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OPTIMAL POLICIES FOR A DETERMINISTIC CONTINUOUS-TIME INVENTORY MODEL WITH SEVERAL SUPPLIERS: A HYPER-GENERALIZED (s, S) POLICY

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Abstract. A deterministic continuous-time continuous-state inventory model is studied. In the absence of intervention, the level of stock evolves by a process governed by a differential equation. The inventory level is monitored continuously, and can be adjusted upwards at any time. The decision maker can order from several suppliers, each of which charges a different ordering and purchasing cost. The problem of selecting the supplier and the size of the order to minimize the total inventory cost over an infinite planning horizon is formulated as the solution of a quasi-variational inequality (QVI). It is shown that the QVI has a unique solution. This corresponds to a generalized (s, S) policy under amenable conditions, which have been characterized in an earlier work by the present authors. Under the complementary conditions a new type of optimal control policy emerges. This leads to the concept of a hyper-generalized (s, S) policy. The theory behind a policy of this type is exposed.

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1. Introduction

The present paper is concerned with a deterministic continuous-time continuous-state integrated-procurement inventory model involving several suppliers. The level of stock is continuously monitored, and a policy that identifies an appropriate supplier together with the quantity of stock that is to be supplied is sought. The goal is to minimize the total cost of the inventory over an infinite planning horizon..

With a single supplier, the decision problem reduces to the search for an optimal (s, S) policy as described in [1, 7, 21, 22, 24]. The optimality of such a policy has been shown for more sophisticated models [4, 9]. When several suppliers are available, an (s, S) policy need not be optimal, and the problem of identifying an optimal policy turns out to be intricate. Indeed, under the assumption that each supplier maintains a fixed set-up cost and fixed cost per item, it has been shown [5] that the following alternatives are mutually exclusive.

– There is an optimal (s, S) policy involving only one predetermined supplier.
– There is an optimal generalized (s, S) policy involving more than one supplier.

Keywords. Optimal inventory policy, quasi-variational inequality, (s,S) policy, generalized (s,S) policy, hyper-generalized (s,S) policy.

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– There is no optimal generalized (s, S) policy, let alone an optimal (s, S) policy.

The generalized (s, S) policy entails N suppliers and stock levels $s_{(N)} < s_{(N-1)} < \cdots < s_{(1)} < S_{(1)} < S_{(2)} < \cdots < S_{(N)}$ for some natural number $N \geq 2$. If the level of stock is greater than $s_{(1)}$ one does not replenish it. If the level is between $s_{(2)}$ and $s_{(1)}$ one orders from supplier (1) up to the level $S_{(1)}$. If the level is between $s_{(3)}$ and $s_{(2)}$ one orders from supplier (2) up to the level $S_{(2)}$. This pattern continues down to a level of stock between $s_{(N)}$ and $s_{(N-1)}$. If the level of stock is less than $s_{(N)}$ one orders from supplier (N) up to the level $S_{(N)}$. The policy may deliberately exclude a selection of available suppliers. For instance, the optimal generalized policy could involve just three of five available suppliers that are ranked 1 to 5 according to some rational criterion, with supplier (1) being number 4 in the original pecking order, supplier (2) being number 3, supplier (3) being number 1, and, suppliers 2 and 5 not included.

The above trichotomy has been established using quasi-variational techniques, and the precise conditions leading to each alternative have been identified [5]. In reality, companies do not have control over various factors affecting their inventory systems, such as demand, interest rate, and incidental inventory costs. Therefore, they have limited influence on the satisfaction of the requirements for an optimal (s, S) or generalized (s, S) policy. The failure to satisfy these requirements leaves the decision maker with no perceivably adoptable policy. Needless to say, no action is not an option in today's competitive market environment where profit margins are tight. A rescue plan has to be developed to address this situation.

The rescue plan begins by admitting the possibility that the optimal policy may be neither an (s, S) policy nor a generalized (s, S) policy. In so doing, a new type of policy emerges. This will be called a hyper-generalized (s, S) policy.

A hyper-generalized (s, S) policy involves, like a generalized (s, S) policy, more than one supplier and stock levels $s_{(N)} < s_{(N-1)} < \cdots < s_{(1)} < S_{(1)} < S_{(2)} < \cdots < S_{(N)}$. However, in addition, it contains stock levels $s_{(N)} \leq r_{(N-1)} \leq s_{(N-1)} \leq r_{(N-2)} \leq s_{(N-2)} \leq \cdots \leq s_{(2)} \leq r_{(1)} \leq s_{(1)}$. If the level of stock is greater than $s_{(1)}$ one does not replenish. If the level is between $r_{(1)}$ and $s_{(1)}$ one orders from supplier (1) up to the level $S_{(1)}$. If the level is between $r_{(2)}$ and $s_{(2)}$ one orders from supplier (2) up to the level $S_{(2)}$. This is the case down to a level between $r_{(N-1)}$ and $s_{(N-1)}$. As with a generalized (s, S) policy, one orders from supplier (N) up to the level $S_{(N)}$ when the level of stock is less than $s_{(N)}$. The distinction is that if the level of stock is between $s_{(2)}$ and $r_{(1)}$, or between $s_{(3)}$ and $r_{(2)}$, and so on, then one does not replenish it, in the same manner as when the stock level is greater than $s_{(1)}$.

In layman's terms, a hyper-generalized (s, S) policy opens an additional course of action for an inventory manager. This applies when he or she is confronted with a level of shortage for which a generalized (s, S) policy offers a choice between two unappealing alternatives. If the backlog were less, then there would be a clear optimal course of action in placing an order with a supplier with a low set-up cost. However, at the encountered shortage level, the supplier's high cost per item makes placement of the order with that supplier unattractive. On the other hand, if the backlog were greater, there would be a clear optimal course of action in placing an order with a supplier with a low cost per item. However, at the encountered level, the supplier's high set-up cost makes placement of the order with that supplier unattractive. The best thing for the manager to do is to let the backlog further accumulate until the amount that has to be ordered is so great that is it is indeed worthwhile to replenish from the supplier with a low cost per item. Hereby, the high set-up cost is set off against the size of the order. A hyper-generalized (s, S) policy accommodates this option.

A generalized (s, S) policy is recovered from a hyper-generalized (s, S) policy when $r_{(N-1)} = s_{(N)}$, $r_{(N-2)} = s_{(N-1)}, \ldots, r_{(1)} = s_{(2)}$ in the notation employed.

The aim of the present paper is provide a technical foundation for a hyper-generalized (s, S) policy and substantiate its functionality. It will be demonstrated that the concept leads to both the existence and uniqueness of a viable optimal control policy.

The existence and uniqueness of a hyper-generalized (s, S) policy reducing to a conventional generalized (s, S) policy, and possibly even to an (s, S) policy, under amenable circumstances, indicates that the previously presented intuitive motivation for the consideration of such a policy based upon opening a further course of action for a manager is in fact one of necessity. There fails to be a conventional generalized (s, S) policy precisely because under the relevant conditions the optimal strategy is not to remove a shortage that would be removed were the backlog less, but, instead, let the backlog further accrue until an appropriate order can be placed with an alternative supplier.

The literature on generalized (s, S) policies is scant. The classical multi-period finite-horizon discrete-time stochastic inventory model with piecewise-linear increasing concave ordering cost has been considered in [19,20]. At each discrete-time epoch, the decision maker is faced with the decision of either not ordering or ordering a quantity to be decided upon. In the latter event, a choice between the available suppliers has to be made. An (s, S) policy is optimal when each available supplier is considered in isolation. Under the standard hypothesis that per period the immediate expected inventory cost expressed as a function of the current level of stock is quasi-convex and coercive, and some additional technical requirements, a generalized (s, S) policy is optimal. The additional technical requirements are met when the demand distribution for the item in question belongs to the Pólya or to the uniform family of distributions. A stochastic inventory model with two suppliers, one with a high purchasing cost and no set-up cost, and the other with a low purchasing cost and a definite set-up cost was studied in [12]. Optimality of a generalized (s, S) policy was deduced for both a finite and infinite planning horizon. In the same spirit, for an arbitrary demand distribution, it has been shown that a generalized (s, S) policy is optimal outside a bounded region of the state space [3].

The above short review of the literature supports the observation that in general a generalized (s, S) policy need not be optimal. It moreover reflects the dearth of research papers on the subject. This is perhaps due to the technical complexities encountered in dealing with such a policy.

In the present paper, the optimality of a hyper-generalized (s, S) policy will be established based on techniques from the study of quasi-variational inequalities (QVI). Such inequalities were introduced by Bensoussan and Lions. For details, readers are referred to the monograph [8]. Supplementary illustrations of the power of QVI techniques applied to a single-supplier inventory model may be found in [7,10].

The next section contains the formal statement of the inventory control problem to be studied, culminating in its QVI formulation. A number of pertinent results from [5] are collected in Section 3. In particular, these results summarize the theory for the model with a single supplier producing an optimal (s, S) policy, and the theory for the model with several suppliers leading to the identification of a necessary and sufficient condition for the optimality of a generalized (s, S) policy. Section 4 is devoted to the proof of the optimality of a hyper-generalized (s, S) policy and its computation. Examples illustrating various aspects are included throughout. A conclusion and some general remarks are provided in Section 5.

2. Problem statement

Consider a stock of a single item. The level of stock at time t is given by $x(t)$, with a value $x \geq 0$ denoting the number of items held, and a value $x < 0$ indicating a shortage of $-x$ items. In the absence of intervention, changes in stock level are governed by the evolution equation

$$\dot{x}(t) = -G(x(t)), \tag{2.1}$$

where G is a positive continuous function defined on $\mathbb{R}$. From the viewpoint of modelling, the latter accounts for stock-dependent demand and deterioration [2,23].

Stock can be replenished by ordering from J suppliers, for some natural number J. Setting

$$\mathcal{J} := \{1, 2, \dots, J\},$$

ordering from supplier $j \in \mathcal{J}$ entails a fixed cost k_j and a cost c_j per unit item. No one supplier is always preferable to another, as

$$k_1 > k_2 > \cdots > k_J > 0 \tag{2.2}$$

and

$$c_1 < c_2 < \cdots < c_J. \tag{2.3}$$

The holding cost depends on the stock level, and is given by a continuous nonnegative function f defined on $\mathbb{R}$. This amalgamates the cost of storage, handling, obsolescence, depreciation, deterioration, insurance, taxation, loss of revenue due to tied-up capital, and miscellaneous other transactions when there is stock in hand. When there is a shortage, it incorporates the cost of lost sales, lost production, loss of good will, overtime, extraordinary administration, and other penalties that might be incurred. The expression

$$f(x) = \begin{cases} -px & \textit{for } x < 0 \\ qx & \textit{for } x \geq 0 \end{cases} \tag{2.4}$$

with constants $p > 0$ and $q > 0$ is classical.

Variability of money-value in time is considered by the exponential discount of costs at a constant rate

$$\alpha > 0. \tag{2.5}$$

An admissible replenishment strategy consists of a sequence

$$V_n = \{(t_i, \xi_i, j_i) : 0 \leq t_1 < t_2 < \cdots < t_n\},$$

where n denotes the number of interventions, t_i the time of the ith control (order), and $\xi_i > 0$ the quantity ordered at that time. There is nothing to be gained from spreading the order among two or more suppliers, and $j_i \in \mathcal{J}$ labels the envisaged supplier. The problem of finding an optimal impulse control policy is to determine a sequence

$$V = \lim_{n\to\infty} V_n,$$

that solves

$$u(x(0)) = \min_V \left\{ \int_0^\infty f(x(t))\, \mathrm{e}^{-\alpha t}\, \mathrm{d}t + \sum_{i=1}^\infty \left(k_{j_i} + c_{j_i}\xi_i\right) \mathrm{e}^{-\alpha t_i} \right\}. \tag{2.6}$$

This contrasts with the familiar Economic Order Quantity (EOQ) model, in which the discounting of future costs is neglected, and the objective is to minimize the long-term average cost. In the long-term, the initial level of stock is irrelevant. In the present problem, the objective is to minimize the total future cost over an infinite time horizon, and specifically to ascertain the dependence of this cost upon the initial level of stock.

As far as the occurrence and nature of an optimal policy are concerned, the model with a stock-dependent demand rate G presents no more difficulty to deal with than the model with a constant demand rate of unit magnitude. If x satisfies (2.1) then

$$\tilde{x} = \int_0^x \frac{\mathrm{d}\eta}{G(\eta)}$$

satisfies the equation

$$\dot{x}(t) = -1. \tag{2.7}$$

Moreover, if u satisfies (2.6) then $\tilde{u}(\tilde{x}) = u(x)$ satisfies (2.6) with f replaced by $\tilde{f}$ where $\tilde{f}(\tilde{x}) = f(x)$. Conversely, provided that

$$\int_0^x \frac{\mathrm{d}\eta}{G(\eta)} \to \pm\infty \quad \text{as } x \to \pm\infty, \tag{2.8}$$

if $\tilde{x}$ is governed by (2.7) and $\tilde{u}$ satisfies (2.6) with f replaced by $\tilde{f}$ then the reverse transformation gives rise to (2.1) and (2.6). Commonly used expressions for G satisfy (2.8). Consequently, with nominal loss of generality, the problem of finding an optimal impulse control policy may be simplified by replacing (2.1) with (2.7).

Following [5], the optimal total future cost $u(x)$ evolving according to equation (2.7) from a starting level of stock x will satisfy the equation $\big(Au\big)(x) = f(x)$, where

$$\big(Au\big)(x) = u'(x) + \alpha u(x).$$

On the other hand, the like cost $u(x)$ associated with a level of stock x that has been obtained by placing an order with supplier j will be such that $u(x) = \big(M_j u\big)(x)$, where

$$\big(M_j u\big)(x) = k_j + \min\{u(x+\xi) + c_j\xi : \xi \geq 0\}.$$

Hence, defining

$$\big(Mu\big)(x) = \min\big\{\big(M_j u\big)(x) : j \in \mathcal{J}\big\},$$

the optimal impulse control policy is given by a solution of the QVI

$$\begin{cases} Au \leq f \\ u \leq Mu \\ (Au - f)\,(u - Mu) = 0. \end{cases} \tag{2.9}$$

3. Previous results

Currently the following is known about the solution of the QVI (2.9). We refer to [5] for the omitted proofs.

3.1. Single supplier

When there is a single supplier, there is an optimal inventory control policy based on the following premise.

Ansatz 3.1. The solution of (2.9) is a differentiable real function u such that $u = Mu$ on $(-\infty, s]$, and, $Au = f$ and $u < Mu$ in (s, ∞), for some number s.

The above ansatz suitably restricts the admissible structure of a solution of (2.9).

Proposition 3.2. *Suppose that f is continuous on $\mathbb{R}$. Then, under Ansatz 3.1, a solution of (2.9) with $M = M_j$ is given by*

$$u(x) = \begin{cases} y(s) + c_j\,(s - x) & \text{for } \; x < s \\ y(x) & \text{for } \; x \geq s, \end{cases}$$

where y is a solution of the differential equation

$$y' + \alpha y = f \quad \text{in } \mathbb{R} \tag{3.1}$$

satisfying

$$y'(s) = y'(S) = -c_j \tag{3.2}$$

and

$$y(s) = y(S) + k_j + c_j\,(S - s) \tag{3.3}$$

for some number $S > s$.

A function y described by Proposition 3.2 exists and is unique when the auxiliary function f_j defined by

$$f_j(x) = f(x) + \alpha c_j x \quad \text{ for } x \in \mathbb{R} \tag{3.4}$$

satisfies the next hypothesis. This hypothesis reflects the conducive attributes of (2.4).

Hypothesis 3.3. The function f_j is a continuous function on $\mathbb{R}$, strictly decreasing on $(-\infty, \gamma_j]$ and strictly increasing on $[\gamma_j, \infty)$ for some $\gamma_j \in \mathbb{R}$, $f_j(x) \to \infty$ as $x \to -\infty$, and

$$\int_{\gamma_j}^{\infty} \mathrm{e}^{\alpha\eta}\,\mathrm{d}f_j(\eta) \geq -\int_{-\infty}^{\gamma_j} \mathrm{e}^{\alpha\eta}\,\mathrm{d}f_j(\eta).$$

The last-mentioned condition is automatically satisfied if $f_j(x) \to \infty$ as $x \to \infty$.

Lemma 3.4. *Equation* (3.1) *has a solution satisfying* (3.2) *and* (3.3) *for* $S > s$ *if and only if* $s = s_j$ *and* $S = S_j$ *for a unique combination* $s_j < \gamma_j < S_j$*. Furthermore, this solution* y_j *is unique,*

$$y_j' < -c_j \quad \text{on } (s_j, S_j), \qquad \text{and} \qquad y_j' > -c_j \quad \text{on } (-\infty, s_j) \cup (S_j, \infty). \tag{3.5}$$

Taking Lemma 3.4 together with Proposition 3.2 leads to the existence and uniqueness of a solution of the QVI (2.9).

Proposition 3.5. *Suppose that* f_j *satisfies Hypothesis* 3.3*. Then* (2.9) *with* $M = M_j$ *has a unique solution satisfying Ansatz* 3.1*.*

The above delivers an (s, S) policy in which the optimal strategy for an inventory manager is to order up to level S when the inventory level is s or less, *i.e.* the level of shortage has reached $-s$ or more, and to freely let the inventory level decrease and a shortage accrue otherwise.

The optimal (s, S) policy can be computed as follows.

Algorithm 3.6. Step 1. Find the unique solution of the simultaneous equations

$$\int_{s_j}^{S_j} \mathrm{e}^{\alpha\eta}\, \mathrm{d}f_j(\eta) = 0 \quad \text{and} \quad f_j(s_j) = f_j(S_j) + k_j \quad \text{with } s_j < \gamma_j < S_j. \tag{3.6}$$

Step 2. Set

$$B_j = \{f_j(s_j) + c_j\}/\alpha. \tag{3.7}$$

Step 3. Output $u(x) = B_j - c_j x$ for $x < s_j$, and

$$u(x) = \mathrm{e}^{-\alpha x}\left\{(B_j - c_j s_j)\,\mathrm{e}^{\alpha s_j} + \int_{s_j}^{x} \mathrm{e}^{\alpha\eta} f(\eta)\, \mathrm{d}\eta\right\} \tag{3.8}$$

for $x \geq s_j$. **End**.

When f assumes the classical form (2.4), the auxiliary function f_j satisfies Hypothesis 3.3 if and only if $p > \alpha c_j > -q$, in which case $\gamma_j = 0$. In this propitious situation, the right-hand equation in (3.6) can be solved for S_j explicitly, yielding

$$S_j = -\{\alpha k_j + (p - \alpha c_j)\, s_j\}/(q + \alpha c_j). \tag{3.9}$$

Consequently, S_j can be eliminated from the left-hand equation in (3.6). The upshot is that s_j can be found as the unique solution in $(-\infty, 0)$ of the transcendental equation

$$\alpha\,(p - \alpha c_j)\, s_j + (q + \alpha c_j) \ln\big\{[p + q - (p - \alpha c_j)\,\mathrm{e}^{\alpha s_j}]/(q + \alpha c_j)\big\} + \alpha^2 k_j = 0. \tag{3.10}$$

It may be of interest to note the connection with the longstanding EOQ model [11, 16]. By the Taylor Theorem, (3.9) and (3.10) can be written as

$$S_j = -p s_j/q + \mathrm{O}(\alpha) \quad \text{and} \quad \alpha^2\left\{p\,(p+q)\, s_j^2/(2q) + k_j\right\} = \mathrm{O}(\alpha^3) \qquad \text{as } \alpha \to 0.$$

Dividing the latter equation by α^2 and passing to the limit in both yields

$$s_j = -\sqrt{\frac{2qk_j}{p\,(p+q)}} \quad \text{and} \quad S_j = \sqrt{\frac{2pk_j}{q\,(p+q)}}.$$

These values agree with those of the EOQ model.

Concrete illustrations, with data that will be encountered again in the coming sections, can be found below. The numerical computations involved have been carried out using readily available propriety software. Any similar package could have been employed.

Example 3.7. Take $p = q = 3$, $\alpha = 1$, $k_j = 1$, and $c_j = 2$. Then, by (3.10), $s_j \approx -1.7672$. By (3.9), $S_j \approx 0.1534$. By (3.4) and (3.7), $B_j \approx 3.767$. Algorithm 3.6 yields $u(x) = B_j - 2x$ for $x < s_j$, $u(x) \approx -0.171\,\mathrm{e}^{-x} - 3x + 3$ for $s_j \le x \le 0$, and $u(x) \approx 5.829\,\mathrm{e}^{-x} + 3x - 3$ for $x > 0$.

Example 3.8. Take $p = q = 3$, $\alpha = 1$, $k_j = 2$, and $c_j = 1$. Then, by (3.10), $s_j \approx -1.6831$. By (3.9), $S_j \approx 0.3415$. By (3.4) and (3.7), $B_j \approx 4.366$. Algorithm 3.6 yields $u(x) = B_j - x$ for $x < s_j$, $u(x) \approx -0.372\,\mathrm{e}^{-x} - 3x + 3$ for $s_j \le x \le 0$, and $u(x) \approx 5.628\,\mathrm{e}^{-x} + 3x - 3$ for $x > 0$.

Example 3.9. Substitute $k_j = 4$ in Example 3.8. Then $s_j \approx -2.7687$, $S_j \approx 0.3843$, $B_j \approx 6.537$, $u(x) = B_j - x$ for $x < s_j$, $u(x) \approx -0.125\,\mathrm{e}^{-x} - 3x + 3$ for $s_j \le x \le 0$, and $u(x) \approx 5.875\,\mathrm{e}^{-x} + 3x - 3$ for $x > 0$.

Example 3.10. Substitute $k_j = 6$ in Example 3.8. Then $s_j \approx -3.7959$, $S_j \approx 0.3979$, $B_j \approx 8.592$, $u(x) = B_j - x$ for $x < s_j$, $u(x) \approx -0.045\,\mathrm{e}^{-x} - 3x + 3$ for $s_j \le x \le 0$, and $u(x) \approx 5.955\,\mathrm{e}^{-x} + 3x - 3$ for $x > 0$.

3.2. Several suppliers

When several suppliers are approachable, the QVI (2.9) need not have a solution satisfying Ansatz 3.1. Minor relaxation of the accepted regularity provides an outcome.

Ansatz 3.11. The solution of (2.9) is a continuous real function u such that $u = Mu$ on $(-\infty, s]$, and, u is differentiable, $Au = f$, and $u < Mu$ in (s, ∞), for some number s.

Proposition 3.12. *Suppose that f is continuous on $\mathbb{R}$. Then, under Ansatz 3.11, a solution of (2.9) is given by*

$$u(x) = \begin{cases} v(x) & \text{for } x < s_j \\ y(x) & \text{for } x \ge s_j, \end{cases} \tag{3.11}$$

where y is a solution of the differential equation (3.1) satisfying (3.2) and (3.3) for some $j \in \mathcal{J}$ and $S > s$, and

$$v(x) = \min\bigl\{\bigl(M_\ell y\bigr)(s) + c_\ell\,(s - x) : \ell \in \mathcal{J}\bigr\}. \tag{3.12}$$

Furthermore, $y \le u$ on $\mathbb{R}$, and

$$y(s) = \bigl(M_j y\bigr)(s) < \bigl(M_\ell y\bigr)(s) \quad \text{for every } \ell \in \mathcal{J} \setminus \{j\}. \tag{3.13}$$

Further characterization is attainable under the assumption that f_j satisfies Hypothesis 3.3 for every $j \in \mathcal{J}$. In the light of (2.3) and (2.5), this requires

$$\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_J. \tag{3.14}$$

Lemma 3.4 then applies, giving rise to a unique solution y_j of (3.1) satisfying (3.2) and (3.3) for $S > s$ if and only if $s = s_j$ and $S = S_j$, for every $j \in \mathcal{J}$. The implication is that each supplier accommodates a unique optimal (s, S) policy in the absence of competitors. With this supposition, Proposition 3.12 states that any solution of the QVI with the properties exposed is such that $s = s_j$ and $y = y_j$ for some $j \in \mathcal{J}$.

The solutions y_ℓ for $\ell \in \mathcal{J}$ possess a greatest minimizer in the following sense.

Lemma 3.13. *There is a $j \in \mathcal{J}$ such that $y_j(x) \le y_\ell(x)$ for all $x \in \mathbb{R}$ and $\ell \in \mathcal{J}$ with equality only if $j \ge \ell$.*

Proof. Since any two functions y_j and y_ℓ with $j \in \mathcal{J}$ and $\ell \in \mathcal{J}$ solve (3.1), their difference is a solution of $y' + \alpha y = 0$ on $\mathbb{R}$. Consequently, $\bigl(y_j - y_\ell\bigr)(x) = \mathcal{C}_{j,\ell}\,\mathrm{e}^{-\alpha x}$ for all $x \in \mathbb{R}$, for some constant $\mathcal{C}_{j,\ell}$. It follows that if $y_j(\zeta) < y_\ell(\zeta)$ for any $\zeta \in \mathbb{R}$, then $y_j(x) < y_\ell(x)$ for all $x \in \mathbb{R}$. Likewise, if $j \ge \ell$ and $y_j(\zeta) = y_\ell(\zeta)$ for any $\zeta \in \mathbb{R}$, then $y_j(x) = y_\ell(x)$ for all $x \in \mathbb{R}$. ☐

The concept of a greatest minimizer is relevant to the following.

Lemma 3.14. *There is a function u such that $y_j \le u \le Mu$ on $(-\infty, s_j]$, and $u = y_j < Mu$ in (s_j, ∞) only if j is the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$. Moreover, if $\ell \in \{1, 2, \dots, j-1\}$ is a lesser minimizer, necessarily $s_\ell < s_j < S_j < S_\ell$.*

Lemma 3.15. *If j is the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$ then $M_\ell y_j$ is well defined on $\mathbb{R}$ for every $\ell \in \mathcal{J}$, $y_j < My_j$ in (s_j, ∞), and (3.13) holds. Furthermore,*

$$y_j' < -c_\ell \text{ in } (s_j, S_{j,\ell}), \quad y_j' > -c_\ell \text{ in } (S_{j,\ell}, \infty), \tag{3.15}$$

and

$$\big(M_\ell y_j\big)(s_j) = y_j(S_{j,\ell}) + k_\ell + c_\ell \left(S_{j,\ell} - s_j\right)$$

for $1 \le \ell \le j$, for a sequence of numbers

$$S_{j,j} = S_j < S_{j,j-1} < S_{j,j-2} < \cdots < S_{j,1}.$$

The aforegoing leads to the following uniqueness result.

Proposition 3.16. *Suppose that f_ℓ satisfies Hypothesis 3.3 for every $\ell \in \mathcal{J}$. Then (2.9) has at most one solution satisfying Ansatz 3.11. To be specific, if u is such a solution, then u is given by (3.11) with $y = y_j$, where j is the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$,*

$$v(x) = \min\big\{v_\ell(x) : 1 \le \ell \le j\big\}, \tag{3.16}$$

and

$$v_\ell(x) = y_j(S_{j,\ell}) + k_\ell + c_\ell \left(S_{j,\ell} - x\right). \tag{3.17}$$

Given that a solution of the QVI satisfying Ansatz 3.11 is of the form (3.11) with v prescribed by (3.16) and (3.17), there is a natural number N and there are real numbers

$$\sigma_1 = s_j > \sigma_2 > \cdots > \sigma_N \tag{3.18}$$

demarcating intervals

$$I_1 = (\sigma_2, \sigma_1),\ I_2 = (\sigma_3, \sigma_2),\ \dots,\ I_N = (-\infty, \sigma_N) \tag{3.19}$$

with the property that v is affine in each and not in the union of any two. Necessarily,

$$v = v_j \text{ in } I_1 \quad \text{ and } \quad v = v_1 \text{ in } I_N. \tag{3.20}$$

The function u given by Proposition 3.16 is not differentiable at σ_m for $2 \le m \le N$ when $N \ge 2$. This means that Au is not well defined at such x. The inequality $Au \le f$ in (2.9) has therefore to be interpreted with a little tolerance. Denoting the right and left derivative of u by D^+u and D^-u respectively, we shall take it to mean $D^+u + \alpha u \le f$ and $D^-u + \alpha u \le f$. This interpretation is equivalent to that employed in [5], which, in comparison, has the drawback of involving the intervals (3.19).

With the above understanding, the existence and nonexistence of a solution of the QVI is provided by the next result, in which

$$T_m = \big(f - D^-v - \alpha v\big)(\sigma_m) \tag{3.21}$$

for $1 \le m \le N$. Necessarily,

$$T_1 = 0. \tag{3.22}$$

Proposition 3.17. *Suppose that f_ℓ satisfies Hypothesis 3.3 for every $\ell \in \mathcal{J}$. Let j be the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$. Then the function u given by (3.11) with $y = y_j$, and (3.15)–(3.17) satisfies Ansatz 3.11. Moreover, if $j = 1$, then u solves (2.9) and satisfies Ansatz 3.1. On the other hand, if $j \ge 2$, then $N \ge 2$, and u solves (2.9) if and only if $T_m \ge 0$ for every $m \in \{2, 3, \dots, N\}$.*

Propositions 3.16 and 3.17 reveal the following procedure for identifying whether the QVI (2.9) has a solution corresponding to an (s, S) policy, a solution corresponding to a generalized (s, S) policy, or no such solution, and, moreover, computing the pertinent solution should it exist.

Algorithm 3.18. Step 1. For every $j \in \mathcal{J}$, find the unique solution of the simultaneous equations (3.6).
Step 2. Pick a convenient $\zeta \in \mathbb{R}$, and compute

$$\Upsilon_\ell = c_\ell + f(\zeta) - \mathrm{e}^{-\alpha\zeta} \int_{s_\ell}^{\zeta} \mathrm{e}^{\alpha\eta}\, \mathrm{d}f_\ell(\eta) \quad \text{for } \ell \in \mathcal{J}.$$

Step 3. Define j as the largest number in $\mathcal{J}$ with the property that $\Upsilon_j \leq \Upsilon_\ell$ for every $\ell \in \mathcal{J}$, and thereafter B_j by (3.7).

Step 4. If $j = 1$, then the QVI has a solution satisfying Ansatz 3.1 given by Step 3 of Algorithm 3.6. **End**. Otherwise, continue to Step 5.

Step 5. For $\ell = 1, 2, \ldots, j-1$, determine $S_{j,\ell} > S_j$ from the equation

$$\int_{S_j}^{S_{j,\ell}} \mathrm{e}^{\alpha\eta}\, \mathrm{d}f_j(\eta) = (c_j - c_\ell)\, \mathrm{e}^{\alpha S_{j,\ell}}, \tag{3.23}$$

and thereafter set

$$B_\ell = k_\ell + \{f_\ell(S_{j,\ell}) + c_\ell\}/\alpha. \tag{3.24}$$

Step 6. Let $\sigma_1 = s_j$, $\kappa(1) = j$, and $m = 2$.
- (a) Define $\mathcal{K} = \{1, 2 \ldots, \kappa(m-1) - 1\}$.
- (b) For every $\ell \in \mathcal{K}$, compute $\rho_\ell = \left(B_{\kappa(m-1)} - B_\ell\right) / \left(c_{\kappa(m-1)} - c_\ell\right)$.
- (c) Define $\sigma_m = \max\{\rho_\ell : \ell \in \mathcal{K}\}$ and $\kappa(m) = \min\{\ell \in \mathcal{K} : \rho_\ell = \sigma_m\}$.
- (d) Let $T_m = f_{\kappa(m)}(\sigma_m) + c_{\kappa(m)} - \alpha B_{\kappa(m)}$.
- (e) If $T_m < 0$, then (2.9) has no solution satisfying Ansatz 3.11. **End**. Otherwise, continue to Step 6(f).
- (f) If $\kappa(m) = 1$, set $N = m$ and proceed to Step 7. Otherwise, increase m by 1 and return to Step 6(a).

Step 7. The QVI (2.9) has a solution satisfying Anstaz 3.11 given by $u(x) = B_1 - c_1 x$ for $x < \sigma_N$, $u(x) = B_{\kappa(m)} - c_{\kappa(m)} x$ for $\sigma_{m+1} \leq x < \sigma_m$ and $1 \leq m \leq N-1$, and (3.8) for $x \geq \sigma_1$. **End**.

Algorithm 3.18 is a refinement of Corollary 4.20 of [5]. In the aforementioned corollary the only candidate considered for the test value in Step 2 is $\zeta = 0$. Step 3 yields the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$ because $\Upsilon_\ell = y_\ell(\zeta)/\alpha$ for every $\ell \in \mathcal{J}$. The number B_ℓ defined for $\ell = j$ in Step 3 and for $\ell = 1, 2, \ldots, j-1$ in Step 5 is contrived so that (3.17) can be succinctly expressed as $v_\ell(x) = B_\ell - c_\ell x$ for $\ell \in \{1, 2, \ldots, j\}$. Step 6 extracts the partition (3.18) with the property that v defined by (3.16) is affine in each of the intervals (3.19) and not in the union of any two of them. The function $\kappa : \mathcal{M} = \{1, 2, \ldots, N\} \to \{1, 2, \ldots, j\}$ introduced with this step is a device for recording that $\ell \in \{1, 2, \ldots, j\}$ for which $v = v_\ell$ in I_m for every $m \in \mathcal{M}$.

When Algorithm 3.18 fulfils Step 7, the output is a generalized (s, S) policy as described in Section 1, in which the number N of suppliers equals N, and, $s_{(m)} = \sigma_m$ and $S_{(m)} = S_{j,\kappa(m)}$ for $m = 1, 2, \ldots, N$. Caution needs to be exercised in not confusing the labelling of the suppliers (m) with their ordering in $\mathcal{J}$ using the criteria (2.2) and (2.3). Supplier (m) is supplier ℓ where $\ell = \kappa(m)$ in the original line-up. In particular, supplier (1) is supplier j. Possibly $j < J$, in which case suppliers $j+1$ to J are excluded from the optimal policy. Also, possibly $\mathrm{N} < j$, in which case another selection of suppliers 2 to $j-1$ is excluded. Both of these possibilities are documented features of a generalized (s, S) policy [21], and corroborated by Example 4.25 in [5]. In general, the most that can be said with certainty is that suppliers (1) to (N) are to be found in the reverse order in the ranking (2.2) and (2.3), and that supplier (N) is supplier 1 in this ranking.

The pivotal supplier j in Propositions 3.16 and 3.17 and Algorithm 3.18 is the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$. This means that j is the supplier with the (s, S) policy that has the least cost when there is stock in hand and no orders are made, or, should there be more than one such supplier, that supplier among these that has the greatest value of s. Once j is known, so too is the optimal policy for inventory levels $x \geq s_j$, where s_j is the value of s in the (s, S) policy of supplier j. Therewith, the inventory level $S_{j,\ell}$ to which a possible order from supplier ℓ is necessarily made is determinable for every $\ell \in \{1, 2, \ldots, j\}$. These levels are unique and strictly decreasing with respect to ℓ. This is a consequence of the ordering of the values c_ℓ and the monotonicity properties of the auxiliary holding-cost functions f_ℓ embodied in Hypothesis 3.3. Each function v_ℓ gives the cost of placing an order with supplier ℓ to bring the inventory level from x up to $S_{j,\ell}$. Thus, in defining v as the minimum of v_ℓ one is identifying the least cost of replenishing from the inventory level x with respect to every supplier $\ell \in \{1, 2, \ldots, j\}$, and, the supplier (or suppliers for the cut-off levels σ_m for $m = 2, 3, \ldots, N$) that delivers this least cost. It may turn out that certain suppliers do not to deliver the least cost for any inventory level $x \leq s_j$. Such suppliers play no subsequent role in the optimal policy. The remaining suppliers, $\kappa(m)$ for $m = 1, 2, \ldots, N$, are so ordered by m that their set-up cost increases and their cost per item decreases as one proceeds through the sequence. Thus, within the regime of placing an order from a position of shortage, the optimal policy entails ordering from a supplier with a high set-up cost and low cost per item in preference to another with a lower set-up cost and higher cost per item if and only if the backlog is greater. This additionally explains why suppliers $j + 1$, $j + 2$, $\ldots$, J are excluded from the generalized (s, S) policy.

In the classical case that f is given by (2.4) with $p > \alpha c_J > \alpha c_1 > -q$, one can streamline the calculations of Step 1 of Algorithm 3.18 by using (3.9) and (3.10). In Step 2 it is easy to take $\zeta = 0$, leading to

$$\Upsilon_\ell = \{p - (p - \alpha c_\ell)\, \mathrm{e}^{\alpha s_\ell}\} / \alpha \quad \text{for } \ell \in \mathcal{J}. \tag{3.25}$$

Moreover, one can explicitly solve (3.23) to deduce

$$S_{j,\ell} = S_j + \ln\big\{(q + \alpha c_j) / (q + \alpha c_\ell)\big\}/\alpha. \tag{3.26}$$

The following concrete examples based on the above considerations confirm that the QVI (2.9) may indeed have a solution corresponding to an (s, S) policy, a solution corresponding to a generalized (s, S) policy, or no such solution.

Example 3.19. Let f be given by (2.4) with $p = q = 3$, $\alpha = 1$, $J = 2$, $k_1 = c_2 = 2$, and $k_2 = c_1 = 1$. Examples 3.7 and 3.8 provide $s_1 \approx -1.6831$, $S_1 \approx 0.3415$, $s_2 \approx -1.7672$, and $S_2 \approx 0.1534$. Formula (3.25) subsequently gives $\Upsilon_1 \approx 2.628$ and $\Upsilon_2 \approx 2.829$. Therefore $\Upsilon_1 < \Upsilon_2$, and, following Algorithm 3.18, $j = 1$. Thus, the QVI has a unique solution satisfying Ansatz 3.1. This corresponds to the (s, S) policy reported in Example 3.8.

Example 3.20. Substitute $k_1 = 6$ in Example 3.19. Example 3.10 gives $s_1 \approx -3.7959$, whereupon (3.25) yields $\Upsilon_1 \approx 2.955$. The values of s_2, S_2, and Υ_2 remain the same as in Example 3.19. Now, however, $\Upsilon_2 < \Upsilon_1$. Therefore, following Algorithm 3.18, $j = 2$. By (3.7) $B_2 \approx 3.767$, by (3.26) $S_{2,1} \approx 0.3766$, and, by (3.4) and (3.24) $B_1 \approx 8.506$. Running through Step 6 of Algorithm 3.18 subsequently delivers $\sigma_1 = s_2$, $\kappa(1) = 2$, $\sigma_2 = (B_2 - B_1) / (c_2 - c_1) \approx -4.7391$, $\kappa(2) = 1$, and $T_2 = f_1(\sigma_2) + c_1 - \alpha B_1 \approx 1.972$. Thus, the QVI has a unique solution satisfying Ansatz 3.11, which does not satisfy Ansatz 3.1. It is given by $u(x) = B_1 - x$ for $x < \sigma_2$, $u(x) = B_2 - 2x$ for $\sigma_2 \leq x < \sigma_1$, $u(x) \approx -0.171\, \mathrm{e}^{-x} - 3x + 3$ for $\sigma_1 \leq x \leq 0$, and $u(x) \approx 5.829\, \mathrm{e}^{-x} + 3x - 3$ for $x > 0$ (in accord with Example 3.7). This corresponds to a generalized (s, S) policy with two suppliers. In the notation used to describe such a policy in Section 1, supplier (1) is supplier 2 in the ordering (2.2) and (2.3), supplier (2) is supplier 1, $s_{(2)} = \sigma_2$, $s_{(1)} = \sigma_1$, $S_{(1)} = S_2$, and $S_{(2)} = S_{2,1}$.

Example 3.21. Substitute $k_1 = 4$ in Example 3.19. Example 3.9 gives $s_1 \approx -2.7687$, whereupon (3.25) yields $\Upsilon_1 \approx 2.875$, while the values of s_2, S_2, and Υ_2 are the same as in Example 3.19. As in Example 3.20, $\Upsilon_2 < \Upsilon_1$, and therefore $j = 2$. Furthermore, since c_1, c_2 and k_2 are the same as in Example 3.20, so too are B_2 and $S_{2,1}$. However, with the different value of k_1, (3.4) and (3.24) give $B_1 \approx 6.506$. Now running through Step 6 of Algorithm 3.18 supplies $\sigma_1 = s_2$, $\kappa(1) = 2$, $\sigma_2 = (B_2 - B_1) / (c_2 - c_1) \approx -2.7391$, $\kappa(2) = 1$, and $T_2 = f_1(\sigma_2) + c_1 - \alpha B_1 \approx -0.028$. It follows that the QVI has no solution satisfying Ansatz 3.11, and therefore no optimal control in the form of a generalized (s, S) policy let alone an (s, S) policy.

In the next section it will be shown that admission of a hyper-generalized (s, S) policy resolves the unsatisfactory outcome of Example 3.21.

4. The hyper-generalized (s, S) policy

Ansatz 3.1 epitomizes an (s, S) policy and Ansatz 3.11 a generalized (s, S) policy. The fact, stated in Proposition 3.17 and corroborated by Example 3.21, that there are situations in which the QVI (2.9) does not admit a solution satisfying either ansatz raises a number of questions. Does this mean that the QVI is unsolvable? Are the ansätze really not a good premise? If so, what could sensibly take their place? Will a more general premise deliver a unique solution? In short, is there an optimal control policy, that has features acceptable for practical applications, and that applies in those situations where Proposition 3.17 fails to deliver a solution of the QVI? We shall answer these questions in this section by extending the concept of a generalized (s, S) policy.

4.1. Ansatz

We entertain an optimal inventory control policy u with an interval (s, ∞) for some number s, in which no replenishment is necessary and thus $u < Mu$ and $Au = f$, as in the cases of an (s, S) policy and a generalized (s, S) policy, and a complementary interval $(-\infty, s]$ comprising stopping intervals in which $u = Mu$ interspersed with a limited number of bounded subintervals in which replenishment does not take place. Such subintervals are excluded in a conventional generalized (s, S) policy [21].

Ansatz 4.1. The solution of (2.9) is a continuous real function u such that $u = Mu$ on $(-\infty, s] \setminus \mathcal{S}$, where $\mathcal{S}$ is the union of a finite number of bounded open subintervals of $(-\infty, s)$, and, u is differentiable, $Au = f$, and $u < Mu$ in $\mathcal{S} \cup (s, \infty)$, for some number s.

The previous Ansatz 3.11 may be viewed as the confinement of the new ansatz to the case that the number of subintervals comprising $\mathcal{S}$ is zero.

If u satisfies Ansatz 4.1 and not Ansatz 3.11, there is a natural number L with real numbers

$$a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_L < b_L \leq a_{L+1} = s, \tag{4.1}$$

such that

$$u < Mu \quad \text{in } \mathcal{S} = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_L, b_L), \tag{4.2}$$

and

$$u = Mu \quad \text{in } (-\infty, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_L, a_{L+1}]. \tag{4.3}$$

Using Ansatz 4.1, we shall realize the following counterpart to Proposition 3.12.

Theorem 4.2. *Suppose that f is continuous on $\mathbb{R}$. Then, under Ansatz 4.1, a solution u of (2.9) is such that $u = y$ in $[s, \infty)$, $y \leq u \leq v$ in $(-\infty, s)$, and $u < v$ in $\mathcal{S}$, where y and v are as in Proposition 3.12.*

The proof of Theorem 4.2 proceeds along the lines of that of Theorem 4.2 in [5] through a sequence of lemmas. The first four are independent of admission of the set $\mathcal{S}$ in the ansatz, and taken directly from [5], where their proof may be found. Throughout this subsection, u will be assumed to be a given solution of (2.9) satisfying Ansatz 4.1. For $\ell \in \mathcal{J}$ and $x \in \mathbb{R}$, we define

$$u_\ell(x) = u(x) + c_\ell x. \tag{4.4}$$

Lemma 4.3. *If $u(x) = \big(M_\ell u\big)(x)$ then u_ℓ has a least absolute minimum on $[x, \infty)$ at an $X > x$ such that $u_\ell(x) = u_\ell(X) + k_\ell$ and $u_\ell(z) \leq u_\ell(x)$ for all $z \leq X$.*

Lemma 4.4. *Suppose that $u = Mu$ in the proper interval $[b, a]$. Then u is concave on $[b, a]$.*

Lemma 4.5. *Suppose that $u = Mu$ on $[\eta, x]$ for some $\eta < x$. Let ℓ be the smallest number in $\mathcal{J}$ such that $u(x) = \big(M_\ell u\big)(x)$. Then u_ℓ is constant and $u = M_\ell u$ on $[w, x]$ for some $w \in [\eta, x)$.*

Lemma 4.6. *Suppose that $u = Mu$ on $[x, \eta]$ for some $\eta > x$. Let ℓ be the largest number in $\mathcal{J}$ such that $u(x) = \big(M_\ell u\big)(x)$. Then u_ℓ is constant and $u = M_\ell u$ on $[x, z]$ for some $z \in (x, \eta]$.*

Lemma 4.7. *Suppose that $u(a) = \big(Mu\big)(a)$ and $u < Mu$ in (a, b) for some $a < b$. Then u is the restriction to $[a, b]$ of a solution of (3.1), and is continuously differentiable in (w, b) for some $w < a$.*

Proof. Because u satisfies $Au = f$ in (a, b), it is a solution of (3.1) there. Furthermore, as f is continuous on $\mathbb{R}$, this solution can be extended to one, y say, on $\mathbb{R}$. The continuity of u and y on $\mathbb{R}$ gives $u = y$ on $[a, b]$. To proceed, we distinguish between the cases $u < Mu$ and $u = Mu$ in (η, a) for some $\eta < a$. In the first case, $Au = f$ in (η, a). As solutions of the initial-value problem for equation (3.1) are unique, this necessitates $u = y$ in (η, b). Hence, u is continuously differentiable in (η, b). For the alternative case, let ℓ be the smallest number in $\mathcal{J}$ for which $u(a) = \big(M_\ell u\big)(a)$. Then Lemma 4.5 tells us that u_ℓ is constant on $[w, a]$ for some $w \in [\eta, a)$. Hence, $y'(x) + \alpha y(x) = f(x) \geq \big(Au\big)(x) = u'(x) + \alpha u(x) = \alpha u(x) - c_\ell$ for $x \in (w, a)$. Passage to the limit $x \to a$ gives $y'(a) \geq -c_\ell$. On the other hand, by Lemma 4.3, there is an $X > a$ such that a is a maximum of u_ℓ in $[a, X]$. This implies that $y'(a) = \big(D^+u\big)(a) = \big(D^+u_\ell\big)(a) - c_\ell \leq -c_\ell$. Hence, combining the two inequalities comparing $y'(a)$ to $-c_\ell$, we obtain $y'(a) = -c_\ell$, from which it follows that u is continuously differentiable in (w, b). ☐

Lemma 4.8. *Further to Lemma 4.7, $u(a) = \big(M_\ell u\big)(a)$ for a unique $\ell \in \mathcal{J}$, and $u'(a) = -c_\ell$.*

Proof. Let $\ell \in \mathcal{J}$ be such that $u(a) = \big(M_\ell u\big)(a)$. By Lemma 4.3, there is an $X > a$ such that a is a maximum of u_ℓ in $(-\infty, X]$. By Lemma 4.7, u is differentiable at a. Consequently, the Fermat Theorem says that $u_\ell'(a) = 0$. In other words, $u'(a) = -c_\ell$. This conclusion cannot be true for more than one $\ell \in \mathcal{J}$. ☐

Proof of Theorem 4.2. By Lemma 4.7, equation (3.1) has a solution y such that $u = y$ on $[s, \infty)$. By Lemma 4.8, (3.13) holds for some $j \in \mathcal{J}$ for which $y'(s) = -c_j$. By Lemma 4.3, u_j has a minimum in $[s, \infty)$ at $S > s$ satisfying $u_j(s) = u_j(S) + k_j$. Inasmuch u_j is differentiable at S, the Fermat Theorem tells us that $u_j'(S) = 0$. Hence, (3.2) and (3.3) hold. The argument used to prove Lemma 4.9 of [5] yields $y \leq u$ in $(-\infty, s]$. It subsequently remains to show that $u \leq v$ in $(-\infty, s)$, and $u < v$ in $\mathcal{S}$, where v is defined by (3.12). To this end, let $x < s$. Then $\big(M_\ell u\big)(x) = k_\ell - c_\ell x + \min\{u_\ell(\eta) : \eta \geq x\} \leq k_\ell - c_\ell x + \min\{u_\ell(\eta) : \eta \geq s\} = \big(M_\ell u\big)(s) + c_\ell\,(s - x)$ for every $\ell \in \mathcal{J}$. Hence,

$$u(x) \leq \big(Mu\big)(x) = \min\{\big(M_\ell u\big)(x) : \ell \in \mathcal{J}\} \leq v(x). \tag{4.5}$$

Moreover, the first inequality in (4.5) is strict for $x \in \mathcal{S}$. In view of the arbitrariness of x, this delivers the desired result. ☐

Comparing Theorem 4.2 to Proposition 3.12, the price that has been paid for the enlargement of Ansatz 3.11 to Ansatz 4.1 is weakening of the conclusion $u = v$ in $(-\infty, s)$, to, $u \leq v$ in $(-\infty, s) \setminus \mathcal{S}$. The next lemma – which we introduce because we shall employ it further on, and not just to make the present point – indicates that ignorance of the concavity of u on $(-\infty, s]$ is the heart of the matter. In the event that u satisfies Ansatz 3.11, this ignorance is removed by Lemma 4.4.

Lemma 4.9. *If u is concave on $(-\infty, s]$ then $u = v$ in $(-\infty, s] \setminus \mathcal{S}$.*

Proof. Fix $x \in (-\infty, s] \setminus \mathcal{S}$. Let ℓ be the largest number in $\mathcal{J}$ such that $u(x) = \big(M_\ell u\big)(x)$. Because u is concave on $(-\infty, s]$, so too is u_ℓ. By Lemmas 4.6 and 4.8, $\big(D^+u_\ell\big)(x) = 0$. Hence, u_ℓ is non-increasing on $[x, s]$. This implies that $u(x) = \big(M_\ell u\big)(x) = k_\ell - c_\ell x + \min\{u_\ell(\eta) : \eta \geq x\} = k_\ell - c_\ell x + \min\{u_\ell(\eta) : \eta \geq s\} = \big(M_\ell u\big)(s) + c_\ell\,(s - x) \geq v(x)$. Consequently, recalling (4.5), $u(x) = v(x)$. ☐

Although just as superfluous to the proof of Theorem 4.2 as Lemma 4.9, we further record the next lemma for future use.

Lemma 4.10. *Suppose that $u < Mu$ in (a, b) and $u(b) = \big(Mu\big)(b)$ for some $a < b$. Then u has a right derivative D^+u and a left derivative D^-u at b. Furthermore, $\big(D^+u\big)(b) = -c_{\ell^+} \leq -c_{\ell^-} \leq \big(D^-u\big)(b)$, where ℓ^+ and ℓ^- are respectively the largest and smallest $\ell \in \mathcal{J}$ for which $u(b) = \big(M_\ell u\big)(b)$.*

Proof. If $u < Mu$ in (b, η) for some $\eta > b$, then the conclusions are provided by Lemmas 4.7 and 4.8. On the other hand, if $u = Mu$ in such an interval, Lemma 4.6 implies that $\big(D^+u\big)(b)$ exists and equals $-c_{\ell+}$. Furthermore, by Lemma 4.7, u is the restriction to $[a, b]$ of a solution y of (3.1). However, Lemma 4.3 tells us that b is a maximum of $u_{\ell-}$ in $(-\infty, b]$. Consequently, $\big(D^-u\big)(b) = y'(b) = \big(D^-u_{\ell-}\big)(b) - c_{\ell-} \geq -c_{\ell-}$. We obtain $-c_{\ell-} \geq -c_{\ell+}$ from (2.3). $\square$

4.2. Uniqueness

To proceed, we revive the supposition that each of the auxiliary functions f_j defined by (3.4) satisfies Hypothesis 3.3. Hereupon, Theorem 4.2 tells us that a solution u of the QVI fulfilling Ansatz 4.1 is such that $s = s_j$ and $u = y_j$ on $[s_j, \infty)$ for some $j \in \mathcal{J}$. Furthermore,

$$y_j \leq u \leq v \quad \text{ in } (-\infty, s_j), \tag{4.6}$$

where v is defined by (3.12). By Lemma 3.14, there is only one $j \in \mathcal{J}$ for which these properties and $u \leq Mu$ on $\mathbb{R}$ can hold, namely the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$.

By (3.12), v is piecewise linear and concave on $\mathbb{R}$. Furthermore, (3.12) and (3.13) imply that $D^+v = D^-v = -c_j$ in (w, s_j) for some $w < s_j$. Consequently, $\ell \in \{j+1, j+2, \ldots, J\}$ plays no role in the definition of $v(x)$ for $x < s_j$. It follows that with no loss of generality we may suppose that v is defined by (3.15)–(3.17). Subsequently, there is a partition (3.18) with the property that v is affine in each of the intervals (3.19) and not in the union of any two of them. Furthermore, (3.20) holds.

The function v possesses additional characteristics that are relevant. We make these the content of the next four lemmas, in which

$$\mathcal{M} = \{1, 2 \ldots, N\}.$$

Lemma 4.11. *Let $m \in \mathcal{M}$ and T_m be as in (3.21). Then the following alternatives are mutually exclusive.*

(a) *$T_m \geq 0$, and $Av < f$ in I_m*
(b) *$T_m < 0$, $m \geq 2$, $Av < f$ in $I_m \cap (-\infty, \eta_m)$, $Av = f$ at η_m, and $Av > f$ in (η_m, σ_m) for some $\eta_m \in I_m$.*
(c) *$T_m < 0$, $2 \leq m \leq N-1$, and $Av > f$ in I_m.*

Proof. Let $\ell \in \{1, 2, \ldots, j\}$ be such that $v = v_\ell$ in I_m. Then

$$\big(f - Av_\ell\big)(x) = f(x) + c_\ell - \alpha v_\ell(x) = f_\ell(x) - f_\ell(\sigma_m) + T_m$$

for all $x \in \mathbb{R}$. By Hypothesis 3.3, f_ℓ is strictly decreasing on $(-\infty, \gamma_\ell]$, and, $f_\ell(x) \to \infty$ as $x \to -\infty$. By (3.14) and (3.18), $\sigma_m \leq s_j < \gamma_j \leq \gamma_\ell$. Thus, $f - Av_\ell$ is continuous and strictly decreasing on $(-\infty, \sigma_m]$, and $\big(f - Av_\ell\big)(x) \to \infty$ as $x \to -\infty$. It follows that if $T_m \geq 0$ then $Av_\ell < f$ in $(-\infty, \sigma_m)$. On the other hand, if $T_m < 0$, then there is an $\eta_m < \sigma_m$ such that $Av_\ell < f$ in $(-\infty, \eta_m)$, $Av_\ell = f$ at η_m, and $Av_\ell > f$ in (η_m, σ_m). This case can be further divided into the subcases $\eta_m \in I_m$ and $\eta_m \notin I_m$. In view of (3.19) and (3.22), the three alternatives in the statement of the lemma follow. $\square$

Lemma 4.12. *A solution y of equation (3.1) is strictly concave on the interval $(-\infty, b]$ if $b \leq s_j$ and $y'(b) \geq -c_j$*

Proof. As y solves (3.1),

$$y(x) = \mathrm{e}^{\alpha(z-x)} y(z) - \int_x^z \mathrm{e}^{\alpha(\eta - x)} f(\eta)\, \mathrm{d}\eta$$

for all $x < z$. Multiplying by α, eliminating αy using (3.1), and, then integrating by parts, yields

$$y'(x) = \mathrm{e}^{\alpha(z-x)} y'(z) - \int_x^z \mathrm{e}^{\alpha(\eta - x)}\, \mathrm{d}f(\eta).$$

Hence

$$y'(x) = y'(z) + \{y'(z) + c_j\}\left\{e^{\alpha(z-x)} - 1\right\} - \int_x^z e^{\alpha(\eta-x)}\,df_j(\eta). \tag{4.7}$$

Since f_j is strictly decreasing on $(-\infty, \gamma_j]$ by Hypothesis 3.3, $b \le s_j < \gamma_j$, and $y'(b) \ge -c_j$, substitution of $z = b$ in (4.7) gives $y'(x) \ge -c_j$ for all $x \le b$. Whence, applying (4.7) with arbitrary $x < z \le b$ in the knowledge that $y'(z) \ge -c_j$, we deduce that $y'(x) > y'(z)$ for such x and z. This confirms the strictly concavity of y on $(-\infty, b]$. □

Lemma 4.13. *Suppose that $Av = f$ at $\eta_m \in I_m$ for some $m \in \mathcal{M}$. Let Y_m be the solution of (3.1) satisfying*

$$Y_m(\eta_m) = v(\eta_m). \tag{4.8}$$

Then

$$Y_m'(\eta_m) = v'(\eta_m). \tag{4.9}$$

Furthermore, given that $v(\eta_m) \ge y_j(\eta_m)$, there exists a $\zeta_m \in (\sigma_m, s_j]$ such that

$$Y_m < v \text{ in } (\eta_m, \zeta_m), \quad Y_m = v \text{ at } \zeta_m, \tag{4.10}$$

Y_m is strictly concave on $[\eta_m, \zeta_m]$, and

$$Y_m'(\zeta_m) \ge \big(D^- v\big)(\zeta_m). \tag{4.11}$$

Proof. By Lemma 4.11, necessarily $T_m < 0$ and $m \ge 2$. Hence, $j \ge 2$ and $v'(\eta_m) = -c_\ell > -c_j$ for some $\ell \in \{1, 2, \ldots, j-1\}$. Inasmuch Y_m solves (3.1), $Y_m'(\eta_m) = \big(f - \alpha Y_m\big)(\eta_m)$. Thus, by the opening supposition of the lemma, $Y_m'(\eta_m) = \big(Av - \alpha Y_m\big)(\eta_m)$. Whence, by (4.8), $Y_m'(\eta_m) = \big(Av - \alpha v\big)(\eta_m)$. In other words, (4.9) holds. Consequently, by the continuous differentiability of Y_m, there is a $b \in (\eta_m, \sigma_m]$ such that $Y_m'(b) > -c_j$. Lemma 4.12 then tells us that Y_m is strictly concave on $(-\infty, b]$. As v is affine in I_m, and, (4.8) and (4.9) hold, this implies that $Y_m < v$ in $(\eta_m, b]$. On the other hand, given that $v(\eta_m) \ge y_j(\eta_m)$, (4.8) implies $Y_m(\eta_m) \ge y_j(\eta_m)$. Moreover, because $Y_m - y_j$ is a solution of $y' + \alpha y = 0$ on $\mathbb{R}$, there holds $\big(Y_m - y_j\big)(x) = \big(Y_m - y_j\big)(\eta_m)\,e^{\alpha(\eta_m - x)}$ for all $x \in \mathbb{R}$. Thus, $Y_m \ge y_j$ on $\mathbb{R}$. Recalling that $v(s_j) = y_j(s_j)$, it follows from the continuity of Y_m, v, and y_j that there has to be a $\zeta_m \in (b, s_j]$ such that (4.10) holds. This necessitates (4.11). Hence, $Y_m'(\zeta_m) \ge -c_j$. So, by Lemma 4.12, Y_m is strictly concave on $[\eta_m, \zeta_m]$. Since v is affine on $[\eta_m, \sigma_m]$ and (4.8) and (4.9) hold, this excludes $\zeta_m \le \sigma_m$. □

Lemma 4.14. *Further to Lemma 4.13,*

$$Y_m'(\zeta_m) \ge \big(D^+ v\big)(\zeta_m) \tag{4.12}$$

with equality if and only if v is differentiable and $Av = f$ at ζ_m.

Proof. The concavity of v and (4.11) imply (4.12). Moreover, they imply strictness if $\zeta_m = \sigma_\lambda$ for some $\lambda \in \mathcal{M} \setminus \{1\}$. On the other hand, if $\zeta_m = s_j$ or $\zeta_m \in I_\lambda$ for some $\lambda \in \mathcal{M}$, then v is differentiable at ζ_m. In this event, by (4.10) and (4.12), $Av = D^+ v + \alpha v \le Y_m' + \alpha Y_m = f$ at ζ_m with equality if and only if (4.12) holds with equality. □

Let us now again pick up the main thread, supposing that u is a solution of the QVI (2.9) satisfying Ansatz 4.1.

Lemma 4.15. *The function u is concave on $(-\infty, s_j]$.*

Proof. As Lemma 4.4 covers the result when u satisfies Ansatz 3.11, let us assume that u does not. In this circumstance, (4.1)–(4.3) apply for some $L \geq 1$. By Lemma 4.4, u is concave on $(-\infty, a_1]$, and, on $[b_\nu, a_{\nu+1}]$ for every $\nu \in \{1, 2, \ldots, L\}$. By Lemma 4.7, u is differentiable at a_ν for every $\nu \in \{1, 2, \ldots, L+1\}$. On the other hand, by Lemma 4.10,

$$\big(D^- u\big)(b_\nu) \geq \big(D^+ u\big)(b_\nu) \tag{4.13}$$

for every $\nu \in \{1, 2, \ldots, L\}$. Armed with this information, we prove that u is concave on $(-\infty, s_j]$ with the aid of induction. The seed is the observation that $a_{L+1} = s_j$. Supposing that u is concave on $[a_{\nu+1}, s_j]$ for some $\nu \in \{1, 2, \ldots, L\}$, the concavity of u on $[b_\nu, a_{\nu+1}]$ and the existence of $u'(a_{\nu+1})$ imply that u is concave on $[b_\nu, s_j]$. Hence, $\big(D^+ u\big)(b_\nu) \geq u'(s_j) = -c_j$. *Via* (4.13), this leads to $\big(D^- u\big)(b_\nu) \geq -c_j$. However, by Lemma 4.7, u is the restriction to $[a_\nu, b_\nu]$ of a solution y of (3.1) with $y'(b_\nu) = \big(D^- u\big)(b_\nu)$. So, $y'(b_\nu) \geq -c_j$. Consequently, Lemma 4.12 says that y is concave on $(-\infty, b_\nu]$. Thus, u is concave on $[a_\nu, b_\nu]$. Whereupon, the concavity of u on $[a_\nu, b_\nu]$, (4.13), and the concavity of u on $[b_\nu, s_j]$ imply that u is concave on $[a_\nu, s_j]$. This argument can be repeated until we arrive at the conclusion that u is concave on $[a_1, s_j]$. Hereupon, the concavity of u on $(-\infty, a_1]$ and the existence of $u'(a_1)$ imply that u is concave on $(-\infty, s_j]$. ☐

Together Theorem 4.2, Lemma 4.9 and Lemma 4.15 provide a crucial characterization of the subset of $(-\infty, s]$ where $u = Mu$, and its complement. To be precise, they tell us that

$$u = v \text{ in } (-\infty, s] \setminus \mathcal{S} \quad \text{ and } \quad u < v \text{ in } \mathcal{S}. \tag{4.14}$$

We use this precision to fully identify $\mathcal{S}$. This identification entails four more lemmas.

Lemma 4.16. *Let $x < s_j$ be such that $u(x) = v(x)$. Then*

$$\big(D^- u\big)(x) \geq \big(D^- v\big)(x) \geq \big(D^+ v\big)(x) = \big(D^+ u\big)(x).$$

Proof. Since $u \leq v$ in $(-\infty, s_j]$ and $u(x) = v(x)$,

$$\big(D^- u\big)(x) \geq \big(D^- v\big)(x) \quad \text{ and } \quad \big(D^+ v\big)(x) \geq \big(D^+ u\big)(x). \tag{4.15}$$

By the concavity of v,

$$\big(D^- v\big)(x) \geq \big(D^+ v\big)(x). \tag{4.16}$$

If now $u = v$ in (x, z) for some $z \in (x, s_j]$, we have equality on the right-hand side of (4.15). On the other hand, if $u < v$ in (x, z) for some $z \in (x, s_j]$, Lemma 4.7 tells us that $\big(D^- u\big)(x) = \big(D^+ u\big)(x)$. So we have equality throughout (4.15) and (4.16). ☐

Lemma 4.17. *Suppose that $u(a) = v(a)$ and $u < v$ in (a, b) for some $a < b \leq s_j$. Then $a = \eta_m$ for some $m \in \mathcal{M}$ for which alternative (b) of Lemma 4.11 applies, and, $u = Y_m$ in $[a, \zeta_m]$, where ζ_m and Y_m are as in Lemma 4.13.*

Proof. If $a = \sigma_m$ for some $m \geq 2$, then $u(a) = \big(M_\ell u\big)(a)$ for at least two $\ell \in \{1, 2, \ldots, j\}$. This contradicts Lemma 4.8. Hence, $a \in I_m$ for some $m \in \mathcal{M}$, which implies that v is differentiable at a. By Lemma 4.7, u is continuously differentiable at a, and the restriction to $[a, b]$ of a solution y of (3.1) with $y(a) = v(a)$. Lemma 4.16 subsequently implies $y'(a) = v'(a)$. Because y is a solution of (3.1), there holds $Ay = f$ on $\mathbb{R}$. So, $Av = Ay = f$ at a. Consequently, alternative (b) of Lemma 4.11 must apply, and necessarily $a = \eta_m$. Whence, by (4.6) and Lemma 4.13, $y = Y_m$. Whereupon, we conclude that $u = Y_m$ in $[\eta_m, \zeta_m]$. ☐

Lemma 4.18. *Suppose that $u(x) = v(x)$ for some $x \in I_m$ and $m \in \mathcal{M}$. If $T_m \geq 0$ then $u = v$ in $[x, \sigma_m]$. On the other hand, if $T_m < 0$ then alternative (b) of Lemma 4.11 applies, $x \leq \eta_m$, $u = v$ in $[x, \eta_m]$, and $u = Y_m$ in $(\eta_m, \zeta_m]$, where ζ_m and Y_m are as in Lemma 4.13.*

Proof. Let $\Omega = \{z \in [x, \sigma_m) : u(z) = v(z)\}$. By Lemma 4.16 and the affinity of v in I_m, $D^+u = v'$ in Ω. However, inasmuch u is a solution of (2.9), $D^+u + \alpha u \leq f$ everywhere. So, $Av \leq f$ in Ω. By Lemma 4.11 this is possible only if either $T_m \geq 0$, or, $T_m < 0$, $Av = f$ at $\eta_m \in I_m$, and $\Omega \subseteq [x, \eta_m]$. On the other hand, by Lemma 4.17, there is an $a \in \Omega$ and a $b \in (a, s_j]$ such that $u(a) = v(a)$ and $u < v$ in (a, b) only if $T_m < 0$ and $a = \eta_m$. Thus, if $T_m \geq 0$ then Ω is an interval with right endpoint σ_m, whereas if $T_m < 0$ then it is an interval with right endpoint η_m. In the former case, the continuity of u and v yields $u = v$ in $\overline{\Omega}$. In the latter case, $u = Y_m$ in $(\eta_m, \zeta_m]$, by Lemma 4.17. □

Lemma 4.19. *Lemma 4.18 extends to $x = \sigma_{m+1}$ for $m \in \mathcal{M} \setminus \{N\}$.*

Proof. Lemma 4.17 tells us that if $u(\sigma_{m+1}) = v(\sigma_{m+1})$ for $m \in \mathcal{M} \setminus \{N\}$, then $u = v$ in $[\sigma_{m+1}, z]$ for some $z \in I_m$. Subsequent application of Lemma 4.18 with z in the place of x gives the result. □

We are now in a position to establish the uniqueness of a solution u of the QVI (2.9) fulfilling Ansatz 4.1. By the opening remarks of this subsection, necessarily $s = s_j$ and $u = y_j$ in $[s_j, \infty)$, where j is the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$. Defining v by (3.16) and (3.17), the partition (3.18) such that v is affine in each of the intervals (3.19) and not in the union of any two of them, and T_m by (3.21) for $m \in \mathcal{M}$; (4.14) specifies that $u \leq v$ in $(-\infty, s)$ with equality if and only if $u = Mu$. By Ansatz 4.1, $u(x) = \big(Mu\big)(x)$ for large enough $-x$. Consequently, Lemmas 4.18 and 4.19 imply that $u = v$ in $(-\infty, s_j]$ if $T_m \geq 0$ for every $m \in \mathcal{M}$. If not, $u = v$ in $(-\infty, a_1]$, $u = Y_{\mu(1)} < v$ in (a_1, b_1), and $u(b_1) = v(b_1)$; where $a_1 = \eta_{\mu(1)} < \sigma_{\mu(1)} < b_1 = \zeta_{\mu(1)} \leq s_j$ and $\mu(1)$ is the largest $m \in \mathcal{M}$ for which $T_m < 0$. Subsequently, if $T_m \geq 0$ for every $m \in \mathcal{M}$ with $\sigma_m \geq b_1$, these lemmas imply that $u = v$ in $[b_1, s_j]$. If not, they imply that $u = v$ in $[b_1, a_2]$, $u = Y_{\mu(2)} < v$ in (a_2, b_2), and $u(b_2) = v(b_2)$, where $b_1 \leq a_2 = \eta_{\mu(2)} < \sigma_{\mu(2)} < b_2 = \zeta_{\mu(2)} \leq s_j$ and $\mu(2)$ is the largest $m \in \mathcal{M}$ for which $\sigma_m \geq b_1$ and $T_m < 0$. Proceeding in this fashion, we find a combination (4.1) such that $u = v$ in $(-\infty, s_j] \setminus \mathcal{S}$, where $\mathcal{S}$ is as in (4.2), and $u = Y_{\mu(\nu)}$ in (a_ν, b_ν) for $1 \leq \nu \leq L$. The property (3.22) ensures that the inductive process terminates neatly. For completeness, one may take $L = 0$ and $\mathcal{S} = \emptyset$ when $u = v$ on $(-\infty, s_j]$.

The essence of the preceding paragraph is the following.

Theorem 4.20. *Suppose that f_ℓ satisfies Hypothesis 3.3 for every $\ell \in \mathcal{J}$. Then (2.9) has at most one solution u satisfying Ansatz 4.1.*

Corollary 4.21. *Except in a finite subset, u is continuously differentiable on $\mathbb{R}$. This subset comprises those b_ν for which $b_\nu < a_{\nu+1}$ and those σ_m with $m \geq 2$ for which $u(\sigma_m) = v(\sigma_m)$. The derivative of u has a jump discontinuity at such places.*

Proof. By Lemma 4.14, $D^+u = D^-u$ at b_ν for $1 \leq \nu \leq L$ if and only if v is differentiable and $Av = f$ at b_ν. However, by Lemmas 4.11 and 4.18, this is so if and only if $a_{\nu+1} = b_\nu$. Lemma 4.16 yields $D^+u < D^-u$ at those $\sigma_m < s_j$ for which $u(\sigma_m) = v(\sigma_m)$. Conversely, u is continuously differentiable in (w, ∞) for some $w < s_j$ by Theorem 4.2 and Lemma 4.7, in (w, b_ν) for some $w < a_\nu$ for every $1 \leq \nu \leq L$ by Lemma 4.7, and, in any open subinterval of I_m in which $u = v$ for $m \in \mathcal{M}$ by the piecewise linearity of v. Since places where u is not continuously differentiable are such that $D^+u \neq D^-u$ and isolated, standard calculus leads to the deduction that D^+u is the limit of u' from the right, and D^-u is the limit of u' from the left at each such place. □

Corollary 4.22. *There holds $Au < f$ in $(I_1 \cup I_2 \cup \cdots \cup I_N) \setminus \mathcal{S}$.*

Proof. This follows from Lemmas 4.11, 4.18, and 4.19. □

4.3. Existence

Thanks to the content of the paragraph running up to Theorem 4.20, we know precisely how to construct a solution of the QVI (2.9) assuming that such exists. To prove existence, it subsequently suffices to attend to two issues. The first is to ascertain that the construction can be carried out without *a priori* assuming that the resulting function is a solution. The second is to show that the constructed function verily solves the QVI.

With regard to the first issue, Lemma 3.15 holds irrespective of any other considerations. Hence, v is definable *via* (3.16) and (3.17). As this leads to piecewise linearity and concavity on $\mathbb{R}$, the partition (3.18) with the property that v is affine in each of the intervals (3.19) and not in the union of any two of them, the set $\mathcal{M}$, and the number (3.21) for every $m \in \mathcal{M}$ follow. Thus there is no obstacle to equating $u = v$ where warranted. The difficulty is in nailing the sequence (4.1). We muster two lemmas to accomplish this.

Lemma 4.23. *Suppose that $x < s_j$ is such that $\sigma_m \geq x$ and $T_m < 0$ for some $m \in \mathcal{M}$. Denote the largest such m by μ. Then alternative (b) of Lemma 4.11 applies to μ, and $x \leq \eta_\mu$, in the following circumstances.*

(i) *The function v is differentiable and $Av \leq f$ at x.*
(ii) *There is a solution y of equation (3.1) such that $y < v$ in (w, x) for some $w < x$, $y(x) = v(x)$, and $y'(x) \geq \big(D^- v\big)(x)$.*

Proof. In case (ii), if v is differentiable at x, then $Av \leq Ay = f$ at x, leading to case (i). On the other hand, if v is not differentiable at x, then $x = \sigma_\lambda$ for some $\lambda \in \mathcal{M} \setminus \{1\}$, with $\lambda \leq \mu$ and $T_\lambda = \big(f - D^- v - \alpha v\big)(\sigma_\lambda) \geq \big(f - y' - \alpha y\big)(x) = 0$. Hence, $\lambda < \mu$, and $\big(f - D^+ v - \alpha v\big)(\sigma_\lambda) > 0$. Therefore, there is a $z \in I_{\lambda-1}$ for which $\big(f - Av\big)(z) > 0$ and μ is the largest $m \in \mathcal{M}$ with $\sigma_m \geq z$ and $T_m < 0$. So case (i) applies to z. Consequently, it suffices to prove the lemma in case (i). Suppose therefore that (i) is the case, and, to the contrary of the statement of the lemma, that alternative (c) of Lemma 4.11 applies to μ. Then $x < \sigma_{\mu+1}$, and, $\big(f - D^+ v - \alpha v\big)(\sigma_{\mu+1}) \leq 0$. Hence, $T_{\mu+1} = \big(f - D^- v - \alpha v\big)(\sigma_{\mu+1}) < 0$, which contradicts the definition of μ. Thus alternative (b) of Lemma 4.11 must apply. The details of that alternative imply $x \leq \eta_\mu$. □

Lemma 4.24. *There holds $v \geq y_j$ in $(-\infty, s_j]$.*

Proof. Let $x \leq s_j$ and $\ell \in \mathcal{M}$ be such that $v(x) = v_\ell(x)$. Then, by (3.17),

$$v(x) = y_j(z) + k_\ell + c_\ell\,(z - x)\,, \tag{4.17}$$

where $z = S_{j,\ell}$. By (3.5), the function $\eta \mapsto y_\ell(\eta) + c_\ell \eta$ is strictly increasing on $(-\infty, s_\ell]$, strictly decreasing on $[s_\ell, S_\ell]$, and strictly increasing on $[S_\ell, \infty)$. Therefore,

$$\{y_\ell(x) + c_\ell x\} - \{y_\ell(z) + c_\ell z\} \leq \{y_\ell(s_\ell) + c_\ell s_\ell\} - \{y_\ell(S_\ell) + c_\ell S_\ell\} = k_\ell. \tag{4.18}$$

Using (4.18) to eliminate k_ℓ from (4.17) gives $v(x) \geq y_j(z) + y_\ell(x) - y_\ell(z)$. Hence,

$$v(x) - y_j(x) \geq y_j(z) - y_\ell(z) - y_j(x) + y_\ell(x) = \int_x^z \big(y_j' - y_\ell'\big)(\eta)\,\mathrm{d}\eta.$$

Inasmuch y_j and y_ℓ are both solutions of (3.1), the above is equivalent

$$v(x) - y_j(x) \geq \alpha \int_x^z \big(y_\ell - y_j\big)(\eta)\,\mathrm{d}\eta.$$

Because y_j is the greatest minimizer of y_ℓ with respect to $\ell \in \mathcal{J}$, the last integral is nonnegative. In view of the arbitrariness of x, this provides the proof. □

By Lemma 4.11 parts (a) and (b), $f > Av$ in $(-\infty, x)$ for $x < \sigma_N$ if $-x$ is sufficiently large. Hence, if $T_m < 0$ for some $m \in \mathcal{M}$, Lemma 4.23 implies that alternative (b) of Lemma 4.11 applies to $\mu(1)$. Thus $a_1 = \eta_{\mu(1)}$ is well defined. Supposing that $a_\nu = \eta_{\mu(\nu)}$ exists for some $\nu \geq 1$, Lemmas 4.13 and 4.24 yield the existence of $b_\nu = \zeta_{\mu(\nu)} \in (a_\nu, s_j]$ and of the solution $Y_{\mu(\nu)}$ of equation (3.1) satisfying the conclusions of Lemma 4.13. Subsequently, either $T_m \geq 0$ for every $m \in \mathcal{M}$ with $\sigma_m \geq \mu$, or, Lemma 4.23 yields the existence of $\mu(\nu + 1)$ for which $a_{\nu+1}$ can be defined. Thus the construction leading up to Theorem 4.20 can be carried out without knowing that this leads to a solution of the QVI (2.9).

Let u denote the function obtained from the construction set out in the paragraph preceding Theorem 4.20. By this construction, u is continuous on $\mathbb{R}$, satisfies the conclusions of Corollary 4.21, and is such that $Au = f$ in (s_j, ∞) and those subintervals of $(-\infty, s_j)$ where $u < v$, while $Au \leq f$ where u is differentiable. By Lemma 3.15, $M_\ell u$ is well defined on $\mathbb{R}$ for every $\ell \in \mathcal{J}$, $u < Mu$ in (s_j, ∞), and $u = Mu$ at s_j. To confirm that u is a solution of the QVI (2.9) it therefore remains to prove that $u \leq Mu$ in $(-\infty, s_j)$. Moreover, to confirm that u satisfies Ansatz 4.1 it remains to prove that $u = Mu$ on those subintervals of $(-\infty, s_j)$ where $u = v$, and $u < Mu$ in those subintervals where $u < v$. The next lemma accomplishes these tasks simultaneously.

Lemma 4.25. *There holds $u \leq Mu$ in $(-\infty, s_j)$ with equality if and only if $u = v$.*

Proof. We first note that by (3.16), (3.20) and Lemma 4.13, u is concave on $(-\infty, s_j]$. Hence, reintroducing the notation (4.4), u_ℓ is likewise concave for every $\ell \in \mathcal{J}$. It follows that for every $x < s_j$ and $\ell \in \mathcal{J}$,

$$\begin{aligned}\big(M_\ell u\big)(x) &= k_\ell - c_\ell x + \min\{u_\ell(\eta) : \eta \geq x\} \\ &= k_\ell - c_\ell x + \min\{u_\ell(\eta) : \eta \geq s_j \text{ or } \eta = x\} \\ &= \min\{\big(M_\ell y_j\big)(s_j) + c_\ell\,(s_j - x)\,, k_\ell + u(x)\}.\end{aligned}$$

By Lemma 3.15, $\big(M_\ell y_j\big)(s_j) + c_\ell\,(s_j - x) = v_\ell(x)$ if $\ell \leq j$. On the other hand, by (2.3) and (3.13) as a consequence of Lemma 3.15, $\big(M_\ell y_j\big)(s_j) + c_\ell\,(s_j - x) > \big(M_j y_j\big)(s_j) + c_j\,(s_j - x)$ if $\ell > j$. Hence,

$$\begin{aligned}\big(Mu\big)(x) &= \min\big\{\min\{v_\ell(x) : 1 \leq \ell \leq j\}, \min\{k_\ell + u(x), \ell \in \mathcal{J}\}\big\} \\ &= \min\{v(x), k_J + u(x)\}.\end{aligned}$$

If $u(x) = v(x)$ this gives $\big(Mu\big)(x) = u(x)$; whereas if $u(x) < v(x)$ it gives $\big(Mu\big)(x) > u(x)$. $\square$

We have thus arrived at our crowning result.

Theorem 4.26. *Suppose that f_ℓ satisfies Hypothesis 3.3 for every $\ell \in \mathcal{J}$. Then (2.9) has a unique solution satisfying Ansatz 4.1.*

Theorem 4.26 leads to an optimal control policy that extends the concept of a generalized (s, S) policy. It involves N suppliers and numbers

$$s_{(\mathrm{N})} < s_{(\mathrm{N}-1)} < \cdots < s_{(1)} < S_{(1)} < S_{(2)} < \cdots < S_{(\mathrm{N})}$$

and

$$s_{(\mathrm{N})} \leq\ r_{(\mathrm{N}-1)} \leq s_{(\mathrm{N}-1)} \leq \cdots \leq s_{(2)} \leq r_{(1)} \leq s_{(1)}.$$

The strategy of an inventory manager with inventory level x is as follows. If $x > s_{(1)}$ then do not intervene. If $r_{(1)} < x \leq s_{(1)}$, then order to the level $S_{(1)}$ from supplier (1). If $s_{(2)} < x < r_{(1)}$ do not intervene. If $r_{(2)} < x < s_{(2)}$, then order to the level $S_{(2)}$ from supplier (2), if $s_{(3)} < x < r_{(2)}$ do not intervene, and, so on. So that if $r_{(\mathrm{N}-1)} < x < s_{(\mathrm{N}-1)}$, the manager should order from supplier $(\mathrm{N}-1)$ to the level $S_{(\mathrm{N}-1)}$, and, if $s_{(\mathrm{N})} < x < r_{(\mathrm{N}-1)}$ then the manager should not intervene. If $x \leq s_{(\mathrm{N})}$, then order to the level $S_{(\mathrm{N})}$ from supplier (N). For the isolated borderline inventory levels not mentioned in this explanation, there is some ambiguity about whether to order from one supplier or another, or whether to order or not to intervene. Should $r_{(m)} < s_{(m)} < r_{(m-1)}$ for some $m \in \{2, 3, \ldots, \mathrm{N}-1\}$, then there is no ambiguity for $x = s_{(m)}$, and an order to level $S_{(m)}$ from supplier (m) is called for. Otherwise, the manager can pursue any of the options presented by the ambiguity.

Note that suppliers (1) and (N) in the above explanation of a hyper-generalized (s, S) policy are respectively suppliers j and 1 in the mathematical notation used in this section prior to said explanation. The mathematical notation used in developing the theory pertains to the J available suppliers in $\mathcal{J}$ ranked by criteria (2.2) and (2.3). Suppliers (1), (2), ..., (N) constitute a strictly decreasing subsequence within this ranking. As with a conventional generalized (s, S) policy, there may be suppliers that are not party to the final policy. Possibly, $j < J$. Possibly, $\mathrm{N} < j$, where N is the number of suppliers playing a role in the piecewise-linear characterization of the function v. Finally, possibly, $\mathrm{N} < N$. Each of these possibilities denotes a reason why a supplier might be excluded.

The pivotal supplier in a hyper-generalized (s, S) policy is supplier j for which the (s, S) policy has the least cost when there is stock in hand, or, if there is more than one such supplier, that one of these suppliers for which the (s, S) policy has the greatest value of s. When $j < J$, suppliers $j + 1$, $j + 2$, ..., J are automatically excluded from the final policy. In other words, every supplier with a greater cost per item than the pivotal supplier is dismissed. This constitutes the first reason for excluding a supplier. In the subsequent determination of the number N, one is counting the numbers of intervals in which the function v is affine. As such, for all inventory levels x from which a replenishment can take place, one is identifying which of the j remaining suppliers potentially leads to the least cost. When $\mathrm{N} < j$, there is one or more suppliers for each of which another supplier (dependent on x) with a lesser or equal cost can be found for all such x. This is the second reason for excluding a supplier. Finally, among those N suppliers still remaining, there may be one or more for which, over the entire range of inventory levels x for which replenishing from that supplier entails the least cost, the cost of not replenishing and letting the shortage further accumulate is even less. Thus, one ends with $\mathrm{N} < N$. This provides the third and final reason for not including a supplier. An example illustrating every one of these three reasons is presented in the next subsection.

A hyper-generalized (s, S) policy involving N suppliers is a conventional generalized (s, S) policy if and only if $s_{(m+1)} = r_{(m)}$ for every $m \in \{1, 2, \dots, \mathrm{N} - 1\}$. It is an (s, S) policy if and only if $\mathrm{N} = 1$.

4.4. Computation

The optimal control policy expounded by Theorem 4.26 can be computed as follows.

Algorithm 4.27. Steps 1 to 6. Follow Steps 1 to 6 of Algorithm 3.18 omitting Step 6(e).

Step 7. Define $\mathcal{L} = \{2, 3, \dots, \mathrm{N}\}$. If $T_m \geq 0$ for every $m \in \mathcal{L}$, then u satisfies Ansatz 3.11, and one may proceed to Step 9 with $L = 0$. If $T_m < 0$ for some $m \in \mathcal{L}$, then u does not satisfy Ansatz 3.11, and one should continue to Step 8.

Step 8. Take $\nu = 1$.

(a) Define $\mu(\nu) = \max\{m \in \mathcal{L} : T_m < 0\}$ and subsequently $\ell = \kappa(\mu(\nu))$.

(b) Determine $a_\nu < \sigma_{\mu(\nu)}$ from the equation

$$f_\ell(a_\nu) = \alpha B_\ell - c_\ell. \tag{4.19}$$

(c) Set

$$\mathsf{y}_\nu(x) = \mathrm{e}^{-\alpha x} \left\{ (B_\ell - c_\ell a_\nu)\, \mathrm{e}^{\alpha a_\nu} + \int_{a_\nu}^{x} \mathrm{e}^{\alpha \eta} f(\eta)\, \mathrm{d}\eta \right\}. \tag{4.20}$$

(d) Working through the sequence $\mu(\nu) - 1$, $\mu(\nu) - 2$, ..., 1 in that order, let λ be first such number encountered for which the equation

$$\mathsf{y}_\nu(x) = B_{\kappa(\lambda)} - c_{\kappa(\lambda)} x \tag{4.21}$$

has a solution in $(\sigma_{\lambda+1}, \sigma_\lambda]$.

(e) Let b_ν be the least solution of (4.21) in $(\sigma_{\lambda+1}, \sigma_\lambda]$.

(f) If $\lambda = 1$, set $L = \nu$ and proceed to Step 9. Otherwise, continue to Step 8(g).

(g) Redefine $\mathcal{L} = \{2, 3, \dots, \lambda\}$. If $T_m \geq 0$ for every $m \in \mathcal{L}$, set $L = \nu$ and proceed to Step 9. Otherwise, increase ν by 1 and return to Step 8(a).

Step 9. Output $u(x) = \mathsf{y}_\nu(x)$ for $a_\nu < x < b_\nu$ and $1 \leq \nu \leq L$. For all other $x < \sigma_1$, $u(x) = B_1 - c_1 x$ if $x \leq \sigma_N$, and $u(x) = B_{\kappa(m)} - c_{\kappa(m)} x$ if $\sigma_{m+1} < x \leq \sigma_m$ and $1 \leq m \leq N - 1$. For $x \geq \sigma_1$, $u(x)$ is given by (3.8). **End**.

Step 8 of Algorithm 4.27 reproduces the proof of the uniqueness of u in Subsection 4.3, with L and $\mu(\nu)$ for $\nu \in \{1, 2, \ldots, L\}$ defined in exactly the same manner. In terms of the notation employed in said subsection, $a_\nu = \eta_m$, $\mathsf{y}_\nu = Y_m$, and $b_\nu = \zeta_m$, where $m = \mu(\nu)$. In this context, a_ν is the unique number in I_m at which $Av = f$. By the proof of Lemma 4.11, this is also the unique number in $(-\infty, \sigma_m)$ at which $Av_\ell = f$ where $\ell = \kappa(m)$. Explication of this criterion leads to (4.19). Formula (4.20) delivers the explicit solution of (3.1) satisfying $y(a_\nu) = v(a_\nu) = v_\ell(a_\nu)$. Subsequently, b_ν is the unique number in $(a_\nu, s_j]$ such that $\mathsf{y}_\nu < v$ on (a_ν, b_ν) and $\mathsf{y}_\nu(b_\nu) = v(b_\nu)$. This is equivalent to requiring b_ν to be the least solution of $\mathsf{y}_\nu(x) = v_{\kappa(\lambda)}(x)$, *i.e.* (4.21), in $(\sigma_{\lambda+1}, \sigma_\lambda]$, where λ is the largest number in $\{1, 2, \ldots, \mu(\nu) - 1\}$ for which such a solution exists.

Should (4.21) have three solutions, $z_1 < z_2 < z_3$ say, for some $\lambda \in \{1, 2 \ldots, N\}$, then there would be an $\mathsf{s} \in [z_1, z_2)$ and an $\mathsf{S} \in (z_2, z_3]$ with $\mathsf{y}'_\nu(\mathsf{s}) = \mathsf{y}'_\nu(\mathsf{S}) = -c_{\kappa(\lambda)}$. However, by Lemma 3.4 of [5], this necessitates $\mathsf{s} < \gamma_{\kappa(\lambda)} < \mathsf{S}$. So, $z_3 > \gamma_{\kappa(\lambda)} \geq \gamma_j > s_j$. It follows that (4.21) actually has either no solution, a unique solution, or precisely two solutions in $(-\infty, s_j]$. Hence, the same applies in $(\sigma_{\lambda+1}, \sigma_\lambda]$. Consequently, 'least' in Step 8(e) of Algorithm 4.27 really means 'unique or lesser'.

When f takes the classical form (2.4) with $p > \alpha c_J > \alpha c_1 > -q$, one can simplify the calculations within Steps 1 to 6 as outlined in Subsection 3.2. Furthermore, where applicable, one can explicitly solve (4.19), to obtain

$$a_\nu = (c_\ell - \alpha B_\ell) / (p - \alpha c_\ell). \tag{4.22}$$

With the above in mind, let us take the opportunity to complete Example 3.21.

Example 4.28. Suppose that f is given by (2.4) with $p = q = 3$, $\alpha = 1$, $J = 2$, $k_1 = 4$, $k_2 = c_1 = 1$, and $c_2 = 2$. According to Example 3.21, Steps 1 to 5 of Algorithm 3.18 yield $j = 2$, $s_2 \approx -1.7672$, $S_2 \approx 0.1534$, $S_{2,1} \approx 0.3766$, $B_1 \approx 6.506$, and $B_2 \approx 3.767$. Repeating Step 6 with Step 6(e) omitted gives $\sigma_1 = s_2$, $\kappa(1) = 2$, $\sigma_2 \approx -2.7391$, $\kappa(2) = 1$, $T_2 \approx -0.028$, and $N = 2$. Step 7 of Algorithm 4.27 subsequently confirms that the QVI indeed has no solution satisfying Ansatz 3.11. Running through Step 8, one finds $a_1 \approx -2.7532$ from (4.22) with $\ell = 1$, $\mathsf{y}_1(x) \approx -0.127\,\mathrm{e}^{-x} - 3x + 3$ for $x \leq 0$ from (4.20) with $\ell = 1$, that the equation $\mathsf{y}_1(x) = B_2 - c_2 x$ has a unique solution in $(\sigma_2, \sigma_1]$ given by $b_1 \approx -2.7389$, and $L = 1$. Thus the QVI has a unique solution satisfying Ansatz 4.1, which does not satisfy Ansatz 3.11. It is $u(x) = B_1 - x$ for $x < a_1$, $u(x) = \mathsf{y}_1(x)$ for $a_1 \leq x < b_1$, $u(x) = B_2 - 2x$ for $b_1 \leq x < \sigma_1$, $u(x) \approx -0.171\,\mathrm{e}^{-x} - 3x + 3$ for $\sigma_1 \leq x \leq 0$, and $u(x) \approx 5.829\,\mathrm{e}^{-x} + 3x - 3$ for $x > 0$ (in accord with Example 3.7). This corresponds to a hyper-generalized (s, S) policy with two suppliers. In the notation used to introduce such a policy in Section 1, supplier (1) is supplier 2 in the ordering (2.2) and (2.3), supplier (2) is supplier 1, $s_{(2)} = a_1$, $r_{(1)} = b_1$, $s_{(1)} = \sigma_1$, $S_{(1)} = S_2$, and $S_{(2)} = S_{2,1}$. The optimal control policy is not to intervene when the inventory level $x > \sigma_1$, to order from supplier 2 up to the inventory level S_2 when $b_1 < x \leq \sigma_1$, not to intervene when $a_1 < x < b_1$, and to order up to the level $S_{2,1}$ from supplier 1 when $x \leq a_1$. When $x = b_1$, one may either order up to the level S_1 from supplier 2, or not intervene.

It is informative to examine hyper-generalized (s, S) policies further to Example 4.28. The emphasis in Example 4.28 has been on elucidating the mathematical theory leading to the existence and uniqueness of a hyper-generalized (s, S) policy, and the method of calculating such a policy. The coming example focusses on the manner in which suppliers' costs influence the character of a hyper-generalized (s, S) policy. Recall that a hyper-generalized (s, S) policy could include every available supplier. *The first reason for excluding a supplier is that the supplier has a greater cost per item than the pivotal supplier*, the pivotal supplier being the supplier for which the (s, S) policy has the least cost when there is stock in hand, or, if there is more than one such supplier, that one of these suppliers whose (s, S) policy has the greatest value of s. *The second reason for excluding a supplier is that, for all inventory levels from which an order might be placed, there is is another supplier with which placement of an order leads to a lesser or equal cost. The third and final reason for excluding a supplier is that, for the range of inventory levels from which it costs least to order from this supplier, it costs even less not to replenish.* With reference to the structure of a hyper-generalized (s, S) policy outlined in Section 1 and expanded upon after Theorem 4.26, recall that such a policy is a generalized (s, S) policy if and only if $s_{(m+1)} = r_{(m)}$ for every $m \in \{1, 2, \ldots, \mathrm{N} - 1\}$, and, an (s, S) policy if and only if $\mathrm{N} = 1$.

TABLE 1. Features of hyper-generalized (s, S) policies presented in Example 4.29.

Problem	Set-up costs				N	Supplier				Suppliers excluded for reason stated			$s_{(m+1)} < r_{(m)}$?		
	k_1	k_2	k_3	k_4		(1)	(2)	(3)	(4)	1st	2nd	3rd	$m = 1$	$m = 2$	$m = 3$
1	24	15	7	1	4	4	3	2	1	–	–	–	Yes	Yes	Yes
2	24	15	7	0.4	4	4	3	2	1		–	–	Yes	Yes	No
3	24	15	7	2.0	3	3	2	1	–	4	–	–	Yes	Yes	–
4	24	15	5.0	1	2	3	1	–	–	4	2	–	No	–	–
5	24	15	6.0	1	2	3	1	–	–	4	2	–	Yes	–	–
6	24	15	6.2	1	3	4	3	1	–	–	2	–	Yes	Yes	–
7	24	15	6.5	1	4	4	3	2	1	–	–	–	Yes	No	Yes
8	24	15	7.5	1	3	4	2	1	–	–	–	3	Yes	Yes	–
9	24	15	8.0	1	3	4	2	1	–	–	3	–	Yes	Yes	–
10	24	13.0	7	1	2	2	1	–	–	3, 4	–	–	No	–	–
11	24	13.2	7	1	3	4	2	1	–	–	3	–	Yes	No	–
12	24	14.0	7	1	3	4	2	1	–	–	–	3	Yes	No	–
13	24	14.5	7	1	4	4	3	2	1	–	–	–	Yes	Yes	No
14	24	15.3	7	1	3	4	3	1	–	–	–	2	Yes	Yes	–
15	24	16.0	7	1	3	4	3	1	–	–	2	–	Yes	Yes	–
16	21.0	15	7	1	1	1	–	–	–	2, 3, 4	–	–	–	–	–
17	22.0	15	7	1	2	4	1	–	–	–	2	3	Yes	–	–
18	23.2	15	7	1	3	4	3	1	–	–	2	–	Yes	Yes	–
19	23.5	15	7	1	3	4	3	1	–	–	–	2	Yes	Yes	–
20	25.0	15	7	1	4	4	3	2	1	–	–	–	Yes	Yes	No

Example 4.29. Suppose that f takes the classical form (2.4) with $p = q = 9$, the discount rate $\alpha = 1$, and there are four suppliers available with a respective cost per item of $c_1 = 2$, $c_2 = 4$, $c_3 = 6$, and $c_4 = 8$. So the auxiliary function f_ℓ satisfies Hypothesis 3.3 with $\gamma_\ell = 0$ for every $\ell \in \mathcal{J} = \{1, 2, 3, 4\}$. For twenty different combinations of the set-up costs $\{k_\ell : \ell \in \mathcal{J}\}$, features of the hyper-generalized (s, S) policy are summarized in Table 1. Problem 1 constitutes the benchmark. The hyper-generalized (s, S) policy for this problem involves all four suppliers and is such that $s_{(4)} < r_{(3)} < s_{(3)} < r_{(2)} < s_{(2)} < r_{(1)} < s_{(1)} < 0 < S_{(1)} < S_{(2)} < S_{(3)} < S_{(4)}$. Therefore, given four suppliers, it possesses the maximal number of intervals of inventory levels for which the optimal strategy is not to intervene. In the other problems, a value of k_ℓ for $\ell \in \mathcal{J}$ has been changed. The table presents the number N of suppliers included in the resulting hyper-generalized (s, S) policy, the original index of these suppliers, the reason for the exclusion of suppliers, and whether $s_{(m+1)} < r_{(m)}$ for $m \in \{1, 2, \ldots, \mathrm{N}-1\}$.

Example 4.29 shows that given a set $\mathcal{J}$ of J suppliers with costs satisfying (2.2) and (2.3), decreasing the set-up cost k_ℓ of supplier ℓ increases the likelihood of the inclusion of that supplier in the optimal hyper-generalized (s, S) policy. Conversely, increasing k_ℓ decreases the likelihood of this inclusion. Arguably, this behaviour could have been predicted, just as decreasing the cost c_ℓ per item of supplier ℓ will increase the likelihood of supplier ℓ being included in the optimal policy, and increasing c_ℓ will decrease this likelihood. However, the examples also show that altering the costs of a single supplier can have a knock-on effect with regard to the inclusion or exclusion of other suppliers. Apart from when there are only two suppliers, this effect appears to be difficult to gauge. Likewise it seems difficult to predict when a hyper-generalized (s, S) policy will be a generalized (s, S) policy. There is evidently an intricate interplay between the cost parameters of the different suppliers.

5. Discussion/Conclusions

A discounted deterministic continuous-time continuous-state inventory model with several suppliers has been studied. Past attempts at determining an optimal inventory policy had demonstrated that, within the (s, S) paradigm, such a policy exists in certain cases and not in others. The distinguishing characteristic had moreover been determined. In the present work, the elusive case, with no known existence, has been fully resolved. This resolution introduces a new class of inventory policy, which has been termed a hyper-generalized (s, S) policy. This policy allows the possibility of, when a propitious market moment has passed, waiting, if required, until more favorable market conditions occur. This provides companies in today's competitive market where profit margins are becoming tighter with greater flexibility. It moreover captures a real practice in inventory management that was missing from (s, S) and generalized (s, S) policies.

Stochastic inventory control is arguably more realistic than its deterministic counterpart. Nevertheless, the elusive case resolved in the current deterministic setting is also observed in classical stochastic inventory models [3,21]. The concept of a hyper-generalized (s, S) policy could be the answer to the hiatuses in the understanding of such models.

It is of further interest to investigate whether the analysis presented extends to models with more sophisticated demand processes, such as the jump processes in [6], the diffusion processes in [13], and both in [18]. The adopted approach could serve as a road map and a kick start for researchers wishing to pursue more elaborate models.

Finally, we note that the exhibited hyper-generalized (s, S) policy delivers an optimal management strategy for every level within the complete conceivable range of levels of stock. In a dynamic situation this involves either placing a specific order with one of a select group of suppliers or waiting for a certain time before placing such an order. Adoption of this tactic would eventually lead to the situation in which one cyclically orders from the stock level $s_{(1)}$ to the stock level $S_{(1)}$ from the key supplier (1). In other words, the dynamics evolve into those of the (s, S) policy for the single supplier (1). There is convincing mathematical evidence that in the limit that the discount factor α goes to 0, the key supplier (1) would be that prescribed by the optimal policy minimizing the long-term average cost per unit time for the same set of suppliers. Furthermore, the corresponding values of s and S would agree in the limit. The perfunctory explanation is that the cost accrued before the dynamic situation has evolved into a cyclic pattern is finite and therefore negligible in the long-term. For a formal presentation of the long-term cost criterion, we refer to [14, 15]; and for a formal presentation of the QVI for the problem with no discount, to [17].

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