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SOME DEGREE CONDITIONS FOR $\mathcal{P}_{\geq k}$-FACTOR COVERED GRAPHS

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Abstract. A spanning subgraph of a graph G is called a path-factor of G if its each component is a path. A path-factor is called a $\mathcal{P}_{\geq k}$-factor of G if its each component admits at least k vertices, where $k \geq 2$. (Zhang and Zhou, *Discrete Math.* **309** (2009) 2067–2076) defined the concept of $\mathcal{P}_{\geq k}$-factor covered graphs, *i.e.*, G is called a $\mathcal{P}_{\geq k}$-factor covered graph if it has a $\mathcal{P}_{\geq k}$-factor covering e for any $e \in E(G)$. In this paper, we firstly obtain a minimum degree condition for a planar graph being a $\mathcal{P}_{\geq 2}$-factor and $\mathcal{P}_{\geq 3}$-factor covered graph, respectively. Secondly, we investigate the relationship between the maximum degree of any pairs of non-adjacent vertices and $\mathcal{P}_{\geq k}$-factor covered graphs, and obtain a sufficient condition for the existence of $\mathcal{P}_{\geq 2}$-factor and $\mathcal{P}_{\geq 3}$-factor covered graphs, respectively.

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1. Introduction

The graphs considered here are finite and simple, unless explicitly stated. Let $G = (V(G), E(G))$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. A spanning subgraph of G is a subgraph H of G such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called an induced subgraph of G if every pair of vertices in H which are adjacent in G are also adjacent in H. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of v and the set of vertices adjacent to v in G, respectively. For $S \subseteq V(G)$, we write $N_G(S) = \cup_{v \in S} N_G(v)$. We use $\delta(G)$ to denote the minimum degree of a graph G. We refer to [5] for the notation and terminologies not defined here.

For a family of connected graphs $\mathcal{F}$, a spanning subgraph of a graph G is called an $\mathcal{F}$-factor of G if its each component is isomorphic to some graph in $\mathcal{F}$. In particular, an $\mathcal{F}$-factor is called a $\mathcal{P}_{\geq k}$-factor of G if every component in $\mathcal{F}$ is a path of order at least k, where $k \geq 2$. A graph G is called a $\mathcal{P}_{\geq k}$-factor covered graph if it has a $\mathcal{P}_{\geq k}$-factor covering e for any $e \in E(G)$.

Since Tutte proposed the well known Tutte 1-factor theorem [15], there are many results on graph factors [1,3,8,10,16] and $\mathcal{P}_{\geq k}$-factors in claw-free graphs and cubic graphs [4,12,13]. More results on graph factors can be found in the survey papers and books in [1,14,19]. We use $\omega(G)$, $i(G)$ to denote the number of components and isolated vertices of a graph G, respectively. For a subset $X \subseteq V(G)$, $G-X$ denotes the graph obtained

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from G by deleting all the vertices of X. Akiyama, Avis and Era [2] proved the following theorem, which is a criterion for a graph to have a $\mathcal{P}_{\geq 2}$-factor.

Theorem 1.1. *(Akiyama et al. [1]) A graph G has a $\mathcal{P}_{\geq 2}$-factor if and only if $i(G-X) \leq 2|X|$ for all $X \subseteq V(G)$.*

By introducing the concept of a sun, Kaneko [9] gave a criterion for a graph with a $\mathcal{P}_{\geq 3}$-factor. Recently, a simpler proof for Kaneko's theorem [9] was presented by Kano *et al.* [11].

A graph H is called factor-critical if $H-\{v\}$ has a 1-factor for each $v \in V(H)$. Let H be a factor-critical graph and $V(H) = \{v_1, v_2, \ldots, v_n\}$. By adding new vertices $\{u_1, u_2, \ldots, u_n\}$ together with new edges $\{v_iu_i : 1 \leq i \leq n\}$ to H, the resulting graph is called a sun. Note that, according to Kaneko [9], we regard K_1 and K_2 also as a sun, respectively. Usually, the suns other than K_1 are called big suns. It is called a sun component of $G-X$ if the component of $G-X$ is isomorphic to a sun. We denote by $sun(G-X)$ the number of sun components in $G-X$.

Theorem 1.2. *(Kaneko [9]) A graph G has a $\mathcal{P}_{\geq 3}$-factor if and only if $sun(G-X) \leq 2|X|$ for all $X \subseteq V(G)$.*

Zhang and Zhou [20] proposed the concept of path-factor covered graph, which is a generalization of matching cover. They also obtained a characterization for $\mathcal{P}_{\geq 2}$-factor and $\mathcal{P}_{\geq 3}$-factor covered graphs, respectively.

Theorem 1.3. *(Zhang and Zhou [20]) Let G be a connected graph. Then G is a $\mathcal{P}_{\geq 2}$-factor covered graph if and only if $i(G-S) \leq 2|S| - \varepsilon_1(S)$ for all $S \subseteq V(G)$, where $\varepsilon_1(S)$ is defined by*

$$\varepsilon_1(S) = \begin{cases} 2 & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set;} \\ 1 & \text{if } S \text{ is a nonempty independent set and there exists} \\ & \text{a nontrivial component of } G\text{-}S; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.4. *(Zhang and Zhou [20]) Let G be a connected graph. Then G is a $\mathcal{P}_{\geq 3}$-factor covered graph if and only if $sun(G-S) \leq 2|S| - \varepsilon_2(S)$ for all $S \subseteq V(G)$, where $\varepsilon_2(S)$ is defined by*

$$\varepsilon_2(S) = \begin{cases} 2 & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set;} \\ 1 & \text{if } S \text{ is a nonempty independent set and there exists a} \\ & \text{non-sun component of } G\text{-}S; \\ 0 & \text{otherwise.} \end{cases}$$

For a connected graph G, its *isolated toughness*, denoted by $I_t(G)$, was first introduced by Yang *et al.* [18] as follows. If G is complete, then $I_t(G) = +\infty$; otherwise,

$$I_t(G) = \min \left\{ \frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2 \right\}.$$

The *binding number* is introduced by Woodall [17] and defined as

$$bind(G) = \min \left\{ \frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G) \right\}.$$

Recently, Zhou [21] and Dai [7] obtained some classes of $\mathcal{P}_{\geq 2}$-factor covered graphs, respectively.

Theorem 1.5. *(Zhou [21]) Let G be a connected graph. Then G is a $\mathcal{P}_{\geq 2}$-factor covered graph if $bind(G) > 2/3$.*

Theorem 1.6. *(Dai [7]) Let G be a connected graph of order at least two. Then G is a $\mathcal{P}_{\geq 2}$-factor covered graph if one the following holds:* (i) *G is claw-free and $\delta(G) \geq 2$;* (ii) *$I_t(G) > 2/3$.*

For a connected graph G, its *toughness*, denoted by $t(G)$, was first introduced by Chvátal [6] as follows. If G is complete, then $t(G) = +\infty$; otherwise,

$$t(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \geq 2\right\}.$$

Using Theorem 1.4, Zhou *et al.* [22] and Dai [7] obtained some classes of $\mathcal{P}_{\geq 3}$-factor covered graphs, respectively.

Theorem 1.7. *(Zhou et al. [22], Zhou [21]) Let G be a connected graph of order at least three. Then G is a $\mathcal{P}_{\geq 3}$-factor covered graph if one the following holds:* (i) *$bind(G) \geq 3/2$;* (ii) *$t(G) > 2/3$;* (iii) *$I_t(G) > 5/3$;* (iv) *G is r-regular where $r \geq 2$.*

Theorem 1.8. *(Dai [7]) Let G be a connected graph of order at least three. Then G is a $\mathcal{P}_{\geq 3}$-factor covered graph if one the following holds:* (i) *G is claw-free and $\delta(G) \geq 3$;* (ii) *G is a 3-connected planar graph.*

In this paper, we proceed to investigate $\mathcal{P}_{\geq k}$-factor covered graphs. We respectively obtain two special classes of $\mathcal{P}_{\geq 2}$-factor covered graphs and $\mathcal{P}_{\geq 3}$-factor covered graphs. Our main results will be shown in Sections 2 and 3, respectively.

2. Minimum degree for $\mathcal{P}_{\geq k}$-factor covered planar graphs

In this section, we study the relationship between planar graphs and $\mathcal{P}_{\geq k}$-factor covered graphs, and obtain a minimum degree condition for a planar graph being a $\mathcal{P}_{\geq 2}$-factor and $\mathcal{P}_{\geq 3}$-factor covered graph, respectively.

To prove our results, we will use an important lemma as following.

Lemma 2.1. *[5] Let G be a connected planar graph with at least three vertices. If G does not contain triangles, then $|E(G)| \leq 2|G| - 4$.*

Theorem 2.2. *Let G be a connected planar graph of order at least two. If $\delta(G) \geq 3$, then G is a $\mathcal{P}_{\geq 2}$-factor covered graph.*

Proof. Suppose G is not a $\mathcal{P}_{\geq 2}$-factor covered graph. By Theorem 1.3, there exists a subset $S \subseteq V(G)$ such that $i(G-S) > 2|S| - \varepsilon_1(S)$. According to the integrality of $i(G-S)$, we obtain that $i(G-S) \geq 2|S| - \varepsilon_1(S) + 1$.

Claim 2.3. *$S \neq \emptyset$.*

Proof. Suppose $S = \emptyset$, by the definition of $\varepsilon_1(S)$, we have $\varepsilon_1(S) = 0$. Then $i(G) = i(G-S) \geq 2|S| - \varepsilon_1(S) + 1 = 1$. On the other hand, $i(G) \leq \omega(G) = 1$ since G is a connected graph. So, we obtain that G is an isolated vertex, a contradiction. This completes the proof of Claim 2.3. □

By Claim 2.3, $S \neq \emptyset$. Then by the definition of $\varepsilon_1(S)$, we obtain $\varepsilon_1(S) \leq 2$. It follows immediately that

$$i(G-S) \geq 2|S| - \varepsilon_1(S) + 1 \geq 2|S| - 1.$$

Set $|S| = s$. We denote by $I(G-S)$ the set of isolated vertices in $G-S$. Then we construct a simple bipartite graph $H = H[X, Y]$ as follows. Let $X = S$ and $Y \subseteq I(G-S)$ such that $|Y| = 2s-1$. For any $s \in X$ and $y \in Y$, $sy \in E(H)$ if and only if $sy \in E(G)$. Since $\delta(G) \geq 3$, it is clear that for each $y \in Y$, we have $|N_H(y)| \geq 3$. Hence, $|H| \geq s + (2s-1) = 3s-1 \geq 5$ and

$$|E(H)| \geq 3 \times (2s-1) = 6s-3. \tag{2.1}$$

As G is a connected planar graph, it is easy to see that H is also a connected planar graph. According to the fact that a bipartite graph does not contain any odd cycles, Lemma 2.1 implies that $|E(H)| \leq 2|H| - 4 = 2 \times (3s-1) - 4 = 6s-6$, which is a contradiction to (2.1). This completes the proof of Theorem 2.2. □

It is not hard to find that the conditions in Theorem 2.2 is not sufficient for a graph to be a $\mathcal{P}_{\geq 3}$-factor covered graph. However, if we strengthen the conditions on connectivity and minimum degree, then we could obtain a minimum degree condition for the existence of $\mathcal{P}_{\geq 3}$-factor covered planar graphs.

Theorem 2.4. *Let G be a connected planar graph. If $\delta(G) \geq 4$, then G is a $\mathcal{P}_{\geq 3}$-factor covered graph.*

Proof. Suppose G is not a $\mathcal{P}_{\geq 3}$-factor covered graph. By Theorem 1.4, there exists a subset $S \subseteq V(G)$ such that $sun(G-S) > 2|S| - \varepsilon_2(S)$. According to the integrality of $sun(G-S)$, we obtain that $sun(G-S) \geq 2|S| - \varepsilon_2(S) + 1$. We distinguish three cases below to show that G is a $\mathcal{P}_{\geq 3}$-factor covered graph.

Case 1. $S = \emptyset$.

In this case, by the definition of $\varepsilon_2(S)$, we have $\varepsilon_2(S) = 0$. Since G is a connected graph, $sun(G) \leq \omega(G) = 1$. On the other hand, we obtain that

$$sun(G) = sun(G-S) \geq 2|S| - \varepsilon_2(S) + 1 = 1.$$

It follows easily that $sun(G) = 1$, *i.e.*, G is a big sun. By the definition of sun, it contradicts the fact that $\delta(G) \geq 4$. This completes the proof of Case 1.

Case 2. $|S| = 1$.

In this case, we obtain $\varepsilon_2(S) \leq 1$ by the definition of $\varepsilon_2(S)$. It follows immediately that

$$sun(G-S) \geq 2|S| - \varepsilon_2(S) + 1 \geq 2.$$

Let C be sun component of $G-S$ and x a vertex of $V(C)$ such that $d_C(x) \leq 1$. Since $\delta(G) \geq 4$, we have

$$|S| \geq d_G(x) - d_C(x) \geq \delta(G) - 1 \geq 3.$$

This contradiction completes the proof of Case 2.

Case 3. $|S| \geq 2$.

In this case, we obtain $\varepsilon_2(S) \leq 2$ by the definition of $\varepsilon_2(S)$. It follows immediately that

$$sun(G-S) \geq 2|S| - \varepsilon_2(S) + 1 \geq 2|S| - 1.$$

Set $|S| = s$. We denote by $Sun(G-S)$ the set of sun components in $G-S$. Since $sun(G-S) \geq 2|S| - 1$, let $C_1, C_2, \ldots, C_{2s-1}$ be $2s-1$ distinct sun components where $C_i \in Sun(G-S)$ for any $1 \leq i \leq 2s-1$.

Then we construct a simple bipartite graph $H = H[X, Y]$ as follows. For each $i \in [1, 2s-1]$, choose vertex $c_i \in V(C_i)$ such that $d_{C_i}(c_i) \leq 1$. Let $X = S$ and $Y = \{c_1, c_2, \ldots, c_{2s-1}\}$. For any $s \in X$ and $c_i \in Y$, $sc_i \in E(H)$ if and only if $sc_i \in E(G)$. Since $\delta(G) \geq 4$, it is clear that for each $1 \leq i \leq 2s-1$, we have $|N_H(c_i)| \geq 3$. Hence, $|H| = s + (2s-1) = 3s - 1 \geq 5$ and

$$|E(H)| \geq 3 \times (2s-1) = 6s - 3. \tag{2.2}$$

As G is a connected planar graph, it is easy to see that H is also a connected planar graph. According to the fact that a bipartite graph does not contain any odd cycles, Lemma 2.1 implies that

$$|E(H)| \leq 2|H| - 4 = 2 \times (3s-1) - 4 = 6s - 6,$$

which is a contradiction to (2.2). This completes the proof of Case 3.

Combining Case 1–3, Theorem 2.4 is proved. □

3. Degree conditions for $\mathcal{P}_{\geq k}$-factor covered graphs

In this section, we mainly investigate the relationship between the maximum degree of any pairs of non-adjacent vertices and $\mathcal{P}_{\geq k}$-factor covered graph, and obtain a degree condition for the existence of $\mathcal{P}_{\geq 2}$-factor and $\mathcal{P}_{\geq 2}$-factor covered graphs, respectively.

Theorem 3.1. *Let G be a connected graph of order at least two. If*

$$\max\{d_G(u), d_G(v)\} > \left\lceil \frac{n+1}{3} \right\rceil$$

for all pairs of non-adjacent vertices u and v of G, then G is a $\mathcal{P}_{\geq 2}$-factor covered graph.

Proof. Suppose G is not a $\mathcal{P}_{\geq 2}$-factor covered graph. By Theorem 1.3, there exists a subset $S \subseteq V(G)$ such that $i(G{-}S) > 2|S|{-}\varepsilon_1(S)$. Let $I(G{-}S)$ be the set of isolated vertices of $G{-}S$. According to the integrality of $i(G{-}S)$, we obtain that

$$i(G{-}S) \geq 2|S|{-}\varepsilon_1(S) + 1. \tag{3.1}$$

Claim 3.2. $|S| \geq 2$.

Proof. If $S = \emptyset$, then $\varepsilon_1(S) = 0$. By (3.1), $i(G) = i(G{-}S) \geq 1$. On the other hand, $i(G) \leq \omega(G) = 1$. So, we obtain that G is an isolated vertex, a contradiction.

Thus, we may assume $|S| = 1$, then $\varepsilon_1(S) \leq 1$. By (3.1), we have that $i(G{-}S) \geq 2|S|{-}\varepsilon_1(S) + 1 \geq 2|S| \geq 2$. As $I(G{-}S)$ is independent in G, there is a vertex $x \in I(G{-}S)$ such that $d_G(x) > \lceil \frac{n+1}{3} \rceil \geq \frac{n+1}{3}$. Then we have that $|S| \geq d_G(x) > \frac{n+1}{3}$ since $N_G(x) \subseteq S$. It follows that $i(G{-}S) \geq 2|S| > \frac{2n+2}{3}$ and thus

$$n \geq |S| + i(G{-}S) > \frac{n+1}{3} + \frac{2n+2}{3} = n + 1,$$

a contradiction. This completes the proof of Claim 3.2. □

By Claim 3.2 and (3.1), we have $\varepsilon_1(S) \leq 2$ and

$$i(G{-}S) \geq 2|S|{-}\varepsilon_1(S) + 1 \geq 2|S|{-}1 \geq 3. \tag{3.2}$$

Since $I(G{-}S)$ is an independent set of G, there exists $x \in I(G{-}S)$ such that $d_G(x) > \lceil \frac{n+1}{3} \rceil \geq \frac{n+1}{3}$. Then we have $|S| \geq d_G(x) > \frac{n+1}{3}$ since $N_G(x) \subseteq S$. It follows from (3.2) that $i(G{-}S) \geq 2|S| - 1 > \frac{2n-1}{3}$ and thus

$$n \geq |S| + i(G{-}S) > \frac{n+1}{3} + \frac{2n{-}1}{3} = n.$$

This contradiction completes the proof of Theorem 3.1. □

Theorem 3.3. *Let G be a connected graph of order $n \geq 7$. Then G is a $\mathcal{P}_{\geq 3}$-factor covered graph if*

$$\max\{d_G(u), d_G(v)\} > \left\lceil \frac{n+2}{3} \right\rceil$$

for all pairs of non-adjacent vertices u and v of G.

Proof. Suppose G is not a $\mathcal{P}_{\geq 3}$-factor covered graph. By Theorem 1.4, there exists a subset $S \subseteq V(G)$ such that $sun(G{-}S) > 2|S| - \varepsilon_2(S)$. According to the integrality of $sun(G{-}S)$, we obtain that

$$sun(G{-}S) \geq 2|S|{-}\varepsilon_2(S) + 1. \tag{3.3}$$

Claim 3.4. $S \neq \emptyset$.

Proof. Suppose $S = \emptyset$, then $\varepsilon_2(S) = 0$. By (3.3), $sun(G) = sun(G-S) \geq 1$. On the other hand, $sun(G) \leq \omega(G) = 1$. So, we obtain that G is a big sun containing at least 7 vertices. It follows that there exist two vertices of degree one, denoted by $\{u, v\}$, which contradicts that $\max\{d_G(u), d_G(v)\} > \lceil \frac{n+2}{3} \rceil \geq 3$. This completes the proof of Claim 3.4. □

By Claim 3.4 and (3.3), we have $|S| \geq 1$. If $|S| = 1$, then $\varepsilon_2(S) = 1$ and

$$sun(G-S) \geq 2|S| - \varepsilon_2(S) + 1 \geq 2|S| \geq 2. \tag{3.4}$$

If $|S| = 1$, then $\varepsilon_2(S) = 2$ and

$$sun(G-S) \geq 2|S| - \varepsilon_2(S) + 1 \geq 2|S| - 1 \geq 3. \tag{3.5}$$

Case 1. $i(G-S) \geq 2$.

Let $\{x, y\}$ be two distinct isolated vertices of $G-S$. Since $\max\{d_G(x), d_G(y)\} > \lceil \frac{n+2}{3} \rceil \geq \frac{n+2}{3}$ and $N_G(x) \cup N_G(y) \subseteq S$, we have that

$$|S| \geq \max\{d_G(x), d_G(y)\} > \frac{n+2}{3}.$$

It follows from (3.4) and (3.5) that $sun(G-S) \geq 2|S| - 1 > \frac{2n+1}{3}$ and thus

$$n \geq |S| + sun(G-S) > \frac{n+2}{3} + \frac{2n+1}{3} = n + 1,$$

a contradiction.

Case 2. $i(G-S) \leq 1$.

In this case, by (3.4) and (3.5), there exist at least two suns of $G-S$, denoted by $C_1, C_2, \ldots, C_t$ where $t \geq 2$. We choose $c_i \in V(C_i)$ such that $d_{C_i}(c_i) \leq 1$, where $i = 1, 2$. Obviously, $c_1c_2 \notin E(G)$. Then $\max\{d_G(c_1), d_G(c_2)\} > \lceil \frac{n+2}{3} \rceil \geq \frac{n+2}{3}$. Without of generality, we assume $d_G(c_1) > \frac{n+2}{3}$. Since $d_S(c_1) = d_G(c_1) - d_{C_1}(c_1) > \frac{n+2}{3} - 1 = \frac{n-1}{3}$, we have that $|S| \geq d_S(c_1) > \frac{n-1}{3}$. It follows from (3.4) and (3.5) that

$$sun(G-S) \geq 2|S| - 1 > \frac{2n-2}{3} - 1,$$

and thus

$$\begin{aligned} n &\geq |S| + 2 \times sun(G-S) - i(G-S) \\ &> \frac{n-1}{3} + 2 \times \left(\frac{2n-2}{3} - 1 \right) - 1 \\ &= \frac{5n-5}{3} - 3 \geq n. \end{aligned}$$

This contradiction completes the proof of Theorem 3.3. □

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