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# INTEGRATED INVENTORY MODEL INVOLVING QUALITY IMPROVEMENT INVESTMENT AND ADVANCE-CASH-CREDIT PAYMENTS

Chih-Te Yang[1,*], Chien-Hsiu Huang[2] and Liang-Yuh Ouyang[2]

**Abstract.** This paper investigates the effects of investment and inspection policies on an integrated production–inventory model involving defective items and upstream advance-cash-credit payment provided by the supplier. In this model, retailers offer customers a downstream credit period. Furthermore, the defective rate of the item can be improved through capital co-investment by the supplier and retailer. The objective of this study was to determine the optimal shipping quantity, order quantity, and investment alternatives for maximizing the supply chain's joint total profit per unit time. An algorithm was developed to obtain the optimal solution for the proposed problem. Several numerical examples are used to demonstrate the proposed model and analyze the effects of parameters changes on the optimal solutions. Finally, management implications for relevant decision makers are obtained from the numerical examples.

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## 1. Introduction

Product quality in real life is not always perfect and is dependent on the manufacturer's production process or the quality of raw materials. Therefore, retailers will assess product quality first instead of immediately stocking it when an order is received. Inventory models of defective products have been studied extensively. Rosenblatt and Lee [34] investigated the influence of imperfect production processes on the economic production quantity (EPQ) model. Kim and Hong [19] extended Rosenblatt and Lee's [34] model to determine the optimal production run length for deteriorating production processes. Salameh and Jaber [35] also modified an EPQ model by accounting for items with imperfect quality. Further they assumed that poor-quality items are sold as a single batch at a discounted price after the end of the screening process. After this, many studies on imperfect production processes have been published, such as [1, 5, 8, 9, 14, 26, 32, 37, 40].

The production process can be realistically controlled which implies the defective rate of items can be reduced by investing capital to improve the equipment. Porteus [33] first introduced the option of investing capital for improving the quality of the production process and developed an EPQ model with defective items. Hong [10] extended Rosenblatt and Lee's model [34] by considering the investment for setup reduction and process

*Keywords.* Inventory, integrated model, defective items, capital investment, advance-cash-credit payment.

[1] Department of International Business, Chien Hsin University of Science and Technology, Taoyuan City, Taiwan.
[2] Department of Management Sciences, Tamkang University, New Taipei City, Taiwan.
*Corresponding author: ctyang@uch.edu.tw

quality improvement. Ouyang and Chang [28] considered a modified lot size reorder point model with imperfect production process and investigated the effect of quality improvement on the proposed model. Hou and Lin [12] also explored the effects of an imperfect production process on the optimal production run length when capital was invested for process quality improvement. Yang and Pan [44] proposed an integrated inventory model involving variable lead time and quality improvement investment with normally distributed demand. Lai *et al.* [21] developed an inventory system that incorporated the quality improvement cost to reduce the proportion of defects. Recently, an integrated production-inventory model with imperfect production system with partial backlogging was presented by Khanna *et al.* [17]. Other studies related to quality improvement include [6, 11, 15, 16, 20, 30, 41, 46]. Capital investment for improving imperfect production processes is generally provided by the manufacturer. However, if retailers agree to provide a part of the investment capital, they are not required to inspect the goods upon receipt, because the defective rate reaches a low level. Ouyang *et al.* [31] referred to this as "non-inspect." In this situation, all received products are treated as non-defective to stock and then sell to customers. Therefore, a penalty cost may be incurred for the defective items returned by customers. Consequently, when investment becomes an option, the retailer may trade the inspection cost for the penalty cost. This issue must be considered when analyzing inventory problems.

The traditional economic order quantity (EOQ) and EPQ models do not investigate payment methods and assume that the payment is made immediately upon receiving the consignment. However, in real business transactions, the supplier usually allows the retailer an extended period to provide full payment to attract new customers and increase sales and market share, because it benefits both the supplier and retailer. Goyal [7] incorporated trade credit into the EOQ model. Aggarwal and Jaggi [2] extended Goyal's model to deteriorating items. Chang *et al.* [3] established an EOQ model for deteriorating items under conditionally permissible delay in payments. Ouyang *et al.* [29] and Sharma *et al.* [39] presented inventory models for non-instantaneous deteriorating items with permissible delay in payments. Lashgari *et al.* [22] and Mukherjee and Mahata [27] investigated inventory control problems for deteriorating items with a two-level trade credit. Recently, Sarkar *et al.* [38] obtained the optimal decision of a retailer for time-varying deterioration items with selling-price and credit-period dependent demand to maximize the retailer's profit. Related articles include studies by [13, 18, 36]. However, when the purchase amount is large, to avoid customer defaults and to stimulate consumption, the manufacturer usually agrees with the retailer on the following payment method. The retailer is required to prepay a fraction of the procurement cost as a contract to buy items, then pay another fraction of the procurement cost in cash upon receiving the order and receive a short interest-free credit term to pay the remainder of the procurement cost. This is called an advance-cash-credit (ACC) payment scheme [23]. ACC payment schemes are commonly used in real-world sales finance situations. Li *et al.* [23] developed an inventory model for perishable products in which the retailer receives an upstream ACC payment from the supplier and in return offers a downstream cash-credit payment to customers. Wu *et al.* [43] considered an EOQ model including perishable products with expiration dates and ACC payment schemes. Tsao *et al.* [42] developed an EPQ model for perishable products under the ACC payment scheme using a discounted cash flow analysis. Other studies related to inventory model with ACC payment include [4, 23, 24] and so on. Although many scholars have investigated inventory problems with ACC payment, all of them developed EOQ/EPQ models from the perspective of retailers or suppliers.

Therefore, this paper presents an integrated inventory model developed based on the aforementioned studies for defective items in which the defective rate can be improved through capital investment from the supplier and retailer under an ACC payment scheme. Mathematical analyses are used to determine the optimal shipping quantity, order quantity, and defective rate to maximize the supply chain's joint total profit per unit time. An algorithm is presented that was developed to determine the optimal solution. Numerical examples are used to demonstrate the proposed model and examine the effects of parameter changes on the optimal solutions. Several management implications for relevant decision makers are obtained from the numerical examples.

## 2. Notation and assumptions

The following notation and assumptions are used in this paper.

**Notation**

| | |
|---|---|
| $M$ | Upstream credit period provided by the supplier to the retailer, $M \geq 0$. |
| $N$ | Downstream credit period provided by the retailer to customers, $N \geq 0$. |
| $l$ | Time within which the prepayments are made, $l > 0$. |
| $\alpha$ | Fraction of the procurement cost to be paid in advance, $0 \leq \alpha \leq 1$. |
| $\beta$ | Fraction of the procurement cost to be paid at the time of delivery, $0 \leq \beta \leq 1$. |
| $\gamma$ | Fraction of the procurement cost granted by the supplier as permissible delay to the retailer, $0 \leq \gamma \leq 1$ and $\alpha + \beta + \gamma = 1$. |
| $D$ | Demand rate of the market. |
| $P$ | Production rate of the supplier. |
| $v$ | Production cost per unit for the supplier. |
| $c$ | Procurement cost per unit for the retailer, $c > v$. |
| $p$ | The selling price of the retailer per unit, $p > c$. |
| $A$ | Retailer's ordering cost per order. |
| $S$ | Supplier's setup cost per setup. |
| $h_{b_1}$ | Retailer's holding cost, excluding the interest charged, per non-defective item per unit time. |
| $h_{b_2}$ | Retailer's holding cost, excluding the interest charged, per defective item per unit time, where $h_{b_2} < h_{b_1}$. |
| $h_{v_1}$ | Supplier's holding cost excluding the interest charged per item per unit time. |
| $h_{v_2}$ | Supplier's treatment cost per defective item. |
| $x$ | Retailer's inspection rate per order. |
| $C_s$ | Retailer's inspection cost per unit. |
| $C_p$ | Retailer's penalty cost (including the treatment cost) per defective item returned by the customer. |
| $C_T$ | Supplier's fixed cost of transportation per shipment. |
| $C_t$ | The supplier's variable cost of transportation per unit. |
| $I_c$ | Interest charged per dollar per unit time. |
| $I_e$ | Interest earned per dollar per unit time. |
| $\theta$ | Opportunity cost of the capital investment per dollar per unit time. |
| $\rho$ | the proposition of capital that the retailer should invest in production process. |
| $\lambda_U$ | Proportion of defective items before improving the production process, where $\lambda_U < 1$. |
| $\lambda_L$ | Proportion of defective items that become "non-inspect", $0 < \lambda_L < \lambda_U$. |
| $\lambda$ | Proportion of defective items, $\lambda \in (0, \lambda_U]$ is a decision variable. |
| $Q$ | Retailer's order quantity, which is a decision variable. |
| $T$ | Length of the retailer's replenishment cycle, which is a decision variable. |
| $n$ | Number of shipments provided by the supplier to the retailer per production cycle (integer decision variable). |
| $q$ | Size of each shipment provided by the supplier to the retailer in a production batch (decision variable). |
| $\mathrm{JTP}(\lambda, q, n)$ | Joint total profit per unit time, as a function of $\lambda$, $q$, and $n$. |
| $*$ | Superscript represents the optimal value. |

**Assumptions**

(1) There is single-supplier and single-retailer for a single product in this system.

(2) The supplier's production rate of non-defective item is finite and greater than the demand rate, *i.e.*, $P(1-\lambda) > D$. Otherwise, there will be no inventory problems.

(3) The retailer orders a large product quantity ($Q$) (of non-defective items) per order asks the supplier to deliver $q$ units for $n$ shipments.

(4) Before product quality improvement, the retailer may make a full inspection with an inspection rate $x$ as soon as the order is received to examine the product quality. When the proportion of defective items becomes equal to or less than a certain low rate ($\lambda_L$) through capital investment, the retailer no longer conducts checks on the received items. In this situation, all the items received from the supplier are treated

as non-defective products to stock and sell to customers, which results in $\lambda q(\lambda \leq \lambda_L)$ defective items returned by the customers. All the defective items that have been inspected or returned by customers are stored and returned to the supplier at the end of each replenishment cycle.

(5) All defective items cannot be repaired or reworked. These items have no salvage value.

(6) The capital investment $[I(\lambda)]$ for improving the production process quality to reduce the defective rate of the product is expressed as a logarithmic function of $\lambda$.

$$I(\lambda) = \frac{1}{\delta} \ln\left(\frac{\lambda_U}{\lambda}\right), \quad 0 < \lambda \leq \lambda_U,$$

where $\lambda_U$ is the proportion of defective items before improving the production processes, and $\delta$ denotes the percentage decrease in $\lambda$ per dollar increase in $I(\lambda)$ (please see [15,30,33]).

(7) Capital investment is jointly shared by the retailer and supplier in the integrated supply chain system. Thus, the proportions of capital that the retailer and supplier should invest in the production process are $\rho$ and $1-\rho$, respectively $(0 \leq \rho \leq 1)$. When $\rho = 0$, it means that the supplier is fully responsible for product quality.

(8) An ACC payment scheme is considered between the retailer and supplier in this model. That is, the retailer prepays $\alpha$ fraction of the pre-determined procurement cost for non-defective at time $-l$ when making a replenishment, pays $\beta$ fraction of procurement cost at the time of delivery (*i.e.*, time 0), and receives an upstream credit period of $M$ on the remaining $\gamma$ fraction of procurement cost for each replenishment cycle, where $\alpha + \beta + \gamma = 1$.

(9) The inspection is nondestructive and error-free.

## 3. Model formulation

### 3.1. Problem description

In this paper, a single supplier and a single retailer is considered in the supply chain production-inventory system. The integrated inventory model involving defective items and ACC payment schemes. The operation of this production-inventory system is as follows: The retailer orders $Q$ units (of non-defective items) per order, and asks the supplier to deliver $q$ units in $n$ shipments. Each received shipment contains a percentage of defective items with a defective rate $\lambda$, and the retailer may inspect all the received items before investing capital to improve the production process. In this situation, the number of defective items $\lambda q$ in each shipment will be checked out immediately, and hence the length of the replenishment cycle is $T = (1-\lambda)q/D$. Alternatively, if the retailer and supplier joint to co-invest capital for improving the production process and the defective rate of the product $(\lambda)$ is reduced to the threshold $(\lambda_L)$ or lower, the retailer does not inspect the received items and stocks them for later use. In this situation, the supplier's shipment size is $q$, and the length of replenishment cycle is $T = q/D$. Furthermore, the supplier provides an ACC payment scheme and allows the retailer to prepay $\alpha$ fraction of the procurement cost prior to shipment, $\beta$ fraction of the procurement cost upon receipt of the goods, and obtain an upstream credit period of $M$ on the remaining payment $\gamma$, where $\alpha + \beta + \gamma = 1$.

In the following text, we first establish the total profit per unit time for the retailer and supplier and subsequently determine the joint total profit per unit time of the integrated inventory system.

### 3.2. Retailer's total profit per unit time

When the investment option is available, the retailer's total profit per replenishment cycle with an ACC payment scheme, a downstream trade credit policy, and defective items in each arriving shipment is composed of the following elements.

(a) Sales revenue.
The retailer's sales revenue per cycle is $pDT$, where

$$T = \begin{cases} (1-\lambda)q/D, & \text{if } \lambda_L < \lambda \leq \lambda_U, \\ q/D, & \text{if } 0 < \lambda \leq \lambda_L. \end{cases} \tag{3.1}$$

(b) Procurement cost.
The retailer's procurement cost per replenishment cycle is $c(1-\lambda)q$.

(c) Ordering cost.
The retailer's ordering cost per replenishment cycle is $A$.

(d) Opportunity cost of capital investment.
Capital investment for improving the production process quality creates an opportunity cost $[\theta I(\lambda)]$. The amount of capital investment is shared between the retailer and supplier, and the ratio shared by the retailer is $\rho(0 \leq \rho \leq 1)$. Therefore, the opportunity cost incurred by the retailer for improving the production process quality per cycle is $\rho\theta I(\lambda)T = \rho(\theta/\delta)T\ln(\lambda_U/\lambda)$, where $T$ is as given in (3.1).

(e) Inspection cost.
The initial defective rate for the items received by the retailer is $\lambda_U$, which can be improved through capital investment. If the defective rate ($\lambda$) is lowered to a certain threshold ($\lambda_L$) (*i.e.*, $0 < \lambda \leq \lambda_L$) or below, the retailer does not inspect the received items, and the inspection cost per cycle is zero. However, if $\lambda_L < \lambda \leq \lambda_U$, the retailer inspects the items after receipt. The unit inspection cost is $C_s$, and the retailer receives a shipment of size $q$. The inspection cost per replenishment cycle for the retailer is as follows:

$$\begin{cases} C_s q, & \text{if } \lambda_L < \lambda \leq \lambda_U, \\ 0, & \text{if } 0 < \lambda \leq \lambda_L. \end{cases}$$

(f) Holding cost of non-defective items.
Defective items with a defective rate $\lambda$ are inspected by the retailer (the inspection rate is $x$) if $\lambda_L < \lambda \leq \lambda_U$. Therefore, the holding cost of non-defective items (including defective items before identification) per cycle is as follows:

$$h_{b_1}[(1-\lambda)qT/2 + \lambda q^2/(2x)] = h_{b_1}q^2[(1-\lambda)^2/(2D) + \lambda/(2x)].$$

However, if $0 < \lambda \leq \lambda_L$, free inspection is adopted, and all the received items are treated as non-defective items for stocking and sales. Therefore, the holding cost per cycle is $h_{b_1}qT/2 = h_{b_1}q^2/(2D)$, and the total holding cost of the non-defective items per replenishment cycle is given as follows:

$$\begin{cases} h_{b_1}q^2[(1-\lambda)^2/(2D) + \lambda/(2x)], & \text{if } \lambda_L < \lambda \leq \lambda_U, \\ h_{b_1}q^2/(2D), & \text{if } 0 < \lambda \leq \lambda_L. \end{cases}$$

(g) Holding cost of defective items.
There are $\lambda q$ defective items in each shipment. When $\lambda_L < \lambda \leq \lambda_U$, the defective items will be inspected and returned to the supplier at the end of each shipment cycle. In this situation, the holding cost of the defective items per replenishment cycle is $h_{b_2}[\lambda qT - \lambda q^2/(2x)] = h_{b_2}\lambda q^2[(1-\lambda)/D - 1/(2x)]$. By contrast, if $0 < \lambda \leq \lambda_L$, the defective items are sequentially returned by customers because of the free inspection policy. These defective items will be stored and returned to the supplier at the end of each cycle. Thus, the holding cost for defective items per cycle is $h_{b_2}\lambda qT/2 = h_{b_2}\lambda q^2/(2D)$. Therefore, the holding cost for defective items per replenishment cycle is as follows:

$$\begin{cases} h_{b_2}\lambda q^2[(1-\lambda)/D - 1/(2x)], & \text{if } \lambda_L < \lambda \leq \lambda_U, \\ h_{b_2}\lambda q^2/(2D), & \text{if } 0 < \lambda \leq \lambda_L. \end{cases}$$

(h) Penalty cost for defective items (external failure cost).
Although adopting a free inspection policy can reduce inspection cost, all units of received items treated as non-defective products are sold, and the defective items ($\lambda q$ items) are returned by customers. This results in the retailer incurring a penalty cost of $C_p$ per unit. By contrast, when the defective rate ($\lambda$) is above a certain threshold (*i.e.*, $\lambda_L < \lambda \leq \lambda_U$), the retailer inspects all the received items. Therefore, no penalty cost is incurred. The penalty cost of defective items per cycle is as follows.

$$\begin{cases} 0, & \text{if } \lambda_L < \lambda \leq \lambda_U, \\ C_p \lambda q, & \text{if } 0 < \lambda \leq \lambda_L. \end{cases}$$

(i) Interest charged and interest earned.

There are two possible credit payment cases based on the values of the upstream ($M$) and downstream ($N$) credit periods: $N \leq M$ and $N \geq M$.

**Case 1.** $N \leq M$.
There exist two subcases according to the value of the downstream credit period ($N$) and time at which the retailer receives the payment from the last customer.
**Subcase 1.** $N \leq M \leq T + N$.
In this subcase, the interest charged for credit payment per replenishment cycle is given as follows:

$$cI_c(1-\lambda)q\left\{\alpha(N+l)+\beta N+\frac{(\alpha+\beta)T}{2}\right\}+\frac{cI_cD\gamma(T+N-M)^2}{2}\cdot$$

The interest earned for credit payment per cycle is as follows:

$$\frac{pI_e\gamma D(M-N)^2}{2}\cdot$$

The retailer's total profit per unit time is the combination of the aforementioned elements divided by the length of the replenishment cycle ($T$), where $T$ is as given in (3.1).

$$\mathrm{TPB}_1(\lambda,q)=\begin{cases} \mathrm{TPB}_{11}(\lambda,q), & \text{if } \lambda_L<\lambda\leq\lambda_U, \\ \mathrm{TPB}_{12}(\lambda,q), & \text{if } 0<\lambda\leq\lambda_L, \end{cases} \tag{3.2}$$

where

$$\begin{aligned}\mathrm{TPB}_{11}(\lambda,q)=(p-c)D-\frac{\rho\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right)-\frac{D}{(1-\lambda)}\left\{\frac{A}{q}+C_s+h_{b_1}q\left[\frac{(1-\lambda)^2}{2D}+\frac{\lambda}{2x}\right]\right.\\ +h_{b_2}\lambda q\left[\frac{1-\lambda}{D}-\frac{1}{2x}\right]+cI_c(1-\lambda)\left[\alpha(N+l)+\beta N+\frac{(\alpha+\beta)(1-\lambda)q}{2D}\right]\\ \left.+\frac{cI_cD\gamma[(1-\lambda)q/D+N-M]^2}{2q}-\frac{pI_e\gamma D(M-N)^2}{2q}\right\},\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\mathrm{TPB}_{12}(\lambda,q)=pD-\frac{\rho\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right)-D\left\{c(1-\lambda)+\frac{A}{q}+\frac{h_{b_1}q}{2D}+\frac{h_{b_2}\lambda q}{2D}+C_p\lambda+cI_c(1-\lambda)\right.\\ \left.\times\left[\alpha(N+l)+\beta N+\frac{(\alpha+\beta)q}{2D}\right]+\frac{cI_cD\gamma(q/D+N-M)^2}{2q}-\frac{pI_e\gamma D(M-N)^2}{2q}\right\}\cdot\end{aligned} \tag{3.4}$$

**Sub-case 2.** $M \geq T + N$.

In this subcase, the interest charged for credit payment per replenishment cycle is given as follows:

$$cI_c(1-\lambda)q\left[\alpha(N+l)+\beta N+\frac{(\alpha+\beta)T}{2}\right].$$

The interest earned for credit payment per cycle is as follows:

$$pI_e\gamma D\left[\frac{T^2}{2}+T(M-T-N)\right].$$

The retailer's total profit per unit time is the combination of the aforementioned elements divided by the length of the replenishment cycle ($T$), where $T$ is as given in (3.1).

$$\mathrm{TPB}_2(\lambda,q)=\begin{cases}\mathrm{TPB}_{21}(\lambda,q), & \text{if } \lambda_L<\lambda\leq\lambda_U,\\ \mathrm{TPB}_{22}(\lambda,q), & \text{if } 0<\lambda\leq\lambda_L,\end{cases} \tag{3.5}$$

where

$$\begin{aligned}\mathrm{TPB}_{21}(\lambda,q)=(p-c)D-\frac{\rho\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right)-\frac{D}{(1-\lambda)}\Bigg\{&\frac{A}{q}+C_s+h_{b_1}q\left[\frac{(1-\lambda)^2}{2D}+\frac{\lambda}{2x}\right]\\ &+h_{b_2}\lambda q\left[\frac{1-\lambda}{D}-\frac{1}{2x}\right]+cI_c(1-\lambda)\left[\alpha(N+l)+\beta N+\frac{(\alpha+\beta)(1-\lambda)q}{2D}\right]\\ &-pI_e(1-\lambda)\gamma\left[M-N-\frac{(1-\lambda)q}{2D}\right]\Bigg\},\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\mathrm{TPB}_{22}(\lambda,q)=pD-\frac{\rho\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right)-D\Bigg\{&c(1-\lambda)+\frac{A}{q}+\frac{h_{b_1}q}{2D}+\frac{h_{b_2}\lambda q}{2D}+C_p\lambda+cI_c(1-\lambda)\\ &\times\left[\alpha(N+l)+\beta N+\frac{(\alpha+\beta)q}{2D}\right]-pI_e\gamma\left(M-N-\frac{q}{2D}\right)\Bigg\}.\end{aligned} \tag{3.7}$$

**Case 2.** $N \geq M$.

In this case, no interest is earned for credit payment. The interest charged for credit payment per replenishment cycle is given as follows:

$$cI_c(1-\lambda)q\left[\alpha(N+l)+\beta N+\gamma(N-M)+\frac{T}{2}\right].$$

The retailer's total profit per unit time is the combination of the aforementioned elements divided by the length of the replenishment cycle ($T$), where $T$ is as given in (3.1).

$$\mathrm{TPB}_3(\lambda,q)=\begin{cases}\mathrm{TPB}_{31}(\lambda,q), & \text{if } \lambda_L<\lambda\leq\lambda_U,\\ \mathrm{TPB}_{32}(\lambda,q), & \text{if } 0<\lambda\leq\lambda_L,\end{cases} \tag{3.8}$$

where

$$\begin{aligned}\mathrm{TPB}_{31}(\lambda,q)=(p-c)D-\frac{\rho\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right)-\frac{D}{(1-\lambda)}\Bigg\{&\frac{A}{q}+C_s+h_{b_1}q\left[\frac{(1-\lambda)^2}{2D}+\frac{\lambda}{2x}\right]+h_{b_2}\lambda q\\ &\times\left[\frac{1-\lambda}{D}-\frac{1}{2x}\right]+cI_c(1-\lambda)\left[\alpha(N+l)+\beta N+\gamma(N-M)+\frac{(1-\lambda)q}{2D}\right]\Bigg\},\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\mathrm{TPB}_{32}(\lambda,q)=pD-\frac{\rho\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right)-D\Bigg\{&c(1-\lambda)+\frac{A}{q}+\frac{h_{b_1}q}{2D}+\frac{h_{b_2}\lambda q}{2D}+C_p\lambda+cI_c(1-\lambda)\\ &\times\left[\alpha(N+l)+\beta N+\gamma(N-M)+\frac{q}{2D}\right]\Bigg\}.\end{aligned} \tag{3.10}$$

### 3.3. Supplier's total profit per unit time

The supplier's total profit per production cycle consists of the following elements:

(a) Sales revenue.
The supplier produces $nq$ units in each production run and delivers $q$ units to the retailer in each shipment. However, a certain proportion of the defective items are returned by the retailer ($\lambda$). Therefore, the supplier's sales revenue per production cycle is $c(1-\lambda)nq$, where

$$n = \begin{cases} n_1, & \text{if } \lambda_L < \lambda \leq \lambda_U, \\ n_2, & \text{if } 0 < \lambda \leq \lambda_L. \end{cases} \tag{3.11}$$

(b) Production cost.
The supplier produces $nq$ units in each production run. Therefore, the supplier's production cost per production cycle is $vnq$, where $n$ is as given in (3.11).

(c) Setup cost.
The supplier's setup cost per production cycle is $S$.

(d) Transportation cost.
The supplier's transportation cost per shipment includes the fixed per-lot transportation cost ($C_T$) and variable transportation cost ($C_t q$). Therefore, the total transportation cost per production cycle can be calculated as $n(C_T + C_t q)$, where $n$ is as given in (3.11).

(e) Opportunity cost of capital investment.
The capital investment is shared between the retailer and supplier, with the proportion of the supplier's investment being $1-\rho(0 \leq \rho \leq 1)$. Therefore, the opportunity cost of capital investment per production cycle for the supplier is $(1-\rho)\theta I(\lambda)nT$, where $T$ and $n$ are as given in (3.1) and (3.11), respectively.

(f) Holding cost.
Once the first $q$ units are produced, the supplier delivers them to the retailer immediately. Following, the supplier schedules successive deliveries every $(1-\lambda)q/D$ units of time until the inventory level decreases to zero if $\lambda_L < \lambda \leq \lambda_U$. The behavior of the inventory level for the supplier is illustrated in Figure 1a. The cumulative inventory per production cycle for the supplier is as follows:

$$\left[n_1 q\left(\frac{q}{P} + \frac{(n_1-1)(1-\lambda)q}{D}\right) - \frac{n_1^2 q^2}{2P}\right] - \left[\frac{(1-\lambda)q^2}{D}\left(1+2+\cdots+(n_1-1)\right)\right]$$
$$= n_1 q^2\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right].$$

However, if $0 < \lambda \leq \lambda_L$, the retailer does not inspect the received items, and all the items are stored directly. In this case, the cumulative inventory per production cycle for the supplier is as follows (see Fig. 1b):

$$\left[n_2 q\left(\frac{q}{P} + (n_2-1)\frac{q}{D}\right) - \frac{n_2^2 q^2}{2P}\right] - \left[\frac{q^2}{D}\left(1+2+\cdots+(n_2-1)\right)\right] = n_2 q^2\left[\frac{1}{P} + \frac{n_2-1}{2D} - \frac{n_2}{2P}\right].$$

Therefore, the holding cost per production cycle is as follows:

$$\begin{cases} h_{v_1} n_1 q^2\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right], & \text{if } \lambda_L < \lambda \leq \lambda_U, \\ h_{v_1} n_2 q^2\left[\frac{1}{P} + \frac{n_2-1}{2D} - \frac{n_2}{2P}\right], & \text{if } 0 < \lambda \leq \lambda_L. \end{cases}$$

(g) Treatment cost for defective items.
For each shipment of size $q$, $\lambda q$ defective items are returned by the retailer at the end of the shipment cycle. The treatment cost for the returned defective items per production cycle is $h_{v_2} n\lambda q$, where $n$ is as given in (3.11).

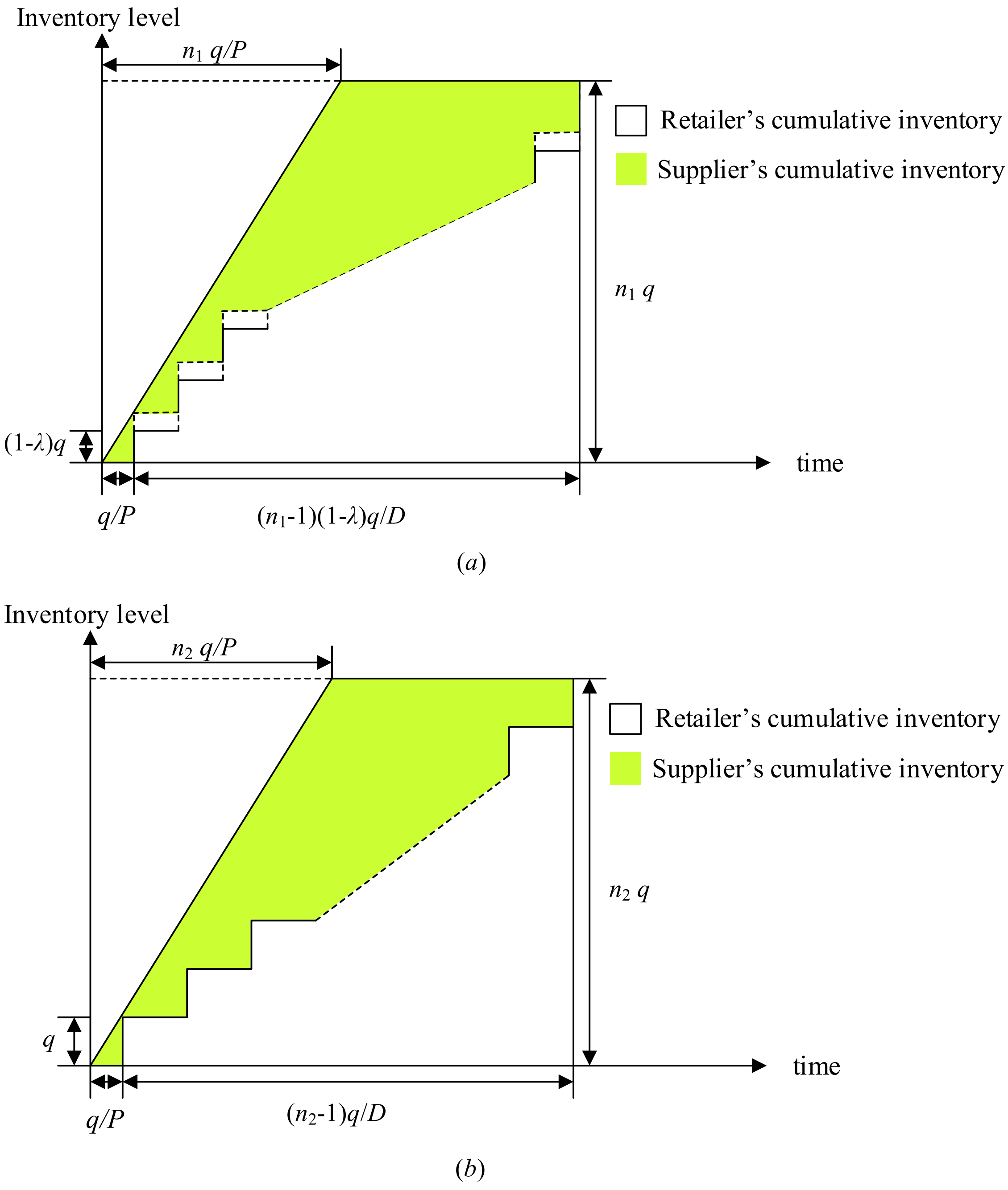


FIGURE 1. Supplier's inventory levels per production run. (a) $\lambda_L < \lambda \leq \lambda_U$. (b) $0 < \lambda \leq \lambda_L$.

(h) Interest charged and interest earned.
Under the ACC payment policy, the supplier receives $\alpha$ fraction of the procurement cost at time $-l$ and provides a credit period of $M$ on the remaining $\gamma$ portion of the procurement cost for each replenishment cycle. Therefore, the interest charged and interest earned per replenishment cycle are given by $\gamma v I_c(1-\lambda)nqM$ and $\alpha c I_e(1-\lambda)nql$, respectively, where $n$ is as given in (3.11).
The supplier's total profit per unit time is the combination of the aforementioned elements divided by the length of production cycle ($nT$), where $T$ and $n$ are as given in (3.1) and (3.11), respectively.

$$\mathrm{TPV}(\lambda,n)=\begin{cases}\mathrm{TPV}_1(\lambda,n_1), & \text{if } \lambda_L<\lambda\le\lambda_U,\\ \mathrm{TPV}_2(\lambda,n_2), & \text{if } 0<\lambda\le\lambda_L,\end{cases}\tag{3.12}$$

where

$$\begin{aligned}\mathrm{TPV}_1(\lambda,n_1)=cD-\frac{D}{(1-\lambda)}\Bigg\{&v+\frac{S}{n_1q}+\frac{C_T}{q}+C_t+\frac{(1-\rho)\theta(1-\lambda)}{\delta D}\ln\left(\frac{\lambda_U}{\lambda}\right)+h_{v_1}q\\ &\times\left[\frac{1}{P}+\frac{(n_1-1)(1-\lambda)}{2D}-\frac{n_1}{2P}\right]+h_{v_2}\lambda+\gamma vI_c(1-\lambda)M-\alpha cI_e(1-\lambda)l\Bigg\},\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}\mathrm{TPV}_2(\lambda,n_2)=D\Bigg\{&c(1-\lambda)-v-\frac{S}{n_2q}-\frac{C_T}{q}-C_t-\frac{(1-\rho)\theta}{\delta D}\ln\left(\frac{\lambda_U}{\lambda}\right)-h_{v_1}q\\ &\times\left[\frac{1}{P}+\frac{n_2-1}{2D}-\frac{n_2}{2P}\right]-h_{v_2}\lambda-\gamma vI_c(1-\lambda)M+\alpha cI_e(1-\lambda)l\Bigg\}.\end{aligned}\tag{3.14}$$

### 3.4. Joint total profit per unit time

When the retailer and supplier build a long-term strategic partnership, they together determine the optimal policy. Therefore, the joint total profit per unit time is the sum of the retailer's and supplier's total profits per unit time. From the values of the upstream and downstream credit periods ($M$ and $N$, respectively), the joint total profit per unit time [$\mathrm{JTP}(\lambda,q,n)$] can be obtained as follows:

$$\mathrm{JTP}(\lambda,q,n)=\begin{cases}\mathrm{JTP}_1(\lambda,q,n), & \text{if } N\le M\le T+N,\\ \mathrm{JTP}_2(\lambda,q,n), & \text{if } M\ge T+N,\\ \mathrm{JTP}_3(\lambda,q,n), & \text{if } N\ge M,\end{cases}\tag{3.15}$$

where

$$\begin{aligned}\mathrm{JTP}_i(\lambda,q,n)&=\begin{cases}\mathrm{JTP}_{i1}(\lambda,q,n_1), & \text{if } \lambda_L<\lambda\le\lambda_U,\\ \mathrm{JTP}_{i2}(\lambda,q,n_2), & \text{if } 0<\lambda\le\lambda_L,\end{cases}\\ &=\begin{cases}\mathrm{TPB}_{i1}(\lambda,q)+\mathrm{TPV}_1(\lambda,n_1), & \text{if } \lambda_L<\lambda\le\lambda_U,\\ \mathrm{TPB}_{i2}(\lambda,q)+\mathrm{TPV}_2(\lambda,n_2), & \text{if } 0<\lambda\le\lambda_L,\end{cases}\end{aligned}\tag{3.16}$$

$i=1,2,3$.

## 4. Theoretical results

The objective of the proposed model was to determine the optimal batch quantity ($q^*$), proportion of defective items ($\lambda^*$), and number of shipments per production cycle ($n^*$) for maximizing the joint total profit per unit time. It is found that the problem to maximize the joint total profit per unit time $\mathrm{JTP}_i(\lambda,q,n)$ is mixed integer non-linear program problem, where $n$ is a integer, $\lambda$ and $q$ are real numbers. To solve this problem, we first considered the following two situations: (i) $\lambda_L<\lambda\le\lambda_U$ and (ii) $0<\lambda\le\lambda_L$ and explained the concavities of the total profits per unit time with respect to $n$ and $q$ under the various situations. Then an algorithm was developed to the optimal solutions $(\lambda^*, q^*, n^*)$ for the whole problem.

**Situation 1.** $\lambda_L < \lambda \le \lambda_U$.

In this situation, consider the following three joint total profit functions:

$$\begin{aligned}\mathrm{JTP}_{11}(\lambda,q,n_1) = pD &- \frac{\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right) - \frac{D}{1-\lambda}\left\{\frac{n_1(A+C_T)+S}{n_1 q} + C_s + v + C_t + h_{b_1} q\left[\frac{(1-\lambda)^2}{2D} + \frac{\lambda}{2x}\right]\right.\\ &+ h_{b_2}\lambda q\left[\frac{1-\lambda}{D} - \frac{1}{2x}\right] + cI_c(1-\lambda)\left[\alpha(N+l) + \beta N + \gamma(N-M) + \frac{(1-\lambda)q}{2D}\right]\\ &+ h_{v_1} q\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right] + h_{v_2}\lambda + \gamma v I_c(1-\lambda)M - \alpha c I_e(1-\lambda)l\\ &\left. + \frac{(cI_c - pI_e)\gamma D(M-N)^2}{2q}\right\} \end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathrm{JTP}_{21}(\lambda,q,n_1) = pD &- \frac{\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right) - \frac{D}{1-\lambda}\left\{\frac{n_1(A+C_T)+S}{n_1 q} + C_s + v + C_t + h_{b_1} q\left[\frac{(1-\lambda)^2}{2D} + \frac{\lambda}{2x}\right]\right.\\ &+ h_{b_2}\lambda q\left[\frac{1-\lambda}{D} - \frac{1}{2x}\right] + cI_c(1-\lambda)\left[\alpha(N+l) + \beta N + \frac{(\alpha+\beta)(1-\lambda)q}{2D}\right]\\ &- pI_e(1-\lambda)\gamma\left[M - N - \frac{(1-\lambda)q}{2D}\right] + \gamma v I_c(1-\lambda)M - \alpha c I_e(1-\lambda)l\\ &\left. + h_{v_1} q\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right] + h_{v_2}\lambda\right\} \end{aligned} \tag{4.2}$$

$$\begin{aligned}\mathrm{JTP}_{31}(\lambda,q,n_1) = pD &- \frac{\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right) - \frac{D}{1-\lambda}\left\{\frac{n_1(A+C_T)+S}{n_1 q} + C_s + v + C_t + h_{b_1} q\left[\frac{(1-\lambda)^2}{2D} + \frac{\lambda}{2x}\right]\right.\\ &+ h_{b_2}\lambda q\left[\frac{1-\lambda}{D} - \frac{1}{2x}\right] + cI_c(1-\lambda)\left[\alpha(N+l) + \beta N + \gamma(N-M) + \frac{(1-\lambda)q}{2D}\right]\\ &\left. + h_{v_1} q\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right] + h_{v_2}\lambda + \gamma v I_c(1-\lambda)M - \alpha c I_e(1-\lambda)I\right\}. \end{aligned} \tag{4.3}$$

Firstly, for fixed $q$ and $\lambda \in (\lambda_L, \lambda_U]$, the effect of $n_1$ on the joint total profit per unit time $\mathrm{JTP}_{11}(\lambda, q, n_1)$ in (4.1) needs to be checked. By taking the second-order derivative of $\mathrm{JTP}_{11}(\lambda, q, n_1)$ with respect to $n_1$, we obtain the following:

$$\frac{\mathrm{d}^2\mathrm{JTP}_{11}(\lambda,\ q, n_1)}{\mathrm{d}n_1^2} = \frac{-2DS}{n_1^3(1-\lambda)q} < 0,$$

which implies the function $\mathrm{JTP}_{11}(\lambda, q, n_1)$ is a concave function of $n_1$. Consequently, the search for the optimal value of $n_1$ (denoted by $n_{11}^*$) is reduced to a local maximum.

Then, for a given $n_1$ and $\lambda \in (\lambda_L, \lambda_U]$, the condition $\partial\mathrm{JTP}_{11}(\lambda, q, n_1)/\partial q = 0$ should be satisfied to maximize the joint total profit per unit time $[\mathrm{JTP}_{11}(\lambda, q, n_1)]$. This implies the following:

$$\begin{aligned}&\frac{n_1[2(A+C_T) - (pI_e - cI_c)\gamma D(M-N)^2] + 2S}{2n_1 q^2} - \frac{(h_{b_1} + cI_c)(1-\lambda)^2}{2D}\\ &\quad - \frac{(h_{b_1} - h_{b_2})\lambda}{2x} - \frac{h_{b_2}\lambda(1-\lambda)}{D} - h_{v_1}\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right] = 0.\end{aligned} \tag{4.4}$$

For notational convenience let

$$\Delta_1 \equiv n_1[2(A+C_T) - (pI_e - cI_c)\,\gamma\, D(M-N)^2] + 2S$$

and

$$\Delta_2 \equiv \frac{(h_{b_1}+cI_c)(1-\lambda)^2}{2D} + \frac{(h_{b_1}-h_{b_2})\lambda}{2x} + \frac{h_{b_2}\lambda(1-\lambda)}{D} + h_{v_1}\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right]$$

then we have the following result.

**Lemma 4.1.** *For any given $n_1$ and $\lambda \in (\lambda_L, \lambda_U]$, if $n_1[2(A+C_T) - (pI_e - cI_c)\gamma D(M-N)^2] + 2S \le 0$, then $\mathrm{JTP}_{11}(\lambda, q, n_1)$ has a maximum value at the point*

$$q = q_{11,n_1,\lambda} = \sqrt{\frac{\Delta_1}{2n_1\Delta_2}}. \tag{4.5}$$

*Otherwise, $q = 0$.*

*Proof.* See the Appendix A. □

Similarly, for fixed $q$ and $\lambda \in (\lambda_L, \lambda_U]$, we can also show that $\mathrm{JTP}_{21}(\lambda, q, n_1)$ and $\mathrm{JTP}_{31}(\lambda, q, n_1)$ are concave functions of $n_1$ because

$$\frac{\mathrm{d}^2\mathrm{JTP}_{i1}(\lambda,\ q, n_1)}{\mathrm{d}n_1^2} = \frac{-2DS}{n_1^3(1-\lambda)q} < 0, \quad i = 2, 3.$$

Therefore, the search for the optimal number of shipments ($n_{21}^*$ and $n_{31}^*$) is simplified to finding the local maximum. Let

$$\Delta_3 \equiv \frac{[h_{b_1} + cI_c(\alpha+\beta) + pI_e\gamma](1-\lambda)^2}{2D} + \frac{(h_{b_1}-h_{b_2})\lambda}{2x} + \frac{h_{b_2}\lambda(1-\lambda)}{D} + h_{v_1}\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right],$$

and

$$\Delta_4 \equiv \frac{(h_{b_1}+cI_c)(1-\lambda)^2}{2D} + \frac{(h_{b_1}-h_{b_2})\lambda}{2x} + \frac{h_{b_2}\lambda(1-\lambda)}{D} \quad + h_{v_1}\left[\frac{1}{P} + \frac{(n_1-1)(1-\lambda)}{2D} - \frac{n_1}{2P}\right],$$

and we have the following result.

**Lemma 4.2.** *For any given $n_1$ and $\lambda \in (\lambda_L, \lambda_U]$, we have $\mathrm{JTP}_{21}(\lambda, q, n_1)$ and $\mathrm{JTP}_{31}(\lambda, q, n_1)$ has maximum values at the point*

$$q_{21,n_1,\lambda} = \sqrt{\frac{n_1(A+C_T)+S}{n_1\Delta_3}}, \tag{4.6}$$

*and*

$$q_{31,n_1,\lambda} = \sqrt{\frac{n_1(A+C_T)+S}{n_1\Delta_4}}, \tag{4.7}$$

*respectively.*

*Proof.* See the Appendix B. □

**Situation 2.** $0 < \lambda \le \lambda_L$.

In this situation, from (3.16), (3.4), (3.7), (3.10), and (3.14), we will consider the following three joint total profit functions:

$$\mathrm{JTP}_{12}(\lambda, q, n_2) = pD - \frac{\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right) - D\left\{\frac{n_2(A+C_T)+S}{n_2 q} + C_p\,\lambda + v + C_t + \frac{(h_{b_1}+h_{b_2}\lambda)q}{2D}\right.$$
$$\left. + cI_c(1-\lambda)\left[\alpha(N+l) + \beta N + \frac{(\alpha+\beta)q}{2D}\right] + \frac{cI_c\gamma\, q}{2D} + cI_c\gamma(N-M)\right.$$

$$+ h_{v_1} q \left[\frac{1}{P} + \frac{n_2 - 1}{2D} - \frac{n_2}{2P}\right] + h_{v_2}\lambda + \gamma\, vI_c(1-\lambda)M - \alpha cI_e(1-\lambda)l$$
$$+ \left.\frac{(cI_c - pI_e)\gamma D(M-N)^2}{2q}\right\}, \tag{4.8}$$

$$\mathrm{JTP}_{22}(\lambda, q, n_2) = pD - \frac{\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right) - D\left\{\frac{n_2(A+C_T)+S}{n_2 q} + C_p\,\lambda + v + C_t + \frac{(h_{b_1}+h_{b_2}\lambda)q}{2D}\right.$$
$$+ cI_c(1-\lambda)\left[\alpha(N+l) + \beta N + \frac{(\alpha+\beta)q}{2D}\right] - pI_e\gamma\left(M - N - \frac{q}{2D}\right)$$
$$\left. + h_{v_1} q \left[\frac{1}{P} + \frac{n_2 - 1}{2D} - \frac{n_2}{2P}\right] + h_{v_2}\lambda + \gamma\, vI_c(1-\lambda)M - \alpha cI_e(1-\lambda)l\right\}, \tag{4.9}$$

$$\mathrm{JTP}_{32}(\lambda, q, n_2) = pD - \frac{\theta}{\delta}\ln\left(\frac{\lambda_U}{\lambda}\right) - D\left\{\frac{n_2(A+C_T)+S}{n_2 q} + C_p\,\lambda + v + C_t + \frac{(h_{b_1}+h_{b_2}\lambda)q}{2D}\right.$$
$$+ cI_c(1-\lambda)\left[\alpha(N+l) + \beta N + \gamma(N-M) + \frac{q}{2D}\right]$$
$$\left. + h_{v_1} q \left[\frac{1}{P} + \frac{n_2 - 1}{2D} - \frac{n_2}{2P}\right] + h_{v_2}\lambda + \gamma\, vI_c(1-\lambda)M - \alpha cI_e(1-\lambda)l\right\}. \tag{4.10}$$

For a fixed $q$ and $\lambda \in (0, \lambda_L]$, the effect of $n_2$ on the joint total profit per unit time $\mathrm{JTP}_{i2}(\lambda, q, n_2)$, for $i = 1, 2, 3$, is explained in (4.8)–(4.10). The second-order derivative of $\mathrm{JTP}_{i2}(\lambda, q, n_2)$, $(i = 1, 2, 3)$ with respect to $n_2$ provides the following equation:

$$\frac{\mathrm{d}^2\mathrm{JTP}_{i2}(\lambda,\ q, n_2)}{\mathrm{d}n_2^2} = \frac{-2DS}{n_2^3 q} < 0, \quad i = 1, 2, 3.$$

Therefore, $\mathrm{JTP}_{i2}(\lambda, q, n_2)$ is a concave function of $n_2$ for $i = 1, 2, 3$ and the search for the optimal number of shipments ($n_{i2}^*$, $i = 1, 2, 3$) is simplified to finding the local maximum. Let

$$\Delta_5 \equiv n_2[2(A + C_T) - (pI_e - cI_c)\gamma D(M-N)^2] + 2S,$$
$$\Delta_6 \equiv \frac{h_{b_1} + h_{b_2}\lambda}{2D} + \frac{cI_c[1 - \lambda(\alpha+\beta)]}{2D} + h_{v_1}\left[\frac{1}{P} + \frac{n_2 - 1}{2D} - \frac{n_2}{2P}\right],$$
$$\Delta_7 \equiv \frac{h_{b_1} + h_{b_2}\lambda + pI_e\gamma + cI_c(\alpha+\beta)(1-\lambda)}{2D} + h_{v_1}\left[\frac{1}{P} + \frac{n_2 - 1}{2D} - \frac{n_2}{2P}\right],$$

and

$$\Delta_8 \equiv \frac{h_{b_1} + h_{b_2}\lambda + cI_c(1-\lambda)}{2D} + h_{v_1}\left[\frac{1}{P} + \frac{n_2 - 1}{2D} - \frac{n_2}{2P}\right],$$

and then we have the following result.

**Lemma 4.3.** *For any given $n_2$ and $\lambda \in (0, \lambda_L]$, we have $\mathrm{JTP}_{i2}(\lambda, q, n_2)$ for $i = 1, 2, 3$ has maximum values at the points*

$$q_{12,n_2,\lambda} = \sqrt{\frac{\Delta_5}{2n_2\Delta_6}}, \tag{4.11}$$

$$q_{22,n_2,\lambda} = \sqrt{\frac{n_2(A+C_T)+S}{n_2\Delta_7}}, \tag{4.12}$$

*and*

$$q_{32,n_2,\lambda} = \sqrt{\frac{n_2(A+C_T)+S}{n_2\Delta_8}}, \tag{4.13}$$

*respectively.*

*Proof.* See the Appendix C. □

Next, for any given $n_1$, $n_2$, and $q$, it is obvious that $\mathrm{JTP}_{i1}(\lambda, q, n_1)$ and $\mathrm{JTP}_{i2}(\lambda, q, n_2), i = 1, 2, 3$, are smooth curves of $\lambda \in (\lambda_L, \lambda_U]$ and $\lambda \in (0, \lambda_L]$, respectively. Therefore, the following iterative algorithm was developed to search the optimal solution $(\lambda^*, q^*, n^*)$ for the whole problem.

**Algorithm**

**Step 1.** Compare $M$ with $N$. If $N \le M$, perform Step 2. If $N > M$, skip to Step 5.

**Step 2.** Set $n_1 = 1$.

**Step 2.1.** Divide the interval $(\lambda_L, \lambda_U]$ into $m$ equal subintervals, and let $\lambda_j = \lambda_L + j(\lambda_U - \lambda_L)/m$, $j = 1, 2, \dots, m$, where $m$ is sufficiently large.

**Step 2.2.** If $n_1[2(A+C_T)-(pI_e-cI_c)\gamma D(M-N)^2]+2S \le 0$, let $q_{11,n_1,\lambda_j} = 0$ and $\mathrm{JTP}_{11}(\lambda_j, q_{11,n_1,\lambda_j}, n_1) = 0$. for each $\lambda_j, j = 1, 2, \dots, m$. Otherwise, for each $\lambda_j$, $j = 1, 2, \dots, m$, determine $q_{11,n_1,\lambda_j}$ from (4.5). Furthermore, if $D(M-N) \le (1-\lambda_j)\, q_{11,n_1,\lambda_j}$, calculate $\mathrm{JTP}_{11}(\lambda_j, q_{11,n_1,\lambda_j}, n_1)$ from (4.1). Otherwise, set $\mathrm{JTP}_{11}(\lambda_j, q_{11,n_1,\lambda_j}, n_1) = 0$.

**Step 2.3.** For each $\lambda_j$, $j = 1, 2, \dots, m$, determine $q_{21,n_1,\lambda_j}$ from (4.6). If $D(M-N) \ge (1-\lambda_j)\, q_{21,n_1,\lambda_j}$, calculate $\mathrm{JTP}_{21}(\lambda_j, q_{21,n_1,\lambda_j}, n_1)$ from (4.2). Otherwise, set $\mathrm{JTP}_{21}(\lambda_j, q_{21,n_1,\lambda_j}, n_1) = 0$.

**Step 2.4.** Find $\mathrm{Max}_{i=1,2;\, j=1,2,\dots,m}\mathrm{JTP}_{i1}(\lambda_j, q_{i1,n_1,\lambda_j}, n_1)$, and let $\mathrm{JTP}^{(1)}(\lambda_{(n_1)}, q_{n_1,\lambda_{(n_1)}}, n_1) = \mathrm{Max}_{i=1,2;\, j=1,2,\dots,m}\mathrm{JTP}_{i1}(\lambda_j, q_{i1,n_1,\lambda_j}, n_1)$

**Step 2.5.** Set $n_1 = n_1 + 1$, and repeat Steps 2.1–2.4 to obtain $\mathrm{JTP}^{(1)}(\lambda_{(n_1)}, q_{n_1,\lambda_{(n_1)}}, n_1)$

**Step 2.6.** If $\mathrm{JTP}^{(1)}(\lambda_{(n_1)}, q_{n_1,\lambda_{(n_1)}}, n_1) \le \mathrm{JTP}^{(1)}(\lambda_{(n_1-1)}, q_{n_1-1,\lambda_{(n_1-1)}}, n_1-1)$, set $\mathrm{JTP}^{(1)}(\lambda_1^*, q_1^*, n_1^*) = \mathrm{JTP}^{(1)}(\lambda_{(n_1-1)}, q_{n_1-1,\lambda_{(n_1-1)}}, n_1-1)$, where $(\lambda_1^*, q_1^*, n_1^*) = (\lambda_{(n_1-1)}, q_{n_1-1,\lambda_{(n_1-1)}}, n_1-1)$ is the optimal solution for Situation 1. Otherwise, return to Step 2.5.

**Step 3.** Set $n_2 = 1$.

**Step 3.1.** Divide the interval $(0, \lambda_L]$ into $m$ equal subintervals, and let $\lambda_j = j\lambda_L/m$, $j = 1, 2, \dots, m$, where $m$ is sufficiently large.

**Step 3.2.** If $n_2[2(A + C_T) - (pI_e - cI_c)\gamma D(M - N)^2] + 2S \le 0$, for each $\lambda_j, j = 1, 2, \dots, m$, let $q_{12,n_2,\lambda_j} = 0$ and $\mathrm{JTP}_{12}(\lambda_j, q_{12,n_2,\lambda_j}, n_2) = 0$. Otherwise, for each $\lambda_j$, $j = 1, 2, \dots, m$, determine $q_{12,n_2,\lambda_j}$ from (4.11). If $D(M-N) \le q_{12,n_2,\lambda_j}$, calculate $\mathrm{JTP}_{12}(\lambda_j, q_{12,n_2,\lambda_j}, n_2)$ from (4.8). Otherwise, set $\mathrm{JTP}_{12}(\lambda_j, q_{12,n_2,\lambda_j}, n_2) = 0$.

**Step 3.3.** For each $\lambda_j$, $j = 1, 2, \dots, m$, determine $q_{22,n_2,\lambda_j}$ from (4.12). If $D(M-N) \ge q_{22,n_2,\lambda_j}$, calculate the corresponding joint total profit per unit time $\mathrm{JTP}_{22}(\lambda_j, q_{22,n_2,\lambda_j}, n_2)$ from (4.9). Otherwise, set $\mathrm{JTP}_{22}(\lambda_j, q_{22,n_2,\lambda_j}, n_2) = 0$.

**Step 3.4.** Find $\mathrm{Max}_{i=1,2;\, j=1,2,\dots,m}\mathrm{JTP}_{i2}(\lambda_j, q_{i2,n_2,\lambda_j}, n_2)$, and let $\mathrm{JTP}^{(2)}(\lambda_{(n_2)}, q_{n_2,\lambda_{(n_2)}}, n_2) = \mathrm{Max}_{i=1,2;\, j=1,2,\dots,m}\mathrm{JTP}_{i2}(\lambda_j, q_{i2,n_2,\lambda_j}, n_2)$

**Step 3.5.** Set $n_2 = n_2 + 1$, and repeat Steps 3.1–3.4 to obtain $\mathrm{JTP}^{(2)}(\lambda_{(n_2)}, q_{n_2,\lambda_{(n_2)}}, n_2)$

**Step 3.6.** If $\mathrm{JTP}^{(2)}(\lambda_{(n_2)}, q_{n_2,\lambda_{(n_2)}}, n_2) \le \mathrm{JTP}^{(2)}(\lambda_{(n_2-1)}, q_{n_2-1,\lambda_{(n_2-1)}}, n_2-1)$, set $\mathrm{JTP}^{(2)}(\lambda_2^*, q_2^*, n_2^*) = \mathrm{JTP}^{(2)}(\lambda_{(n_2-1)}, q_{n_2-1,\lambda_{(n_2-1)}}, n_2-1)$, where $(\lambda_2^*, q_2^*, n_2^*) = (\lambda_{(n_2-1)}, q_{n_2-1,\lambda_{(n_2-1)}}, n_2-1)$ is the optimal solution for Situation 2. Otherwise, return to Step 3.5.

**Step 4.** Find $\mathrm{Max}_{k=1,2}\mathrm{JTP}^{(k)}(\lambda_k^*, q_k^*, n_k^*)$. Let $\mathrm{JTP}(\lambda^*, q^*, n^*) = \mathrm{Max}_{k=1,2}\mathrm{JTP}^{(k)}(\lambda_k^*, q_k^*, n_k^*)$, where $(\lambda^*, q^*, n^*)$ is the optimal solution. Then, skip to Step 8.

**Step 5.** Set $n_1 = 1$.

**Step 5.1.** Divide the interval $(\lambda_L, \lambda_U]$ into $m$ equal subintervals, and let $\lambda_j = \lambda_L + j(\lambda_U - \lambda_L)/m$, $j = 1, 2, \dots, m$, where $m$ is sufficiently large.

**Step 5.2.** For each $\lambda_j$, $j = 1, 2, \dots, m$, find $q_{31,n_1,\lambda_j}$ from (4.7), and calculate $\mathrm{JTP}_{31}(\lambda_j, q_{31,n_1,\lambda_j}, n_1)$ from (4.3).

**Step 5.3.** Find $\mathrm{Max}_{j=1,2,\dots,m}\mathrm{JTP}_{31}(\lambda_j, q_{31,n_1,\lambda_j}, n_1)$, and let $\mathrm{JTP}^{(1)}(\lambda_{(n_1)}, q_{n_1,\lambda_{(n_1)}}, n_1) = \mathrm{Max}_{j=1,2,\dots,m}\mathrm{JTP}_{31}(\lambda_j, q_{31,n_1,\lambda_j}, n_1)$

**Step 5.4.** Set $n_1 = n_1 + 1$, and repeat Steps 5.1–5.3 to obtain $\mathrm{JTP}^{(1)}(\lambda_{(n_1)}, q_{n_1,\lambda_{(n_1)}}, n_1)$.

**Step 5.5.** If $\mathrm{JTP}^{(1)}(\lambda_{(n_1)}, q_{n_1,\lambda_{(n_1)}}, n_1) \leq \mathrm{JTP}^{(1)}(\lambda_{(n_1-1)}, q_{n_1-1,\lambda_{(n_1-1)}}, n_1 - 1)$, set $\mathrm{JTP}^{(1)}(\lambda_1^*, q_1^*, n_1^*) = \mathrm{JTP}^{(1)}(\lambda_{(n_1-1)}, q_{n_1-1,\lambda_{(n_1-1)}}, n_1 - 1)$, where $(\lambda_1^*, q_1^*, n_1^*) = (\lambda_{(n_1-1)}, q_{n_1-1,\lambda_{(n_1-1)}}, n_1 - 1)$ is the optimal solution for Situation 1. Otherwise, return to Step 5.4.

**Step 6.** Set $n_2 = 1$.

**Step 6.1.** Divide the interval $(0, \lambda_L]$ into $m$ equal subintervals, and let $\lambda_j = j\lambda_L/m$, $j = 1, 2, \ldots, m$, where $m$ is sufficiently large.

**Step 6.2.** For each $\lambda_j$, $j = 1, 2, \ldots, m$, find $q_{32,n_2,\lambda_j}$ from (4.13), and calculate $\mathrm{JTP}_{32}(\lambda_j, q_{32,n_2,\lambda_j}, n_2)$ from (4.10).

**Step 6.3.** Find $\mathrm{Max}_{j=1,2,\ldots,m}\mathrm{JTP}_{32}(\lambda_j, q_{32,n_2,\lambda_j}, n_2)$, and let $\mathrm{JTP}^{(2)}(\lambda_{(n_2)}, q_{n_2,\lambda_{(n_2)}}, n_2) = \mathrm{Max}_{j=1,2,\ldots,m}\mathrm{JTP}_{32}(\lambda_j, q_{32,n_2,\lambda_j}, n_2)$

**Step 6.4.** Set $n_2 = n_2 + 1$, and repeat Steps 6.1–6.3 to obtain $\mathrm{JTP}^{(2)}(\lambda_{(n_2)}, q_{n_2,\lambda_{(n_2)}}, n_2)$

**Step 6.5.** If $\mathrm{JTP}^{(2)}(\lambda_{(n_2)}, q_{n_2,\lambda_{(n_2)}}, n_2) \leq \mathrm{JTP}^{(2)}(\lambda_{(n_2-1)}, q_{n_2-1,\lambda_{(n_2-1)}}, n_2 - 1)$, set $\mathrm{JTP}^{(2)}(\lambda_2^*, q_2^*, n_2^*) = \mathrm{JTP}^{(2)}(\lambda_{(n_2-1)}, q_{n_2-1,\lambda_{(n_2-1)}}, n_2 - 1)$, where $(\lambda_2^*, q_2^*, n_2^*) = (\lambda_{(n_2-1)}, q_{n_2-1,\lambda_{(n_2-1)}}, n_2 - 1)$ is the optimal solution for Situation 2. Otherwise, return to Step 6.4.

**Step 7.** Find $\mathrm{Max}_{k=1,2}\mathrm{JTP}^{(k)}(\lambda_k^*, q_k^*, n_k^*)$, and let $\mathrm{JTP}(\lambda^*, q^*, n^*) = \mathrm{Max}_{k=1,2}\mathrm{JTP}^{(k)}(\lambda_k^*, q_k^*, n_k^*)$, where $(\lambda^*, q^*, n^*)$ is the optimal solution.

**Step 8.** Stop.

The aforementioned algorithm can be implemented using a computer-oriented numerical technique for any given set of parameter values. Once the optimal value $(\lambda^*, q^*, n^*)$ is obtained, it can obtain the joint capital investment $I(\lambda^*) = (1/\delta)\ln(\lambda_U/\lambda^*)$ and $T^* = (1 - \lambda^*)q^*/D$ or $q^*/D$ according to the value of $\lambda^*$ that belongs to the interval $(\lambda_L, \lambda_U]$ or $(0, \lambda_L]$. Further, $\mathrm{JTP}^* = \mathrm{JTP}(\lambda^*, q^*, n^*)$ can be found.

## 5. Numerical examples

The above theoretical results and algorithm can be applied to the following numerical example.

**Example 5.1.** Consider an inventory system with the following data: $A = 50$, $P = 2000$, $D = 1000$, $S = 200$, $v = 10$, $c = 20$, $p = 40$, $h_{b_1} = 2$, $h_{b_2} = 0.5$, $h_{v_1} = 1.5$, $h_{v_2} = 0.5$, $x = 3000$, $C_s = 0.3$, $C_T = 10$, $C_t = 0.3$, $C_p = 10$, $\theta = 0.01$, $\delta = 0.0003$, $\rho = 0.5$, $\lambda_U = 0.05$, $\lambda_L = 0.005$, $\alpha = 0.2$, $\beta = 0.3$, $\gamma = 0.5$, $M = 45/365\,(= 0.123288)$, $N = 30/365\,(= 0.082192)$, $l = 10/365\,(= 0.027397)$, $I_c = 0.03$ and $I_e = 0.01$ in appropriate units. Further, we set $m = 500$. Using the aforementioned algorithm, we obtain the computational results presented in Table 1.

Table 1 reveals that the optimal number of shipments per production cycle for this example is $n^* = 3$, the batch quantity per shipment is $q^* = 228.608$ units, and the proportion of defective items is $\lambda^* = 0.00318 < 0.005 = \lambda_L$, which implies that the retailer does not inspect the received items. In this situation, the retailer's optimal order quantity is $Q^* = n^*q^* = 685.824$ units, optimal length of the replenishment cycle is $T^* = q^*/D = 0.2286$, and optimal joint total profit per unit time is $\mathrm{JTP}^* = \$28\,433.008$. To understand the effects of capital investment, we determined the optimal number of shipments per production cycle $(n_{\mathrm{WI}}^*)$, batch quantity per shipment $(q_{\mathrm{WI}}^*)$, and retailer's optimal order quantity $(Q_{\mathrm{WI}}^*)$ without capital investment. For the joint total profit per unit time without capital investment $\mathrm{JTP}_{\mathrm{WI}}^*$, $n_{\mathrm{WI}}^* = 4$, $q_{\mathrm{WI}}^* = 205.886$, and $\mathrm{JTP}_{\mathrm{WI}}^* = 27\,867.647$. A comparison of the results with and without capital investment indicates that the supply chain benefits when capital is jointly invested for quality improvement of the product.

**Example 5.2.** In this example, the effects of parameter changes on the optimal solutions can be analyzed. We divide the parameters into four segments for discussion: retailer's parameters, supplier's parameters, investing parameters, and trade credit parameters. The comparison results are represented in Tables 2–5.

The following observations can be made from the results presented in Table 2:

TABLE 1. Results of using the algorithm for Example 5.1.

| $n_1$ | $\lambda_{(n_1)}$ | $q_{n_1,\lambda(n_1)}$ | JTP$^{(1)}$ $(\lambda_{(n_1)}, q_{n_1,\lambda_{(n_1)}}, n_1)$ | $n_2$ | $\lambda_{(n_2)}$ | $q_{n_2,\lambda(n_2)}$ | JTP$^{(2)}$ $(\lambda_{(n_2)}, q_{n_2,\lambda(n_2)}, n_2)$ |
|---|---|---|---|---|---|---|---|
| 1 | 0.00510 | 395.158 | 28 111.974 | 1 | 0.00318 | 394.011 | 28 221.714 |
| 2 | 0.00510 | 280.351 | 28 287.191 | 2 | 0.00318 | 279.424 | 28 396.086 |
| **3** | **0.00510** | **229.430** | **28 324.489** | **3** | **0.00318** | **228.608** | **28 433.008** |
| 4 | 0.00510 | 199.025 | 28 323.164 | 4 | 0.00318 | 198.271 | 28 431.458 |

**Notes.** Boldface type expresses the optimal solution of Situations 1 and 2, respectively.

TABLE 2. Optimal solutions under various retailer's parameters.

| Parameters | Value | $n^*$ | $\lambda^*$ | $q^*$ | $Q^*$ | JTP* |
|---|---|---|---|---|---|---|
| $A$ | 40 | 4 | 0.0031826 | 189.051 | 756.205 | 28 483.10 |
| | 45 | 4 | 0.0031825 | 193.716 | 774.864 | 28 456.97 |
| | 50 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 55 | 3 | 0.0031813 | 233.073 | 699.220 | 28 411.35 |
| | 60 | 3 | 0.0031812 | 237.455 | 712.364 | 28 390.10 |
| $D$ | 800 | 3 | 0.0039759 | 201.367 | 604.100 | 22 608.95 |
| | 900 | 3 | 0.0035346 | 215.210 | 645.631 | 25 518.65 |
| | 1000 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 1100 | 4 | 0.0028932 | 210.799 | 843.197 | 31 356.44 |
| | 1200 | 4 | 0.0026523 | 223.276 | 893.102 | 34 285.74 |
| $h_{b_1}$ | 1.6 | 3 | 0.0031811 | 238.66 | 715.980 | 28 479.71 |
| | 1.8 | 3 | 0.0031813 | 233.472 | 700.416 | 28 456.11 |
| | 2 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 2.2 | 4 | 0.0031824 | 194.823 | 779.292 | 28 411.81 |
| | 2.4 | 4 | 0.0031825 | 191.549 | 766.195 | 28 392.49 |
| $h_{b_2}$ | 0.4 | 3 | 0.0031849 | 228.616 | 685.847 | 28 433.04 |
| | 0.45 | 3 | 0.0031832 | 228.612 | 685.835 | 28 433.03 |
| | 0.5 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.55 | 3 | 0.0031797 | 228.604 | 685.813 | 28 432.99 |
| | 0.6 | 3 | 0.0031780 | 228.601 | 685.802 | 28 432.97 |
| $x$ | 2400 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 2700 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 3000 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 3300 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 3600 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| $C_s$ | 0.08 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.09 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.1 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.11 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.12 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| $C_p$ | 8 | 3 | 0.0039320 | 228.605 | 685.814 | 28 440.07 |
| | 9 | 3 | 0.0035171 | 228.606 | 685.819 | 28 436.35 |
| | 10 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 11 | 3 | 0.0029042 | 228.609 | 685.828 | 28 429.97 |
| | 12 | 3 | 0.0026715 | 228.61 | 685.831 | 28 427.18 |
| $p$ | 32 | 3 | 0.0031814 | 228.638 | 685.915 | 20 432.86 |
| | 36 | 3 | 0.0031814 | 228.623 | 685.869 | 24 432.93 |
| | 40 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |

TABLE 2. continued.

| Parameters | Value | $n^*$ | $\lambda^*$ | $q^*$ | $Q^*$ | JTP$^*$ |
|---|---|---|---|---|---|---|
| | 44 | 3 | 0.0031814 | 228.593 | 685.778 | 32 433.08 |
| | 48 | 3 | 0.0031814 | 228.577 | 685.732 | 36 433.16 |
| $c$ | 16 | 3 | 0.0031776 | 231.438 | 694.315 | 28 449.89 |
| | 18 | 3 | 0.0031795 | 230.01 | 690.03 | 28 441.43 |
| | 20 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 22 | 3 | 0.0031833 | 227.232 | 681.696 | 28 424.63 |
| | 24 | 4 | 0.0031858 | 196.229 | 784.915 | 28 416.50 |

TABLE 3. Optimal solutions under various supplier's parameters.

| Parameters | Value | $n^*$ | $\lambda^*$ | $q^*$ | $Q^*$ | JTP$^*$ |
|---|---|---|---|---|---|---|
| $P$ | 1600 | 4 | 0.0031821 | 205.262 | 821.047 | 28 469.28 |
| | 1800 | 4 | 0.0031822 | 201.289 | 805.154 | 28 448.11 |
| | 2000 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 2200 | 3 | 0.0031815 | 227.018 | 681.054 | 28 425.24 |
| | 2400 | 3 | 0.0031815 | 225.718 | 677.154 | 28 418.81 |
| $S$ | 160 | 3 | 0.0031818 | 216.250 | 648.750 | 28 493.10 |
| | 180 | 3 | 0.0031816 | 222.515 | 667.544 | 28 462.56 |
| | 200 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 220 | 4 | 0.0031822 | 202.724 | 810.895 | 28 406.52 |
| | 240 | 4 | 0.0031821 | 207.081 | 828.322 | 28 382.12 |
| $h_{v_1}$ | 1.2 | 4 | 0.0031820 | 209.829 | 839.315 | 28 492.62 |
| | 1.35 | 4 | 0.0031822 | 203.805 | 815.218 | 28 461.61 |
| | 1.5 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 1.65 | 3 | 0.0031816 | 223.483 | 670.45 | 28 407.58 |
| | 1.8 | 3 | 0.0031817 | 218.689 | 656.066 | 28 382.71 |
| $h_{v_2}$ | 0.4 | 3 | 0.0032121 | 228.608 | 685.824 | 28 433.33 |
| | 0.45 | 3 | 0.0031967 | 228.608 | 685.824 | 28 433.17 |
| | 0.5 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.55 | 3 | 0.0031663 | 228.608 | 685.824 | 28 432.85 |
| | 0.6 | 3 | 0.0031513 | 228.608 | 685.824 | 28 432.69 |
| $C_T$ | 8 | 3 | 0.0031815 | 226.797 | 680.392 | 28 441.79 |
| | 9 | 3 | 0.0031814 | 227.704 | 683.113 | 28 437.39 |
| | 10 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 11 | 3 | 0.0031814 | 229.508 | 688.524 | 28 428.64 |
| | 12 | 3 | 0.0031814 | 230.404 | 691.213 | 28 424.29 |
| $C_t$ | 0.24 | 3 | 0.0031814 | 228.608 | 685.824 | 28 493.01 |
| | 0.27 | 3 | 0.0031814 | 228.608 | 685.824 | 28 463.01 |
| | 0.3 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.33 | 3 | 0.0031814 | 228.608 | 685.824 | 28 403.01 |
| | 0.36 | 3 | 0.0031814 | 228.608 | 685.824 | 28 373.01 |
| $v$ | 8 | 3 | 0.0031803 | 228.608 | 685.824 | 30 436.69 |
| | 9 | 3 | 0.0031809 | 228.608 | 685.824 | 29 434.85 |
| | 10 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 11 | 3 | 0.0031820 | 228.608 | 685.824 | 27 431.16 |
| | 12 | 3 | 0.0031825 | 228.608 | 685.824 | 26 429.32 |

Table 4. Optimal solutions under various investing parameters.

| Parameters | Value | $n^*$ | $\lambda^*$ | $q^*$ | $Q^*$ | JTP* |
|---|---|---|---|---|---|---|
| $\delta$ | 0.00024 | 3 | 0.0039768 | 228.604 | 685.813 | 28 411.02 |
| | 0.00027 | 3 | 0.0035349 | 228.606 | 685.819 | 28 423.00 |
| | 0.0003 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.00033 | 3 | 0.0028922 | 228.609 | 685.828 | 28 441.50 |
| | 0.00036 | 3 | 0.0026512 | 228.585 | 685.756 | 28 448.80 |
| $\lambda_U$ | 0.04 | 3 | 0.0031814 | 228.608 | 685.824 | 28 440.45 |
| | 0.045 | 3 | 0.0031814 | 228.608 | 685.824 | 28 436.52 |
| | 0.05 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.055 | 3 | 0.0031814 | 228.608 | 685.824 | 28 429.83 |
| | 0.06 | 3 | 0.0031814 | 228.608 | 685.824 | 28 426.93 |
| $\lambda_L$ | 0.004 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.0045 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.005 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.0055 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.006 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| $\theta$ | 0.008 | 3 | 0.0025451 | 228.584 | 685.753 | 28 452.09 |
| | 0.009 | 3 | 0.0028633 | 228.609 | 685.828 | 28 442.36 |
| | 0.01 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.011 | 3 | 0.0034996 | 228.607 | 685.820 | 28 423.97 |
| | 0.012 | 3 | 0.0038177 | 228.605 | 685.815 | 28 415.27 |

(1) When the retailer's ordering cost ($A$) or penalty cost ($C_p$) increases, the optimal proportion of defective items ($\lambda^*$) and joint total profit (JTP*) decrease, whereas the optimal shipping quantity ($q^*$) and order quantity ($Q^* = n^*q^*$) (for the same $n^*$) increase.
(2) As the market demand ($D$) increases, $\lambda^*$ decreases; however, $q^*$, $Q^*$ (for the same $n^*$), and JTP* increase.
(3) The retailer's holding cost per non-defective item ($h_{b_1}$) and procurement cost ($c$) positively affect $\lambda^*$ but negatively affect $q^*$, $Q^*$ (for the same $n^*$), and JTP*.
(4) When the retailer's holding cost per defective item ($h_{b_2}$) increases, $\lambda^*$, $q^*$, $Q^*$, and JTP* decrease.
(5) When the retailer's unit selling price ($p$) increases, $q^*$ and $Q^*$ decrease, but JTP* increases. Furthermore, the selling price ($p$) has no effect on $\lambda^*$.
(6) The inspection rate ($x$) and inspection cost ($C_s$) have no effect on the optimal solutions under the "non-inspect" situation in which the retailer no longer conducts any checks on the received items.

From the results of Table 3, we have: (1) All the supplier parameters negatively affect JTP*. (2) When the supplier's production rate ($P$), holding cost ($h_{v_1}$) (excluding interest charge), or production cost ($v$) increases, $\lambda^*$ increases (for the same $n^*$). (3) When the supplier's setup cost ($S$), treatment cost per defective item ($h_{v_2}$), or fixed cost of transportation ($C_T$) increases, $\lambda^*$ decreases (for the same $n^*$). (4) The optimal shipping quantity and order quantity decrease (for the same $n^*$) with an increase in the supplier's production rate ($P$) but increase (for the same $n^*$) with an increase in the supplier's setup cost ($S$) and fixed cost of transportation ($C_T$).

From the results shown in Table 4, the following observations can be made: (1) The percentage decrease in the defective rate per dollar increase in the capital investment ($\delta$) has a negative effect on $\lambda^*$. However, the opportunity cost of the capital investment ($\theta$) has a positive effect on $\lambda^*$. (2) The value of $\delta$ has a positive effect on JTP*, whereas $\theta$ and $\lambda_U$ negatively affect JTP*. (3) As the percentage decrease in the defective rate per dollar increase in capital investment $\delta$ or the opportunity cost of the capital investment $\theta$ increase, both the optimal shipping quantity $q^*$ and the order quantity $Q^* = n^*q^*$ increase firstly and then decrease.

From Table 5, it is obtained that: (1) When the interest charged $I_c$, the downstream credit period by the retailer to customers $N$ increases, the optimal proportion of defective items $\lambda^*$ increases but the shipping

TABLE 5. Optimal solutions under various credit parameters.

| Parameters | Value | $n^*$ | $\lambda^*$ | $q^*$ | $Q^*$ | JTP* |
|---|---|---|---|---|---|---|
| $I_c$ | 0.024 | 3 | 0.0031764 | 231.438 | 694.315 | 28 453.8 |
| | 0.027 | 3 | 0.0031789 | 230.010 | 690.030 | 28 443.4 |
| | 0.03 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.033 | 3 | 0.0031839 | 227.232 | 681.696 | 28 422.7 |
| | 0.036 | 4 | 0.0031870 | 196.229 | 784.915 | 28 412.6 |
| $I_e$ | 0.008 | 3 | 0.0031815 | 228.638 | 685.915 | 28 432.6 |
| | 0.009 | 3 | 0.0031815 | 228.623 | 685.869 | 28 432.8 |
| | 0.01 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.011 | 3 | 0.0031814 | 228.593 | 685.778 | 28 433.2 |
| | 0.012 | 3 | 0.0031814 | 228.577 | 685.732 | 28 433.4 |
| $M$ | 0.09863 | 3 | 0.0031803 | 228.544 | 685.632 | 28 429.6 |
| | 0.110959 | 3 | 0.0031809 | 228.569 | 685.707 | 28 431.3 |
| | 0.123288 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.135616 | 3 | 0.0031820 | 228.660 | 685.981 | 28 434.6 |
| | 0.147945 | 3 | 0.0031825 | 228.727 | 686.180 | 28 436.1 |
| $N$ | 0.065753 | 3 | 0.0031799 | 228.681 | 686.043 | 28 442.5 |
| | 0.073973 | 3 | 0.0031807 | 228.641 | 685.924 | 28 437.8 |
| | 0.082192 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.090411 | 3 | 0.0031822 | 228.581 | 685.742 | 28 428.2 |
| | 0.09863 | 3 | 0.0031829 | 228.559 | 685.678 | 28 423.4 |
| $l$ | 0.021918 | 3 | 0.0031813 | 228.608 | 685.824 | 28 433.4 |
| | 0.024658 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.2 |
| | 0.027397 | 3 | 0.0031814 | 228.608 | 685.824 | 28 433.01 |
| | 0.030137 | 3 | 0.0031815 | 228.608 | 685.824 | 28 432.8 |
| | 0.032877 | 3 | 0.0031816 | 228.608 | 685.824 | 28 432.6 |

quantity $q^*$, the order quantity $Q^* = n^*q^*$ (for the same $n^*$) and the joint total profit JTP* decrease. (2) When the length of time during which the prepayments are paid $l$ increases, the optimal proportion of defective items $\lambda^*$ increases but the joint total profit JTP* decreases. (3) The interest earned $I_e$ has a positive impact on the optimal joint total profit JTP*, but has negative impacts on the optimal proportion of defective items $\lambda^*$, the shipping quantity $q^*$ and the order quantity $Q^* = n^*q^*$. (4) The upstream credit period by the supplier to the retailer $M$ has positive impacts on the optimal proportion of defective items $\lambda^*$, the shipping quantity $q^*$, the order quantity $Q^* = n^*q^*$ and the joint total profit JTP*.

## 6. Conclusions

In this study, an integrated production-inventory model is presented involving defective items and ACC payment. The product quality can be improved through capital investment from the supplier and retailer. The theorems proposed in this paper ensure the existence and uniqueness of the optimal solutions and add rigor to the model. An algorithm is used to determine the optimal solutions. Several numerical examples are used to demonstrate the model and examine the effects of parameter changes on the optimal solutions. Furthermore, the optimal solutions with and without capital investment are determined and compared. The comparison results indicate that the supply chain benefits when capital is jointly invested for the quality improvement of a product. Moreover, it reveals that the optimal shipping, order, investment, and inspection policies are determined by trading off the opportunity cost of the capital investment and inspection cost for the penalty cost from the numerical examples.

The proposed model can be extended in several ways. For example, it may be used to study the demand rate as a function of factors such as the selling price, time, and stock. Furthermore, the model can be generalized to account for shortages, quantity discounts, and inflation.

## Appendix A.

*Proof of Lemma 4.1.* Taking the first-order derivative of $\mathrm{JTP}_{11}(\lambda, q, n_1)$ with respect to $q$ gives the following:

$$\frac{\mathrm{dJTP}_{11}(\lambda, q, n_1)}{\mathrm{d}q} = \frac{D}{1-\lambda}\left\{\frac{n_1[2(A+C_T)-(pI_e-cI_c)\gamma D(M-N)^2]+2S}{2n_1q^2} - \frac{(h_{b_1}+cI_c)(1-\lambda)^2}{2D} - \frac{(h_{b_1}-h_{b_2})\lambda}{2x} - \frac{h_{b_2}\lambda(1-\lambda)}{D} - h_{v_1}\left[\frac{1}{P}+\frac{(n_1-1)(1-\lambda)}{2D}-\frac{n_1}{2P}\right]\right\}. \tag{A.1}$$

Then, the second-order derivative of $\mathrm{JTP}_{11}(\lambda, q, n_1)$ with respect to $q$ can be obtained as follows:

$$\frac{\mathrm{d}^2\mathrm{JTP}_{11}(\lambda, q, n_1)}{\mathrm{d}q^2} = \frac{-D\{n_1[2(A+C_T)-(pI_e-cI_c)\gamma D(M-N)^2]+2S\}}{(1-\lambda)n_1q^3}.$$

From (A.1), if $n_1[2(A+C_T)-(pI_e-cI_c)\gamma D(M-N)^2]+2S \leq 0$, $\mathrm{dJTP}_{11}(\lambda, q, n_1)/\mathrm{d}q < 0$. This implies that for a fixed $n_1$ and $\lambda \in (\lambda_L, \lambda_U]$, $\mathrm{JTP}_{11}(\lambda, q, n_1)$ is a decreasing function of $q$. In this case, the optimal size of each shipment from the supplier to the retailer is $q = 0$, which is not true in reality. Therefore, it is reasonable to assume that $n_1[2(A+C_T)-(pI_e-cI_c)\gamma D(M-N)^2]+2S > 0$. In this case, $\mathrm{d}^2\mathrm{JTP}_{11}(\lambda, q, n_1)/\mathrm{d}q^2 < 0$. Consequently, if $n_1[2(A+C_T)-(pI_e-cI_c)\gamma D(M-N)^2]+2S > 0$, $\mathrm{JTP}_{11}(\lambda, q, n_1)$ is a concave function of $q$ for any given $n_1$ and $\lambda \in (\lambda_L, \lambda_U]$. Thus, a unique value of $(q_{11,n_1,\lambda})$ is obtained when solving the equation $\mathrm{dJTP}_{11}(\lambda, q, n_1)/\mathrm{d}q = 0$ that maximizes $\mathrm{JTP}_{11}(\lambda, q, n_1)$ as $q_{11,n_1,\lambda} = \sqrt{\frac{\Delta_1}{2n_1\Delta_2}}$. This completes the proof. □

## Appendix B.

*Proof of Lemma 4.2.* For any given $n_1$ and $\lambda \in (\lambda_L, \lambda_U]$, $\mathrm{JTP}_{21}(\lambda, q, n_1)$ and $\mathrm{JTP}_{31}(\lambda, q, n_1)$ are concave functions of $q$, because

$$\frac{\mathrm{d}^2\mathrm{JTP}_{i1}(\lambda, q, n_1)}{\mathrm{d}q^2} = \frac{-2D\,[n_1(A+C_T)+S]}{(1-\lambda)n_1q^3} < 0, \quad i = 2, 3.$$

Thus, there exist unique values of ($q_{21,n_1,\lambda}$ and $q_{31,n_1,\lambda}$) that maximize $\mathrm{JTP}_{21}(\lambda, q, n_1)$ and $\mathrm{JTP}_{31}(\lambda, q, n_1)$ as follows:

$$q_{21,n_1,\lambda} = \sqrt{\frac{n_1(A+C_T)+S}{n_1\Delta_3}},$$

and

$$q_{31,n_1,\lambda} = \sqrt{\frac{n_1(A+C_T)+S}{n_1\Delta_4}}.$$

This completes the proof. □

## Appendix C.

*Proof of Lemma 4.3.* Using a similar approach shown as in Lemma 4.1, the following equation is obtained for any given $n_2$ and $\lambda \in (0, \lambda_L]$.

$$\frac{\mathrm{d}^2\mathrm{JTP}_{12}(\lambda, q, n_2)}{\mathrm{d}q^2} = -\frac{D\left\{n_2[2(A+C_T)-(pI_e-cI_c)\,\gamma D\,(M-N)^2]+2S\right\}}{n_2q^3}.$$

We may assume without loss of generality (WLOG) that $n_2[2(A+C_T)-(pI_e-cI_c)\,\gamma D\,(M-N)^2]+2S>0$, then $\mathrm{d}^2\mathrm{JTP}_{12}(\lambda,q,n_2)/\mathrm{d}q^2<0$ and hence the optimal solution of $q$ (denoted by $q_{12,n_2,\lambda}$) that maximizes the joint total profit per unit time $\mathrm{JTP}_{12}(\lambda,q,n_2)$ can be obtained by solving the equation $\mathrm{dJTP}_{12}(\lambda,q,n_2)/\mathrm{d}q=0$ as $q_{12,n_2,\lambda}=\sqrt{\frac{\Delta_5}{2n_2\Delta_6}}$.

Similarly, the following equation can be obtained:

$$\frac{\mathrm{d}^2\mathrm{JTP}_{i2}(\lambda,q,n_2)}{\mathrm{d}q^2}=\frac{-2D[n_2(A+C_T)+S]}{n_2q^3}<0,\quad i=2,3.$$

Hence the optimal solution of $q$ (denoted by $q_{i2,n_2,\lambda}, i=2,3$) that maximizes $\mathrm{JTP}_{i2}(\lambda,q,n_2), i=2,3$, can be obtained by solving the equation $\mathrm{dJTP}_{i2}(\lambda,q,n_2)/\mathrm{d}q=0, i=2,3$, as follows:

$$q_{22,n_2,\lambda}=\sqrt{\frac{n_2(A+C_T)+S}{n_2\Delta_7}},$$

and

$$q_{32,n_2,\lambda}=\sqrt{\frac{n_2(A+C_T)+S}{n_2\Delta_8}}.$$

This completes the proof. □

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