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ON OPTIMALITY AND DUALITY IN INTERVAL-VALUED VARIATIONAL PROBLEM WITH B-(p, r)-INVEXITY

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Abstract. In this paper, we consider a class of interval-valued variational optimization problem. We extend the definition of B-(p, r)-invexity which was originally defined for scalar optimization problem to the interval-valued variational problem. The necessary and sufficient optimality conditions for the problem have been established under B-(p, r)-invexity assumptions. An application, showing utility of the sufficiency theorem in real-world problem, has also been provided. In addition to this, for the interval-optimization problem Mond–Weir and Wolfe type duals are presented and related duality theorems have been proved. Non-trivial examples verifying the results have also been presented throughout the paper.

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1. Introduction

Interval-valued optimization has a huge number of real-world applications such as in fuzzy logic, artificial intelligence, robotics, genetic algorithm, optimal control, neural computing, image restoration problem, optimal shape design problems, robust optimization, power unit problems, molecular distance geometry problems, engineering designs etc. Also, the variational optimization problems are of much importance in various fields such as structural optimization, epidemic, production and inventory, shape optimization in fluid mechanics, optimal control of processes and material inversion in geophysics. Motivated and inspired by the above observations, in this paper we extend the notion of B-(p, r)-invexity to the class of interval-valued variational optimization problem.

In variational calculus, the classical solutions to minimization problem are prescribed by boundary value problems involving Euler–Lagrange equations. Euler was the first to develop the general formula for finding the curve along which a given integral expression has its greatest value. Later, Lagrange proposed a method of multipliers that allows for solutions to the problem without having to solve the conditions or constraints explicitly. These two techniques changed the way of solving the optimization problem and have the last influence on how partial differential equations are viewed. The technique of handling such optimization problem is fundamental in many wide areas of mathematics, physics and engineering. [for more details, refer [7, 14]]

Keywords. Interval-valued variational problem, optimality, duality, LU optimal, B-(p, r)-invexity.

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In field of operation research, three kinds of optimization techniques, namely deterministic optimization, stochastic optimization and interval-valued optimization are mostly used to solve any optimization problem. Among these techniques, interval-valued optimization is an emerging branch of mathematics which deals with uncertainty in optimization problem. In an interval optimization problem, uncertainty may appear in the form of closed intervals either in the objective function or in at least one of the constraints or in both. Various solution concepts have been introduced for interval-valued optimization problems. Ishibuchi and Tanaka [17] used UC-nondominated solution concept for an interval valued optimization problem. In this paper, LU-optimality solution concept is used which was originally defined by Wu [23] and it follows from the concept of a nondominated solution employed in vector optimization problems.

Recently, many researchers have specifically driven attention towards the study of optimality conditions and duality results for interval-valued optimization problems under various generalized convexity assumptions. Wu [23] studied Wolfe-type dual and derived duality theorems for an interval valued optimization problem using nondominated solution concept [24–26]. Bhurjee and Panda [11] studied whether the solution of an interval optimization problem exists. Using the concept of generalized preinvexity, Zhang *et al.* [28] established the optimality conditions and duality theorems for an interval-valued optimization problem. For an interval-valued optimization problem, Ahmad *et al.* [2] obtained optimality conditions and duality results by using generalized (p, r)-ρ-(η, θ) invexity. Debnath and Gupta [13] derived the Fritz John and KKT optimality conditions for a nondifferentiable fractional interval-valued programming problem utilizing the concept of LU optimal solution. For an interval optimization problem with generalized convex function (pseudo-invex function), the sufficient optimality conditions and duality theorems were investigated by Jayswal *et al.* [18].

In recent times, many authors have worked in establishing the relation between mathematical problems and calculus of variation which was first proposed by Hanson [15]. After Hanson's work, this problem became of much interest for many researchers and several contributions have been made in this direction. Till the times when Hanson [16] introduces the concept of invexity, optimality conditions and duality results were proved for the optimization problems where the functions involved were convex. Hanson was the first to prove optimality conditions and duality results for a wider class of invex functions. Mond *et al.* [21] extended this invexity concept to continuous case and proved duality results for a class of variational problem. Bector and Husain [8] extended the concept of duality used in vector optimization problem to variational problem. Thereafter, many researchers derived duality relations for multiobjective problem under generalized invexity assumptions [1, 3–5, 8–10, 12, 19, 28].

Optimality and duality have various applications in optimization. They are of interest as they play an important role in analysing the behaviour of original problem and also helps in obtaining an optimal solution. Antczak and Jiménez [4, 5] established optimality conditions and duality results for a multiobjective variational problem utilizing B-(p, r)-invexity and (ϕ, ρ)-invexity. Bhatia and Mehra [10] derived optimality and duality results under generalized B-invexity assumptions. Kim [19] obtained duality results for a multiobjective variational problem using concept of generalized type I invexity. Mishra and Mukherjee [20] used V-invexity assumptions to prove various results for a multiobjective variational problem. Ahmad and Sharma [1] discussed the sufficiency and duality for a multiobjective variational control problem involving $(F, \alpha, \rho, \theta)$-$V$-convexity. Aran-Jiménez [6] provided new pseudoinvexity conditions for a multiobjective variational problem using generalized convexity. Zhang *et al.* [27] derived various duality results for a multiobjective variational control problem with G-invexity. Very recently, Ahmad *et al.* [3] studied sufficiency and duality for an interval-valued variational problem under invexity assumption.

In this paper, we consider a class of variational problem where the objective function is interval-valued. Our main focus is to investigate the optimality conditions and duality results for the considered problem under B-(p, r)-invexity assumptions. The paper is divided section-wise as follows. In Section 2, some basic definitions and concepts regarding the interval-valued optimization problem are given. In Section 3, sufficient optimality conditions for a class of interval-valued variational problem are studied. In Section 4, some duality results for defined Mond–Weir dual are obtained. Duality results for defined Wolfe dual are established in Section 5 and finally, in Section 6, we conclude our paper.

2. Definitions and preliminaries

Let $\mathbb{J}$ be the class of all closed and bounded interval $[a, b]$ with $a \leq b$ in $\mathbb{R}$. For a closed interval B in $\mathbb{R}$, using standard notation $B = [b^L, b^U]$ with b^L and b^U being the lower and upper endpoints of B, respectively. If $b^L = b^U = b$, then $B = [b^L, b^U] = b$ is a real number. Let $B = [b^L, b^U] \in \mathbb{J}$ and $C = [c^L, c^U] \in \mathbb{J}$. Then, we have the following operations of interval analysis by Moore [22]:

(1) $B = C \implies b^L = c^L$ and $b^U = c^U$.
(2) $-B = \{-b : b \in B\} = \left[-b^U, -b^L\right]$.
(3) $B + C = \{b + c : b \in B, c \in C\} = \left[b^L + c^L, b^U + c^U\right]$.
(4) $\lambda + B = \{\lambda + b : b \in B\} = \left[\lambda + b^L, \lambda + b^U\right]$.
(5) $\lambda B = \lambda[b^L, b^U] = \begin{cases} \left[\lambda b^L, \lambda b^U\right], & \text{if } \lambda \geq 0 \\ \left[\lambda b^U, \lambda b^L\right], & \text{if } \lambda < 0 \end{cases}$,
where λ is a real number.

For $B = \left[b^L, b^U\right]$ and $C = \left[c^L, c^U\right]$, following ordering relation is used to rank interval numbers:

$B \preceq_{\mathrm{LU}} C$ if and only if, $b^L \leq c^L, b^U \leq c^U$.

It can be easily seen that $\preceq_{\mathrm{LU}}$ is a partial ordering on $\mathbb{J}$. Further, $B \prec_{\mathrm{LU}} C$ if and only if $B \leq C$ and $B \neq C$ or equivalently, $B \prec_{\mathrm{LU}} C$ if and only if:

$b^L < c^L, b^U \leq c^U$, or
$b^L \leq c^L, b^U < c^U$, or
$b^L < c^L, b^U < c^U$.

Let $\mathbb{R}^q$ be the q-dimensional Euclidean space and $\mathbb{A} = [a, b]$ be a real interval. We assume that $C(\mathbb{A}, \mathbb{R}^q)$ be the space of piecewise smooth functions $u : \mathbb{A} \to \mathbb{R}^q$ with norm $\|u\| = \|u\|_\infty + \|Du\|_\infty$, where $\|\cdot\|_\infty$ is the uniform norm and the differentiation operator D is given by $z = Du \iff u(t) = u(a) + \int_a^b z(s)\,\mathrm{d}s$, with $u(a)$ being a prescribed boundary value. Thus, $\frac{\mathrm{d}}{\mathrm{d}t} \equiv D$ excluding discontinuities.

A function $\Phi : \mathbb{A} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{J}$ is said to be interval-valued if range of the function is an interval in $\mathbb{R}$ such that $\Phi^L(t, u(t), \dot{u}(t)) \leq \Phi^U(t, u(t), \dot{u}(t))$. Thus, an interval-valued function can be written as:

$$\Phi(t, u(t), \dot{u}(t)) = \left[\Phi^L(t, u(t), \dot{u}(t)), \Phi^U(t, u(t), \dot{u}(t))\right],$$

where $u : \mathbb{A} \to \mathbb{R}^q$ is piecewise smooth function of t and $\dot{u}(t)$ is derivative of u with respect to t and $\Phi^L(t, u(t), \dot{u}(t))$ and $\Phi^U(t, u(t), \dot{u}(t))$ are real valued functions.

We denote the first partial derivative of Φ with respect to u and $\dot{u}$ by Φ_u and $\Phi_{\dot{u}}$, respectively, such that

$$\Phi_u = \begin{pmatrix} \frac{\partial \Phi}{\partial u_1} \\ \frac{\partial \Phi}{\partial u_2} \\ \vdots \\ \frac{\partial \Phi}{\partial u_q} \end{pmatrix} \quad \text{and} \quad \Phi_{\dot{u}} = \begin{pmatrix} \frac{\partial \Phi}{\partial \dot{u}_1} \\ \frac{\partial \Phi}{\partial \dot{u}_2} \\ \vdots \\ \frac{\partial \Phi}{\partial \dot{u}_q} \end{pmatrix}.$$

Now, we extend the definition of B-(p, r)-invexity for the interval-valued variational optimization case which was originally defined for a scalar optimization problem by Antczak and Jiménez [5].

Definition 2.1. Assume that $\Phi : \mathbb{A} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{J}$ be an interval-valued function defined as $\Phi(t, u, \dot{u}) = [\Phi^L(t, u, \dot{u}), \Phi^U(t, u, \dot{u})]$. Also, let $\phi^L, \phi^U : \mathbb{A} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$ are continuously differentiable functions. If their exists real numbers p, r, a function $\eta : \mathbb{A} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q$ with $\eta(t, u(t), \bar{u}(t)) = 0$ for all $u \in C(\mathbb{A}, \mathbb{R}^q)$ and $t \in A$ with $u(t) = \bar{u}(t)$ and a function $b : C(\mathbb{A}, \mathbb{R}^q) \times C(\mathbb{A}, \mathbb{R}^q) \to \mathbb{R}_+ \backslash \{0\}$ such that for $u \in C(\mathbb{A}, \mathbb{R}^q)$:

Case (i): if $p \neq 0, r \neq 0$.

$$\left.\begin{array}{l}\frac{1}{r} b\left(u, \bar{u}\right)\left(e^{r\left(\int_a^b \Phi^L(t,u,\dot{u})\,\mathrm{d}t-\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)}-1\right) \\ \qquad \geq \frac{1}{p}\int_a^b\left(e^{p\eta(t,u,\bar{u})}-1\right)^T\left(\Phi_{\bar{u}}^L\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{\bar{u}}}^L\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t, \\ \text{and} \\ \frac{1}{r} b\left(u, \bar{u}\right)\left(e^{r\left(\int_a^b \Phi^U(t,u,\dot{u})\,\mathrm{d}t-\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)}-1\right) \\ \qquad \geq \frac{1}{p}\int_a^b\left(e^{p\eta(t,u,\bar{u})}-1\right)^T\left(\Phi_{\bar{u}}^U\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{\bar{u}}}^U\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t.\end{array}\right\} \tag{2.1}$$

Case (ii): if $p = 0, r \neq 0$.

$$\left.\begin{array}{l}\frac{1}{r} b(u, \bar{u})\left(e^{r\left(\int_a^b \Phi^L(t,u,\dot{u})\,\mathrm{d}t-\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)}-1\right) \\ \qquad \geq \int_a^b\left(\eta\left(t,u,\bar{u}\right)\right)^T\left(\Phi_{\bar{u}}^L\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{\bar{u}}}^L\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t, \\ \text{and} \\ \frac{1}{r} b(u, \bar{u})\left(e^{r\left(\int_a^b \Phi^U(t,u,\dot{u})\,\mathrm{d}t-\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)}-1\right) \\ \qquad \geq \int_a^b\left(\eta\left(t,u,\bar{u}\right)\right)^T\left(\Phi_{\bar{u}}^U\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{\bar{u}}}^U\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t.\end{array}\right\} \tag{2.2}$$

Case (iii): if $p \neq 0, r = 0$.

$$\left.\begin{array}{l}b\left(u, \bar{u}\right)\left(\int_a^b \Phi^L\left(t,u,\dot{u}\right)\mathrm{d}t-\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right) \\ \qquad \geq \frac{1}{p}\int_a^b\left(e^{p\eta(t,u,\bar{u})}-1\right)^T\left(\Phi_{\bar{u}}^L\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{\bar{u}}}^L\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t, \\ \text{and} \\ b\left(u, \bar{u}\right)\left(\int_a^b \Phi^U\left(t,u,\dot{u}\right)\mathrm{d}t-\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right) \\ \qquad \geq \frac{1}{p}\int_a^b\left(e^{p\eta(t,u,\bar{u})}-1\right)^T\left(\Phi_{\bar{u}}^U\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{\bar{u}}}^U\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t.\end{array}\right\} \tag{2.3}$$

Case (iv): if $p = 0, r = 0$.

$$\left.\begin{array}{l}b\left(u, \bar{u}\right)\left(\int_a^b \Phi^L\left(t,u,\dot{u}\right)\mathrm{d}t-\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right) \\ \qquad \geq \int_a^b\left(\eta\left(t,u,\bar{u}\right)\right)^T\left(\Phi_{\bar{u}}^L\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{\bar{u}}}^L\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t, \\ \text{and} \\ b\left(u, \bar{u}\right)\left(\int_a^b \Phi^U\left(t,u,\dot{u}\right)\mathrm{d}t-\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right) \\ \qquad \geq \int_a^b\left(\eta\left(t,u,\bar{u}\right)\right)^T\left(\Phi_{\bar{u}}^U\left(t,\bar{u},\dot{\bar{u}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{\bar{u}}}^U\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t.\end{array}\right\} \tag{2.4}$$

Then, Φ is said to be B-(p,r)-invex funciton with respect to η at $\bar{u}$ in $C(\mathbb{A},\mathbb{R}^q)$.

Definition 2.2. The function Φ is said to be strict B-(p,r)-invex function with respect to η at $\bar{u}$ in $C(\mathbb{A},\mathbb{R}^q)$, if all the inequalities (2.1)–(2.4) in the Definition 2.1 are strict inequalities.

Example 2.3. Let $\mathbb{A} = [0,1]$ and $C(\mathbb{A},[0,1])$ be the space of all piecewise smooth functions $u : \mathbb{A} \to [0,1]$. Consider the interval-valued function $\Phi : \mathbb{A} \times [0,1] \times [0,1] \to \mathbb{J}$, $\phi\left(t, u(t), \dot{u}(t)\right) = \left[\phi^L\left(t, u(t), \dot{u}(t)\right), \phi^U\left(t, u(t), \dot{u}(t)\right)\right]$, where ϕ^L and ϕ^U are continuously differentiable functions, as follows:

$$\Phi\left(t, u, \dot{u}\right) = \left[u^2 + 1, u^2 + 2u + 5\right].$$

Let $\eta : \mathbb{A} \times [0,1] \times [0,1] \to \mathbb{R}$ and $b : C(\mathbb{A},[0,1]) \times C(\mathbb{A},[0,1]) \to \mathbb{R}_+ \backslash \{0\}$ be given by $\eta\left(t, u, \bar{u}\right) = \frac{1}{4}(\bar{u}^2 - u^2)$ and $b(u,\bar{u}) = u\bar{u} + 9$, respectively. Then, the function ϕ defined above is B-$(4,3)$ invex at $\bar{u} = 0$.

Consider the following interval-valued variational problem, in which the state vector $u(t)$ travels from a defined initial state $u(a) = \gamma$ to some defined final state $u(b) = \delta$ in such a manner so that a given functional can be minimized:

$$\text{(IVP)} \qquad \min \int_a^b \Phi\left(t, u(t), \dot{u}(t)\right) \, \mathrm{d}t = \left[\int_a^b \Phi^L\left(t, u(t), \dot{u}(t)\right) \, \mathrm{d}t, \int_a^b \Phi^U\left(t, u(t), \dot{u}(t)\right) \, \mathrm{d}t \right]$$

subject to

$$u(a) = \gamma, u(b) = \delta,$$
$$h\left(t, u(t), \dot{u}(t)\right) \leq 0, t \in \mathbb{A},$$

where $\Phi : \mathbb{A} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{J}$ is an interval-valued function and $\Phi^L, \Phi^U : \mathbb{A} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}$ are continuously differentiable function and $h : \mathbb{A} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^s$ is continuously differentiable s-dimensional function.

Let ω denotes the set of all feasible points of the interval-valued variational problem. Then

$$\omega = \left\{ u \in C\left(\mathbb{A}, \mathbb{R}^q\right) : u(a) = \gamma, u(b) = \delta \quad \textit{and} \quad h\left(t, u, \dot{u}\right) \leq 0, t \in \mathbb{A} \right\}.$$

Definition 2.4. Let $\bar{u} \in \omega$ be a feasible point of the problem (IVP). Then, $\bar{u}$ is said to be an LU optimal point of (IVP), if there exists no feasible point $u \in \omega$ such that

$$\int_a^b \Phi\left(t, u, \dot{u}\right) \, \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \Phi\left(t, \bar{u}, \dot{\bar{u}}\right) \, \mathrm{d}t.$$

3. Optimality conditions

In this section, we give the necessary and sufficient optimality conditions for the considered interval-valued variational problem (IVP). Throughout the paper, all the proofs are presented for the $p \neq 0$ and $r \neq 0$ case of B-(p, r)-invexity. The proofs for the other cases of B-(p, r)-invexity can be completed in same lines.

Theorem 3.1 (Necessary conditions)**.** *Let $\bar{u}$ be an* LU *optimal point for the (IVP) and assume that Kuhn–Tucker constraint qualification is satisfied at $\bar{u}$. Then, there exists piecewise smooth functions $\sigma : \mathbb{A} \to \mathbb{R}^2, \sigma(t) = \left(\sigma^L(t), \sigma^U(t)\right) \geq 0, \sigma \neq 0$ and $\xi : \mathbb{A} \to \mathbb{R}^s, \xi(t) \geq 0$, such that*

$$\sigma^L \Phi^L_{\bar{u}}\left(t, \bar{u}, \dot{\bar{u}}\right) + \sigma^U \Phi^U_{\bar{u}}\left(t, \bar{u}, \dot{\bar{u}}\right) + \xi^T h_{\bar{u}}\left(t, \bar{u}, \dot{\bar{u}}\right) = \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \sigma^L \Phi^L_{\dot{\bar{u}}}\left(t, \bar{u}, \dot{\bar{u}}\right) + \sigma^U \Phi^U_{\dot{\bar{u}}}\left(t, \bar{u}, \dot{\bar{u}}\right) + \xi^T h_{\dot{\bar{u}}}\left(t, \bar{u}, \dot{\bar{u}}\right) \right\},$$
$$\xi^T h\left(t, \bar{u}, \dot{\bar{u}}\right) = 0, t \in \mathbb{A}.$$

Theorem 3.2 (Sufficiency)**.** *Let $\bar{u}$ be a feasible point for (IVP). Assume that there exists piecewise smooth functions $\sigma : \mathbb{A} \to \mathbb{R}^2, \sigma(t) = (\sigma^L(t), \sigma^U(t)) \geq 0, \sigma \neq 0$ and $\xi : \mathbb{A} \to \mathbb{R}^s, \xi(t) \geq 0$, such that the following conditions*

$$\sigma^L \Phi^L_{\bar{u}}\left(t, \bar{u}, \dot{\bar{u}}\right) + \sigma^U \Phi^U_{\bar{u}}\left(t, \bar{u}, \dot{\bar{u}}\right) + \xi^T h_{\bar{u}}\left(t, \bar{u}, \dot{\bar{u}}\right) = \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \sigma^L \Phi^L_{\dot{\bar{u}}}\left(t, \bar{u}, \dot{\bar{u}}\right) + \sigma^U \Phi^U_{\dot{\bar{u}}}\left(t, \bar{u}, \dot{\bar{u}}\right) + \xi^T h_{\dot{\bar{u}}}\left(t, \bar{u}, \dot{\bar{u}}\right) \right\}, \quad (3.1)$$
$$\xi^T h\left(t, \bar{u}, \dot{\bar{u}}\right) = 0, t \in \mathbb{A}, \quad (3.2)$$

hold. Further, assume that Φ is strictly B-(p, r)-invex function with respect to η at $\bar{u}$ on ω and $\xi^T h$ is B-(p, r)-invex function with respect to same η at $\bar{u}$ on ω. Then, $\bar{u}$ is an LU *optimal point for (IVP).*

Proof. Assume that $\bar{u}$ is not an LU optimal point. Then, there exists another feasible point u such that

$$\int_a^b \Phi\left(t, u, \dot{u}\right) \, \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \Phi\left(t, \bar{u}, \dot{\bar{u}}\right) \, \mathrm{d}t.$$

This means

$$\begin{cases} \int_a^b \Phi^L(t,u,\dot{u})\ \mathrm{d}t < \int_a^b \Phi^L(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t \\ \int_a^b \Phi^U(t,u,\dot{u})\ \mathrm{d}t < \int_a^b \Phi^U(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t \end{cases}$$

or

$$\begin{cases} \int_a^b \Phi^L(t,u,\dot{u})\ \mathrm{d}t \leq \int_a^b \Phi^L(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t \\ \int_a^b \Phi^U(t,u,\dot{u})\ \mathrm{d}t < \int_a^b \Phi^U(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t \end{cases} \tag{3.3}$$

or

$$\begin{cases} \int_a^b \Phi^L(t,u,\dot{u})\ \mathrm{d}t < \int_a^b \Phi^L(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t \\ \int_a^b \Phi^U(t,u,\dot{u})\ \mathrm{d}t \leq \int_a^b \Phi^U(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t. \end{cases}$$

From feasibility of u for (IVP) with hypothesis (3.2) of Theorem 3.2, it implies that

$$\int_a^b \xi^T h(t,u,\dot{u})\ \mathrm{d}t \leq \int_a^b \xi^T h(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t. \tag{3.4}$$

By assumption Φ is strictly B-(p,r)-invex function with respect to η at $\bar{u}$ on ω. Then, by Definition 2.2, we have

$$\frac{1}{r}b(u,\bar{u})\left(e^{r\left(\int_a^b \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)} - 1\right) > \frac{1}{p}\int_a^b \left(e^{p\eta(t,u,\bar{u})} - 1\right)^T \left(\Phi^L_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^L_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right)\mathrm{d}t, \tag{3.5}$$

and

$$\frac{1}{r}b(u,\bar{u})\left(e^{r\left(\int_a^b \Phi^U(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)} - 1\right) > \frac{1}{p}\int_a^b \left(e^{p\eta(t,u,\bar{u})} - 1\right)^T \left(\Phi^U_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^U_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right)\mathrm{d}t. \tag{3.6}$$

Also, $\xi^T h$ is B-(p,r)-invex function with respect to η at $\bar{u}$ on ω. Thus, following inequality holds

$$\begin{aligned} &\frac{1}{r}b(u,\bar{u})\left(e^{r\left(\int_a^b \xi^T h(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \xi^T h\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)} - 1\right) \\ &\quad \geq \frac{1}{p}\int_a^b \left(e^{p\eta(t,u,\bar{u})} - 1\right)^T \left(\xi^T h_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\xi^T h_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right)\mathrm{d}t. \end{aligned} \tag{3.7}$$

Taking into account that $b(u,\bar{u}) > 0$ and combining (3.3) and (3.5), (3.3) and (3.6), (3.4) and (3.7), respectively, we obtain

$$\frac{1}{p}\int_a^b \left(e^{p\eta(t,u,\bar{u})} - 1\right)^T \left(\Phi^L_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^L_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right)\mathrm{d}t < 0 \tag{3.8}$$

$$\frac{1}{p}\int_a^b \left(e^{p\eta(t,u,\bar{u})} - 1\right)^T \left(\Phi^U_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^U_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right)\mathrm{d}t < 0 \tag{3.9}$$

$$\frac{1}{p}\int_a^b (e^{p\eta(t,u,\bar{u})} - 1)^T \left(\xi^T h_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\xi^T h_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right)\mathrm{d}t \leq 0. \tag{3.10}$$

Multiplying (3.8) by σ^L, (3.9) by σ^U and then adding both sides of resultant inequalities and (3.10), we have

$$\frac{1}{p}\int_a^b \left(e^{p\eta(t,u,\bar{u})} - 1\right)^T \left[\left(\sigma^L\Phi^L_{\bar{u}} + \sigma^U\Phi^U_{\bar{u}} + \xi^T h_{\bar{u}}\right)(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\sigma^L\Phi^L_{\dot{\bar{u}}} + \sigma^U\Phi^U_{\dot{\bar{u}}} + \xi^T h_{\dot{\bar{u}}}\right)(t,\bar{u},\dot{\bar{u}})\right)\right]\mathrm{d}t < 0,$$

which is contradiction to (3.1). Hence, $\bar{u}$ is an LU optimal point of (IVP) and this completes the proof. □

Example 3.3. Let $\mathbb{A} = [0,1]$ and $C(\mathbb{A},[0,1])$ be the space of all piecewise smooth functions $u : \mathbb{A} \to [0,1]$. Consider the following interval-valued variational problem:

$$\text{(P1)} \qquad \min \int_0^1 \Phi\left(t,u,\dot{u}\right)\,\mathrm{d}t = \left[\int_0^1 (u^3 + e^{2u})\,\mathrm{d}t, \int_0^1 \left(u^2 + e^{2u} + 5\right)\,\mathrm{d}t\right]$$

subject to

$$u(0) = 0, u(1) = 1,$$
$$u - 2\ln(u+1) \leq 0, t \in \mathbb{A}.$$

From the formulation of (P1) it is clear that, $\phi^L\left(t,u,\dot{u}\right) = (u^3 + e^{2u})$, $\phi^U\left(t,u,\dot{u}\right) = (u^2 + e^{2u} + 5)$ and $h\left(t,u,\dot{u}\right) = u - 2\ln(u+1)$.

Let $\omega = \{u \in C(\mathbb{A},[0,1]) : u(0) = 0, u(1) = 1, u - 2\ln(u+1) \leq 0, t \in \mathbb{A}\}$ be the feasible set for (P1). Then, $\bar{u} = 0$ is a feasible point for (P1) and the function ϕ is an interval-valued function.

We check hypothesis (3.1) of Theorem 3.2 holds at $\bar{u} = 0$:

$$\sigma^L \Phi^L_{\bar{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + \sigma^U \Phi^U_{\bar{u}}\left(t,\bar{u},\dot{\bar{u}}\right) + \xi^T h_{\bar{u}}\left(t,\bar{u},\dot{\bar{u}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left\{\sigma^L \Phi^L_{\dot{\bar{u}}}\left(t,\bar{u},\dot{\bar{u}}\right) + \sigma^U \Phi^U_{\dot{\bar{u}}}\left(t,\bar{u},\dot{\bar{u}}\right) + \xi^T h_{\dot{\bar{u}}}\left(t,\bar{u},\dot{\bar{u}}\right)\right\} = 0.$$
$$\Rightarrow 2\sigma^L + 2\sigma^U = \xi.$$

From this, let $\sigma^L = 1/4, \sigma^U = 1/4, \xi = 1$.

Next, we check hypothesis (3.2) of Theorem 3.2 holds at $\bar{u} = 0$:

$$\begin{aligned}\xi h\left(t,\bar{u},\dot{\bar{u}}\right) &= \xi(\bar{u} - 2\ln(\bar{u}+1)) \\ &= 0, (\bar{u} = 0).\end{aligned}$$

Let $\eta : \mathbb{A} \times [0,1] \times [0,1] \to \mathbb{R}$ and $b : C(\mathbb{A},[0,1]) \times C(\mathbb{A},[0,1]) \to \mathbb{R}_+ \backslash \{0\}$ be defined as $\eta\left(t,u,\bar{u}\right) = 2\sin(u - \bar{u})$ and $b(u,\bar{u}) = u\bar{u} + 2$, respectively.

Now, we check whether the function Φ is strictly B-(p,r)-invex and ξh is B-(p,r)-invex at $\bar{u} = 0$!

Here, computations are presented for function ϕ^L:

Case (i): $p = 1/2, r = 2$.

$$\begin{aligned}&\frac{1}{r} b(u,\bar{u})\left(e^{r\left(\int_0^1 \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_0^1 \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)} - 1\right) \\ &\quad - \frac{1}{p}\int_0^1 \left(e^{p\eta(t,u,\bar{u})} - 1\right)^T \left(\Phi^L_{\bar{u}}\left(t,\bar{u},\dot{\bar{u}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^L_{\dot{\bar{u}}}\left(t,\bar{u},\dot{\bar{u}}\right)\right)\right)\mathrm{d}t, \\ &\quad = \frac{1}{2}\left(u\bar{u} + 2\right)\left(e^{2\left(\int_0^1 \left(u^3 + e^{2u}\right)\mathrm{d}t - \int_0^1 \left(\bar{u}^3 + e^{2\bar{u}}\right)\mathrm{d}t\right)} - 1\right) \\ &\quad\quad - \frac{1}{1/2}\int_0^1 \left(e^{(1/2)(2\sin(u-\bar{u}))} - 1\right)\left(3\bar{u}^2 + 2e^{\bar{u}} - \frac{\mathrm{d}}{\mathrm{d}t}(0)\right)\mathrm{d}t, \\ &\quad = \left(e^{2\left(\int_0^1 \left(u^3 + e^{2u} - 1\right)\mathrm{d}t\right)} - 1\right) - 4\int_0^1 \left(e^{\sin(u)} - 1\right)\mathrm{d}t > 0.\end{aligned}$$

Case (ii): $p = 0, r = 2$.

$$\frac{1}{r} b\left(u,\bar{u}\right)\left(e^{r\left(\int_0^1 \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_0^1 \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)} - 1\right)$$

$$- \int_0^1 (\eta(t,u,\bar{u}))^T \left(\Phi^L_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^L_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right) \mathrm{d}t,$$
$$= \left(e^{2\left(\int_0^1 \left(u^3+e^{2u}-1\right)\mathrm{d}t\right)} - 1\right) - 4\int_0^1 \sin(u)\,\mathrm{d}t > 0.$$

Case (iii): $p = 1/2, r = 0$**.**

$$b(u,\bar{u})\left(\int_0^1 \Phi^L(t,u,\dot{u})\ \mathrm{d}t - \int_0^1 \Phi^L(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t\right)$$
$$- \frac{1}{p}\int_0^1 \left(e^{p\eta(t,u,\bar{u})} - 1\right)^T \left(\Phi^L_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^L_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right) \mathrm{d}t,$$
$$= 2\int_0^1 \left(u^3 + e^{2u} - 1\right)\ \mathrm{d}t - 4\int_0^1 \left(e^{\sin(u)} - 1\right)\ \mathrm{d}t > 0.$$

Case (iv): $p = 0, r = 0$**.**

$$b(u,\bar{u})\left(\int_0^1 \Phi^L(t,u,\dot{u})\ \mathrm{d}t - \int_0^1 \Phi^L(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t\right)$$
$$- \int_0^1 (\eta(t,u,\bar{u}))^T \left(\Phi^L_{\bar{u}}(t,\bar{u},\dot{\bar{u}}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^L_{\dot{\bar{u}}}(t,\bar{u},\dot{\bar{u}})\right)\right) \mathrm{d}t,$$
$$= 2\int_0^1 \left(u^3 + e^{2u} - 1\right)\ \mathrm{d}t - 4\int_0^1 \sin(u)\,\mathrm{d}t > 0.$$

It is clear from cases (i) to (iv) that function ϕ^L satisfy condition of strict B-$(1/2, 2)$-invexity with respect to η at $\bar{u} = 0$. Now, to show that function ϕ is strict B-$(1/2, 2)$-invex function, we have to show that ϕ^U also satisfies the conditions of Definition 2.2. For this, we have listed the calculations of all four cases for function ϕ^U in Table 1. Then, from calculations of ϕ^L and ϕ^U, it is clear that function ϕ is strict B-$(1/2, 2)$-invex function with respect to η at $\bar{u} = 0$.

Again, all four cases for ξh have been listed in Table 1 and it can be seen that ξh is B-$(1/2, 2)$-invex with respect to η at $\bar{u} = 0$.

Next, we prove that $\bar{u} = 0$ is an LU optimal point for (P1). If possible, suppose $\bar{u} = 0$ is not an LU optimal point for (P1), then there exists another feasible point $v \in \omega$ such that

$$\int_0^1 \Phi(t,v,\dot{v})\ \mathrm{d}t \prec_{\mathrm{LU}} \int_0^1 \Phi(t,\bar{u},\dot{\bar{u}})\ \mathrm{d}t.$$

Then, at $\bar{u} = 0$ this means

$$\begin{cases} \int_0^1 \Phi^L(t,v,\dot{v})\ \mathrm{d}t < 1 \\ \int_0^1 \Phi^U(t,v,\dot{v})\ \mathrm{d}t < 6 \end{cases}$$

or

$$\begin{cases} \int_0^1 \Phi^L(t,v,\dot{v})\ \mathrm{d}t \leq 1 \\ \int_0^1 \Phi^U(t,v,\dot{v})\ \mathrm{d}t < 6 \end{cases}$$

or

$$\begin{cases} \int_0^1 \Phi^L(t,v,\dot{v})\ \mathrm{d}t < 1 \\ \int_0^1 \Phi^U(t,v,\dot{v})\ \mathrm{d}t \leq 6. \end{cases}$$

TABLE 1. Resultant inequalities for functions ϕ^U and ξh.

ϕ^U	$p=1/2, r=2$	$\left(e^{2\int_0^1 \left(u^2+e^{2u}-1\right) \mathrm{d}t}-1\right)-4\int_0^1 (e^{\sin(u)}-1)\,\mathrm{d}t>0$
	$p=0, r=2$	$\left(e^{2\int_0^1 \left(u^2+e^{2u}-1\right) \mathrm{d}t}-1\right)-4\int_0^1 \sin(u)\,\mathrm{d}t>0$
	$p=1/2, r=0$	$2\int_0^1 \left(u^2+e^{2u}-1\right)\,\mathrm{d}t-4\int_0^1 (e^{\sin(u)}-1)\,\mathrm{d}t>0$
	$p=0, r=0$	$2\int_0^1 \left(u^2+e^{2u}-1\right)\,\mathrm{d}t-4\int_0^1 \sin(u)\,\mathrm{d}t>0$
ξh	$p=1/2, r=2$	$\left(e^{2\int_0^1 (u-2\ln(u+1))\, \mathrm{d}t}-1\right)+2\int_0^1 (e^{\sin(u)}-1)\,\mathrm{d}t\geq 0$
	$p=0, r=2$	$\left(e^{2\int_0^1 (u-2\ln(u+1))\, \mathrm{d}t}-1\right)+2\int_0^1 \sin(u)\,\mathrm{d}t\geq 0$
	$p=1/2, r=0$	$2\int_0^1 (u-2\ln(u+1))\,\mathrm{d}t+2\int_0^1 (e^{\sin(u)}-1)\,\mathrm{d}t\geq 0$
	$p=0, r=0$	$2\int_0^1 (u-2\ln(u+1))\,\mathrm{d}t+2\int_0^1 \sin(u)\,\mathrm{d}t\geq 0$

But there does not exists any other feasible point v such that

$$\int_0^1 \Phi\left(t, v, \dot{v}\right)\,\mathrm{d}t \prec_{\mathrm{LU}} \int_0^1 \Phi\left(t, \bar{u}, \dot{\bar{u}}\right)\,\mathrm{d}t.$$

Hence, $\bar{u}=0$ is an LU optimal point for (P1) and thus sufficiency theorem is verified.

4. MOND–WEIR TYPE DUALITY

We present the Mond–Weir type dual for the considered (IVP) as follows:

$$\text{(IMD)}\qquad \max\int_a^b \Phi\left(t, v, \dot{v}\right)\,\mathrm{d}t=\left[\int_a^b \Phi^L\left(t, v, \dot{v}\right)\,\mathrm{d}t, \int_a^b \Phi^U\left(t, v, \dot{v}\right)\,\mathrm{d}t\right]$$

subject to

$$v(a)=\gamma, v(b)=\delta,$$

$$\sigma^L\Phi_v^L\left(t, v, \dot{v}\right)+\sigma^U\Phi_v^U\left(t, v, \dot{v}\right)+\xi^T h_v\left(t, v, \dot{v}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sigma^L\Phi_{\dot{v}}^L\left(t, v, \dot{v}\right)+\sigma^U\Phi_{\dot{v}}^U\left(t, v, \dot{v}\right)+\xi^T h_{\dot{v}}\left(t, v, \dot{v}\right)\right\}, \tag{4.1}$$

$$\int_a^b \xi^T h\left(t, v, \dot{v}\right)\geq 0, t\in\mathbb{A}, \tag{4.2}$$

$$\sigma=(\sigma^L, \sigma^U)\geq 0, \sigma\neq 0, \xi\geq 0.$$

Let ω_{MD} denotes the set of all feasible points of (IMD). Then, $\omega_{\mathrm{MD}}=\{(v, \sigma, \xi): v\in C(\mathbb{A}, \mathbb{R}^q), \sigma\in\mathbb{R}^2, \xi\in\mathbb{R}^s,$ satisfying the constraints of (IMD), $\forall t\in\mathbb{A}\}$.

We denote the set $W=\{v\in C(\mathbb{A}, \mathbb{R}^q):(v, \sigma, \xi)\in\omega_{\mathrm{MD}}\}$ and $\Omega=\omega\cup W$.

Definition 4.1. A feasible point $(\bar{v}, \bar{\sigma}, \bar{\xi})$ is said to be an LU optimal point of a maximum type for (IMD), if there exists no feasible point (v, σ, ξ) such that

$$\int_a^b \Phi(t, \bar{v}, \dot{\bar{v}})\ \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \Phi(t, v, \dot{v})\ \mathrm{d}t.$$

Theorem 4.2 (Weak duality)**.** *Let u and (v, σ, ξ) be any feasible points for (IVP) and (IMD), respectively. Assume that $\sigma = (\sigma^L, \sigma^U) > 0$, Φ and $\xi^T h$ are B-(p, r)-invex functions with respect to same η at v on Ω. Then, the following can not hold*

$$\int_a^b \Phi(t, u, \dot{u})\ \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \Phi(t, v, \dot{v})\ \mathrm{d}t.$$

Proof. Suppose that the following hold

$$\int_a^b \Phi(t, u, \dot{u})\ \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \Phi(t, v, \dot{v})\ \mathrm{d}t.$$

Then, this implies

$$\begin{cases} \int_a^b \Phi^L(t, u, \dot{u})\ \mathrm{d}t - \int_a^b \Phi^L(t, v, \dot{v})\ \mathrm{d}t < 0 \\ \int_a^b \Phi^U(t, u, \dot{u})\ \mathrm{d}t - \int_a^b \Phi^U(t, v, \dot{v})\ \mathrm{d}t < 0 \end{cases}$$

or

$$\begin{cases} \int_a^b \Phi^L(t, u, \dot{u})\ \mathrm{d}t - \int_a^b \Phi^L(t, v, \dot{v})\ \mathrm{d}t \leq 0 \\ \int_a^b \Phi^U(t, u, \dot{u})\ \mathrm{d}t - \int_a^b \Phi^U(t, v, \dot{v})\ \mathrm{d}t < 0 \end{cases}$$

or

$$\begin{cases} \int_a^b \Phi^L(t, u, \dot{u})\ \mathrm{d}t - \int_a^b \Phi^L(t, v, \dot{v})\ \mathrm{d}t < 0 \\ \int_a^b \Phi^U(t, u, \dot{u})\ \mathrm{d}t - \int_a^b \Phi^U(t, v, \dot{v})\ \mathrm{d}t \leq 0. \end{cases}$$

The above inequalities are further equivalent to

$$\begin{cases} \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0 \\ \left(e^{r\left(\int_a^b \Phi^U(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^U(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0 \end{cases}$$

or

$$\begin{cases} \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) \leq 0 \\ \left(e^{r\left(\int_a^b \Phi^U(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^U(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0 \end{cases}$$

or

$$\begin{cases} \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0 \\ \left(e^{r\left(\int_a^b \Phi^U(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^U(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) \leq 0. \end{cases}$$

Since, $(\sigma^L, \sigma^U) > 0$, the above inequalities implies

$$\sigma^L \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) + \sigma^U \left(e^{r\left(\int_a^b \Phi^U(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \Phi^U(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0. \tag{4.3}$$

From the feasibility of u for (IVP) and from the feasibility of (v, σ, ξ) for (IMD), it follows that

$$\int_a^b \xi^T h(t, u, \dot{u})\ \mathrm{d}t \leq \int_a^b \xi^T h(t, v, \dot{v})\ \mathrm{d}t.$$

Then, the above inequality implies that

$$\left(e^{r\left(\int_a^b \xi^T h(t,u,\dot u)\,\mathrm{d}t - \int_a^b \xi^T h(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) \leq 0. \tag{4.4}$$

On adding both sides of (4.3) and (4.4), we get

$$\sigma^L \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot u)\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) + \sigma^U \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot u)\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) + \left(e^{r\left(\int_a^b \xi^T h(t,u,\dot u)\,\mathrm{d}t - \int_a^b \xi^T h(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) < 0. \tag{4.5}$$

From assumption that Φ is B-(p,r)-invex function with respect to η at v on Ω. Then, we have the following

$$\frac{1}{r} b(u,v) \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot u)\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) \geq \frac{1}{p} \int_a^b \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\Phi_v^L(t,v,\dot v) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot v}^L(t,v,\dot v)\right)\right) \mathrm{d}t \tag{4.6}$$

and

$$\frac{1}{r} b(u,v) \left(e^{r\left(\int_a^b \Phi^U(t,u,\dot u)\,\mathrm{d}t - \int_a^b \Phi^U(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) \geq \frac{1}{p} \int_a^b \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\Phi_v^U(t,v,\dot v) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot v}^U(t,v,\dot v)\right)\right) \mathrm{d}t. \tag{4.7}$$

Also, $\xi^T h$ is B-(p,r)-invex function with respect to same η at v on Ω, which implies

$$\frac{1}{r} b(u,v) \left(e^{r\left(\int_a^b \xi^T h(t,u,\dot u)\,\mathrm{d}t - \int_a^b \xi^T h(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) \geq \frac{1}{p} \int_a^b \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\xi^T h_v(t,v,\dot v) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\xi^T h_{\dot v}(t,v,\dot v)\right)\right) \mathrm{d}t. \tag{4.8}$$

Multiplying (4.6) and (4.7) by σ^L and σ^U and adding resultant inequalities, we have

$$\frac{1}{r} b(u,v) \left[\sigma^L \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot u)\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) + \sigma^U \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot u)\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right)\right] \geq \frac{1}{p} \int_a^b \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\left(\sigma^L \Phi_v^L + \sigma^U \Phi_v^U\right)(t,v,\dot v) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\sigma^L \Phi_{\dot v}^L + \sigma^U \Phi_{\dot v}^U\right)(t,v,\dot v)\right)\right) \mathrm{d}t. \tag{4.9}$$

Adding both sides of (4.8) and (4.9) and from the feasibility of (IMD), we have

$$\frac{1}{r} b(u,v) \left[\sigma^L \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot u)\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) + \sigma^U \left(e^{r\left(\int_a^b \Phi^L(t,u,\dot u)\,\mathrm{d}t - \int_a^b \Phi^L(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right) + \left(e^{r\left(\int_a^b \xi^T h(t,u,\dot u)\,\mathrm{d}t - \int_a^b \xi^T h(t,v,\dot v)\,\mathrm{d}t\right)} - 1\right)\right] \geq 0, \tag{4.10}$$

which is contradiction to (4.5).

Hence, $\int_a^b \Phi(t,u,\dot u)\,\mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \Phi(t,v,\dot v)\,\mathrm{d}t$ can not hold. □

Example 4.3. Let $\mathbb{A} = [0,1]$ and $C(\mathbb{A},[0,1])$ be the space of all piecewise smooth functions $u : \mathbb{A} \to [0,1]$. Consider the following interval-valued variational problem:

$$\text{(P2)} \qquad \min \int_0^1 \Phi(t,u,\dot u)\,\mathrm{d}t = \left[\int_0^1 \left(2u^2 + \ln(u+1)\right)\mathrm{d}t, \int_0^1 \left(2u^2 + e^u\right)\mathrm{d}t\right]$$

subject to

$$u(0) = 0, u(1) = 1,$$

$$u^2 - u \le 0, t \in \mathbb{A}.$$

Let $\omega = \{u \in C(\mathbb{A}, [0,1]) : u(0) = 0, u(1) = 1 \text{ and } u^2 - u \le 0, t \in \mathbb{A}\}$ be feasible set for (P2). From the formulation of (P2) and definition of interval-valued function, it is clear that, $\phi^L(t,u,\dot{u}) = 2u^2 + \ln(u+1), \phi^U(t,u,\dot{u}) = 2u^2 + e^u$ and $h(t,u,\dot{u}) = u^2 - u$.

The corresponding Mond–Weir-dual is

$$\text{(MD)} \qquad \max \int_0^1 \Phi(t,v,\dot{v})\,\mathrm{d}t = \left[\int_0^1 \left(2v^2 + \ln(v+1)\right)\mathrm{d}t, \int_0^1 \left(2v^2 + e^v\right)\mathrm{d}t\right]$$

subject to

$$v(0) = 0, v(1) = 1,$$
$$v^2 - v \le 0, t \in \mathbb{A},$$
$$\sigma^L\left(4v + \frac{1}{v+1}\right) + \sigma^U(4v + e^v) + \xi(2v-1) = 0.$$

At $v = 0$,

$$\sigma^L + \sigma^U - \xi = 0,$$
$$\implies \sigma^L = 1/4, \sigma^U = 3/4, \xi = 1,$$
$$\int_0^1 \xi(t)(v^2 - v)\,\mathrm{d}t \ge 0, t \in \mathbb{A}.$$

From the formation of Mond–Weir-dual it is clear that, $\phi^L(t,v,\dot{v}) = 2v^2 + \ln(v+1), \phi^U(t,v,\dot{v}) = 2v^2 + e^v$ and $h(t,v,\dot{v}) = v^2 - v$. Let ω_{MD} be the feasible set of (MD). $\omega_{\mathrm{MD}} = \{(v,\sigma,\xi) : v \in C(\mathbb{A},[0,1]), \sigma \in \mathbb{R}^2, \xi \in \mathbb{R}$, satisfying the constraints of (MD), $t \in \mathbb{A}\}$. Then, we denote the set $W = \{v \in C(\mathbb{A},[0,1]) : (v,\sigma,\xi) \in \omega_{\mathrm{MD}}\}$ and $\Omega = \omega \cup W$.

We observe that $(v(t), \sigma(t), \xi(t)) = (0, (1/4, 3/4), 1)$ is a feasible point for (MD) and objective value at this feasible point is $[0,1]$. Also, $u(t) = t$ is a feasible point for (P2) and the corresponding objective value is $[1.05, 2.38]$.

Let $\eta : \mathbb{A} \times [0,1] \times [0,1] \to \mathbb{R}$ and $b : C(\mathbb{A},[0,1]) \times C(\mathbb{A},[0,1]) \to \mathbb{R}_+ \backslash \{0\}$ be defined as $\eta(t,u,v) = \ln(1+u-v)$ and $b(u,v) = uv + 1$, respectively.

Next, we show that ϕ and ξh are B-(p,r)-invex function at $v = 0$ with respect to above defined η.

We check for function ϕ^L as follows:

Case (i): $p = 1, r = 1/3$.

$$\frac{1}{r} b(u,v)\left(e^{r\left(\int_0^1 \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_0^1 \Phi^L(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) - \frac{1}{p}\int_0^1 \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\Phi^L_v(t,v,\dot{v}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^L_{\dot{v}}(t,v,\dot{v})\right)\right)\mathrm{d}t,$$
$$= \frac{1}{1/3}(uv+1)\left(e^{(1/3)\left(\int_0^1 (2u^2 + \ln(u+1))\,\mathrm{d}t - \int_0^1 \left(2v^2 + \ln(v+1)\right)\mathrm{d}t\right)} - 1\right)$$
$$- \int_0^1 \left(e^{\ln(1+u-v)} - 1\right)^T \left(4v + \frac{1}{v+1} - \frac{\mathrm{d}}{\mathrm{d}t}(0)\right)\mathrm{d}t,$$
$$= 3\left(e^{(1/3)\left(\int_0^1 (2u^2 + \ln(u+1))\,\mathrm{d}t\right)} - 1\right) - \int_0^1 u\,\mathrm{d}t \ge 0.$$

Case (ii): $p = 0, r = 1/3$.

$$\frac{1}{r} b(u,v)\left(e^{r\left(\int_0^1 \Phi^L(t,u,\dot{u})\,\mathrm{d}t - \int_0^1 \Phi^L(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) - \int_0^1 (\eta(t,u,v))^T \left(\Phi^L_v(t,v,\dot{v}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi^L_{\dot{v}}(t,v,\dot{v})\right)\right)\mathrm{d}t,$$

TABLE 2. Resultant inequalities for functions ϕ^U and ξh

ϕ^U	$p=1, r=1/3$	$3\left(e^{\frac{1}{3}\int_0^1\left(2u^2+e^u-1\right)\mathrm{d}t}-1\right)-\int_0^1 u\,\mathrm{d}t \geq 0$
	$p=0, r=1/3$	$3\left(e^{\frac{1}{3}\int_0^1\left(2u^2+e^u-1\right)}-1\right)-\int_0^1 \ln(u+1)\,\mathrm{d}t \geq 0$
	$p=1, r=0$	$\int_0^1\left(2u^2+e^u-1\right)\mathrm{d}t-\int_0^1 u\,\mathrm{d}t \geq 0$
	$p=0, r=0$	$\int_0^1\left(2u^2+e^u-1\right)-\int_0^1 \ln(u+1)\,\mathrm{d}t \geq 0$
ξh	$p=1, r=1/3$	$3\left(e^{\frac{1}{3}\int_0^1\left(u^2-u\right)\mathrm{d}t}-1\right)+\int_0^1 u\,\mathrm{d}t \geq 0$
	$p=0, r=1/3$	$3\left(e^{\frac{1}{3}}\int_0^1\left(u^2-u\right)\mathrm{d}t-1\right)+\int_0^1 \ln(u+1)\,\mathrm{d}t \geq 0$
	$p=1, r=0$	$\int_0^1 u^2\,\mathrm{d}t \geq 0$
	$p=0, r=0$	$\int_0^1\left(u^2-u\right)\mathrm{d}t+\int_0^1(\ln(u+1))\,\mathrm{d}t \geq 0$

$$= 3\left(e^{(1/3)\left(\int_0^1(2u^2+\ln(u+1))\,\mathrm{d}t\right)}-1\right)-\int_0^1 \ln(u+1)\,\mathrm{d}t \geq 0.$$

Case (iii): $p=1, r=0$**.**

$$b(u,v)\left(\int_0^1 \Phi^L(t,u,v)\,\mathrm{d}t-\int_0^1 \Phi^L(t,v,\dot{v})\,\mathrm{d}t\right)-\frac{1}{p}\int_0^1\left(e^{p\eta(t,u,v)}-1\right)^T\left(\Phi_v^L(t,v,\dot{v})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{v}}^L(t,v,\dot{v})\right)\right)\mathrm{d}t,$$
$$=\int_0^1\left(2u^2+\ln(u+1)\right)\mathrm{d}t-\int_0^1 u\,\mathrm{d}t \geq 0.$$

Case(iv): $p=0, r=0$**.**

$$b(u,v)\left(\int_0^1 \Phi^L(t,u,\dot{u})\,\mathrm{d}t-\int_0^1 \Phi^L(t,v,\dot{v})\,\mathrm{d}t\right)-\int_0^1(\eta(t,u,v))^T\left(\Phi_v^L(t,v,\dot{v})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\dot{v}}^L(t,v,\dot{v})\right)\right)\mathrm{d}t,$$
$$=\int_0^1 2u^2\,\mathrm{d}t \geq 0.$$

It is clear from cases (i) to (iv) that function ϕ^L satisfies the conditions of Definition 2.1. Next, to show that ϕ is B-$(1,1/3)$-invex funciton, ϕ^L and ϕ^U must be B-$(1,1/3)$-invex function.We have listed the calulations of all four cases for function ϕ^U and ξh in Table 2. It implies from Table 2 that ϕ is B-$(1,1/3)$-invex function with respect to η at $v=0$. Again from Table 2, it can be seen that ξh is B-$(1,1/3)$-invex with respect to η at $v=0$.

Moreover, $u(t)=t$ and $(v(t),\sigma(t),\xi(t))=(0,(1/4,3/4),1)$ are the feasible points for (P2) and (MD), then we observe that

$$\int_0^1 \Phi(t,u,\dot{u})\,\mathrm{d}t \not\prec \int_0^1 \Phi(t,v,\dot{v})\,\mathrm{d}t.$$

Hence, the weak duality theorem is verified.

Theorem 4.4 (Strong duality)**.** *Let $\bar{u}$ be an* LU *optimal point for (IVP) and further, assume that Kuhn–Tucker constraint qualification is satisfied at $\bar{u}$. Then, there exists piecewise smooth functions $\bar{\sigma}:\mathbb{A}\to\mathbb{R}^2_+, \bar{\sigma}\neq 0$ and $\bar{\xi}:\mathbb{A}\to\mathbb{R}^s, \bar{\xi}\geq 0$ such that $(\bar{u},\bar{\sigma},\bar{\xi})$ is a feasible point for (IMD) and the objective function of (IVP) and (IMD) have the same value at $\bar{u}$ and $(\bar{u},\bar{\sigma},\bar{\xi})$, respectively. If the hypotheses of weak duality theorem are satisfied for all feasible points $(\bar{v},\bar{\sigma},\bar{\xi})$, then $(\bar{u},\bar{\sigma},\bar{\xi})$ is an* LU *optimal point for (IMD).*

Proof. Since $\bar{u}$ be an LU optimal point for (IVP), then by Theorem 3.1, there exists piecewise smooth functions $\bar{\sigma}:\mathbb{A}\to\mathbb{R}^2_+, \bar{\sigma}\neq 0$ and $\bar{\xi}:\mathbb{A}\to\mathbb{R}^s, \bar{\xi}\geq 0$ such that:

$$\bar{\sigma}^L\Phi^L_{\bar{u}}\left(t,\bar{u},\dot{\bar{u}}\right)+\bar{\sigma}^U\Phi^U_{\bar{u}}\left(t,\bar{u},\dot{\bar{u}}\right)+\bar{\xi}^T h_{\bar{u}}\left(t,\bar{u},\dot{\bar{u}}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\bar{\sigma}^L\Phi^L_{\dot{\bar{u}}}\left(t,\bar{u},\dot{\bar{u}}\right)+\bar{\sigma}^U\Phi^U_{\dot{\bar{u}}}\left(t,\bar{u},\dot{\bar{u}}\right)+\bar{\xi}^T h_{\dot{\bar{u}}}\left(t,\bar{u},\dot{\bar{u}}\right)\right\},$$
$$\bar{\xi}^T h\left(t,\bar{u},\dot{\bar{u}}\right)=0, t\in\mathbb{A}.$$

Hence, it follows that $(\bar{u},\bar{\sigma},\bar{\xi})$ is a feasible point for (IMD) and corresponding objective function of (IVP) and (IMD) are equal at $\bar{u}$.

Next, if possible, assume that $(\bar{u},\bar{\sigma},\bar{\xi})$ is not an LU optimal point for (IMD). Then, there exists another feasible point $(\bar{v},\bar{\sigma},\bar{\xi})\in\Omega$ such that

$$\int_a^b \phi\left(t,\bar{u},\dot{\bar{u}}\right)\,\mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \phi\left(t,\bar{v},\dot{\bar{v}}\right)\,\mathrm{d}t. \tag{4.11}$$

Since $\bar{u}$ and $(\bar{v},\bar{\sigma},\bar{\xi})$ are the feasible points of (IVP) and (IMD), respectively, and all the hypotheses of weak duality hold for all feasible points $(\bar{v},\bar{\sigma},\bar{\xi})$, then from the weak duality theorem 4.2, we have

$$\int_a^b \phi\left(t,\bar{u},\dot{\bar{u}}\right)\,\mathrm{d}t \not\prec_{\mathrm{LU}} \int_a^b \phi\left(t,\bar{v},\dot{\bar{v}}\right)\,\mathrm{d}t,$$

which contradicts (4.11). Thus, the proof. □

Theorem 4.5 (Strict converse duality)**.** *Let $\bar{u}$ and $(\bar{v},\bar{\sigma},\bar{\xi})$ are the feasible points of (IVP) and (IMD), respectively, such that*

$$\int_a^b \Phi\left(t,\bar{u},\dot{\bar{u}}\right)\,\mathrm{d}t = \int_a^b \Phi\left(t,\bar{v},\dot{\bar{v}}\right)\,\mathrm{d}t. \tag{4.12}$$

Further, assume that Φ and $\bar{\xi}^T h$ are strictly B-(p,r)-invex at $\bar{v}$ on Ω with respect to η. Then, $\bar{u}=\bar{v}$ and $\bar{v}$ is the LU *optimal point for (IVP).*

Proof. Suppose contrary to the result that $\bar{u}\neq\bar{v}$. From (4.12), we have

$$\begin{cases}\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\,\mathrm{d}t-\int_a^b \Phi^L\left(t,\bar{v},\dot{\bar{v}}\right)\,\mathrm{d}t=0\\ \int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right)\,\mathrm{d}t-\int_a^b \Phi^U\left(t,\bar{v},\dot{\bar{v}}\right)\,\mathrm{d}t=0.\end{cases}$$

From the above inequalities following holds

$$e^{r\left(\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t-\int_a^b \Phi^L\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}=1,$$

and

$$e^{r\left(\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t-\int_a^b \Phi^U\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}=1.$$

Since $(\bar{\sigma}^L,\bar{\sigma}^U)>0$, the above equalities implies that

$$\bar{\sigma}^L\left(e^{r\left(\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t-\int_a^b \Phi^L\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)=0, \tag{4.13}$$

and

$$\bar{\sigma}^U \left(e^{r\left(\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^U\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) = 0. \tag{4.14}$$

From the feasibility of $\bar{u}$ for (IVP) and $(\bar{v}, \bar{\sigma}, \bar{\xi})$ for (IMD), we get

$$\int_a^b \bar{\xi}^T h\left(t, \bar{u}, \dot{\bar{u}}\right) \mathrm{d}t \leq \int_a^b \bar{\xi}^T h\left(t, \bar{v}, \dot{\bar{v}}\right) \mathrm{d}t.$$

That is

$$\left(e^{r\left(\int_a^b \bar{\xi}^T h\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \bar{\xi}^T h\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) \leq 0. \tag{4.15}$$

Adding both sides of (4.13)–(4.15), the following inequality hold

$$\bar{\sigma}^L \left(e^{r\left(\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^L\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) + \bar{\sigma}^U \left(e^{r\left(\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^U\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) + \left(e^{r\left(\int_a^b \bar{\xi}^T h\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \bar{\xi}^T h\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) \leq 0. \tag{4.16}$$

From assumption that Φ is strictly B-(p,r)-invex function with respect to η at v on Ω. Then, we have the following

$$\frac{1}{r} b(\bar{u}, \bar{v}) \left(e^{r\left(\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^L\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) > \frac{1}{p} \int_a^b \left(e^{p\eta(t,\bar{u},\bar{v})} - 1 \right)^T \left(\Phi^L_{\bar{v}}\left(t, \bar{v}, \dot{\bar{v}}\right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\Phi^L_{\dot{\bar{v}}}\left(t, \bar{v}, \dot{\bar{v}}\right) \right) \right) \mathrm{d}t \tag{4.17}$$

and

$$\frac{1}{r} b(\bar{u}, \bar{v}) \left(e^{r\left(\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^U\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) > \frac{1}{p} \int_a^b \left(e^{p\eta(t,\bar{u},\bar{v})} - 1 \right)^T \left(\Phi^U_{\bar{v}}\left(t, \bar{v}, \dot{\bar{v}}\right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\Phi^U_{\dot{\bar{v}}}\left(t, \bar{v}, \dot{\bar{v}}\right) \right) \right) \mathrm{d}t. \tag{4.18}$$

Also, $\bar{\xi}^T h$ is strictly B-(p,r)-invex function with respect to η at v on Ω, which implies

$$\frac{1}{r} b(\bar{u}, \bar{v}) \left(e^{r\left(\int_a^b \bar{\xi}^T h\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \bar{\xi}^T h\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) > \frac{1}{p} \int_a^b \left(e^{p\eta(t,\bar{u},\bar{v})} - 1 \right)^T \left(\bar{\xi}^T h_{\bar{v}}\left(t, \bar{v}, \dot{\bar{v}}\right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\bar{\xi}^T h_{\dot{\bar{v}}}\left(t, \bar{v}, \dot{\bar{v}}\right) \right) \right) \mathrm{d}t. \tag{4.19}$$

Multiplying (4.17) by $\bar{\sigma}^L$ and (4.18) by $\bar{\sigma}^U$, respectively and adding both sides of resultant inequalities, we obtain

$$\frac{1}{r} b(\bar{u}, \bar{v}) \left[\bar{\sigma}^L \left(e^{r\left(\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^L\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) + \bar{\sigma}^U \left(e^{r\left(\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^U\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) \right]$$
$$> \frac{1}{p} \int_a^b \left(e^{p\eta(t,\bar{u},\bar{v})} - 1 \right)^T \left(\left(\sigma^L \Phi^L_{\bar{v}} + \sigma^U \Phi^U_{\bar{v}} \right)\left(t, \bar{v}, \dot{\bar{v}}\right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\left(\sigma^L \Phi^L_{\dot{\bar{v}}} + \sigma^U \Phi^U_{\dot{\bar{v}}} \right)\left(t, \bar{v}, \dot{\bar{v}}\right) \right) \right) \mathrm{d}t. \tag{4.20}$$

Adding both sides of (4.19) and (4.20) and using (4.1), we get the following

$$\frac{1}{r} b\left(\bar{u}, \bar{v}\right) \left[\bar{\sigma}^L \left(e^{r\left(\int_a^b \Phi^L\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^L\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) + \bar{\sigma}^U \left(e^{r\left(\int_a^b \Phi^U\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \Phi^U\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) + \left(e^{r\left(\int_a^b \bar{\xi}^T h\left(t,\bar{u},\dot{\bar{u}}\right) \mathrm{d}t - \int_a^b \bar{\xi}^T h\left(t,\bar{v},\dot{\bar{v}}\right) \mathrm{d}t\right)} - 1 \right) \right] > 0,$$

which is contradiction to (4.16). Thus, $\bar{u} = \bar{v}$.

Next, assume that $\bar{v}$ is not an LU optimal point for (IVP). Then, there exists another feasible point $u \in \omega$ such that

$$\int_a^b \phi(t, u, \dot{u})\, \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \phi(t, \bar{v}, \dot{\bar{v}})\, \mathrm{d}t. \tag{4.21}$$

Since u and $(\bar{v}, \bar{\sigma}, \bar{\xi})$ are the feasible points of (IVP) and (IMD), respectively, then from the weak duality Theorem 4.2, we have

$$\int_a^b \phi(t, u, \dot{u})\, \mathrm{d}t \not\prec_{\mathrm{LU}} \int_a^b \phi(t, \bar{v}, \dot{\bar{v}})\, \mathrm{d}t,$$

which contradicts (4.21). Thus, $\bar{v}$ is an LU optimal point for (IVP) and this completes the proof. □

5. Wolfe-type duality

We present the Wolfe-type dual for the considered (IVP) as follows:

$$\text{(IWD)} \qquad \max \int_a^b \left(\Phi + \xi^T h\right)(t, v, \dot{v})\, \mathrm{d}t$$

subject to

$$\begin{aligned} & v(a) = \gamma, v(b) = \delta. \\ & \sigma^L \Phi_v^L(t, v, \dot{v}) + \sigma^U \Phi_v^U(t, v, \dot{v}) + \xi^T h_v(t, v, \dot{v}) \\ & \qquad = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\sigma^L \Phi_{\dot{v}}^L(t, v, \dot{v}) + \sigma^U \Phi_{\dot{v}}^U(t, v, \dot{v}) + \xi^T h_{\dot{v}}(t, v, \dot{v})\right\}, \\ & \sigma = (\sigma^L, \sigma^U) \geq 0, \sigma^L + \sigma^U = 1, \xi \geq 0. \end{aligned} \tag{5.1}$$

where

$$\int_a^b (\Phi + \xi^T h)(t, v, \dot{v})\, \mathrm{d}t = \left[\int_a^b (\Phi^L + \xi^T h)(t, v, \dot{v})\, \mathrm{d}t, \int_a^b (\Phi^U + \xi^T h)(t, v, \dot{v})\, \mathrm{d}t\right]$$

is an interval-valued function.

Let $\Omega_{\mathrm{WD}} = \{(v, \sigma, \xi) : v \in C(\mathbb{A}, \mathbb{R}^q), \sigma \in \mathbb{R}^2, \xi \in \mathbb{R}^s$, satisfying the constraints of (IWD), $\forall t \in \mathbb{A}\}$ be the set of all feasible points of (IWD).

Definition 5.1. Let $(\bar{v}, \bar{\sigma}, \bar{\xi})$ be a feasible point of the dual problem (IWD). Then, $(\bar{v}, \bar{\sigma}, \bar{\xi})$ is an LU optimal point of the dual problem (IWD) if there exists no (v, σ, ξ) such that

$$\int_a^b \left(\Phi(t, \bar{v}, \dot{\bar{v}}) + \bar{\xi}^T h(t, \bar{v}, \dot{\bar{v}})\right) \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \left(\Phi(t, v, \dot{v}) + \xi^T h(t, v, \dot{v})\right) \mathrm{d}t.$$

Theorem 5.2 (Weak duality). *Let u and (v, σ, ξ) be the feasible points for (IVP) and (IWD), respectively. Assume that $\sigma = (\sigma^L, \sigma^U) > 0$, $\sigma^L + \sigma^U = 1$, $\xi^T \geq 0$ and $\Phi + \xi^T h$ is B-(p, r)-invex function with respect to η at v on Ω_{WD}. Then, the following can not hold*

$$\int_a^b \Phi(t, u, \dot{u})\, \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \left(\Phi(t, v, \dot{v}) + \xi^T h(t, v, \dot{v})\right) \mathrm{d}t.$$

Proof. Suppose contrary to the result that

$$\int_a^b \Phi(t, u, \dot{u})\, \mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \left(\Phi(t, v, \dot{v}) + \xi^T h(t, v, \dot{v})\right) \mathrm{d}t.$$

From (IVP), $h(t,u,\dot{u})\ \mathrm{d}t \leq 0$ and h is continously differential function. Moreover, $\xi^T \geq 0$ then we have

$$\int_a^b \xi^T h(t,u,\dot{u})\ \mathrm{d}t \leq 0.$$

Then, it follows that

$$\begin{cases} \int_a^b \left(\Phi^L(t,u,\dot{u}) + \xi^T h(t,u,\dot{u})\right)\ \mathrm{d}t < \int_a^b (\Phi^L(t,v,\dot{v}) + \xi^T h(t,v,\dot{v}))\,\mathrm{d}t \\ \int_a^b (\Phi^U(t,u,\dot{u}) + \xi^T h(t,u,\dot{u}))\,\mathrm{d}t < \int_a^b (\Phi^U(t,v,\dot{v}) + \xi^T h(t,v,\dot{v}))\,\mathrm{d}t \end{cases}$$

or

$$\begin{cases} \int_a^b (\Phi^L(t,u,\dot{u}) + \xi^T h(t,u,\dot{u}))\,\mathrm{d}t \leq \int_a^b (\Phi^L(t,v,\dot{v}) + \xi^T h(t,v,\dot{v}))\,\mathrm{d}t \\ \int_a^b (\Phi^U(t,u,\dot{u}) + \xi^T h(t,u,\dot{u}))\,\mathrm{d}t < \int_a^b (\Phi^U(t,v,\dot{v}) + \xi^T h(t,v,\dot{v}))\,\mathrm{d}t \end{cases}$$

or

$$\begin{cases} \int_a^b (\Phi^L(t,u,\dot{u}) + \xi^T h(t,u,\dot{u}))\,\mathrm{d}t < \int_a^b (\Phi^L(t,v,\dot{v}) + \xi^T h(t,v,\dot{v}))\,\mathrm{d}t \\ \int_a^b (\Phi^U(t,u,\dot{u}) + \xi^T h(t,u,\dot{u}))\,\mathrm{d}t \leq \int_a^b (\Phi^U(t,v,\dot{v}) + \xi^T h(t,v,\dot{v}))\,\mathrm{d}t. \end{cases}$$

We can equivalently write the above inequalities as

$$\begin{cases} \left(e^{r\left(\int_a^b \left(\Phi^L + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^L + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0 \\ \left(e^{r\left(\int_a^b \left(\Phi^U + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^U + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0 \end{cases}$$

or

$$\begin{cases} \left(e^{r\left(\int_a^b \left(\Phi^L + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^L + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) \leq 0 \\ \left(e^{r\left(\int_a^b \left(\Phi^U + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^U + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0 \end{cases}$$

or

$$\begin{cases} \left(e^{r\left(\int_a^b \left(\Phi^L + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^L + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0 \\ \left(e^{r\left(\int_a^b \left(\Phi^U + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^U + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) \leq 0. \end{cases}$$

Since $(\sigma^L, \sigma^U) > 0$, the above inequalities implies that

$$\sigma^L \left(e^{r\left(\int_a^b \left(\Phi^L + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^L + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) + \sigma^U \left(e^{r\left(\int_a^b \left(\Phi^U + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^U + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) < 0. \tag{5.2}$$

From assumption $\Phi + \xi^T h$ is B-(p,r)-invex function with respect to η at v on Ω, we have the following

$$\begin{aligned} &\frac{1}{r} b(u,v) \left(e^{r\left(\int_a^b \left(\Phi^L + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^L + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) \\ &\qquad \geq \frac{1}{p} \int_a^b \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\left(\Phi_v^L + \xi^T h_v\right)(t,v,\dot{v}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\Phi_{\dot{v}}^L + \xi^T h_{\dot{v}}\right)(t,v,\dot{v})\right)\right) \mathrm{d}t, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} &\frac{1}{r} b(u,v) \left(e^{r\left(\int_a^b \left(\Phi^U + \xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_a^b \left(\Phi^U + \xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) \\ &\qquad \geq \frac{1}{p} \int_a^b \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\left(\Phi_v^U + \xi^T h_v\right)(t,v,\dot{v}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\Phi_{\dot{v}}^U + \xi^T h_{\dot{v}}\right)(t,v,\dot{v})\right)\right) \mathrm{d}t. \end{aligned} \tag{5.4}$$

Multiplying (5.3) by σ^L and (5.4) by σ^U, respectively and then adding the resultant inequalities with the fact that $\sigma^L + \sigma^U = 1$ and using (5.1), we obtain

$$\frac{\sigma^L}{r} b(u,v)\left(e^{r\left(\int_a^b\left(\Phi^L+\xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t-\int_a^b\left(\Phi^L+\xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)}-1\right) + \frac{\sigma^U}{r} b(u,v)\left(e^{r\left(\int_a^b\left(\Phi^U+\xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t-\int_a^b\left(\Phi^U+\xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)}-1\right) \geq 0.$$

Since $b(u,v) > 0$, the above inequality yield

$$\sigma^L\left(e^{r\left(\int_a^b\left(\Phi^L+\xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t-\int_a^b\left(\Phi^L+\xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)}-1\right) + \sigma^U\left(e^{r\left(\int_a^b\left(\Phi^U+\xi^T h\right)(t,u,\dot{u})\,\mathrm{d}t-\int_a^b\left(\Phi^U+\xi^T h\right)(t,v,\dot{v})\,\mathrm{d}t\right)}-1\right) \geq 0,$$

which is contradiction to (5.2). Hence, the proof is complete. □

Example 5.3. Let $\mathbb{A} = [0,1]$ and $C(\mathbb{A},[0,1])$ denotes the space of all piecewise smooth functions $u : \mathbb{A} \to [0,1]$. Consider the following interval-valued variational problem

$$\text{(P3)} \qquad \min \int_0^1 \Phi\,(t,u,\dot{u})\;\mathrm{d}t = \left[\int_0^1 \left(u^2-2u+3\right)\mathrm{d}t, \int_0^1 \left(u^2-3u+5\right)\mathrm{d}t\right]$$

subject to

$$u(0)=0, u(1)=1,$$
$$2(u-1)\leq 0, t\in\mathbb{A}.$$

Let $\omega = \{u \in C(\mathbb{A},[0,1]) : u(0)=0, u(1)=1 \text{ and } 2(u-1)\leq 0, t\in\mathbb{A}\}$ be the feasible set for (P3). From the formulation of (P3) and definition of interval-valued function, it is clear that, $\phi^L(t,u,\dot{u}) = u^2-2u+3, \phi^U(t,u,\dot{u}) = u^2-3u+5$ and $h(t,u,\dot{u}) = 2(u-1)$.

The corresponding Wolfe type-dual is

$$\text{(WD)} \quad \max \int_0^1 (\Phi+\xi h)(t,v,\dot{v})\;\mathrm{d}t = \left[\int_0^1 \left((v^2-2v+3)+2\xi(v-1)\right)\,\mathrm{d}t, \int_0^1 \left((v^2-3v+5)+2\xi(v-1)\right)\,\mathrm{d}t\right],$$

subject to

$$v(0)=0, v(1)=1,$$
$$2(v-1)\leq 0, t\in\mathbb{A},$$
$$\sigma^L(2v-2)+\sigma^U(2v-3)+2\xi = 0.$$

At $v = 0$,

$$-2\sigma^L - 3\sigma^U + 2\xi = 0.$$
$$\implies \sigma^L = 1/2, \sigma^U = 1/2, \xi = 5/4.$$
$$\int_0^1 2\xi(t)(v-1)\,\mathrm{d}t \geq 0, t\in\mathbb{A}.$$

Then,

$$\int_0^1 (\Phi+\xi h)(t,v,\dot{v})\;\mathrm{d}t = \left[\int_0^1 \left(v^2+\frac{v}{2}+\frac{1}{2}\right)\,\mathrm{d}t, \int_0^1 \left(v^2-\frac{v}{2}+\frac{5}{2}\right)\,\mathrm{d}t\right].$$

From the definition of interval-valued function it implies that

$$(\phi^L + \xi)h\,(t, v, \dot{v}) = \left(v^2 + \frac{v}{2} + \frac{1}{2}\right),$$

and

$$(\phi^U + \xi h)\,(t, v, \dot{v}) = \left(v^2 - \frac{v}{2} + \frac{5}{2}\right).$$

Let ω_{WD} be the feasible set of (WD). $\omega_{\text{WD}} = \{(v, \sigma, \xi) : v \in C(\mathbb{A}, [0, 1]), \sigma \in \mathbb{R}^2, \xi \in \mathbb{R}$, satisfying the constraints of (WD), $t \in \mathbb{A}\}$. Then, we denote the set $W = \{v \in C(\mathbb{A}, [0, 1]) : (v, \sigma, \xi) \in \omega_{\text{WD}}\}$ and $\Omega = \omega \cup W$.

We observe that $(v(t), \sigma(t), \xi(t)) = (0, (1/2, 1/2), 5/4)$ is a feasible point for the considered Wolfe type dual and objective value at this feasible point is $[0.5, 2.5]$. Also, $u(t) = t^3$ is a feasible point for the primal problem and the corresponding objective value is $[2.64, 4.39]$.

Let $\eta : \mathbb{A} \times [0, 1] \times [0, 1] \to [0, 1]$ defined as $\eta(t, u, v) = \ln(1 + u - v)$, and $b : C(\mathbb{A}, [0, 1]) \times C(\mathbb{A}, [0, 1]) \to \mathbb{R}_+ \backslash \{0\}$ be defined as $b(u, v) = uv + 1$.

Next, we show that $\phi + \xi h$ is B-(p, r)-invex function at $v = 0$ with respect to above defined η.

First we calculate four cases for function the $\phi^L + \xi h$ as follows:

Case (i): $p = 1, r = 3$.

$$\begin{aligned}
&\frac{1}{r} b\,(u, v) \left(e^{r\left(\int_0^1 \left(\phi^L + \xi h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_0^1 \left(\phi^L + \xi h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) \\
&\quad - \frac{1}{p}\int_0^1 \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\left(\phi_v^L + \xi h_v\right)(t, v, \dot{v}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\phi_{\dot{v}}^L + \xi h_{\dot{v}}\right)(t, v, \dot{v})\right)\right) \mathrm{d}t, \\
&\quad = \frac{1}{3}(uv + 1) \left(e^{3\left(\int_0^1 \left(u^2 + \frac{u}{2} + \frac{1}{2}\right) \mathrm{d}t - \int_0^1 \left(v^2 + \frac{v}{2} + \frac{1}{2}\right) \mathrm{d}t\right)} - 1\right) - \int_0^1 \left(e^{\ln(1+u-v)} - 1\right)^T \left(2v + \frac{1}{2}\right) \mathrm{d}t, \\
&\quad = \frac{1}{3}\left(e^{3\int_0^1 \left(u^2 + \frac{u}{2}\right) \mathrm{d}t} - 1\right) - \frac{1}{2}\int_0^1 u\,\mathrm{d}t \geq 0.
\end{aligned}$$

Case (ii): $p = 0, r = 3$.

$$\begin{aligned}
&\frac{1}{r} b(u, v) \left(e^{r\left(\int_0^1 \left(\phi^L + \xi h\right)(t,u,\dot{u})\,\mathrm{d}t - \int_0^1 \left(\phi^L + \xi h\right)(t,v,\dot{v})\,\mathrm{d}t\right)} - 1\right) \\
&\quad - \int_0^1 (\eta(t, u, v))^T \left(\left(\phi_v^L + \xi h_v\right)(t, v, \dot{v}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\phi_{\dot{v}}^L + \xi h_{\dot{v}}\right)(t, v, \dot{v})\right)\right) \mathrm{d}t, \\
&\quad = \frac{1}{3}\left(e^{3\int_0^1 \left(u^2 + \frac{u}{2}\right) \mathrm{d}t} - 1\right) - \frac{1}{2}\int_0^1 \ln(u + 1)\,\mathrm{d}t \geq 0.
\end{aligned}$$

Case (iii): $p = 1, r = 0$.

$$\begin{aligned}
&b(u, v) \left(\int_0^1 \left(\phi^L + \xi h\right)(t, u, \dot{u})\,\mathrm{d}t - \int_0^1 \left(\phi^L + \xi h\right)(t, v, \dot{v})\,\mathrm{d}t\right) \\
&\quad - \frac{1}{p}\int_0^1 \left(e^{p\eta(t,u,v)} - 1\right)^T \left(\left(\phi_v^L + \xi h_v\right)(t, v, \dot{v}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\phi_{\dot{v}}^L + \xi h_{\dot{v}}\right)(t, v, \dot{v})\right)\right) \mathrm{d}t, \\
&\quad = \int_0^1 u^2\,\mathrm{d}t \geq 0.
\end{aligned}$$

Table 3. Resultant inequalities for functions $\phi^U + \xi h$

$\phi^U + \xi h$	$p = 1, r = 3$	$\frac{1}{3}\left(e^{3\int_0^1 \left(u^2 - \frac{u}{2}\right) \mathrm{d}t} - 1\right) + \frac{1}{2}\int_0^1 u \, \mathrm{d}t \geq 0$
	$p = 0, r = 3$	$\frac{1}{3}\left(e^{3\int_0^1 \left(u^2 - \frac{u}{2}\right) \mathrm{d}t} - 1\right) + \frac{1}{2}\int_0^1 \ln(u+1) \, \mathrm{d}t \geq 0$
	$p = 1, r = 0$	$\int_0^1 u^2 \, \mathrm{d}t \geq 0$
	$p = 0, r = 0$	$\int_0^1 \left(u^2 - \frac{u}{2}\right) \mathrm{d}t + \frac{1}{2}\int_0^1 \ln(u+1) \, \mathrm{d}t \geq 0$

Case (iv): $p = 0, r = 0$**.**

$$
\begin{aligned}
b(u,v) & \left(\int_0^1 \left(\phi^L + \xi h\right)(t,u,\dot{u}) \ \mathrm{d}t - \int_0^1 \left(\phi^L + \xi h\right)(t,v,\dot{v}) \ \mathrm{d}t\right) \\
& - \int_0^1 (\eta(t,u,v))^T \left(\left(\phi_v^L + \xi h_v\right)(t,v,\dot{v}) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\phi_{\dot{v}}^L + \xi h_{\dot{v}}\right)(t,v,\dot{v})\right)\right) \mathrm{d}t, \\
& = \int_0^1 \left(u^2 + \frac{u}{2}\right) \mathrm{d}t - \frac{1}{2}\int_0^1 \ln(u+1) \, \mathrm{d}t \geq 0.
\end{aligned}
$$

From above four cases it is clear that $\phi^L + \xi h$ is B-$(1,3)$-invex function. Now, to show that $\phi + \xi h$ is B-$(1,3)$-invex, we have to show that $\phi^U + \xi h$ is B-$(1,3)$-invex. All the four cases of Definition 2.1 have been calculated for function $\phi^U + \xi h$ and listed in Table 3.

From Table 3 it can be seen that function $\phi^U + \xi h$ is B-$(1,3)$-invex with respect to η at $v = 0$. Thus, $\phi + \xi h$ is B-$(1,3)$-invex function with respect to η at $v = 0$.

Moreover, $u(t) = t^3$ and $(v(t), \sigma(t), \xi(t)) = (0, (1/2, 1/2), 5/4)$ are the feasible points for (P3) and (WD), then we observe that

$$
\int_0^1 \Phi\left(t,u,\dot{u}\right) \ \mathrm{d}t \prec_{\mathrm{LU}} \int_0^1 \left(\Phi\left(t,v,\dot{v}\right) + \xi^T h\left(t,v,\dot{v}\right)\right) \mathrm{d}t.
$$

Hence, the weak duality theorem is verified.

Theorem 5.4 (Strong duality)**.** *Let $\bar{u}$ be an LU optimal point for (IVP) and further, assume that Kuhn–Tucker constraint qualification is satisfied at $\bar{u}$. Then there exists piecewise smooth functions $\bar{\sigma} : \mathbb{A} \to \mathbb{R}_+^2, \bar{\sigma} \neq 0$ and $\bar{\xi} : \mathbb{A} \to \mathbb{R}^s, \bar{\xi} \geq 0$ such that $(\bar{u}, \bar{\sigma}, \bar{\xi})$ is a feasible point for (IWD) and the objective function of (IVP) and (IWD) have the same value at $\bar{u}$ and $(\bar{u}, \bar{\sigma}, \bar{\xi})$, respectively. If also the weak duality hold for all feasible points $(\bar{v}, \bar{\sigma}, \bar{\xi})$ of (IWD), then $(\bar{u}, \bar{\sigma}, \bar{\xi})$ is an LU optimal point for (IWD).*

Proof. Since $\bar{u}$ is an LU optimal point for (IVP), then there exists piecewise smooth functions $\bar{\sigma} : \mathbb{A} \to \mathbb{R}_+^2, \bar{\sigma} \neq 0$ and $\bar{\xi} : \mathbb{A} \to \mathbb{R}^s, \bar{\xi} \geq 0$ such that

$$
\bar{\sigma}^L \Phi_{\bar{u}}^L\left(t,\bar{u},\dot{\bar{u}}\right) + \bar{\sigma}^U \Phi_{\bar{u}}^U\left(t,\bar{u},\dot{\bar{u}}\right) + \bar{\xi}^T h_{\bar{u}}\left(t,\bar{u},\dot{\bar{u}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\bar{\sigma}^L \Phi_{\dot{\bar{u}}}^L\left(t,\bar{u},\dot{\bar{u}}\right) + \bar{\sigma}^U \Phi_{\dot{\bar{u}}}^U\left(t,\bar{u},\dot{\bar{u}}\right) + \bar{\xi}^T h_{\dot{\bar{u}}}\left(t,\bar{u},\dot{\bar{u}}\right)\right\}, \tag{5.5}
$$

$$
\bar{\xi}^T h\left(t,\bar{u},\dot{\bar{u}}\right) = 0, t \in \mathbb{A}. \tag{5.6}
$$

Hence, it follows that $(\bar{u}, \bar{\sigma}, \bar{\xi})$ is a feasible point for (IWD) and corresponding objective function of (IVP) and (IWD) are equal at $\bar{u}$.

Next, we show that $(\bar{u},\bar{\sigma},\bar{\xi})$ is an LU optimal point for (IWD). If possible, assume that $(\bar{u},\bar{\sigma},\bar{\xi})$ is not an LU optimal point for (IWD). Then, there exists another feasible point $(\bar{v},\bar{\sigma},\bar{\xi}) \in \Omega_{\mathrm{WD}}$ such that

$$\int_a^b \left(\Phi\left(t,\bar{u},\dot{\bar{u}}\right)+\bar{\xi}^T h\left(t,\bar{u},\dot{\bar{u}}\right)\right)\,\mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \left(\Phi\left(t,\bar{v},\dot{\bar{v}}\right)+\bar{\xi}^T h\left(t,\bar{v},\dot{\bar{v}}\right)\right)\,\mathrm{d}t. \tag{5.7}$$

Since $\bar{u}$ and $(\bar{v},\bar{\sigma},\bar{\xi})$ are the feasible points of (IVP) and (IWD), respectively, and all the hypotheses of weak duality hold for all feasible points $(\bar{v},\bar{\sigma},\bar{\xi})$, then from the weak duality Theorem 5.2, we have

$$\int_a^b \left(\Phi\left(t,\bar{u},\dot{\bar{u}}\right)+\bar{\xi}^T h\left(t,\bar{u},\dot{\bar{u}}\right)\right)\,\mathrm{d}t \nprec_{\mathrm{LU}} \int_a^b \left(\Phi\left(t,\bar{v},\dot{\bar{v}}\right)+\bar{\xi}^T h\left(t,\bar{v},\dot{\bar{v}}\right)\right)\,\mathrm{d}t,$$

which contradicts (5.7). Thus, the proof. □

Theorem 5.5 (Strict converse duality)**.** *Let $\bar{u}$ and $(\bar{v},\bar{\sigma},\bar{\xi})$ are the feasible points of (IVP) and (IWD), respectively. Assume that $\Phi+\bar{\xi}^T h$ is strictly B-(p,r)-invex function at $\bar{v}$ on Ω with respect to η and further, assume that following holds*

$$\int_a^b \left(\Phi+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\,\mathrm{d}t = \int_a^b \left(\Phi+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\,\mathrm{d}t. \tag{5.8}$$

Then, $\bar{u}=\bar{v}$ and $\bar{v}$ is an optimal point for (IVP).

Proof. Suppose contrary to the result that $\bar{u}\neq\bar{v}$. Then, by (5.8), it follows that

$$\left(\int_a^b \left(\Phi^L+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\,\mathrm{d}t\right)-\left(\int_a^b \left(\Phi^L+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\,\mathrm{d}t\right)=0.$$
$$\left(\int_a^b \left(\Phi^U+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\,\mathrm{d}t\right)-\left(\int_a^b \left(\Phi^U+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\,\mathrm{d}t\right)=0.$$

The above equalities further implies that

$$\left(e^{r\left(\int_a^b\left(\Phi^L+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)-\left(\int_a^b\left(\Phi^L+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)=0, \tag{5.9}$$

and

$$\left(e^{r\left(\int_a^b\left(\Phi^U+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)-\left(\int_a^b\left(\Phi^U+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)=0. \tag{5.10}$$

Multiplying both sides of (5.9) and (5.10) by $\bar{\sigma}^L$ and $\bar{\sigma}^U$, respectively and then adding, we get

$$\begin{aligned}\bar{\sigma}^L &\left(e^{r\left(\int_a^b\left(\Phi^L+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)-\left(\int_a^b\left(\Phi^L+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)\\ &+\bar{\sigma}^U\left(e^{r\left(\int_a^b\left(\Phi^U+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)-\left(\int_a^b\left(\Phi^U+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)=0.\end{aligned} \tag{5.11}$$

Also, $\Phi+\bar{\xi}^T h$ is strict B-(p,r)-invex function at $\bar{v}$ on Ω with respect to η, then we have

$$\begin{aligned}&\frac{1}{r}b\left(\bar{u},\bar{v}\right)\left(e^{r\left(\int_a^b\left(\Phi^L+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t-\int_a^b\left(\Phi^L+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)\\ &\quad>\frac{1}{p}\int_a^b\left(e^{p\eta(t,\bar{u},\bar{v})}-1\right)^T\left(\left(\Phi^L_{\bar{v}}+\bar{\xi}^T h_{\bar{v}}\right)\left(t,\bar{v},\dot{\bar{v}}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\Phi^L_{\dot{\bar{v}}}+\bar{\xi}^T h_{\dot{\bar{v}}}\right)\left(t,\bar{v},\dot{\bar{v}}\right)\right)\right)\mathrm{d}t,\end{aligned} \tag{5.12}$$

and

$$\frac{1}{r}b\left(\bar{u},\bar{v}\right)\left(e^{r\left(\int_a^b\left(\Phi^U+\bar{\xi}^T h\right)\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t-\int_a^b\left(\Phi^U+\bar{\xi}^T h\right)\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)$$

$$> \frac{1}{p}\int_a^b \left(e^{p\eta(t,\bar{u},\bar{v})}-1\right)^T \left(\left(\Phi^U_{\bar{v}}+\bar{\xi}^T h_{\bar{v}}\right)(t,\bar{v},\dot{\bar{v}})-\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\Phi^U_{\dot{\bar{v}}}+\bar{\xi}^T h_{\dot{\bar{v}}}\right)(t,\bar{v},\dot{\bar{v}})\right)\right)\mathrm{d}t. \tag{5.13}$$

Multiplying $\bar{\sigma}^L, \bar{\sigma}^U$ in (5.12) and (5.13), respectively and adding the resultant inequalities with $\bar{\sigma}^L+\bar{\sigma}^U=1$ and using (5.1), the following is the result

$$\frac{1}{r}b(\bar{u},\bar{v})\left[\bar{\sigma}^L\left(e^{r\left(\int_a^b(\Phi^L+\bar{\xi}^T h)\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)-\left(\int_a^b(\Phi^L+\bar{\xi}^T h)\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)\right.$$
$$\left.+\,\bar{\sigma}^U\left(e^{r\left(\int_a^b(\Phi^U+\bar{\xi}^T h)\left(t,\bar{u},\dot{\bar{u}}\right)\mathrm{d}t\right)-\left(\int_a^b(\Phi^U+\bar{\xi}^T h)\left(t,\bar{v},\dot{\bar{v}}\right)\mathrm{d}t\right)}-1\right)\right]>0,$$

which is contraction to (5.11).

Next, assume that $\bar{v}$ is not an LU optimal point for (IVP). Then, there exists another feasible point $u\in\omega$ such that

$$\int_a^b \Phi(t,u,\dot{u})\,\mathrm{d}t \prec_{\mathrm{LU}} \int_a^b \Phi(t,\bar{v},\dot{\bar{v}})\,\mathrm{d}t. \tag{5.14}$$

Since u and $(\bar{v},\bar{\sigma},\bar{\xi})$ are the feasible points of (IVP) and (IWD), respectively, then from the weak duality Theorem 5.2, we have

$$\int_a^b \Phi(t,u,\dot{u})\,\mathrm{d}t \not\prec_{\mathrm{LU}} \int_a^b \left(\Phi(t,\bar{v},\dot{\bar{v}})+\bar{\xi}^T h(t,\bar{v},\dot{\bar{v}})\right)\mathrm{d}t.$$

From the feasibility of u for (IVP), it implies that

$$\int_a^b \left(\Phi(t,u,\dot{u})+\bar{\xi}^T h(t,u,\dot{u})\right)\mathrm{d}t \not\prec_{\mathrm{LU}} \int_a^b \left(\Phi(t,\bar{v},\dot{\bar{v}})+\bar{\xi}^T h(t,\bar{v},\dot{\bar{v}})\right)\mathrm{d}t,$$

which contradicts (5.8). Thus, $\bar{v}$ is an LU optimal point for (IVP) and this completes the proof. □

Application: A production house of a company produces some goods and the company wishes to minimize the production cost. Production house has a range of total production cost as an interval-valued function as follows:

$$\left[\phi^L(t,u(t),\dot{u}(t)),\phi^U(t,u(t),\dot{u}(t))\right]=[u^2(t)-tu(t)+1,u^2(t)-2tu(t)+5],$$

where $u(t)$ is the level of output and $\dot{u}(t)$ is its rate of change. The production cost should be minimized subject to the constraint:

$$h(t,u(t),\dot{u}(t))=t-u(t)\leq 0.$$

The endpoints conditions are $u(0)=0$ and $u(1)=1$.

We have to find suitable output function of time which minimizes production cost. This problem can be formulated as an interval-valued variational problem as follows:

$$\min\int_0^1 \Phi(t,u(t),\dot{u}(t))\,\mathrm{d}t=\left[\int_0^1\left(u^2(t)-tu(t)+1\right)\mathrm{d}t,\int_0^1\left(u^2(t)-2tu(t)+5\right)\mathrm{d}t\right],$$

subject to

$$u(0)=0,u(1)=1,$$
$$t-u(t)\leq 0, t\in[0,1].$$

Let $\mathbb{A}=[0,1]$ and the set $\omega=\{u\in C(\mathbb{A},[0,1]): u(0)=0,u(1)=1,t-u(t)\leq 0,t\in\mathbb{A}\}$ is the feasible set for this problem. Clearly, $\bar{u}(t)=t$ is a feasible point. We see that hypothesis (3.1) and (3.2) of Theorem 3.2 is satisfied at feasible point $\bar{u}=0$ with $(\sigma^L,\sigma^U)=(1,2)$ and $\xi=t$.

Let $\eta:\mathbb{A}\times[0,1]\times[0,1]\to[0,1]$ defined as $\eta(t,u,\bar{u})=\frac{1}{2}(u(t)-t)$, and $b: C(\mathbb{A},[0,1])\times C(\mathbb{A},[0,1])\to\mathbb{R}_+\backslash\{0\}$ be defined as $b(u,\bar{u})=\frac{1}{2}$.

It can be shown that function ϕ is strictly B-$(1/2,1)$-invex and ξh is B-$(1/2,1)$-invex with respect to η at $\bar{u}(t)=t$. From Theorem 3.2 the feasible point $\bar{u}(t)=t$ is an LU optimal point for this problem. That is, to minimize production cost the suitable output function is $\bar{u}(t)=t$.

6. Conclusion

In this paper, the concept of B-(p,r)-invexity is extended to the class of interval-valued variational problem. Using this concept of generalized convexity on the functions involved, the necessary and sufficient optimality conditions are derived for the considered interval-valued variational problem. A real-world problem, explaining application of the sufficiency theorem, has been presented. Moreover, both Wolfe and Mond–Weir type dual problems have been defined for the problem and appropriate duality theorems have been established. Some non-trivial examples have also been presented at suitable places in order to give a better insight to the results established in the paper. To the best of our knowledge, the results derived in this paper are new in the area of interval-valued variational optimization problems.

In future, some interesting topics for further research to be carried out remains. It would be interesting to investigate whether these results are true for different class of interval-valued variational problems, like, nonconvex multiobjective fractional interval-valued variational problems and non-differentiable interval-valued variational problems with the assumptions of generalized invexity. We shall investigate these queries in future subsequent papers.

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