fuel acquisition under contractual and logistic conditions for electric power generation. The demand that the company faces is uncertain, given that thermal electricity generation, as a complement in an electrical system, is highly dependent on renewable sources [23, 29]. While the particular situation of the company focuses on fuel procurement for thermal generation, this paper discusses the formulation and solution of a more general variant of the problem core, which can represent situations in which a product procurement is carried out by selecting distinguishable discrete supply options in the context of uncertain demand.

The uncapacitated lot-sizing problem formulated by Wagner and Whitin [27] and its variant with variable capacity and discrete sizing of Nemhauser and Wolsey [19] is used as a baseline to formulate the problem of this work. For the deterministic uncapacitated and continuous lot-sizing case, Wagner and Whitin [27] and Wagelmans *et al.* [26] showed that the problem has efficient resolution through dynamic programming applied to the original mixed-integer formulation. For this case, there are also formulations with known feasible regions convex hull: the formulation on extended facility location of Krarup and Bilde [16] and the formulation on $(\ell, \mathcal{S})$ valid inequalities of Barany *et al.* [3]. Moreover Bitran and Yanasse [5] established that the lot-sizing problem with discrete sizing is a generalization of the binary knapsack problem and that it belongs to the $\mathcal{NP}$-hard class of complexity. Several authors propose heuristics to solve the capacitated variant of the problem [1, 7, 22].

In the event that data of instances are uncertain the problem may be modeled by stochastic optimization [4]. Ahmed *et al.* [2] established an extended formulation of the stochastic continuous uncapacitated problem, and they proved that for the stochastic case the Wagner–Whitin conditions are not preserved. Guan *et al.* [11] showed that the $(\ell, \mathcal{S})$ inequalities are valid for the stochastic continuous formulation, and they proposed an extension that allows to determine the facets of the convex hull.

Delivery time of the lots is modeled by a deterministic continuous uncapacitated lot-sizing problem. Lee *et al.* [17] presented a formulation with a time interval of demand fulfillment, and they proposed a dynamic programming based efficient solution. Brahimi *et al.* [6] proposed formulations with delivery time where the lots can be distinguishable or not. For the distinguishable case and the undistinguishable case when the lots' delivery intervals are not inclusive, the authors proposed a dynamic programming efficient algorithm. Tight extended formulations of these variants were proposed by Wolsey [28]. For the stochastic case, Huang and Küçükyavuz [14] established that the problem with random lead times and non-overlapping delivery intervals can be efficiently solved. Later, Jiang and Guan [15] established a quadratic polinomial time algorithm for this variant. Liu and Küçükyavuz [18] proposed valid inequalities for the static probabilistic lot-sizing problem. Hosseini and MirHassani [13] generated valid inequalities for tightening a refueling station location model. A stronger formulation based on valid inequalities for simple assembly line balancing was proposed by Ritt and Costa [20].

Testuri *et al.* [24] present a specific application of the proposed problem, which details the complexity of its resolution and motivates the present work. In that work, the authors show the validity of a stochastic optimization model of the problem based on the assessment of the classic VSS and EVPI metrics for some problem instances. The formulation of the problem was presented in a conference paper [25], but it is detailed again in this work to facilitate the deduction of the inequalities introduced.

In the present work the problem under study is modeled by a new extension of the stochastic capacitated discrete lot-sizing problem with corrective actions and lead times. The model is formulated through a multi-stage stochastic mixed-integer linear optimization approach. The formulation is tightened by novel valid inequalities derived of the conventional $(\ell, \mathcal{S})$ inequalities approach. Computational experiments show that the derived inequalities are prominent to improve the resolution of the model.

This work is organized according to the following sections. An algebraic model of the problem is presented in Section 2. In Section 3 valid inequalities for the model formulation are presented. Computational experiments are established to determine the utility of the generated valid inequalities in Section 4. Finally, conclusions and future work are discussed in Section 5.

2. A multistage stochastic model

The model formulation is based on a mathematical programming approach. Basic entities represented by index sets are described as:

T periods, $\{1, \ldots, \hbar\}$ (ordered set),
A already acquired orders,
F possible (future) orders to be acquired,
O orders, $A \cup F$,
N nodes of the scenario tree.

The planning time is represented by the set T of discrete time periods, from initial period 1 up to horizon period $\hbar$. The set O of orders is partitioned in two sets: the set A of already acquired orders – orders established in previous executions of the model – that are pending reception, and the set F of possible (future) orders to be acquired from now on. Acquisition decisions are made on orders in set F, and cancellation or postponement decisions are taken on orders in set A.

The uncertain demand is modeled by a discrete-time stochastic process indexed in the planning periods. A finite probability space is used to define the stochastic process. The first period demand is deterministic, and the demand of the remaining period is discrete random with known probability functions. The process information structure can be modeled by an arborescence structure called *tree of scenarios* [21]. The structure is a directed rooted tree, with the root node in period $t = 1$ and with leaf nodes in period $t = \hbar$ (identifying itself the scenarios).

The state of the process is described by a node of the scenario tree, and it is identified by a index n in a set of nodes N. The root node of the tree, denoted by 1, takes place on period $t = 1$.

Set-valued functions on the nodes of the scenario tree are described as:

$t(n)$ period corresponding to node $n \in N$,
$p(n)$ immediate time predecessor node of node $n \in N$; the predecessor of the root node is defined as the auxiliary node 0, such that $0 \notin N$,
$p(n,k)$ k-th time predecessor of node $n \in N$; defined as $p(n,k) := p(p(n,k-1))$ for $k = 2, \ldots, t(n)-1$, such that $p(n,1) := p(n)$,
$P(n)$ sequence of nodes on the path from the root node to node $n \in N$; defined as $\big(r = p(n,t(n)-1), p(n,t(n)-2), \ldots, p(n,1), n\big)$,
$S(n)$ set of nodes successors of node $n \in N$; defined as $S(n) := \{n' \in N, k = 1, \ldots, \hbar - t(n) | n = p(n',k)\}$,
L set of leaf nodes of the tree; defined as $L := \{n \in N | t(n) = \hbar\}$.

The parameters of the model are:

d_n demand at node $n \in N$,
π_n probability of event node $n \in N$,
s_0 initial inventory level,
$\underline{s}, \overline{s}$ minimum and maximum storage capacities by period,
τ^i period in which already acquired order $i \in A$ is received,
q^i size of order $i \in O$,
γ^i delivery time of order $i \in F$, such that $0 \leq \gamma^i \leq \hbar - 1$,
δ^i cancellation minimum time of already acquired order $i \in A$, such that $0 \leq \delta^i \leq \tau^i - 1$,
ϵ^i postponement minimum time of already acquired order $i \in A$, such that $0 \leq \epsilon^i \leq \hbar - \tau^i$,
ca^i unit acquisition cost of order $i \in O$,
cc^i unit cancellation cost of order $i \in O$,

cp^i unit postponement cost of order $i \in O$,
h_t unit storage cost in period $t \in T$,
a_t amount of already acquired orders received in period $t \in T$.

In the stochastic setting the product demand at each node n is defined as d_n. The probability of the state on each node n is denoted as π_n, such that $\pi_n \geq 0$ and $\sum_{n \in N|t(n)=t} \pi_n = 1$, for each $t \in T$. The demand distribution for each period $t \in T$ is represented by (d_n, π_n) such that $n \in N$ and $t(n) = t$. Due to storage constraints, the inventory of the product at the end of each period is restricted between a minimum size, $\underline{s}$, and a maximum size, $\overline{s}$, and there is an initial inventory size, s_0, at the beginning of the planning horizon.

The reception period τ^i of an already acquired order i is established in previous acquisitions. Each order i has a given size, q^i. Decisions on each order have a delay time of achievement of its results measured in periods. The delivery time of order i, γ^i, establishes the elapsed time in periods between the period of the acquisition decision and the period of arrival of the order. The cancellation minimum time of order i, δ^i, determines the minimum number of periods before the delivery period in which the order may be cancelled. The postponement minimum time of order i, ϵ^i, determines the minimum number of periods after the delivery period in which the posponed order may be received. The fulfilment period of decisions on acquisition, cancellation and postponement must take place within the planning horizon.

The decisions to acquire, cancel and postpone order i incur in unit costs ca^i, cc^i and cp^i, respectively. Associated to each period t there is a unit storage cost h_t. The sum of the units of the orders that are received in each period determine the already acquired amount as

$$a_t := \sum_{\{i \in A|\tau^i = t\}} q^i, \quad t \in T, \tag{2.1}$$

this is an auxiliary summary parameter.

In order to facilitate the formulation, derived subsets of the sets of nodes and periods that are indexed in orders are established as

N^i_γ nodes where order i can be acquired, $\{n \in N|t(n) \leq \hbar - \gamma^i\}$,
N^i_δ nodes where order i can be cancelled and postponed, $\{n \in N|t(n) \leq \tau^i - \delta^i\}$,
T^i_ϵ periods during which the order $i \in A$ can be postponed, $\{t \in T|t \geq \tau^i + \epsilon^i\}$.

These subsets abbreviate the denomination of nodes where, for each order i, it is possible to acquire it, N^i_γ, and where it is possible to cancel and postpone it, N^i_δ. In addition, subsets of periods to where it is possible to postpone each order i are established, T^i_ϵ. The subscripts of this subsets are part of their denomination.

In the stochastic model all decisions depend on the nodes of the tree according to the following definitions of the variables. There are binary variables associated with decisions on order i taken at node n for acquisition, v^i_n, cancellation, x^i_n, and postponement z^i_{nt} towards period t. The postponement of an already acquired order involves two sequential decisions: one on the period when the decision is made and the other of the period where it is postponed. In order to reduce the number of variables by not establishing a specific variable to determine the time of the postponement, the period when the decision is made is modeled as a cancellation decision. There are three continuous auxiliary variables that consolidate the units, at each node n, by type of decision: s_n integrates the inventory level at the end of the period of the node, u_n unifies the acquired units incoming at the node, w_n unites the cancelled units outgoing of the node, and y_n combines the postponed units incoming at the node.

The indexes of periods for already acquired units and storage unit cost parameters are stated on the temporal fulfillment of their associated node n by $t(n)$.

Based on preceding definitions of index sets, parameters and variables (and assuming a risk-neutral setting), a multi-stage stochastic mixed-integer linear optimization formulation of the multi-stage stochastic lot-sizing problem (MSLP) is

$$\min \sum_{n\in N} \pi_n \Bigg[\sum_{\{i\in F | n\in N^i_\gamma\}} ca^i q^i v^i_n \tag{2.2}$$

$$+ \sum_{\{i\in A | n\in N^i_\delta\}} \left(cc^i - ca^i\right) q^i x^i_n \tag{2.3}$$

$$+ \sum_{\{i\in A, t\in T^i_\epsilon | n\in N^i_\delta\}} \left(cp^i + ca^i - cc^i\right) q^i z^i_{nt} \tag{2.4}$$

$$+ \; h_{t(n)} s_n \Bigg], \tag{2.5}$$

s.t.

$$s_{p(n)} + a_{t(n)} + u_n + y_n = d_n + w_n + s_n, \qquad n\in N, \tag{2.6}$$

$$\underline{s} \le s_n \le \overline{s}, \qquad n\in N, \tag{2.7}$$

$$u_n = \sum_{\{i\in F | t(n)\ge \gamma^i+1\}} q^i v^i_{p(n,\gamma^i)}, \qquad n\in N, \tag{2.8}$$

$$\sum_{n'\in P(n)} v^i_{n'} \le 1, \qquad i\in F, n\in N, t(n)=\hbar-\gamma^i, \tag{2.9}$$

$$w_n = \sum_{\{i\in A | t(n)=\tau^i\}} q^i \left(\sum_{\{n'\in P(n) | t(n')\le \tau^i-\delta^i\}} x^i_{n'} \right), \qquad n\in N, \tag{2.10}$$

$$\sum_{n'\in P(n)} x^i_{n'} \le 1, \qquad i\in A, n\in N, t(n)=\tau^i-\delta^i, \tag{2.11}$$

$$x^i_n \ge z^i_{nt}, \qquad i\in A, n\in N^i_\delta, t\in T^i_\epsilon, \tag{2.12}$$

$$y_n = \sum_{\{i\in A | t(n)\ge \tau^i+\epsilon^i\}} q^i \left(\sum_{\{n'\in P(n)\cap N^i_\delta\}} z^i_{n',t(n)} \right), \qquad n\in N, \tag{2.13}$$

$$\sum_{\{n'\in P(n), t\in T^i_\epsilon\}} z^i_{n't} \le 1, \qquad i\in A, n\in N^i_\delta, \tag{2.14}$$

$$s_n, u_n, w_n, y_n \ge 0, \qquad n\in N, \tag{2.15}$$

$$v^i_n \in \{0,1\}, \qquad i\in F, n\in N^i_\gamma, \tag{2.16}$$

$$x^i_n, z^i_{nt} \in \{0,1\}, \qquad i\in A, n\in N^i_\delta, t\in T^i_\epsilon. \tag{2.17}$$

The (MSLP) formulation minimizes the expectation of the acquisition costs (2.2), the cancellation costs less the acquisition costs in case of cancellation (2.3), the postponement costs plus the acquisition costs minus the cancellation costs (2.4) – since a postponement is modeled in conjunction with a cancellation – and the storage costs (2.5).

The main requirement stated by constraints (2.6) is to satisfy demand while maintaining the inventory balance with contributions of the product previously stored, what was already acquired, what is acquired, and what is cancelled, postponed and remains available in storage for each node. Constraints (2.7) determine the lower and upper storage bounds at each node. The units of acquired product that are received at a node are established by the units of previously acquired orders that deliver at the period of the node according to (2.8). Constraints (2.9) state that for each order and each node where it is possible to acquire it, the order is acquired at most once on a single node on the path from the root to the node. Constraints (2.10) establish for each node under consideration the cancelled units of already acquired orders that outgo from the node. This is obtained by the sum of the cancellations of already acquired orders that arrive on the period of the node under consideration. The cancellation of an already acquired order can take place on the nodes – where it is possible to cancel it – in the path from the root node to the node in consideration. Constraints (2.11) establish that for each order and each node where it is possible to cancel it, the order is cancelled at most once on a single node on the path from the root to the node. Constraints (2.12) state that only cancelled orders may be postponed, since the postponement of an order is modeled along with its cancellation. Constraints (2.13) state for each node under consideration the postponed units of already acquired orders that enter the node. This is obtained by the sum of the postponement of already acquired orders that are postponed to the period of the node under consideration. Constraints (2.14) establish that for each order and each node where is possible take a postponement decision of it, the order is postponed at most once on a single node on the path from the root to the node, for periods to where it is possible to postpone the order. Constraints (2.15)–(2.17) state the domain of the variables. The set of feasible solutions of (MSLP) is denoted by X_{MSLP}.

3. MSLP valid inequalities

The problem belongs to the time complexity class $\mathcal{NP}$-hard, since it is an extension of the discrete lot-sizing problem. Therefore, no description of the convex hull of X_{MSLP} is known. One option used to strengthen the formulation is to incorporate specific valid inequalities derived from it. These inequalities may improve the solvers' capabilities to determine solutions.

From the classic $(\ell, \mathcal{S})$ valid inequalities formulation of the deterministic uncapacitated lot-sizing problem of Barany *et al.* [3], valid inequalities for X_{MSLP} are derived considering its stochastic case extension of Guan *et al.* [11]. Two sets of valid inequalities are established for sequence of nodes on the path from the root node to nodes in the scenario tree.

Proposition 3.1. *Let $\ell \in N$ and $\ell \in S(n)$ the inequalities*

$$u_n \leq d_{n\ell}\beta(n) + s_\ell, \tag{3.1}$$

$$u_n \leq d_{n\ell}\beta(n) + w_{n\ell} + s_\ell, \tag{3.2}$$

where $\beta(n) := \sum_{\{i \in F | t(n) \geq \gamma^i + 1\}} v^i_{p(n,\gamma^i)}$, are valid for X_{MSLP}.

Proof. Proof of (3.1).
From equations (2.10) and inequalities (2.11) it is satisfied that

$$w_n \leq \sum_{\{i \in A | t(n) = \tau^i\}} q^i, \quad n \in N.$$

From these inequalities and definition of a_t (*cf.* (2.1)) it holds that

$$w_n \leq a_{t(n)}, \quad n \in N. \tag{3.3}$$

Consider the material balance equation (2.6) of model (MSLP), for all $n \in N$

$$s_{p(n)} + a_{t(n)} + u_n + y_n = d_n + w_n + s_n,$$

given that $s_{p(n)}, a_{t(n)}, y_n \geq 0$ and inequalities (3.3) it follows that

$$u_n \leq d_n + s_n. \tag{3.4}$$

From inequalities (3.4) the following valid inequalities can be established

$$u_n \leq d_n\beta(n) + s_n, \quad \text{for all} \quad n \in N, \tag{3.5}$$

since for all $n \in N$, if $\beta(n) = 0$, then from (2.8) $u_n = \sum_{\{i \in F | t(n) \geq \gamma^i + 1\}} q^i v^i_{p(n,\gamma^i)} = 0$. Otherwise, if $\beta(n) \geq 1$, then (3.5) holds.

In general, from the sum of material balance equation (2.6) between $n \in N$ and $\ell \in S(n)$, the following condition holds

$$u_n \leq d_{n\ell} + s_\ell, \tag{3.6}$$

where $d_{n\ell} := \sum_{n' \in P(\ell) \setminus P(p(n))} d_{n'}$ is the accumulated demand in the nodes in the path from n to ℓ.

From (3.6) the valid inequalities (3.1) can be established, similar to the provision for (3.5).

The aggregated valid inequalities (3.8) are obtained by adding the inequalities (3.1) for each subset of the set of nodes of the path from the root node to the ℓ node.

Proof of (3.2).

Given the material balance equation (2.6) of model (MSLP), for all $n \in N$

$$s_{p(n)} + a_{t(n)} + u_n + y_n = d_n + w_n + s_n,$$

given that $s_{p(n)}, a_{t(n)}, y_n \geq 0$ it follows that

$$u_n \leq d_n + w_n + s_n. \tag{3.7}$$

From inequalities (3.7) and following similar provisions than the previous case the valid inequalities (3.2) can be established.

The aggregated valid inequalities (3.9) are obtained by adding the inequalities (3.2) for each subset of the set of nodes of the path from the root node to the ℓ node. ☐

Theorem 3.2. *Let $\ell \in N$ and $\mathcal{S} \subseteq P(\ell)$ then the MSLP-$(\ell, \mathcal{S})$ inequalities*

$$\text{(i)} \quad \sum_{n \in \mathcal{S}} u_n \leq \sum_{n \in \mathcal{S}} d_{n\ell}\beta(n) + s_\ell, \tag{3.8}$$

$$\text{(ii)} \quad \sum_{n \in \mathcal{S}} u_n \leq \sum_{n \in \mathcal{S}} d_{n\ell}\beta(n) + \sum_{n \in \mathcal{S}} w_{n\ell} + s_\ell, \tag{3.9}$$

are valid for X_{MSLP}.

Proof. The proof of (3.8) is based on the deterministic case presented by Barany *et al.* [3]. Given a point $(s, v, x, z) \in X_{\text{MSLP}}$ there are two cases.

(1) If $\beta(n) = \sum_{\{i \in F | t(n) \geq \gamma^i + 1\}} v^i_{p(n,\gamma^i)} = 0$ for all $n \in \mathcal{S}$, then $u_n = \sum_{\{i \in F | t(n) \geq \gamma^i + 1\}} q^i v^i_{p(n,\gamma^i)} = 0$ for all $n \in \mathcal{S}$ and $s_\ell \geq 0$, therefore the inequality holds.

(2) Otherwise, there exists $n \in \mathcal{S}$ such that $\beta(n) = 1$. Let $n' = \text{argmin}\{t(n) | n \in N, \beta(n) = 1\}$. Then $\beta(n) = 0$ and $u_n = 0$ for all $n \in \mathcal{S} \cap P(p(n'))$. Thus $\sum_{n \in \mathcal{S}} u_n \leq \sum_{n \in P(\ell) \setminus P(p(n'))} u_n \leq d_{n'\ell} + s_\ell \leq \sum_{n \in \mathcal{S}} d_{n\ell}\beta(n) + s_\ell$.

The proof of (3.9) is similar considering that $\sum_{n \in \mathcal{S}} w_{n\ell} \geq 0$. ☐

Lemma 3.3. *The MSLP-$(\ell, \mathcal{S})$ inequalities can be written alternatively as*

$$\text{(i)} \quad \sum_{n \in P(\ell) \setminus \mathcal{S}} u_n + \sum_{n \in \mathcal{S}} d_{n\ell}\beta(n) + \sum_{n \in P(\ell)} (y_n - w_n)$$

TABLE 1. Size of scenario tree structures.

Arity (g)	Horizon ($\hbar$)	Scenarios ($\|L\|$)	Nodes ($\|N\|$)
2	5	16	31
2	6	32	63
2	7	64	127
3	5	81	121
3	6	243	364
3	7	729	1093

$$\geq d_{1\ell} - \sum_{n \in P(\ell)} a_{t(n)} - s_0, \quad \text{for all} \quad \ell \in N, \mathcal{S} \subseteq P(\ell). \tag{3.10}$$

$$\text{(ii)} \quad \sum_{n \in P(\ell) \setminus \mathcal{S}} u_n + \sum_{n \in \mathcal{S}} d_{n\ell} \beta(n) + \sum_{n \in P(\ell)} (y_n - w_n) + \sum_{n \in \mathcal{S}} w_{n\ell}$$
$$\geq d_{1\ell} - \sum_{n \in P(\ell)} a_{t(n)} - s_0, \quad \text{for all} \quad \ell \in N, \mathcal{S} \subseteq P(\ell). \tag{3.11}$$

Proof. The sum of equations (2.6), for all $n \in P(\ell)$ of a given $\ell \in N$, results in

$$s_0 + \sum_{n \in P(\ell)} a_{t(n)} + \sum_{n \in P(\ell)} y_n + \sum_{n \in P(\ell)} u_n = d_{1\ell} + w_{1\ell} + s_\ell,$$

from where it is possible to solve for s_ℓ and substitute it in (3.8) and (3.9), obtaining an alternative representation of the valid inequalities without inventory variables, denoted as MSLP-$(\ell, \mathcal{S})$-i and MSLP-$(\ell, \mathcal{S})$-ii, respectively. □

The formulation variants in which the inequalities (3.10) and (3.11) are added to (MSLP) are called (MSLP-$(\ell, \mathcal{S})$-i) and (MSLP-$(\ell, \mathcal{S})$-ii), respectively.

4. Numerical results

A series of computational experiments and their results for the solution of the (MSLP) stochastic formulation and its variants are presented. The section describes experiments to determine the value of information and solution of the stochastic formulation. It outlines the experimentation performed to determine the performance of formulations derived from valid inequalities.

The computational effect of selecting a subset of the MSLP-$(\ell, \mathcal{S})$ inequalities of the formulation variants is explored. Specially, the (MSLP) formulation and its two variants, where the set $\mathcal{S}$ is fixed with the root node, are tested over a collection of instances, checking the quality of the obtained solutions and the computational effort invested in obtaining them.

The collection of instances is based on six scenario tree structures founded on rooted perfect directed trees. Each tree structure is determined by the number of periods of the planning horizon and the number of immediate time successors of each node (tree arity) as depicted in Table 1. Each tree structure with arity g and time horizon $\hbar$ contains $g^{\hbar-1}$ scenarios and $(g^{\hbar} - 1)/(g - 1)$ nodes.

A distribution of orders by quantity is associated to each tree structure. Each distribution of orders by quantity ($|O|$) is identified by the sum of numbers of already acquired orders ($|A|$) and possible orders to be acquired ($|F|$), as shown in column labelled "Orders" in Table 2. The 3-tuple $\langle$*arity, horizon, distribution of orders*$\rangle$ identifies table rows, denominated as categories of data instances. The table depicts, for each category, the number of constraints $(\ell, \mathcal{S})$ inequalities, variables and binary variables.

TABLE 2. Size of instance categories defined by scenario tree structure and order distribution.

Arity	Horizon	Orders [$\lvert O\rvert$ ($\lvert A\rvert + \lvert F\rvert$)]	Variables	(Binary)	Constraints	$(\ell, \mathcal{S})$-inequalities
2	5	10 (2 + 8)	249	(124)	225	31
2	6	12 (3 + 9)	549	(296)	480	63
2	7	14 (3 + 11)	1223	(714)	1012	127
3	5	10 (3 + 7)	809	(324)	827	121
3	6	12 (4 + 8)	2485	(1028)	2542	364
3	7	14 (4 + 10)	8091	(3718)	7987	1093

TABLE 3. Average results of 30 instances of formulation (MSLP) by instance category.

g	$\hbar$	$\lvert O\rvert$	Time-mean (s)	Time-median (s)	MIP-gap (%)	Nodes	Cuts	LP-gap (%)
2	5	10	1.60	0.87	0	15 261	159	10.31
2	6	12	27.61	19.53	0	111 228	473	11.25
2	7	14[a]	868.58	900.80	0.73	2 752 608	855	9.89
3	5	10	21.99	12.61	0	42 727	444	12.40
3	6	12[b]	821.46	900.39	2.86	868 850	1797	19.69
3	7	14[c]	900.20	900.14	5.81	46 457	2031	24.43

Notes. [a]28 of 30 instances reach the 900 s time limit. [b]27 of 30 instances reach the 900 s time limit. [c]All instances reach the 900 s time limit.

A distribution of orders by quantity is associated to each tree structure. Each distribution of orders by quantity ($|O|$) is identified by the sum of numbers of already acquired orders ($|A|$) and possible orders to be acquired ($|F|$), as shown in column labelled "Orders" in Table 2. The 3-uple $\langle$*arity, horizon, distribution of orders*$\rangle$ identifies table rows, denominated as categories of data instances. The table depicts, for each category, the numbers of variables, binary variables, constraints, and $(\ell, \mathcal{S})$ inequalities.

For each of the six instance categories, thirty data instances were randomly generated, adding up 180 instances. The product demand for each node is evenly distributed, $d_n \sim U[10, 50]$. For each scenario, identified as leaf node $n \in L$, a probability of its state π_n is established from a normalized $Beta(\alpha = 2, \beta = 2)$ distribution. The probability of the remaining nodes is obtained, backwards from the leaf nodes towards the root node, as the sum of the probabilities of their corresponding immediate successor nodes. Each instance has an initial storage, $s_0 = 20$, and a lower and an upper general storage bounds, $\underline{s} = 0$ and $\overline{s} = 80$, respectively. Each already acquired order $i \in A$ has delivery period $\tau^i = 1$ or 2 with equal probability. Each order $i \in O$ has delivery time $\gamma^i = 1$, cancellation minimum time $\delta^i = 1$ and postponement minimum time $\epsilon^i = 1$. Also, each order $i \in O$ has an uniformly distributed size, $q^i \sim U[10, 50]$, and there are costs evenly distributed according to the operations of acquisition, $ca^i \sim U[150, 250]$, cancellation, $cc^i \sim U[30, 50]$, and postponement, $cp^i \sim U[5, 12]$. Finally, the unit storage cost at each period t is $h_t = 1$.

The computational algebraic coding of the stochastic model was carried out by using AMPL [10]. The branch and cut solver of GUROBI 6.5 [12] was used for solving the formulation instances. The calculations were performed on an Intel Core i7 5960X 3.5 GHz computer with 20 MiB cache and 64 GiB RAM, operating with CentOS-7 Linux system.

The instances of the original formulation and its variants were solved up to either no gap between the objective function and its lower bound, or a time limit of 900 s. The instances average results of the original model and its variants, grouped by instance category, are summarized in Tables 3–5, respectively for formulations (MSLP), (MSLP-$(\ell, \mathcal{S})$-i) and (MSLP-$(\ell, \mathcal{S})$-ii).

TABLE 4. Average results of 30 instances of formulation (MSLP-$(\ell, \mathcal{S})$-i) by instance category.

g	$\hbar$	$\vert O\vert$	Time-mean (s)	Time-median (s)	MIP-gap (%)	Nodes	Cuts	LP-gap (%)
2	5	10	0.67	0.36	0	3835	108	7.48
2	6	12	17.65	6.10	0	61 852	431	9.51
2	7	14[a]	542.96	727.28	0.48	1 311 339	1218	7.01
3	5	10	7.94	3.17	0	18 532	318	9.74
3	6	12[b]	728.13	900.23	2.46	495 200	2211	17.26
3	7	14[c]	900.32	900.16	7.15	55 149	2048	26.08

Notes. [a]15 of 30 instances reach the 900 s time limit. [b]22 of 30 instances reach the 900 s time limit. [c]All instances reach the 900 s time limit.

TABLE 5. Average results of 30 instances of formulation (MSLP-$(\ell, \mathcal{S})$-ii) by instance category.

g	$\hbar$	$\vert O\vert$	Time-mean (s)	Time-median (s)	MIP-gap (%)	Nodes	Cuts	LP-gap (%)
2	5	10	1.13	0.88	0	11 346	159	9.51
2	6	12	18.19	12.78	0	56 467	483	11.00
2	7	14[a]	743.11	900.68	0.56	1 766 353	1349	8.89
3	5	10	18.13	5.81	0	30 371	476	11.77
3	6	12[b]	819.11	900.32	2.46	593 253	2387	19.47
3	7	14[c]	900.27	900.17	6.52	39 807	1712	25.18

Notes. [a]23 of 30 instances reach the 900 s time limit. [b]27 of 30 instances reach the 900 s time limit. [c]All instances reach the 900 s time limit.

These summary tables show, for each instance category depicted by columns labeled g, $\hbar$ and $|O|$, average results of the 30 instances of the model (MSLP) and its variants (MSLP-$(\ell, \mathcal{S})$-i) and (MSLP-$(\ell, \mathcal{S})$-ii). The elapsed time results by instance category: mean and median solver elapsed time are provided at columns "Time-mean" and "Time-median", respectively. The solver mean relative mixed-integer optimization gap for instances that reach the time limit of 900 s is depicted at column "MIP-gap (%)". The mean number of nodes of solver branch and cut method is provided at column "Nodes". The mean number of cuts added by solver's branch and cut method is provided by column "Cuts". The mean relative gap of the best objective with respect to the corresponding one under linear optimization relaxation is provided at column "LP-gap (%)". The median elapsed time is shown as a summary statistic since most of the instance categories present outliers for the metric. Due that most of the metrics are bounded from below, the median tends to be smaller than the mean. The types of cuts added by the solver are by descending frequency of occurrence: mixed integer rounding, flow cover, Gomory passes, implied bound, mod-k, zero-half, generalized upper bound cover, implied bound, and cover. There are almost no cases of the following cut types: strong Chvátal–Gomory, infinity proof, network, and lift and project. The distribution of cut type frequency of the original formulation and its variants are similar.

The results of formulation (MSLP-$(\ell, \mathcal{S})$-i) are better than those of formulation (MSLP) except for instance category $(3, 7, 14)$ that is equally hard for both. Formulation (MSLP-$(\ell, \mathcal{S})$-i) has smaller mean values than formulation (MSLP) for the remaining instance categories. For all instance categories where some of its instances reach the 900 s time limit, the formulation (MSLP-$(\ell, \mathcal{S})$-i) obtains a lower or equal number of these instances and a lower MIP-gap average than formulation (MSLP). Specifically in the case of instance category $(2, 7, 14)$, while formulation (MSLP) is solved to optimality for two instances in the allotted time, formulation (MSLP-$(\ell, \mathcal{S})$-i) is solved to optimality for 15 instances. On the other hand, formulation (MSLP-$(\ell, \mathcal{S})$-ii) presents mixed results with respect to formulation (MSLP) depending on instance categories, and overall worse results than formulation (MSLP-$(\ell, \mathcal{S})$-i).

TABLE 6. Risk neutral (expected) objective value and scenario's objective values by percentiles of one instance of formulation (MSLP) by instance category.

g	$\hbar$	$\lvert O \rvert$	Objective	0%	25%	50%	75%	100%
2	5	10	16 117	9308	13 950	17 037	19 123	20 563
2	6	12	8092	3735	7062	8262	10 886	13 288
2	7	14[a]	15 366	5295	12 557	16 660	18 865	22 604
3	5	10	17 109	9898	12 882	15 476	21 840	27 815
3	6	12[a]	6328	117	3224	6435	9453	12 811
3	7	14[a]	11 492	3180	7764	11 734	14 652	22 143

Notes. [a]The instance reaches the 900 s time limit.

A preliminary analysis of the objective value risk is performed for an instance of formulation (MSLP) by each instance category. Table 6 shows (MSLP)'s risk neutral objective value and its associated objective values of scenarios for probability distribution percentiles close to values 0, 25, 50, 75 and 100.

For the different categories, it can be appreciated that there is a wide distribution of the objective value with respect to the risk neutral (expected) objective. In many cases, it may be the case that the decision makers are particularly interested in avoiding extremely bad results, even if they occur with low probability. The variability in the results, and particularly the results of the worst scenarios, show that an alternative can be to employ a risk averse objective function, which can result in less variability in the worst cases (in tradeoff, admitting some impairment in the mean results). Future work could explore the effectiveness of those approaches for the problem under study.

5. Conclusions

A uncertain demand procurement problem with lead times is addressed by a stochastic multi-stage capacitated discrete lot-sizing model. Decisions on procurement, cancellation and postponement of product orders are modeled with delay time. The information uncertain structure is modelled by a discrete time stochastic process with finite probability, outlined in a scenario tree. The model is formulated by stochastic mixed-integer optimization with entities indexed by nodes of the scenario tree. The tree event when decisions on procurement, cancellation and postponement of product orders are effective is modelled by a function of parameters. Two classes of valid inequalities, based in the $(\ell, \mathcal{S})$ inequalities approach, were generated to tighten the formulation.

Computational experiments were performed for several instances of six tree structures and order distributions of different sizes. All of the experiments were solved at optimum for the structures of small and medium sized instance categories. The (MSLP-$(\ell, \mathcal{S})$-i) formulation shows improved performance over the (MSLP) original formulation for execution time, MIP-gap, number of branch and bound nodes, and number of solver cuts. The experimental results show the interest of the (MSLP-$(\ell, \mathcal{S})$-i) formulation, which can be used to solve larger instances in shorter computational time. As future work, the development of reduction and separation algorithms, and the study of formulations that combine the derived $(\ell, \mathcal{S})$ valid inequalities for different node paths in the scenario tree are being considered.

The formulation presented in this work is risk neutral; the objective function is an expected cost value, which can hide the variability of the objective function in the different scenarios. Of particular concern can be the case of "black swan" scenarios (*i.e.*, scenarios with a small probability of occurring and a very high total cost), which can motivate employing risk averse measures. We consider this as an important line of research for future work; in particular, it would be interesting to experiment with time-consistent risk measures as Expected Conditional Value-at-Risk [8] and Expected Conditional Stochastic Dominance [9].

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