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ON THE SHORTAGE CONTROL IN A CONTINUOUS REVIEW (Q, r) INVENTORY POLICY USING α_L SERVICE-LEVEL

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Abstract. Popular measures of product availability in inventory systems seek to control different aspects of stock shortages. However, none of them simultaneously control all aspects of shortages, because stock shortages in inventory systems are complex random events. This paper analyzes the performance of α_L service measure, defined as the probability that stockouts do not occur during a replenishment cycle, to cover different aspects of stock shortages when used to design an optimal continuous review (Q, r) policy. We show that explicitly controlling the frequency of replenishment cycle stockouts, using the α_L service-level, allows to implicitly control the size of the stockouts at an arbitrary time, the size of accumulated backorders at an arbitrary time, and the duration of the replenishment cycle stockouts. However, the cost of controlling the frequency of replenishment cycle stockouts is greater than the cost of controlling the size of stockouts and the duration of the replenishment cycle stockouts.

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1. Introduction

The shortage in inventory systems is an inevitable random phenomena under real demand scenarios. This random phenomena can be classified and combined in a variety of ways. For example, from customer perception, frequent stockouts with large backorders are different from infrequent and long-lasting stockouts with small backorders. Depending on the product and the customer's reaction to the shortage, it is desirable that the decision-maker identify the shortage events that most deteriorate the customer service and that the inventory system should place special emphasis on controlling [9]. On the other hand, stock availability measures in inventory systems seek to control the shortage events in three main dimensions: frequency, size, and duration of stockouts. Shortage events can be seen as the simultaneous combination of these three dimensions, while popular availability measures control only one of these dimensions at a time. Although an availability measure observes only one dimension of the shortage, its control produces effects on the other aspects of the shortage.

Three popular measures of product availability in inventory systems are the probability (α) of not being out of stock at an arbitrary time, the fraction (β) of demand met directly from inventory on-hand in any period, and

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the fraction $(1-\gamma)$ of demand being on backorder each period [7,26,27]. From the definitions of these availability measures, it is easy to infer that the α service-level controls the frequency of the stockouts at an arbitrary time, the β service-level, which is referred to as *fill-rate*, controls the size of backorders at an arbitrary time, and the γ service-level controls the size of accumulated backorders at an arbitrary time. In the continuous review case, the α service-level is often defined in other ways, as the probability (α_L) that stockouts do not occur during a replenishment cycle [26]. Thus, we infer that the α_L service-level controls the frequency of replenishment cycle stockouts.

According to several authors [5,7,29,33] the β service-level definition is the most observed in practical settings. Consequently, it can be inferred that the size of the stockouts is the most important shortage dimension for inventory managers. However, as will be shown later, an optimal inventory policy that ensures a high level of β service does not guarantee a good performance in other dimensions of the shortage.

Other service level measures, such as the *expected duration of the stockouts* [2], and the *customer waiting time* [15,17,30] are considered in the literature. From the definitions of these availability measures, it is easy to infer that the expected duration of the stockouts and the customer waiting time control the stockouts duration.

The occurrence of shortage events can be reduced, in some of its dimensions, by implementing inventory policies that ensure high levels of stock availability. To determine the parameters of these policies, the literature has focused on two major groups of stochastic inventory control problems: the full cost model, and the partial cost model with a service level constraint (SLC). In the full cost model, the objective is to find the optimal parameters of an inventory policy which minimizes the sum of the holding, ordering, and shortages costs. Thus, to ensure high levels of stock availability, full cost models use high penalties for shortages. On the other hand, the service level approach introduces a service level constraint in place of the shortage cost. The goal is to deliver a specified level of service at a minimum ordering and holding costs.

Although most of the literature on stochastic inventory control problems focuses on full cost models, there are several reasons to adopt a SLC approach when the objective of the inventory system is to control stock shortages while minimizing the cost of inventory management. The first one is that full-cost optimization models assume linear cost functions for shortage costs which primary for analytical tractability rather than an accurate representation of reality [6]. Second, imposing a preset service level is a much more direct way to quantify and improve an inventory system's quality service performance [14]. Third, companies are very mindful of maintaining high service levels to satisfy customer demand, and these high levels have been shown to alter customer decisions [9,20]. Fourth, the backlogging cost is often very difficult to quantify in practice [8]. Mechanisms for determining target service levels (mainly fill-rate) have been proposed by Teunter *et al.* [32,33], and Thonemann *et al.* [34].

In this paper, we study the effect of an optimal continuous review (Q,r) policy with full-backorders and deterministic lead time that ensures a high α_L service-level on those shortage dimensions that are not explicitly controlled by the inventory policy. In particular, we studied the effect on the frequency of stockouts at arbitrary times, the size of backorders at arbitrary times, the size of accumulated backorders at arbitrary times, and the duration of stockouts. The converse is also done, *i.e.*, we examine the effect that an optimal continuous review (Q,r) policy that ensures a high service level other than α_L has on the frequency of replenishment cycle stockouts. We consider the fill-rate, the γ service-level, and the *fraction* (δ_L) *of the lead time without stockouts.* We determine the ordering relationship between the α_L service-level and the β, γ, and δ_L service levels. To determine the optimal parameters of the inventory policy, we adopt a SLC approach. Consequently, four SLC problems, one for each service level, are formulated as convex nonlinear optimization problems that ensure optimal solutions. The SLC models, denoted as α_L-SLC, β-SLC, γ-SLC, and δ_L-SLC, are based on the exact formulation of the objective function, as well as the exact expression for the α_L, β, γ, and δ_L service levels, respectively. We determine the ordering relationship between the optimal solution of α_L-SLC and the optimal solutions of β-SLC, γ-SLC, and δ_L-SLC, respectively. Some propositions allow us to infer that ensuring a high α_L service-level also ensures a high β, γ, and δ_L service levels and that the opposite is not true. In other words, and from a practical point of view, strongly limiting the frequency of replenishment cycle stockouts strongly limits the size of backorders at an arbitrary time, the size of accumulated backorders at an arbitrary time, and the duration of stockouts. Furthermore, we study the cost of providing a high α_L service-level and show that it is higher than the cost of providing a high β, γ, and δ_L service levels.

The main contribution of this paper is the following. We make a comparative study between inventory availability measures as well as between the optimal solution of SLC models, which had not been addressed in the literature under exact formulations. In particular, (i) we define a new ordering relationship between α_L and γ service levels, and between α_L and δ_L service levels, and (ii) we define a new ordering relationship between the optimal solutions of α_L-SLC and γ-SLC, and between the optimal solutions of α_L-SLC and δ_L-SLC.

The rest of this paper is structured as follows. In the next section, we review related works. In Section 3, we formulate the α_L, β, γ, and δ_L service levels under a continuous review (Q, r) policy with full-backorders and present the α_L-SLC, β-SLC, γ-SLC, and δ_L-SLC models. In Section 4, we define the ordering relationship between α_L service-level and β, γ and δ_L service levels, respectively. In Section 5, we define the ordering relationship between optimal solution of α_L-SLC and the optimal solution of β-SLC, γ-SLC, and δ_L-SLC models, respectively. In Section 6, we report computational results. Finally, in Section 7, we conclude with managerial insights and future extensions to this work.

2. Literature review

The study of service level measures to control the shortage events in inventory systems are focused on two main aspects: (i) formulation and characterization of these measures as functions of the inventory policy parameters; and (ii) definition of the optimal parameters of the inventory policy that explicitly ensures a particular service level measure.

Schneider [26] studies α, β, and γ service levels under three inventory policies: a continuous review (Q, r) policy, a periodic review (S, T) policy, and a periodic review (s, S, T) policy. For the first policy, the inventory is continuously reviewed, and a quantity Q is ordered when the inventory position (*i.e.*, inventory on-hand plus outstanding orders minus backorders) falls to or below the reorder point r. In the second policy, the inventory is reviewed every T units of time, and an amount S minus the inventory position is ordered, whereas in the third policy, that amount is ordered if the inventory position falls to or below the reorder point s. This author formulates exact and approximate mathematical expressions for α, β, and γ service levels, using discrete (Poisson) and continuous distributions demand and assuming deterministic lead time.

Under a continuous review (Q, r) policy with full-backorders, Zipkin [38] studies the mathematical properties of two often-used service-level measures, which allow defining γ and β service levels, respectively: the *expected backorders in steady-state* and the *average stockouts per unit time*. This author shows that the first service-level measure is jointly convex on the inventory policy parameters when the density function of the lead time demand is non-negative and that the second one is also jointly convex on the inventory policy parameters under some additional conditions. Zipkin [38] also studies some widely used approximations for these measures, which are known to be convex. From computational experiments, the author observes that these approximations are only acceptable when the lead time demand distribution is quite regular, *e.g.*, normal or Poisson, and the quantity order is large.

Zhang [37] uses a simpler approach than [38] to prove the joint convexity of the expected backorders in steady-state. This approach is based on the composition of convex functions instead of showing the non-negative definiteness of the Hessian matrix. Assuming a Poisson demand process, this author also proves the joint convexity of the average stockouts per unit time. The convexity properties of these service-level measures are also studied by Wang and Li [35] under a continuous review (Q, r) policy when the demand is modeled as a discrete process. These authors state that the approaches used by Zipkin [38] and Zhang [37] to prove the convexity of the expected backorders in steady-state and the average stockouts per unit time are not valid in the discrete case since the expressions of these measures for the continuous demand case are adequate approximations only when the order quantity is large. Using the definition and properties of a convex function on an integer lattice, the authors show that both the expected backorders in steady-state and the average stockout per unit time are jointly convex functions on the inventory policy parameters.

Akinniyi and Silver [2] study the duration of stockouts and determine their distribution function when demand is represented by a Wiener or Poisson process under deterministic lead time. They define a non-linear equation to determine the reorder point that ensures that the expected duration of stockouts is not greater than a specified fraction of the lead time. The study of customer waiting time has focused mainly on determining its distribution function under different inventory systems. A comprehensive review of exact and approximate procedures for determining the customers' waiting time distribution function can be found in Tempelmeier and Fischer [31].

With regard to determining the optimal parameters of an inventory policy using SLC problems, Federgruen and Zheng [12] study a (Q, r) policy with full-backorders and deterministic lead time under discrete demand. These authors formulate a full-cost optimization model and exploit the unimodality of the objective function to develop an efficient algorithm to obtain the optimal parameters of the inventory policy. This algorithm can be used for solving the Lagrangian relaxation of β-SLC problem since shortage costs parameters can be obtained from the corresponding Lagrangian multiplier of the service-level constraint.

Considering a (Q, r) policy with either continuous or periodic review, deterministic lead time, and continuous demand, Rosling [25] formulates a full-cost optimization problem and presents an algorithm to determine the optimal parameters of the inventory policy. This algorithm is based on proof that the objective function is pseudo-convex. Furthermore, Rosling [25] studies the α_L-SLC and β-SLC problems. A variation of the algorithm is used to solve the Lagrangian relaxation of these SLC problems, in which case the service-level constraint holds with equality under certain assumptions. From computational experiments and considering Gamma distribution, the author observes that the relative benefit of using the algorithm over the *Economic Order Quantity (EOQ)* solution is more noticeable for low values of the preset service level.

Agrawal and Seshadri [1] formulate a β-SLC model for a continuous review (Q, r) policy with full-backorders, deterministic lead time, and continuous demands. Without making assumptions about the convexity of the backorders, they derive bounds for the optimal solution that are independent of the lead time demand distribution. Chen and Krass [8] determine the optimal parameters of a periodic review (s, S) policy with full-backorders and deterministic lead time using SLC models. They formulate the α, β, and γ availability measure as minimal service-level measures, *i.e.*, as the minimal service level provided by the system in any period. Consequently, they formulate three SLC problems, one for each level of service, which is solved using a modified version of the Federgruen and Zheng [12] algorithm.

Under a continuous review (Q, r) policy with full-backorders, deterministc lead time, and normally distributed demand, Axsäter [3] reformulates the β-SLC problem so that the problem depends on a single parameter, aggregating the ordering cost, the holding cost per unit and unit time, the deterministic lead time, the demand per unit time and the variance. With the fill-rate service level constraint defined to equality, Axsäter [3] proposes to solve the reformulation using the algorithm of Rosling [25] or through a direct search procedure. Thus, this author solves the problem for different values of the parameter and different values of the preset service level, storing the results in a table. In this manner, the optimal parameters of the inventory policy can be obtained by linear interpolation using the data stored in the table or by special approximations for values outside of it.

A comparative study between α_L and β service levels, under a continuous review (Q, r) policy with full-backorders and deterministic lead time, was conducted by Zeng and Hayya [36]. To study the performance of both service level measures, they formulate two sets of optimization models. The first set considers the maximization of the service level as objective, subject to a budget (*budget-constrained problems*). The second set corresponds to SLC models. These formulations are made considering that (i) the expected backorders in steady-state are negligible, which induces a relaxation of α_L-SLC that decomposes the problem into the EOQ model, and the *stochastic reorder point* model; and (ii) a widely used approximation for the provided fill-rate, whereby the optimal parameters of the approximated β-SLC problem can be obtained through Karush–Kuhn–Tucker (KKT) conditions. Therefore, Zeng and Hayya [36] study a relaxation of α_L-SLC, and an approximation of β-SLC, since the latter is neither a relaxation nor a constriction of the original problem. Using four continuous distribution demands, they derive closed-form expressions in order to evaluate the performance of both service level measures. However, their conclusions are limited by the approximations made.

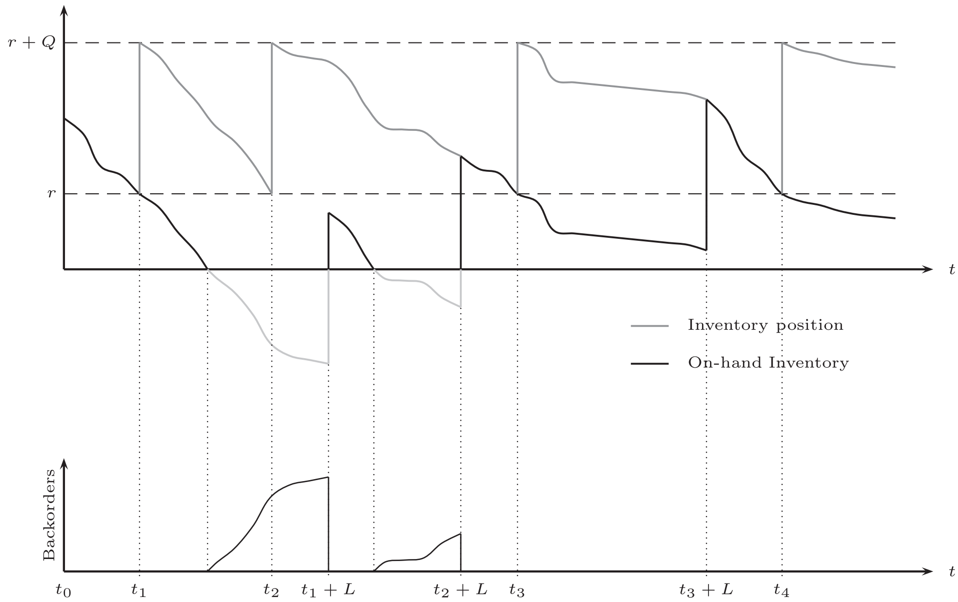

FIGURE 1. Continuous review (Q, r) policy with full-backorder and deterministic lead time.

It should be noted that our study considers an inventory policy under deterministic lead time. The management of situations where the lead time is stochastic is more difficult [4,13]. A comprehensive review on stochastic lead time inventory models can be found in Muthuraman *et al.* [18]. Furthermore, to the best of our knowledge, there are no studies of SLC problems under stochastic lead time. On the other hand, several issues in inventory control systems have gained the interest of many researchers in previous decades. In particular, deterioration, default risk and price sensitive demand (see *e.g.*, [19,21–23]). However, to the best of our knowledge, these issues have not been addressed under a SLC approach when the inventory policy is one of continuous review.

3. INVENTORY MODELS

Consider a wholesaler that supplies a single type of fast-moving customer goods (FMCG) to several retailers (customers) with independent demand, where FMCG are products with high demand volume or items with high inventory turnover. Examples include non-perishable food, toiletries, over-the-counter drugs, cleaning supplies, building supplies, and office supplies.

We assume that each customer follows a *strictly increasing non-negative demand*. Thus, the total demand is represented by a non-decreasing stochastic process with stationary increments and continuous sample paths, because FMCG are more representative of modeling the demand over time with a continuous distribution [4,10,11,24]. Let $D(t, t+\tau)$ be the total demand in the interval $(t, t+\tau]$ and $F_{D(\tau)}(x)$ be the cumulative distribution function of the total demand in the interval $[0, \tau]$. It should be noted that under stationary increments, $D(\tau) := D(0, \tau) = D(t, t+\tau)$ for any $t \geq 0$.

The wholesaler uses a continuous review (Q, r) policy with full-backorder and deterministic lead time, where an order quantity of constant size, Q, is placed whenever the inventory position (*i.e.*, inventory on-hand plus outstanding orders minus backorders) falls below a fixed reorder point, r, which arrives at a fixed $L > 0$ time unit later. Furthermore, all unmet demand is backordered. Figure 1 illustrates the continuous review (Q, r) policy with full-backorder, deterministic lead time, and strictly increasing non-negative demand.

Figure 1 shows that at time t_1 the inventory position reaches r, and a quantity Q is ordered, which arrives at $t_1 + L$. The same happens at times t_2, t_3, and t_4. It should be noted that the inventory position and the on-hand inventory are equal in the interval $[t_0, t_1]$, $[t_2 + L, t_3]$, and $[t_3 + L, t_4]$.

To complete the inventory policy, we assume that the wholesaler defines a single service level policy. In this paper, we consider four kinds of service-level definitions – the α_L service-level, which controls the frequency of the replenishment cycles stockouts and is independent of order quantity Q; the β service-level which controls the backorders at an arbitrary time; the γ service level which controls the accumulated backorders at an arbitrary time; and the δ_L service-level which controls the duration of stockout replenishment cycles. Thus, the objective of the wholesaler is to determine the optimal (Q, r) parameters, which minimize the sum of ordering and holding costs per unit time and to ensure a single service-level.

In what follows, we develop expressions for the sum of ordering and holding costs per unit time, for the α_L, β, γ, and δ_L service levels, and present the α_L, β, γ and δ_L service level constraints problems. The notations used in this section are summarized in Appendix A.

3.1. Average cost per unit time

We seek to minimize the average cost per unit time, which is the sum of ordering and holding costs per unit time. Let μ be the total average demand per unit time, let h be the holding cost per unit and unit time, and let S be the ordering cost. Consequently, the average cost per unit time is

$$\mathrm{AC}(Q, r) = S\frac{\mu}{Q} + h\mathbb{E}(\mathrm{OH}^{\infty}(Q, r)), \tag{3.1}$$

where $\mathbb{E}(\mathrm{OH}^{\infty}(Q, r))$ is the expected on-hand steady-state inventory. The first term of (3.1) corresponds to the ordering cost, and the second term is the holding cost, both measured per unit time.

In a continuous review (Q, r) policy with full-backorders and a deterministic lead time, we have that $\mathrm{OH}(t + L) = (\mathrm{IL}(t + L))^+ = (\mathrm{IP}(t) - D(t, t + L))^+ = \mathrm{IP}(t) - D(t, t + L) + (\mathrm{IP}(t) - D(t, t + L))^-$, where $\mathrm{OH}(t + L)$ denotes on-hand inventory at time $t + L$, $\mathrm{IL}(t + L)$ denotes the inventory level at time $t + L$, and $\mathrm{IP}(t)$ denotes the inventory position at time t. It should be noted that, in steady state, the inventory position is uniformly distributed in the interval $[r, r + Q]$, *i.e.*, $\mathrm{IP}(t) \sim U[r, r + Q]$. Consequently, the expected on-hand steady-state inventory is:

$$\mathbb{E}(\mathrm{OH}^{\infty}(Q, r)) = \frac{Q}{2} + r - \mu L + B(Q, r), \tag{3.2}$$

where $B(Q, r) = \mathbb{E}\left(\mathrm{IP}(t) - D(t, t + L)\right)^-$ is the steady-state backorders. Then, conditioning on the inventory position and changing the order of integration, it is easy to show that the expected backorders in steady-state is $B(Q, r) = \frac{1}{Q}\left(G(r) - G(r + Q)\right)$ with $G(v) = \int_v^{\infty} (x - v)(1 - F_{D(L)}(x))\, \mathrm{d}x$. Substituting in (3.2) and then in (3.1), the average cost per unit time is

$$\mathrm{AC}(Q, r) = S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L + B(Q, r)\right). \tag{3.3}$$

Zipkin [38] has shown that $B(Q, r)$ is jointly convex in Q and r when $f_{D(L)}(x) > 0$ for any $x > 0$, where $f_{D(L)}(x)$ is the density function of the lead time demand. Consequently, it is easy to show that $\mathrm{AC}(Q, r)$ is jointly convex in Q and r because it is the sum of convex functions.

3.2. Provided service levels

The *provided service level* is the availability of stock capable of providing an inventory system that operates under a given inventory policy. The provided service level is expressed as a function of the inventory policy parameters, in our case, as a function of the (Q, r) parameters.

Let $\alpha(Q, r)$ be the provided α service-level, *i.e.*, the probability of not being out of stock at an arbitrary time. The inventory system is in stockout at an arbitrary time $t+L$ if the inventory level in $t+L$ is strictly less than zero, *i.e.*, $\mathrm{IL}(t+L) < 0$. Consequently, the probability of not being out of stock at an arbitrary time under a continuous review (Q, r) policy is defines as $\alpha(Q, r) = \mathbb{P}(\mathrm{IL}(t+L) \geq 0)$ which is also interpreted as the fraction of time with a positive stock-on-hand known as *ready-rate* [4]. It is easy to show that $\alpha(Q, r) = \mathbb{P}(D(L) \leq \mathrm{IP}(t))$ because under a continuous review (Q, r) policy with strictly increasing non-negative demand we have $\mathrm{IL}(t+L) = \mathrm{IP}(t) - D(L)$. Thus, conditioning on the inventory position, $\mathrm{IP}(t)$, the provided α service-level is:

$$\alpha(Q, r) = \frac{1}{Q} \int_{r}^{r+Q} F_{D(L)}(y)\,\mathrm{d}y. \tag{3.4}$$

Let $\alpha_L(r)$ be the provided α_L service-level, *i.e.*, the probability that stockouts do not occur during a replenishment cycle. The inventory system is in stockout during a replenishment cycle if the demand during the lead time is strictly greater than the reorder point, *i.e.*, $D(L) > r$. Consequently, the probability that stockouts do not occur during a replenishment cycle is defined as:

$$\alpha_L(r) = \mathbb{P}(D(L) \leq r) = F_{D(L)}(r). \tag{3.5}$$

Let $\beta(Q, r)$ be the provided fill-rate defined as the fraction of demand met directly from on-hand inventory at an arbitrary time. Following Zipkin [38], the provided fill-rate is defined as

$$\beta(Q, r) = 1 - \frac{A(Q, r)}{\mu}, \tag{3.6}$$

where $A(Q, r)$ is the average stockouts per unit time. Demand per unit time is backorder in $t+L$ if the inventory level in $t+L$ is strictly less than zero. Therefore, assuming independence, the average stockouts per unit time is $A(Q, r) = \mu\mathbb{P}(\mathrm{IL}(t+L) < 0)$. Substituting in (3.6), it can be shown that $\beta(Q, r) = 1 - \mathbb{P}(\mathrm{IL}(t+L) < 0) = \alpha(Q, r)$. The main consequence of this equality is that the provided fill-rate, under a continuous review (Q, r) policy with strictly increasing non-negative demand, simultaneously controls frequency and size of backorders at an arbitrary time.

It should be noted that under pure Poisson and normally distributed demand, the fill-rate is equivalent to the ready-rate [4, 28]. We conclude that under strictly increasing non-negative demand, the fill-rate and ready-rate are also equivalent.

The fill-rate defined as (3.4) is a concave function because $A(Q, r)$ is jointly convex on Q and r when $f_{D(L)}(x)$ is non-increasing for any $x \geq r \geq \mathbb{E}(D(L))$. This is valid, *e.g.*, if $f_{D(L)}(x)$ is unimodal and symmetric (or skewed to the right) and the safety stock is non-negative [38].

Proposition 3.1. *$\beta(Q, r)$ is strictly increasing in Q and r.*

Proof. Proof is provided in Appendix B. □

Let $\gamma(Q, r)$ be the provided γ service-level. Following Schneider [26], the provided γ service-level is defined as:

$$\gamma(Q, r) = 1 - \frac{B(Q, r)}{\mu}. \tag{3.7}$$

It should be noted that $\gamma(Q, r)$ defined by (3.7) is a concave function because $B(Q, r)$ is jointly convex on Q and r when $f_{D(L)}(x) > 0$ for any $x > 0$. Furthermore, function $\gamma(Q, r)$ is increasing in Q and r because $B(Q, r)$ is non-increasing in Q and r [38].

Let $\delta_L(r)$ be the provided δ_L service-level, defined as the fraction of the lead time without stockouts. Consequently,

$$\delta_L(r) = 1 - \frac{\Delta_L(r)}{L}, \tag{3.8}$$

where $\Delta_L(r)$ is the expected duration of stockouts replenishment cycle. To develop an expression for $\Delta_L(r)$ we use the hitting time $\tau^r_{H,D} = \inf\{\tau > 0 : D(\tau) > r\}$ corresponding to the time required for r demands. Thus,

$\Delta_L(r) = \mathbb{E}\left(L - \tau^r_{H,D}\right)$ if $\tau^r_{H,D} < L$; 0 otherwise. Substituting in (3.8) and considering some straightforward calculations we have,

$$\delta_L(r) = F_{D(L)}(r) + \frac{1}{L}\int_0^L \tau f^r_{H,D}(\tau)\partial\tau, \tag{3.9}$$

where $f^r_{H,D}(\tau) = -\frac{\partial F_{D(\tau)}(x)}{\partial \tau}$ is the distribution function of the hitting time $\tau^r_{H,D}$. Under a strictly increasing non-negative demand, we have $\mathbb{P}\left(\tau^r_{H,D} \leq \tau\right) = \mathbb{P}\left(D(\tau) \geq r\right)$ [10, 11]. Considering the following reformulation of the second term of the right-hand side of (3.9): $\frac{1}{L}\int_0^L \tau f^r_{H,D}(\tau)\partial\tau = \frac{1}{L}\int_0^L \left[\int_0^\tau \partial t\right] f^r_{H,D}(\tau)\partial\tau = \frac{1}{L}\int_0^L \left[\int_t^L f^r_{H,D}(\tau)\partial\tau\right]\partial t = \frac{1}{L}\int_0^L \left[\int_t^L \partial f^r_{H,D}(\tau)\right]\partial t = \frac{1}{L}\int_0^L \left[\mathbb{P}\left(\tau^r_{H,D} \leq L\right) - \mathbb{P}(\tau^r_{H,D} \leq t)\right]\partial t = \frac{1}{L}\int_0^L \left[F_{D(t)}(r) - F_{D(L)}(r)\right]\partial t = \frac{1}{L}\int_0^L F_{D(t)}(r)\partial t - F_{D(L)}(r)$, we have,

$$\delta_L(r) = \frac{1}{L}\int_0^L F_{D(t)}(r)\partial t. \tag{3.10}$$

Proposition 3.2. *$\delta_L(r)$ is strictly concave function for any $r \geq \mathbb{E}(D(L))$. Furthermore, $\delta_L(r)$ is strictly increasing in r.*

Proof. Proof is provided in Appendix C. □

3.3. Service level constraint problems

Using (3.3) and (3.5), the α_L service level constraint problem is defined as the following nonlinear problem (NLP):

$$\alpha_L\text{-SLC}: \quad \min_{Q,r} \quad S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L + B(Q,r)\right) \tag{3.11}$$

$$\text{s.t.} \quad \bar{\alpha}_L - F_{D(L)}(r) \leq 0 \tag{3.12}$$

$$r \geq \mathbb{E}(D(L)) \tag{3.13}$$

$$Q \geq Q_{\text{EOQ}}, \tag{3.14}$$

where $\bar{\alpha}_L \in (0,1)$ is the preset α_L service-level, *i.e.*, $\bar{\alpha}_L$ is the minimum frequency of replenishment cycles that the decision-maker wants (or requires) do not fall into shortage, and $Q_{\text{EOQ}} = \sqrt{\frac{2\mu S}{h}}$ is the economic order quantity. Q_{EOQ} is the lower bound of the ordered quantity Q for any SLC problem with objective function (3.11).

The objective is to minimize the sum of ordering and holding costs per unit time. Constraint (3.12) ensures that the provided α_L service-level is greater than or equal to the preset service level. Constraint (3.13) ensures that the safety stock is non-negative, because, by definition, the safety stock in a continuous review (Q,r) policy is $r - \mathbb{E}(D(L))$. Constraint (3.14) avoids expressing the ordered quantity greater than or equal to zero and thus does not produce a contradiction with (3.11). It should be noted that for any $\bar{\alpha}_L \geq F_{D(L)}(\mathbb{E}(D(L)))$, the constraint (3.13) is redundant and can be relaxed from α_L-SLP.

The α_L-SLC model can be reformulated as a convex NLP, changing (3.12) by $r \geq F^{-1}_{D(L)}(\bar{\alpha}_L)$ and consequently, optimally solved through KKT conditions or using a nonlinear convex solver.

In the same way, using (3.3) and (3.4), the β service level constraint problem is defined as the following NLP:

$$\beta\text{-SLC}: \quad \min_{Q,r} \quad S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L + B(Q,r)\right) \tag{3.11}$$

$$\text{s.t.} \quad \bar{\beta} - \frac{1}{Q}\int_r^{r+Q} F_{D(L)}(y)\,\mathrm{d}y \leq 0 \tag{3.15}$$

$$(3.13), (3.14),$$

where $\bar{\beta} \in (0,1)$ is the preset fill-rate, *i.e.*, the minimum fraction of demand that the decision-maker wants (or requires) to meet directly with the available stock at an arbitrary time. Constraint (3.15) ensures that the provided fill-rate is greater than or equal to their preset service level.

Let C_β be the set of all (Q, r) satisfying (3.15), (3.13), and (3.14). It is easy to show that C_β is a convex set because the function of the inequality constraint (3.15) is jointly convex on Q and r when $f_{D(L)}(x)$ is non-increasing for any $x \geq r \geq \mathbb{E}(D(L))$, which is ensured by (3.13). Thus, β-SLC model is a convex NLP because a convex function on a convex set is minimized. Consequently, β-SLC can be solved through KKT conditions or using a nonlinear solver that computes integrals.

Using (3.3) and (3.7), the γ service level constraint problem is defined as the following NLP:

$$\gamma\text{-SLC}: \quad \min_{Q,r} \quad S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L + B(Q,r)\right) \tag{3.11}$$

$$\text{s.t.} \quad \bar{\gamma} - \left\{1 - \frac{1}{Q\mu}\left(\int_r^\infty (x-r)(1-F_{D(L)}(x))\,\mathrm{d}x - \int_{r+Q}^\infty (x-r-Q)(1-F_{D(L)}(x))\,\mathrm{d}x\right)\right\} \leq 0 \tag{3.16}$$

$$(3.13), (3.14),$$

where $\bar{\gamma} \in (0,1)$ is the preset γ service-level. We refer to $(1-\bar{\gamma})$ as the maximum fraction of the demand that the decision-maker allows to be on backorder at an arbitrary time. Constraint (3.16) ensures that the provided γ service-level is greater than or equal to their preset service level.

Let C_γ be the feasible region of γ-SLC model, *i.e.*, the set of all (Q, r) satisfying (3.16), (3.13), and (3.14). It is easy to show that C_γ is a convex set because the function of the inequality constraint (3.16) is jointly convex on Q and r when $f_{D(L)}(x) > 0$ for any $x \geq 0$. Thus, γ-SLC model is a convex NLP because a convex function on a convex set is minimized. Consequently, γ-SLC can be solved through KKT conditions or using a nonlinear solver that computes integrals.

Using (3.3) and (3.10), the δ_L service level constraint problem is defined as the following NLP:

$$\delta_L\text{-SLC}: \quad \min_{Q,r} \quad S\frac{\mu}{Q} + h\left(\frac{Q}{2} + r - \mu L + B(Q,r)\right) \tag{3.11}$$

$$\text{s.t.} \quad \bar{\delta}_L - \frac{1}{L}\int_0^L F_{D(t)}(r)\partial t \leq 0 \tag{3.17}$$

$$(3.13), (3.14),$$

where $\bar{\delta}_L \in (0,1)$ is the preset δ_L service-level, *i.e.*, the minimum fraction of the lead time that the decision-maker allows to be without shortage. Constraint (3.17) ensures that the provided δ_L service-level is greater than or equal to their preset service level.

Let C_{δ_L} be the set of all (Q, r) satisfying (3.17), (3.13), and (3.14). It is easy to show that C_{δ_L} is a convex set because the function of the inequality constraint (3.17) is convex when $f_{D(t)}(r)$ is non-increasing for any $r \geq \mathbb{E}(D(L))$, which is ensured by (3.13). Thus, δ_L-SLC model is a convex NLP. Consequently, δ_L-SLC can be solved through KKT conditions or using a nonlinear solver that computes integrals. On the other hand, δ_L-SLC model can be reformulated as a convex NLP, changing (3.17) by $r \geq r_{\delta_L}$, where $r_{\delta_L} = \min\left\{r : \delta_L(r) \geq \bar{\delta}_L\right\}$ and r_{δ_L} solves $\delta_L(r) = \bar{\delta}_L$ because $\delta_L(r)$ is increasing in r (Prop. 3.2).

4. Order relationship between α_L and β, γ, δ_L

In this section, we study the ordering relationship between the provided service level $\alpha_L(r)$ and the provided service levels $\beta(Q, r)$, $\gamma(Q, r)$, $\delta_L(r)$. We show that ensuring a high α_L service-level, ensures a high β, γ, and δ_L service levels.

Klemm [16] shows that in general $\alpha_L(r)$ is less than or equal to $\alpha(Q,r)$ for a definite reorder point [26]. Unfortunately, we do not have access to Klemm's proof. We conclude that under a continuous review (Q,r) policy with strictly increasing non-negative demand it follows that

$$\alpha_L(r) \leq \alpha(Q,r) = \beta(Q,r) \qquad \forall Q, r \geq 0, \tag{4.1}$$

because $\beta(Q,r) = \alpha(Q,r) = \frac{1}{Q}\int_r^{r+Q} F_{D(L)}(y)\,\mathrm{d}y \geq \frac{1}{Q}\int_r^{r+Q} F_{D(L)}(r)\,\mathrm{d}y = \alpha_L(r)$.

The main consequence of (4.1) is that ensuring a high α_L service-level ensures a high fill-rate and the opposite is not true. In other words, ensuring low stockout frequency during the replenishment cycle ensures low stockout frequency at any arbitrary time and small backorders at any arbitrary time.

To the best of our knowledge, the order relationship between $\alpha_L(r)$ and $\gamma(Q,r)$ is unknown. We propose the following relationship:

Proposition 4.1. *$\alpha_L(r) \leq \gamma(Q,r)$ for any $Q \geq 0$ and $r \geq r^{\gamma}_{\alpha_L}$, where $r^{\gamma}_{\alpha_L}$ solve the following NLP,*

$$P1: \quad \min_r \quad r \tag{4.2}$$

$$\text{s.t.} \quad F_{D(L)}(r) - \gamma(Q_{\text{EOQ}}, r) \leq 0 \tag{4.3}$$

$$r \geq F_{D(L)}(E(D(L))). \tag{4.4}$$

Proof. Proof is provided in Appendix D. □

The main consequence of Proposition 4.1 is that by ensuring a high α_L service-level, with $\bar{\alpha}_L \geq F_{D(L)}\left(r^{\gamma}_{\alpha_L}\right)$, ensures a high γ service-level. In other words, ensuring low stockout frequency during the replenishment cycle ensures low cumulative backorders at any arbitrary time.

Model $P1$ is neither convex nor concave because the inequality constraint function (4.3) is neither convex nor concave. However, model $P1$ is an NLP optimization problem easily solved with a global search method because it is a single variable problem.

To the best of our knowledge, the order relationship between $\alpha_L(r)$ and $\delta_L(r)$ is unknown. We propose the following relationship:

Proposition 4.2. *$\alpha_L(r) \leq \delta_L(r)$ for any $r > 0$.*

Proof. Proof is provided in Appendix E. □

The main consequence of Proposition 4.2 is that ensuring a high α_L service-level ensures a high δ_L service-level, and the opposite is not true. In other words, ensuring a low stockout frequency during the replenishment cycle ensures a low stockout duration during the replenishment cycle.

5. Cost of providing an α_L service-level

In what follows, we describe a number of properties of the SLC models that allow us to establish an ordering among optimal solutions of α_L-SLC and β-SLC, γ-SLC, and δ_L-SLC models.

Let $Z^*_{\alpha_L}$ and Z^*_{β} be the optimal solutions of α_L-SLC and β-SLC models, respectively. The following proposition establishes an ordering relation among the optimal solutions of α_L-SLC and β-SLC models.

Proposition 5.1. *$Z^*_{\beta} \leq Z^*_{\alpha_L}$ for any $\bar{\alpha}_L \geq \bar{\beta}$.*

Proof. Proof is provided in Appendix F. □

The main consequence of Proposition 5.1 is that, under the same preset service levels, the cost of providing a high α_L service-level is greater than or equal to the cost of providing a high fill-rate service level. However, this higher cost has the benefit that ensuring a high level of service α_L also ensures a high level of fill-rate, because $\beta\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right) \geq \alpha_L\left(r^*_{\alpha_L}\right) \geq \bar{\alpha}_L$ according to (4.1) and constraint (3.12), where $\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ are the optimal parameters of the α_L-SLC model.

Let Z^*_γ be the optimal solutions of γ-SLC model. The following proposition establishes an ordering relation among the optimal solutions of α_L-SLC and γ-SLC models.

Proposition 5.2. *$Z^*_\gamma \leq Z^*_{\alpha_L}$ for any $\bar{\alpha}_L \geq F_{D(L)}\left(r^\gamma_{\alpha_L}\right) \geq \bar{\gamma}$, where $r^\gamma_{\alpha_L}$ solve P1.*

Proof. Proof is provided in Appendix G. □

The main consequence of Proposition 5.2 is that, under the same preset service levels, the cost of providing a high α_L service-level, with $\bar{\alpha_L} \geq F_{D(L)}\left(r^\gamma_{\alpha_L}\right)$, is greater than or equal to the cost of providing a high γ service-level. However, this higher cost has the benefit that ensuring a high level of service α_L also ensures a high level of γ service level, because $\gamma\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right) \geq \alpha_L\left(r^*_{\alpha_L}\right) \geq \bar{\alpha}_L$ according to Proposition 4.1 and constraint (3.16).

Let $Z^*_{\delta_L}$ be the optimal solutions of δ_L-SLC model. The following proposition establishes an ordering relation among the optimal solutions of α_L-SLC and δ_L-SLC models.

Proposition 5.3. *$Z^*_{\delta_L} \leq Z^*_{\alpha_L}$ for any $\bar{\alpha}_L \geq \bar{\delta}_L$.*

Proof. Proof is provided in Appendix H. □

The main consequence of Proposition 5.3 is that, under the same preset service levels, the cost of providing a high α_L service-level is greater than or equal to the cost of providing a high δ_L service-level. However, this higher cost has the benefit that ensuring a high level of service α_L also ensures a high level of δ_L service-level, because $\delta_L\left(r^*_{\alpha_L}\right) \geq \alpha_L\left(r^*_{\alpha_L}\right) \geq \bar{\alpha}_L$ according to Proposition 4.2 and constraint (3.17).

6. Computational study

In Section 4, we conclude that ensuring a high α_L service-level ensures a high fill-rate and δ_L service-level, and for any $\bar{\alpha}_L \geq F_{D(L)}\left(r^\gamma_{\alpha_L}\right)$ a high γ service-level is also ensured (Props. 4.1 and 5.2). Consequently, the first objective of the computational study is to determine the threshold $F_{D(L)}\left(r^\gamma_{\alpha_L}\right)$ for which Propositions 4.1 and 5.2 are valid.

On the other hand, the magnitude of the service levels β, γ, and δ_L induced by a continuous review $\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ policy that ensures high α_L service-level are unknown. The opposite is also unknown, *i.e.*, the magnitude of the induced α_L service-level by an optimal continuous review (Q, r) policy under β, γ, and δ_L service-level constraint, respectively. Consequently, the second objective of the computational study is to determine the magnitude of the induced service levels.

In Section 5, we show that the cost of providing a high α_L service-level is greater than or equal to the cost of providing a high fill-rate, γ, and δ_L service levels when $\bar{\alpha}_L \geq \bar{\beta}$, $\bar{\alpha}_L \geq F_{D(L)}\left(r^\gamma_{\alpha_L}\right) \geq \bar{\gamma}$ and $\bar{\alpha}_L \geq \bar{\delta}_L$, respectively. However, the magnitude of this higher cost is unknown. Consequently, the third objective of the computational study is to establish the magnitude of this higher cost under the same preset service levels. Furthermore, a comparison between the optimal parameters of the continuous review (Q, r) policy resulting from models α_L-SLC, β-SLC, γ-SLC, and δ_L-SLC is performed.

To illustrate the objectives of the computational study, we simulated several test problems with the following common criteria and parameters: holding cost per unit and unit time $h = U[0.1, 1.25]$, ordering cost $S = U[100, 500]$, and normal demand distribution, as an approximation to strictly increasing non-negative demand, with coefficient of variation CV $= U[0.1, 0.6]$. Furthermore, we mean high service level values greater than or equal to 0.95.

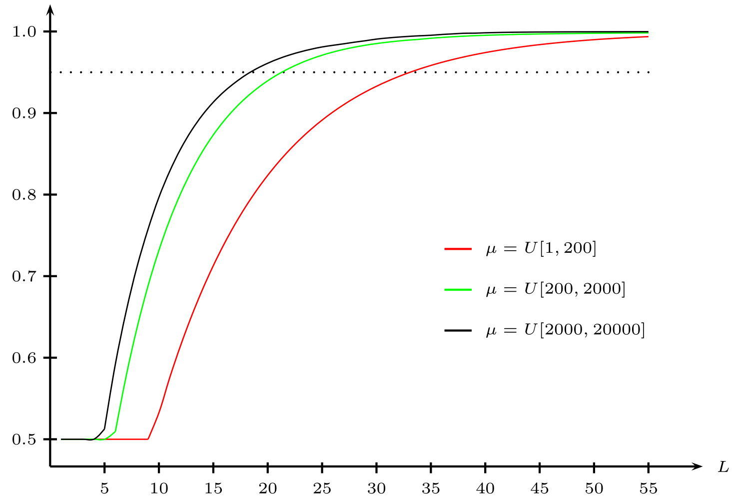

FIGURE 2. Maximum $F_{D(L)}\left(r^{\gamma}_{\alpha_L}\right)$ for different lead times and set of experiments.

Models α_L-SLC, β-SLC, γ-SLC and δ_L-SLC were programmed in Pyton using Sequential Least Squares Programming libraries. All tests were done on a PC with an Intel Core i7 2.3 GHz processor and 16 GB RAM. The time to compute the optimal parameters of the continuous review (Q, r) policy using α_L-SLC, β-SLC, γ-SLC and δ_L-SLC models are, on average 0.44s, 0.66s, 0.87s and 8.12s, respectively.

6.1. Determining the threshold $F_{D(L)}\left(r^{\gamma}_{\alpha_L}\right)$

Propositions 4.1 and 5.2, which define the ordering relationship between $\alpha_L(r)$ and $\gamma(Q, r)$, and the ordering relationship between the optimal solutions of α_L-SLC and γ-SLC models, respectively, depend on the threshold $F_{D(L)}\left(r^{\gamma}_{\alpha_L}\right)$ where $r_{\alpha^{\gamma}_L}$ solves P1 model. To establish the range of $F_{D(L)}\left(r^{\gamma}_{\alpha_L}\right)$ for which Propositions 4.1 and 5.2 are valid we designed a set of three experiments covering a wide range of data. In each experiment, we generated 10 000 random sets of $\{h, S, \mu, \mathrm{CV}\}$ according to the common criteria of h, S and CV, while $\mu = U[1, 200]$ for the first set of experiments, $\mu = U[200, 2000]$ for the second set of experiments, and $\mu = U[2000, 20\,000]$ for the third set of experiments. For each set of randomly generated parameters we solve $P1$ model for $L = \{1, \ldots, 55\}$ and return the maximum $F_{D(L)}\left(r^{\gamma}_{\alpha_L}\right)$ for each L. Figure 2 shows the behavior of the maximum $F_{D(L)}\left(r^{\gamma}_{\alpha_L}\right)$ for each experiment.

From Figure 2, we observe that ensuring a high α_L service-level, with $\bar{\alpha}_L \geq 0.95$, ensures a high γ service-level when $L \leq 33$ and $\mu \in [1, 200]$, $L \leq 21$ and $\mu \in [200, 2000]$, and $L \leq 18$ and $\mu \in [2000, 20\,000]$. Thus, we infer that Propositions 4.1 and 5.2 hold for a wide range of data.

6.2. Induced service levels

The fill-rate, γ, and δ_L service levels induced by the continuous review $\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ policy are defined as $\beta\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$, $\gamma\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ and $\delta_L\left(r^*_{\alpha_L}\right)$ according to (3.4), (3.7) and (3.10), respectively.

To determine the magnitude of the induced service levels, we generated 10 000 random sets of $\{h, S, \mu, \mathrm{CV}, L\}$ according to the common criteria for h, S and CV, while $\mu = U[1, 200]$ and $L = \{1, \ldots, 8\}$ with discrete uniform distribution. We denote this set of instances as *test set*. For each random set, the α_L-SLC model is solved, and the induced service levels are computed. Table 1 shows the average, maximum, and minimum induced β, γ, and δ_L service levels when α_L-SLC is solved using a high preset α_L service-level.

As expected from (4.1), and Propositions 4.1 and 4.2, Table 1 shows that the fill-rate, γ, and δ_L induced by the continuous review $(Q^*_{\alpha_L}, r^*_{\alpha_L})$ policy are strictly greater than the preset α_L service-level. Here, the surprise

TABLE 1. Induced β, γ, and δ_L service levels by the optimal parameters of an α_L-SLC model.

	$\beta\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$			$\gamma\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$			$\delta_L\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$		
$\bar{\alpha_L}$	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.
0.99	0.999	1.000	0.998	1.000	1.000	0.999	0.999	1.000	0.999
0.98	0.999	1.000	0.995	1.000	1.000	0.999	0.999	1.000	0.998
0.97	0.998	1.000	0.992	1.000	1.000	0.998	0.998	0.999	0.997
0.96	0.997	1.000	0.989	0.999	1.000	0.997	0.997	0.999	0.996
0.95	0.996	1.000	0.986	0.999	1.000	0.996	0.996	0.999	0.994

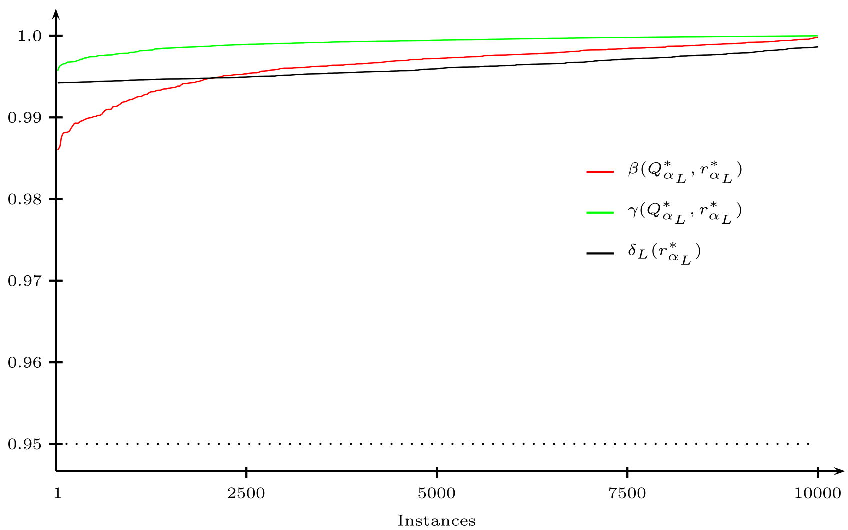

FIGURE 3. Induced service levels when $\bar{\alpha}_L = 0.95$.

is that for all instances tested, the minimum $\beta\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$, $\gamma\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$, and $\delta_L\left(r^*_{\alpha_L}\right)$ are strictly greater than 0.98. Figure 3 shows the behavior of the induced service levels when $\bar{\alpha}_L = 0.95$.

From Figure 3, it is observed that a continuous review $\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ policy that ensures a high α_L service-level induces a high performance of service levels not explicitly defined in the inventory policy. Furthermore, we observe that $\gamma\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ service-level is the best performing and that in 80% of the tested instances it is satisfied that $\bar{\alpha}_L < \delta_L\left(r^*_{\alpha_L}\right) < \beta\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right) < \gamma\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$. Similar behavior is observed for any $\bar{\alpha}_L \in \{0.96, 0.97, 0.98, 0.99\}$.

Let $\left(Q^*_{\beta}, r^*_{\beta}\right)$, $\left(Q^*_{\gamma}, r^*_{\gamma}\right)$, and $\left(Q^*_{\delta_L}, r^*_{\delta_L}\right)$ be the optimal parameters of the β-SLC, γ-SLC, and δ_L-SLC models, respectively. The α_L service-level induced by the continuous review $\left(Q^*_{\beta}, r^*_{\beta}\right)$, $\left(Q^*_{\gamma}, r^*_{\gamma}\right)$, and $\left(Q^*_{\delta_L}, r^*_{\delta_L}\right)$ policies are defined as $\alpha_L\left(r^*_{\beta}\right)$, $\alpha_L\left(r^*_{\gamma}\right)$, and $\alpha_L\left(r^*_{\delta_L}\right)$ according to (3.5). Table 2 shows the average, maximum and minimum induced α_L service-level when β-SLC, γ-SLC, and δ_L-SLC are solved using the same high preset service level.

As expected from (4.1), and Propositions 4.1 and 4.2, Table 2 shows that $\alpha_L\left(r^*_{\beta}\right)$, $\alpha_L\left(r^*_{\gamma}\right)$, and $\alpha_L\left(r^*_{\delta_L}\right)$ are strictly lower than the preset service level $\bar{\beta} = \bar{\gamma} = \bar{\delta_L}$. Here, the surprise is that for all instances tested, the average induced α_L service levels are strictly lower than 0.88, *i.e.*, the continuous review $\left(Q^*_{\beta}, r^*_{\beta}\right)$, $\left(Q^*_{\gamma}, r^*_{\gamma}\right)$,

TABLE 2. Induced $\alpha_L\left(r^*_\beta\right)$, $\alpha_L\left(r^*_\gamma\right)$ and $\alpha_L\left(r^*_{\delta_L}\right)$ service levels.

	$\alpha_L\left(r^*_\beta\right)$			$\alpha_L\left(r^*_\gamma\right)$			$\alpha_L\left(r^*_{\delta_L}\right)$		
$\bar{\beta}=\bar{\gamma}=\bar{\delta_L}$	Ave.	Max.	Min.	Ave.	Max.	Min.	Ave.	Max.	Min.
0.99	0.813	0.953	0.500	0.628	0.885	0.500	0.881	0.922	0.750
0.98	0.709	0.913	0.500	0.555	0.811	0.500	0.797	0.865	0.590
0.97	0.642	0.877	0.500	0.526	0.750	0.500	0.728	0.817	0.500
0.96	0.596	0.842	0.500	0.513	0.698	0.500	0.673	0.773	0.500
0.95	0.565	0.809	0.500	0.506	0.652	0.500	0.629	0.734	0.500

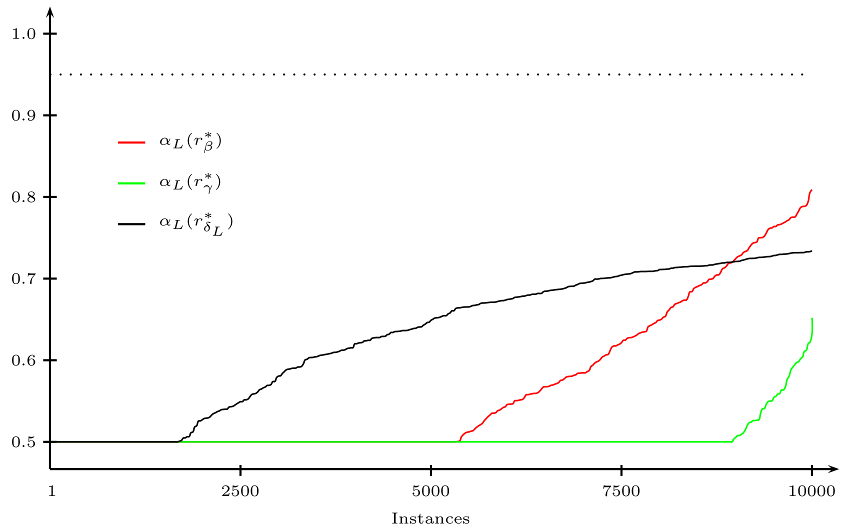

FIGURE 4. Induced α_L service levels when $\bar{\beta}=\bar{\gamma}=\bar{\delta_L}=0.95$.

and $\left(Q^*_{\delta_L}, r^*_{\delta_L}\right)$ policies ensure high β, γ, and δ_L service levels, respectively, perform poorly at α_L service-level. Figure 4 shows the behavior of the induced α_L service-level when $\bar{\beta}=\bar{\gamma}=\bar{\delta_L}=0.95$.

From Figure 4, we observe that $\alpha_L\left(r^*_\gamma\right)$ service-level is the lowest performing and that in the 80% of the instances it is satisfied that $\alpha_L\left(r^*_\gamma\right)\leq\alpha_L\left(r^*_\beta\right)\leq\alpha_L\left(r^*_{\delta_L}\right)$. Similar behavior is observed for any $\bar{\beta}=\bar{\gamma}=\bar{\delta_L}\in\{0.96, 0.97, 0.98, 0.99\}$.

6.3. Cost of providing a high α_L service-level

To quantify how much higher the cost of a continuous review $\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ policy is in relation to the cost of continuous review $\left(Q^*_\beta, r^*_\beta\right)$, $\left(Q^*_\gamma, r^*_\gamma\right)$, and $\left(Q^*_{\delta_L}, r^*_{\delta_L}\right)$ policies, we compute $\triangle\left(Z^*_{\alpha_L}, Z^*_{(\cdot)}\right)=100\times\left(Z^*_{\alpha_L}-Z^*_{(\cdot)}\right)/Z^*_{(\cdot)}$ for each instance of the test set. In other words, we compute the the relative cost of the continuous review $\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ policy with respect to the continuous review $\left(Q^*_\beta, r^*_\beta\right)$, $\left(Q^*_\gamma, r^*_\gamma\right)$, and $\left(Q^*_{\delta_L}, r^*_{\delta_L}\right)$ policies, respectively. Table 3 shows the average, maximum and minimum relative cost.

From Table 3, we observe that $\triangle\left(Z^*_{\alpha_L}, Z^*_{(\cdot)}\right)$ is decreasing in the preset service level and that the lowest and highest relative cost are δ_L and γ service levels, respectively. Thus, the maximum relative cost is 65.9% when $\bar{\alpha_L}=\bar{\gamma}=0.95$. Figure 5 shows the behavior of the relative cost of the continuous review $\left(Q^*_{\alpha_L}, r^*_{\alpha_L}\right)$ policy in

TABLE 3. Relative cost between inventory policies with single service level strategies.

$\bar{\alpha_L} = \bar{\beta} = \bar{\gamma} = \bar{\delta_L}$	$\triangle\left(Z^*_{\alpha_L}, Z^*_{\beta}\right)$ Ave.	Max.	Min.	$\triangle\left(Z^*_{\alpha_L}, Z^*_{\gamma}\right)$ Ave.	Max.	Min.	$\triangle\left(Z^*_{\alpha_L}, Z^*_{\delta_L}\right)$ Ave.	Max.	Min.
0.99	14.3%	19.6%	2.2%	24.8%	45.9%	2.2%	13.6%	32.5%	1.6%
0.98	16.2%	22.5%	2.0%	26.1%	53.4%	2.0%	15.2%	37.8%	1.7%
0.97	17.5%	24.8%	1.8%	26.2%	57.5%	1.8%	16.4%	41.8%	1.8%
0.96	18.3%	26.6%	1.7%	25.7%	62.0%	1.7%	17.3%	45.1%	1.7%
0.95	18.8%	28.3%	1.6%	24.9%	65.9%	1.6%	18.1%	48.1%	1.6%

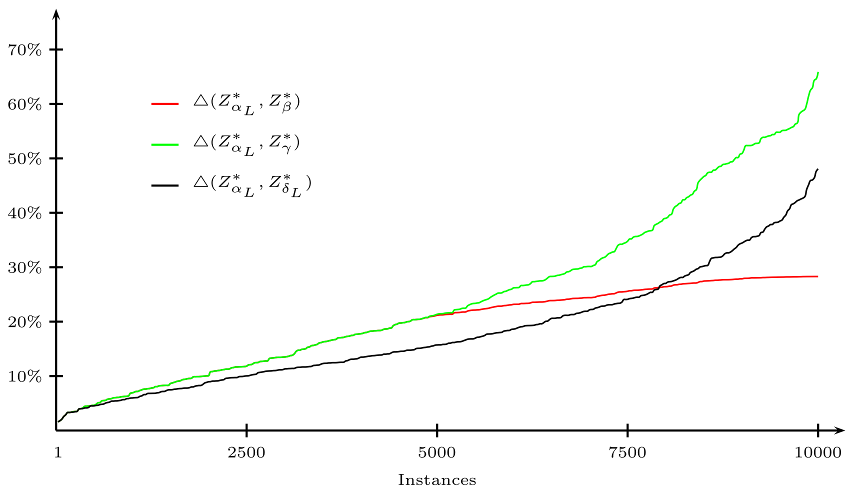

FIGURE 5. $\triangle\left(Z^*_{\alpha_L}, Z^*_{(\cdot)}\right)$ when $\bar{\alpha_L} = \bar{\beta} = \bar{\gamma} = \bar{\delta_L} = 0.95$.

relation to the continuous review $\left(Q^*_{\beta}, r^*_{\beta}\right)$, $\left(Q^*_{\gamma}, r^*_{\gamma}\right)$, and $\left(Q^*_{\delta_L}, r^*_{\delta_L}\right)$ policies, respectively, when $\bar{\beta} = \bar{\gamma} = \bar{\delta_L} = 0.95$.

From Figure 5, we observed that the highest relative cost is in relation to γ service-level for any instances. More precisely, we observed that in 79% of the instances the lowest and highest relative cost is in relation to δ_L and γ service levels, respectively, *i.e.*, $\triangle\left(Z^*_{\alpha_L}, Z^*_{\delta_L}\right) \leq \triangle\left(Z^*_{\alpha_L}, Z^*_{\beta}\right) \leq \triangle\left(Z^*_{\alpha_L}, Z^*_{\gamma}\right)$, and that in the rest of the instances it holds that $\triangle\left(Z^*_{\alpha_L}, Z^*_{\beta}\right) < \triangle\left(Z^*_{\alpha_L}, Z^*_{\delta_L}\right) < \triangle\left(Z^*_{\alpha_L}, Z^*_{\gamma}\right)$.

6.4. Optimal parameters of a continuous review (Q, r) policy under α_L, β, γ, and δ_L service-level constraint

In what follows, we compare the optimal parameters of the continuous review (Q, r) policy resulting from α_L-SLC, β-SLC, γ-SLC and δ_L-SLC models.

For each instance of the test set we compute $Q^*_{\alpha_L}$, Q^*_{β}, Q^*_{γ}, and $Q^*_{\delta_L}$, and determine the ordering relationship between the optimal order quantities. Table 4 shows the percentage of instances associated with each ordering of the optimal order quantities.

From Table 4, it is observed that under the same preset service levels the optimal order quantity resulting from α_L-SLC model is less than or equal to the optimal order quantity resulting from β-SLC, γ-SLC, and δ_L-SLC models. On the contrary, the optimal order quantity resulting from β-SLC is greater than or equal to

TABLE 4. Order of optimal order quantities.

	$\bar{\alpha_L} = \bar{\beta} = \bar{\gamma} = \bar{\delta_L}$				
	0.99	0.98	0.97	0.96	0.95
$Q^*_\beta = Q^*_\gamma = Q^*_{\delta_L} = Q^*_{\alpha_L}$	0.6%	6.6%	11.6%	16.8%	18.4%
$Q^*_\beta > Q^*_\gamma > Q^*_{\delta_L} > Q^*_{\alpha_L}$	25.3%	44.1%	55.1%	55.9%	54.5%
$Q^*_\beta > Q^*_\gamma = Q^*_{\delta_L} = Q^*_{\alpha_L}$	12.8%	10.4%	7.0%	3.0%	0.8%
$Q^*_\beta > Q^*_\gamma > Q^*_{\delta_L} = Q^*_{\alpha_L}$	12.0%	8.6%	5.8%	2.6%	1.2%
$Q^*_\beta > Q^*_\gamma = Q^*_{\delta_L} > Q^*_{\alpha_L}$	0.0%	2.2%	5.8%	10.6%	11.6%
$Q^*_\beta = Q^*_\gamma = Q^*_{\delta_L} > Q^*_{\alpha_L}$	0.0%	0.0%	0.8%	3.6%	10.0%
$Q^*_\beta = Q^*_\gamma > Q^*_{\delta_L} = Q^*_{\alpha_L}$	0.0%	0.0%	0.0%	1.0%	2.2%
$Q^*_\gamma > Q^*_\beta > Q^*_{\delta_L} > Q^*_{\alpha_L}$	36.1%	25.9%	14.0%	6.6%	1.4%
$Q^*_\gamma > Q^*_{\alpha_L} = Q^*_\beta = Q^*_{\delta_L}$	13.2%	2.2%	0.0%	0.0%	0.0%

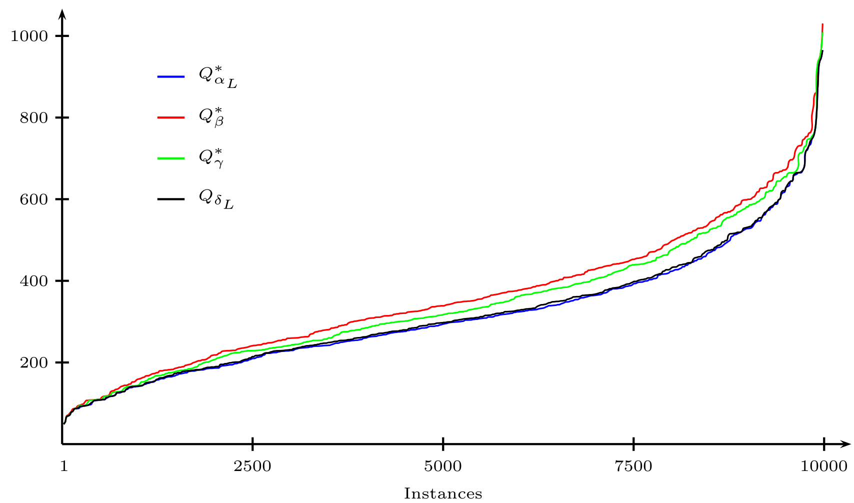

FIGURE 6. $Q^*_{\alpha_L}$, Q^*_β, Q^*_γ, and $Q^*_{\delta_L}$ when $\bar{\alpha_L} = \bar{\beta} = \bar{\gamma} = \bar{\delta_L} = 0.95$.

the optimal order quantity resulting from γ-SLC, δ_L-SLC, and α_L-SLC in 50.7% and 98.6% of the instances when the preset service levels are 0.95 and 0.99, respectively. Figure 6 shows the behavior of the optimal order quantities resulting from α_L-SLC, β-SLC, γ-SLC, and δ_L-SLC when $\bar{\alpha_L} = \bar{\beta} = \bar{\gamma} = \bar{\delta_L} = 0.95$.

From Figure 6, it is observed that the order quantities resulting from α_L-SLC and δ_L-SLC are similar in magnitude and are clearly lower than the order quantities resulting from β-SLC and γ-SLC in most instances. In particular, the average and maximum difference between $Q^*_{\alpha_L}$ and $Q^*_{\delta_L}$ is 1.3% and 11%, respectively. Furthermore, it is observed that the order quantity resulting from β-SLC is larger than the order quantities resulting from α_L-SLC, γ-SLC, and δ_L-SLC models. In particular, the average and maximum difference between $Q^*_{\alpha_L}$ and Q^*_β is 14.6% and 63%, respectively.

In the same way, for each instance of the test set we compute $r^*_{\alpha_L}$, r^*_β, r^*_γ, and $r^*_{\delta_L}$, and determine the ordering relationship between optimal reorder points, respectively. Table 5 shows the percentage of instances associated with each ordering of optimal reorder points.

From Table 5, it is observed that under the same preset service levels the optimal reorder point resulting from α_L-SLC model is strictly greater than the reorder points resulting from β-SLC, γ-SLC, and δ_L-SLC models. It should be noted that the inequality $r^*_\alpha > r^*_\beta$ is hold since $\beta(Q, r)$ is strictly increasing in r and Q (Prop. 3.1),

TABLE 5. Order of optimal reorder points.

	$\bar{\alpha_L} = \bar{\beta} = \bar{\gamma} = \bar{\delta_L}$				
	0.99	0.98	0.97	0.96	0.95
$r^*_{\alpha_L} > r^*_{\beta} > r^*_{\gamma} > r^*_{\delta_L}$	17.0%	20.8%	21.4%	21.2%	20.0%
$r^*_{\alpha_L} > r^*_{\beta} > r^*_{\delta_L} > r^*_{\gamma}$	43.5%	34.1%	26.7%	20.0%	14.4%
$r^*_{\alpha_L} > r^*_{\delta_L} > r^*_{\beta} > r^*_{\gamma}$	38.5%	35.5%	30.7%	24.4%	18.2%
$r^*_{\alpha_L} > r^*_{\delta_L} > r^*_{\beta} = r^*_{\gamma}$	1.0%	5.8%	9.4%	13.4%	15.6%
$r^*_{\alpha_L} > r^*_{\delta_L} = r^*_{\beta} = r^*_{\gamma}$	0.0%	0.8%	4.2%	11.6%	19.2%
$r^*_{\alpha_L} > r^*_{\beta} > r^*_{\delta_L} = r^*_{\gamma}$	0.0%	3.0%	7.6%	9.6%	12.6%

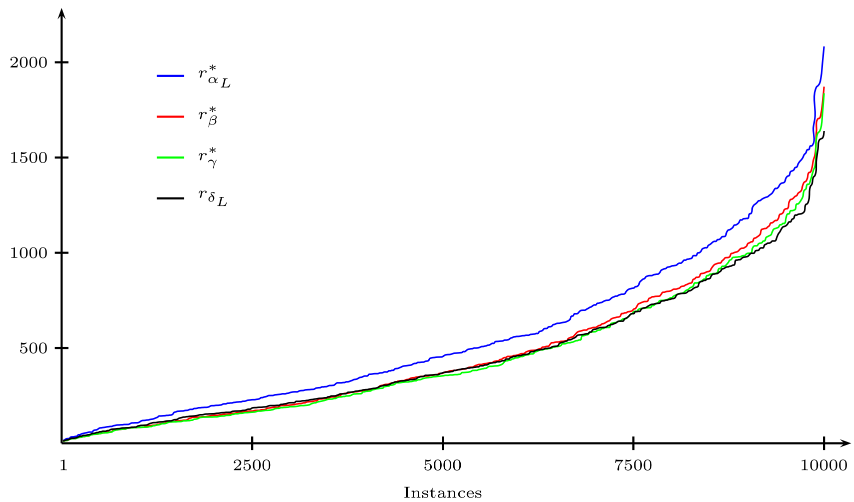

FIGURE 7. $r^*_{\alpha_L}$, r^*_{β}, r^*_{γ}, and $r^*_{\delta_L}$ when $\bar{\alpha_L} = \bar{\beta} = \bar{\gamma} = \bar{\delta_L} = 0.95$.

and $\alpha(r) \leq \beta(Q, r)$ according to (4.1). Furthermore, it is easy to show that $r_{\alpha_L} \geq r_{\delta_L}$ since $\alpha_L(r) \leq \delta_L(r)$ for any $r > 0$ (Prop. 4.2). Figure 7 shows the behavior of reorder points resulting from α_L-SLC, β-SLC, γ-SLC, and δ_L-SLC when $\bar{\alpha_L} = \bar{\beta} = \bar{\gamma} = \bar{\delta_L} = 0.95$.

From Figure 7, it is observed that the optimal reorder points resulting from α_L-SLC are noticeably larger than the optimal reorder points resulting from β-SLC, γ-SLC, and δ_L-SLC. In particular, the average and maximum difference between $r^*_{\alpha_L}$ and r^*_{γ} is 22% and 49.4%, respectively.

7. CONCLUSIONS

In inventory control, stock availability measures seek to control the frequency, size, and duration of stock shortages. However, none of them simultaneously control all dimensions of shortages. Depending on the type of product and consumer reaction to shortages, the decision-maker should implement an inventory policy that ensures a high service level in the shortage dimension that most affects customer service while minimizing inventory costs. However, ensuring a high service level in one shortage dimension does not necessarily guarantee good performance in the other shortage dimensions.

In this paper, we study the effect of an optimal continuous review (Q, r) policy that ensures a low frequency of replenishment cycle stockouts (high α_L service-level) on the size of backorders at an arbitrary time (β service-level), on the size of accumulated backorders at an arbitrary time (γ service-level), and on the duration of stockouts replenishment cycle (δ_L service-level). The converse was also done, *i.e.*, we analyzed the performance of optimal continuous review (Q, r) policies that ensure high β, γ, and δ_L service levels, respectively, on the frequency of replenishment cycle stockouts. In order to determine the optimal parameters of the continuous review (Q, r) policy that ensures a high service level in some shortage dimension, four SLC problems (one for each service level) were formulated as convex nonlinear optimization problems based on exact expressions for each service level and holding costs. Thus, different of previous work, our analysis and results are based on exact formulations.

Several relationships between stock availability measures and optimal solutions of SLC models have been proved, from which we concluded the following managerial insights for a continuous review (Q, r) policy under a single service level constraint.

– The β service-level simultaneously controls frequency and size of backorders at an arbitrary time.
– Ensuring a high α_L service-level also ensures a high level of β and δ_L service levels, and the opposite is not true.
– Ensuring a high α_L service-level also ensures a high γ service-level for a wide range of demand configurations.
– The cost of providing a high α_L service-level is greater than the cost of providing a high level of β and δ_L service levels, and the opposite is not true.
– The cost of providing a high α_L service-level is greater than the cost of providing a high γ service-level for a wide range of demand configurations.

We conducted several test problems from which the following managerial insights are inferred.

– An optimal continuous review (Q, r) policy that ensures an α_L service-level greater or equal to 0.95 induces β, γ, and δ_L service levels greater or equal to 0.98.
– An optimal continuous review (Q, r) policy that ensures a β, γ, and δ_L service-level greater than or equal to 0.95 induces an α_L service-level less than or equal to 0.81, 0.65, and 0.73, respectively.
– The cost of assuring an α_L service-level greater than or equal to 0.95 is on average 24%, 57%, and 41% higher than assuring a β, γ, and δ_L service levels greater than or equal to 0.95, respectively.
– Under the same preset service levels, the continuous review (Q, r) policy causes the lowest order quantity and the highest reorder point.

Thus, based on theoretical and experimental evidence, we infer that the α_L service-level performs better overall than β, γ, and δ_L service levels on frequency, size, and duration of stockouts. However, this better performance has higher holding and ordering costs. Therefore, the fill-rate (the most observed service level definition in practical settings) is an attractive control measure to design a continuous review (Q, r) policy only for its lower cost.

There are a number of questions and issues left for future research. The first issue is to relax the main assumptions made in this study, *i.e.*, deterministic lead time and full-backorder. Thus, it will be possible to verify whether the main conclusions of this study hold under stochastic lead time, partial backorder, and lost sale. Similarly, a question for future work is whether the conclusions obtained in this work hold under deterioration, default risk, and price sensitivity demand. Since each stock availability measures only consider a single shortage aspect, a second issue is to study the relationship between a pure service level policy, *i.e.*, an inventory policy that responds to a single service level explicitly defined in the stochastic inventory model, and a mixed service level policy, *i.e.*, an inventory policy in which more than one service level is explicitly defined in the stochastic inventory model. A third issue is related to exploring other service levels, which by their nature, induce a good performance in those measures of stock availability that are not explicitly defined in the inventory policy.

Appendix A. Glossary of terms

See Table A.1.

TABLE A.1. Spectroscopic parameters of propynethial in MHz – S Reduction.

Parameters	
S	Ordering cost
h	Holding cost per unit and unit time
μ	Average demand per unit time
L	Lead time
$\bar{\alpha}_L$	Preset α_L service-level
$\bar{\beta}$	Preset β service-level
$\bar{\gamma}$	Preset γ service-level
$\bar{\delta}_L$	Preset δ_L service-level
Parameter functions	
$F_{D(\tau)}(x)$	Cumulative demand distribution function in the interval $[0, \tau]$
$f_{D(\tau)}(x)$	Demand density function
Q_{EOQ}	Economic order quantity
$\mathbb{E}(D(L))$	Expected demand during lead time
Variables	
Q	Order quantity
r	Reorder point
Variable functions	
$\text{AC}(Q, r)$	Average cost per unit time
$B(Q, r)$	Expected on-hand steady-state inventory
$\alpha(Q, r)$	Provided α service-level
$\alpha_L(r)$	Provided α_L service-level
$\beta(Q, r)$	Provided fill-rate service-level
$\gamma(Q, r)$	Provided γ service-level
$\delta_L(r)$	Provided δ_L service-level
$A(Q, r)$	Average stockouts per unit time
$\Delta_L(r)$	Expected duration of stockouts replenishment cycle

APPENDIX B. PROOF OF PROPOSITION 3.1

Proof. Using the Leibniz integral rule to obtain partial derivatives of (3.4) with respect to Q and r, respectively, and assuming $Q > 0$ and $r \geq 0$, we have

$$\frac{\partial \beta(Q, r)}{\partial Q} = \frac{1}{Q^2} \left(\int_r^{r+Q} \left(F_{D(L)}(r + Q) - F_{D(L)}(y) \right) \mathrm{d}y \right) > 0,$$

because $r + Q > y$ for any $y \in (r, r + Q)$ and $Q > 0$; and

$$\frac{\partial \beta(Q, r)}{\partial r} = \frac{1}{Q} \left(F_{D(L)}(r + Q) - F_{D(L)}(r) \right) > 0,$$

because $Q > 0$, and $F_{D(L)}(x)$ is a monotonically increasing function in x. □

APPENDIX C. PROOF OF PROPOSITION 3.2

Proof. Deriving (3.10) with respect to r we have that $\frac{\partial \delta_L(r)}{\partial r} = \frac{1}{L} \int_0^L f_{D(t)}(r) \partial t$. We conclude that $\delta_L(r)$ is strictly increasing in r, *i.e.*, $\frac{\partial \delta_L(r)}{\partial r} > 0$, because $f_{D(t)}(r) > 0$ for any $t \in (0, L]$ and $r > 0$.

Taking the second derivative of (3.10) with respect to r we have that $\frac{\partial^2 \delta_L(r)}{\partial r^2} = \frac{1}{L} \int_0^L \frac{\partial f_{D(t)}(r)}{\partial r} \partial t$. We conclude that $\frac{\partial^2 \delta_L(r)}{\partial r^2} < 0$, which implies that $\delta_L(r)$ is strictly concave function, because $\frac{\partial f_{D(t)}(r)}{\partial r} < 0$ for any $t \in (0, L]$ when $r \geq \mathbb{E}(D(L))$. □

Appendix D. Proof of Proposition 4.1

Proof. Because $\gamma(Q,r)$ is increasing in Q we have that $\alpha_L(r) = F_{D(L)}(r) \leq \gamma(Q_{\mathrm{EOQ}}, r) \leq \gamma(Q,r)$ for any $Q \geq Q_{\mathrm{EOQ}}$, $r \geq r_{\alpha_L\gamma}$. □

Appendix E. Proof of Proposition 4.2

Proof. The result follows directly from (3.5) and (3.9). □

Appendix F. Proof of Proposition 5.1

Proof. Let $r_\beta(Q)$ be the minimum reorder point such that the provided β service-level is greater than or equal to the preset service level $\bar{\beta}$, *i.e.*, $r_\beta(Q) = \min\{r : \beta(Q,r) \geq \bar{\beta})\}$. Because function $\beta(Q,r)$ increasing in r (Prop. 3.1) we conclude that $r_\beta(Q)$ solve $\beta(Q,r) = \bar{\beta}$. Furthermore, $r_\beta(Q)$ is jointly convex in (Q,r) because $r_\beta(Q)$ is super-level set of the concave function $\beta(Q,r)$, and $r_\beta(Q)$ is strictly decreasing in Q because $\frac{\partial r_\beta(Q)}{\partial Q} = -\frac{\partial \beta(Q,r)}{\partial Q} / \frac{\partial \beta(Q,r)}{\partial r} < 0$. Once $r_\beta(Q)$ is defined, the feasible region C_β is the intersection of the areas above $r_\beta(Q)$, above $\mathbb{E}(D(L))$ and to the right of Q_{EOQ}. On the other hand, let C_{α_L} be the feasible region of α_L-SLP, *i.e.*, the set of all (Q,r) satisfying (3.12)–(3.14), defined as the intersection of the areas above $F^{-1}_{D(L)}(\bar{\alpha}_L)$, above $\mathbb{E}(D(L))$ and to the right of Q_{EOQ}.

For any $\bar{\alpha}_L \geq \bar{\beta}$ we have $r_\beta(0) = F^{-1}_{D(L)}(\bar{\beta}) \leq F^{-1}_{D(L)}(\bar{\alpha}_L)$ and we conclude that $C_{\alpha_L} \subseteq C_\beta$. Therefore, β-SLC is relaxation of α_L-SLC for any $\bar{\alpha}_L \geq \bar{\beta}$ and consequently, the optimal solution of β-SLC is a lower bound of α_L-SLC, *i.e.*, $Z^*_\beta \leq Z^*_{\alpha_L}$ for any $\bar{\alpha}_L \geq \bar{\beta}$. □

Appendix G. Proof of Proposition 5.2

Proof. Let $r_\gamma(Q) = \min\{r : \gamma(Q,r) \geq \bar{\gamma})\}$. Because function $\gamma(Q,r)$ increasing in r we conclude that $r_\gamma(Q)$ solve $\gamma(Q,r) = \bar{\gamma}$. Furthermore, $r_\gamma(Q)$ is jointly convex in (Q,r) because $r_\gamma(Q)$ is super-level set of the concave function $\gamma(Q,r)$, and $r_\gamma(Q)$ is strictly decreasing in Q because $\frac{\partial r_\gamma(Q)}{\partial Q} = -\frac{\partial \gamma(Q,r)}{\partial Q} / \frac{\partial \gamma(Q,r)}{\partial r} < 0$. Once $r_\gamma(Q)$ is defined, the feasible region C_γ is the intersection of the areas above $r_\gamma(Q)$, above $\mathbb{E}(D(L))$ and to the right of Q_{EOQ}.

For any $\bar{\alpha}_L \geq F_{D(L)}(r_{\alpha_L\gamma}) \geq \bar{\gamma}$ we have $r_\gamma(Q_{\mathrm{EOQ}}) \leq r_{\alpha_L\gamma} \leq F^{-1}_{D(L)}(\bar{\alpha}_L)$, where $r_\gamma(Q_{\mathrm{EOQ}})$ solve $\gamma(Q_{\mathrm{EOQ}}, r) = \bar{\gamma}$ and $r_{\alpha_L\gamma}$ solve $P1$, and we conclude that $C_{\alpha_L} \subseteq C_\gamma$ for any $\bar{\alpha}_L \geq F_{D(L)}(r_{\alpha_L\gamma}) \geq \bar{\gamma}$. Therefore, γ-SLC is relaxation of α_L-SLC for any $\bar{\alpha}_L \geq F_{D(L)}(r_{\alpha_L\gamma}) \geq \bar{\gamma}$ and consequently, the optimal solution of β-SLC is a lower bound of α_L-SLC, *i.e.*, $Z^*_\gamma \leq Z^*_{\alpha_L}$ for any $\bar{\alpha}_L \geq F_{D(L)}(r_{\alpha_L\gamma}) \geq \bar{\gamma}$. □

Appendix H. Proof of Proposition 5.3

Proof. For any $\bar{\alpha}_L \geq \bar{\delta}_L$ we have $r_{\delta_L} \leq F^{-1}_{D(L)}(\bar{\alpha}_L)$, where r_{δ_L} solve $\delta_L(r) = \bar{\delta}_L$, and we conclude that $C_{\alpha_L} \subseteq C_{\delta_L}$. Therefore, δ_L-SLC is relaxation of α_L-SLC for any $\bar{\alpha}_L \geq \bar{\delta}_L$ and consequently, the optimal solution of δ_L-SLC is a lower bound of α_L-SLC, *i.e.*, $Z^*_{\delta_L} \leq Z^*_{\alpha_L}$ for any $\bar{\alpha}_L \geq \bar{\delta}_L$. □

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