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DIRECTIONAL SCALE ELASTICITY CONSIDERING THE MANAGEMENT PREFERENCE OF DECISION-MAKERS

Tiantian Ren[1,2], Zhongbao Zhou[1,2,*], Ruiyang Li[1,2] and Wenbin Liu[1,3,4]

Abstract. Most data envelopment analysis (DEA) studies on scale elasticity (SE) and returns to scale (RTS) of efficient units arise from the traditional definitions of them in economics, which is based on measuring radial changes in outputs caused by the simultaneous change in all inputs. In actual multiple inputs/outputs activities, the goals of expanding inputs are not only to obtain increases in outputs, but also to expect the proportions of such increases consistent with the management preference of decision-makers. However, the management preference is usually not radial changes in outputs. With the latter goal into consideration, this paper proposes the directional SE and RTS in a general formula for multi-output activities, and offers a DEA-based model for the formula of directional SE at any point on the DEA frontier, which is straightforward and requires no simplifying assumptions. Finally, the empirical part employs the data of 16 basic research institutions in Chinese Academy of Sciences (CAS) to illustrate the superiority of the proposed theories and methods.

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1. Introduction

Returns to scale (RTS) is an important issue in the performance analysis of production organizations, which is concerned with the relationship between efficient vectors of inputs and outputs for a given technology production function. Scale elasticity (SE) is a quantitative measure of the strength of RTS characterization [11]. It is defined as the ratio of the proportional change in output to the equal-proportional change in inputs at a (frontier function) unit. Frisch [14] noted that in classical economics, the analysis of RTS and SE is mainly for the production function of the single output technology. In actual production, the real production function is often difficult to obtain, especially in the case of multi-input/multi-output production. Data Envelopment Analysis (DEA), as a data-driven evaluation method, can effectively estimate the real production function. Banker [3] and Banker *et al.* [5] first introduced the definitions of RTS and SE from the classical economics into the DEA framework. Since then, the issues of RTS and SE in the multi-input/multi-output context have been extensively studied using DEA (for a review, see [6, 7]).

Keywords. Data envelopment analysis, returns to scale, management preference, directional scale elasticity.

[1] School of Business Administration, Hunan University, Changsha 410082, P.R. China.

[2] Institute of Data Science and Decision Optimization, Hunan University, Changsha 410082, P.R. China.

[3] Beijing Normal University, Zhuhai 519087, P.R. China.

[4] DBM, BNU-HKBU United International College, Zhuhai 519087, P.R. China.

[*]Corresponding author: Z.B.Zhou@hnu.edu.cn; z.b.zhou@163.com

Within the existing work in DEA, there are two strands of research to providing RTS information of the decision-making units (DMUs), including the qualitative and quantitative approaches. The former is to describe the qualitative characterization of RTS, that is, to distinguish whether DMUs have increasing, constant or decreasing RTS (see, *e.g.*, [4,5,8,9,22,23], among others). The latter is to measure the numerical value of SE, as noted, which can be used to offer a stronger quantitative characterization of RTS than the qualitative approach. Relatively recently, most DEA research efforts in developing the quantitative approach include, *e.g.*, Golany and Yu [15], Førsund and Hjalmarsson [11,12], Podinovski *et al.* [19], Podinovski and Førsund [18], Atici and Podinovski [1], Zelenyuk [27], Mirjaberi and Matin [16], Podinovski *et al.* [17,20,21]. Our development also mainly contributes to the quantitative approach as outline below.

In essence, the above studies mostly arise from the definitions of RTS and SE in DEA by Banker [3], which is based on the idea of measuring the radial change in outputs caused by the radial change in inputs. However, in practice, the decision-maker usually has a preferred (or desirable) change in outputs due to the constraints of actual management needs and objective conditions, and it is referred as the management preference of decision-makers in our paper. Taking the two outputs of banking activities, total loans and total customers served, when the latter approaches saturation, managers will naturally prefer the growth of total loans rather than the radial growth of these two outputs. In most cases, the management preference of decision-makers is not radial, so the results of SE calculated by the traditional approaches with radial measure do not suffice to provide practical guidance for the management. In fact, our numerical examples also verify that the SE and RTS are different under different management preference of the decision-makers. Therefore, it is necessary to measure RTS and SE with respect to the actual management preference of decision-makers. To this end, in the multi-input/multi-output context, this paper proposes a method to define and measure directional RTS and directional SE with the consideration of the management preference of decision-makers. In our method, the management preference of decision-makers is characterized by the specified direction of output changes, and then the amount of output changes satisfying such management preference can be measured by the projection of actual optimal outputs in this direction. The detailed definitions of directional SE and RTS will be introduced in Section 2.

Furthermore, it is worth noting that the projection in the specified direction of output changes is usually related to the direction of input changes, so there is also the problem of optimizing the direction of input changes for the target DMU. As mentioned above, most of the traditional methods assume that all inputs change in an equal proportion, although the input-directional SE is introduced and discussed in Yang *et al.* [26] and Yang and Liu [25] (abbreviated as Yang's method hereafter). Obviously, the equal-proportional change in all inputs may not be the best strategy for decision-makers in the presence of their own management preferences, which is illustrated *via* the following empirical example in this paper. Thus, in this work the direction of input changes is also considered so that one can find out a higher SE along the specific direction of output changes by varying the direction of input changes.

The purpose of this paper is to develop the analysis of SE and RTS for the target DMU along the direction satisfying the manager preference. The contributions of our work are threefold. First, we introduce the direction of output changes to describe the management preference of decision-makers, and extend the concepts of directional SE and directional RTS on this basis. Second, we propose a general DEA-based method to numerically calculate the directional SE at any point on the DEA frontier. Instead of several estimation methods for directional SE (*e.g.*, Yang's method), we provide a straightforward derivation for the formula of the directional SE, and accordingly construct two auxiliary linear programs to obtain the exact values of the right- and left-hand directional SE. This proposed method used for measuring the directional SE not only overcomes the problems of infeasibility and parameter interference that arises Yang's method, but also can be easily extended to more general cases since no simplifying assumptions are imposed. Third, we employ the data from 16 basic research institutions in CAS as the empirical example to illustrate the advantages of the analyses of directional SE and directional RTS. The results show that the directional SE and directional RTS provide more precise information than those of traditional SE and RTS on how to adjust inputs to maximize output increases consistent with the decision maker's management preference.

The structures of these contributions are organized as follows. In Section 2, we propose the generalized definitions of directional RTS and directional SE considering the management preference of decision-makers. In Section 3, we offer a DEA-based approach to derive the formulae for directional SE of the DMU located at the DEA frontier. In Section 4, we provide an empirical example consisting of data from 16 basic institutes of CAS in 2016 to illustrate the proposed theories and models. The last section contains the conclusion and delineates prospects of further research.

2. Definitions of directional RTS and directional SE

In this section, we propose the generalized definitions RTS and SE with consideration of the management preference of the decision-makers. Suppose there are n observed DMUs. $\mathbf{X}_j = (x_{1j}, \ldots, x_{mj})^T$ and $\mathbf{Y}_j = (y_{1j}, \ldots, y_{sj})^T$ represent the input vector and the output vector that of the j-th DMU, respectively. Under the assumption of variable returns to scale (VRS), the production possibility set (PPS) can be formulated as:

$$T = \left\{ (\mathbf{X}, \mathbf{Y}) | \sum_{j=1}^{n} \lambda_j \mathbf{X}_j \leq \mathbf{X},\ \sum_{j=1}^{n} \lambda_j \mathbf{Y}_j \geq \mathbf{Y},\ \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0,\ j = 1, 2, \ldots, n \right\}. \tag{2.1}$$

Before discussing the scale issues, it should be kept in mind that RTS and SE is a local property and applies to boundary (frontier) points. This argument is emphasized in most studies: examples include Banker and Thrall [4], Førsund and Hjalmarsson [11], Yang and Liu [25] and Zelenyuk [27] among others. In the studies, the definition of SE is only associated to the efficient units on the frontier, and therefore it does not depend on the efficiency measure. For an inefficient DMU that is a point in the interior of the PPS is first represented by a benchmark (projection) of the frontier, in which case some explicit efficiency measures are needed, and thereafter the discussion of the scale issue is targeted at this benchmark. Podinovski *et al.* [19] extended the SE to the inefficient DMUs by integrating the input and output radial efficiency measures into the formulae of SE, which is essentially in line with the idea of the typical literature mentioned above. Such benchmarks for inefficient DMUs in Podinovski *et al.* [19] are set as the input and output radial projections.

Analogous to this idea, we employ the Farrell measures of efficiency from input and output orientation [10] to realize the radial projections for a target DMU$(\mathbf{X}_0, \mathbf{Y}_0)$, which have been commonly used as the benchmarks for inefficient DMUs in the discussion of SE using DEA methods. Such projections can be determined by solving the following input- and output-oriented BCC models, respectively.

[BCC-I]

$$\begin{aligned}
&\min\ \theta = \theta_0 - \varepsilon \left(\mathbf{s}^- \mathbf{e}^T + \mathbf{s}^+ \mathbf{e}^T \right) \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j \mathbf{X}_j + \mathbf{s}^- = \theta_0 \mathbf{X}_0, \\
&\quad \sum_{j=1}^{n} \lambda_j \mathbf{Y}_j - \mathbf{s}^+ = \mathbf{Y}_0, \\
&\quad \sum_{j=1}^{n} \lambda_j = 1, \qquad \lambda_j \geq 0,\ \ j = 1, \ldots, n, \\
&\quad \mathbf{s}^-, \mathbf{s}^+ \geq 0.
\end{aligned} \tag{2.2}$$

[BCC-O]

$$\begin{aligned}
&\min\ \varphi = \varphi_0 - \varepsilon \left(\mathbf{s}^- \mathbf{e}^T + \mathbf{s}^+ \mathbf{e}^T \right) \\
&\text{s.t.} \sum_{j=1}^{n} \lambda_j \mathbf{X}_j + \mathbf{s}^- = \mathbf{X}_0,
\end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j \mathbf{Y}_j - \mathbf{s}^+ = \mathbf{Y}_0 + \varphi_0 \delta \mathbf{Y}_0,$$
$$\sum_{j=1}^{n} \lambda_j = 1, \qquad \lambda_j \geq 0, \;\; j = 1, \ldots, n,$$
$$\mathbf{s}^-, \; \mathbf{s}^+ \geq 0. \tag{2.3}$$

Here ε is a non-Archimedean small positive number, and the vectors $\mathbf{s}^- = (s_1^-, \ldots, s_m^-)$ and $\mathbf{s}^+ = (s_1^+, \ldots, s_s^+)$ represent the input slacks and output slacks, respectively. The DMU$(\mathbf{X}_0, \mathbf{Y}_0)$ is called "strongly efficient" if and only if the optimal solution $(\theta^*, \lambda^*, \mathbf{s}^{+*}, \mathbf{s}^{-*})$ $((\varphi^*, \lambda^*, \mathbf{s}^{+*}, \mathbf{s}^{-*}))$ obtained from model (2.2) ((2.3)) satisfies $\theta^* = 1$ $(\varphi^* = 1)$ and has no slacks $(\mathbf{s}^{+*} = 0, \mathbf{s}^{-*} = 0)$. Otherwise, the DMU$(\mathbf{X}_0, \mathbf{Y}_0)$ can be brought into the strongly efficient status by the BCC-I and BCC-O projections as defined by
[BCC-I-P]

$$\mathbf{X}_0^{I*} \leftarrow \theta^* \mathbf{X}_0 - \mathbf{s}^{-*},$$
$$\mathbf{Y}_0^{I*} = \mathbf{Y}_0 + \mathbf{s}^{+*}. \tag{2.4}$$

[BCC-O-P]

$$\mathbf{X}_0^{O*} \leftarrow \mathbf{X}_0 - \mathbf{s}^{-*},$$
$$\mathbf{Y}_0^{O*} = \varphi^* \delta \mathbf{Y}_0 + \mathbf{s}^{+*}. \tag{2.5}$$

For convenience, we define the projection point of DMU$(\mathbf{X}_0, \mathbf{Y}_0)$ derived from different efficiency measures uniformly as DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$, where if the observation point DMU$(\mathbf{X}_0, \mathbf{Y}_0)$ is strongly efficient, then it is consistent with DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$. It is noted that this projection can be also determined by some alternative efficiency measures (*e.g.*, directional radial measures). In fact, the above efficiency measures only provide a measure of the distance from the inefficient DMU to the efficient frontier, but do not serve to explain why a DMU is inefficient [11]. It is not difficult to find that projecting inefficient DMUs to the efficient frontier and exploring the SE and RTS of their projection points are two different processes. The general definitions of RTS and SE can be treated as the idea of measuring the change in outputs caused by the change in inputs on the efficient frontier. This change can be radial or non-radial, depending on the management preferences of decision-makers. This paper is concerned with how to incorporate the management preferences into the SE and RTS of a strongly efficient DMU or the strongly efficient projection of an inefficient DMU, and the resulting SE and RTS are called as the direction SE and directional RTS. To address this problem, in what follows we dwell mainly on both radial projections under input and output orientations of each DMU in the definition and derivation of directional SE and directional RTS.

We now propose the definitions of directional SE and directional RTS accounting for the management preference of decision-makers towards this strongly efficient DMU. As mentioned earlier, when considering the management preference of decision-makers in the analyses of RTS and SE, we need to generalize the concept of SE and RTS to accommodate any proportional change of outputs and not only in the proportional change as in the traditional SE measure. Notionally as Yang and Liu [25], we first introduce the direction of input changes and the direction of output changes to describe the situation that inputs and outputs change non-proportionally. For a strongly efficient DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$, let the vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)^T (\omega_i \geq 0, \; i = 1, \ldots, m)$ represents the direction of input changes, and the vector $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_s)^T (\delta_r \geq 0, \; r = 1, \ldots, s)$ represents the direction of output changes, which essentially describes the management preference of decision-makers. Now, along these given directions, increasing inputs $\mathbf{X}_0^*$ by proportion t in the direction $\boldsymbol{\omega}$, the maximal proportion $\beta(t)$ of increase in outputs $\mathbf{Y}_0^*$ possible in the direction $\boldsymbol{\delta}$ is allowed by the function

$$\beta(t) = \max\{\beta | \; (\boldsymbol{\Omega}_t \mathbf{X}_0^*, \boldsymbol{\Phi}_\beta \mathbf{Y}_0^*) \in T\} \tag{2.6}$$

where, $\mathbf{\Omega}_t = \mathrm{diag}\{1+\omega_1 t, \ldots, 1+\omega_m t\}$, $\mathbf{\Phi}_\beta = \mathrm{diag}\{1+\delta_1\beta, \ldots, 1+\delta_s\beta\}$, $\mathrm{diag}\{*\}$ denotes the diagonal matrix.

Let $\|T\| = \sum_{i=1}^m \omega_i t/m$ and $\|B\| = \sum_{r=1}^s \delta_r \beta(t)/m$, where $T = (\omega_1 t, \omega_2 t, \ldots, \omega_m t)$ and $B = (\delta_1\beta(t), \delta_2\beta(t), \ldots, \delta_s\beta(t))$. Similar to the classic definition of SE, at any boundary point let

$$(\mathbf{X}, \mathbf{Y}) = \left(\mathrm{diag}\left\{1+\omega_1 t, \ldots, 1+\omega_m t\right\} \mathbf{X}_0^*, \mathrm{diag}\{1+\delta_1\beta(t), \ldots, 1+\delta_s\beta(t)\}\mathbf{Y}_0^*\right). \tag{2.7}$$

At this point, its directional SE can be defined as the ratio of the marginal productivity $\frac{\mathrm{d}\|B\|}{\mathrm{d}\|T\|}$ to the average productivity $\frac{\|B\|+1}{\|T\|+1}$ when $t \to 0$ (see, *e.g.*, [25]). Then

$$\begin{aligned}\varepsilon(\mathbf{X}, \mathbf{Y}) &= \lim_{t\to 0} \frac{\mathrm{d}\,\|B\|}{\mathrm{d}\,\|T\|} \times \frac{\|T\|+1}{\|B\|+1} = \lim_{t\to 0} \frac{\mathrm{d}\,\|B\|/\mathrm{d}t}{\mathrm{d}\,\|T\|/\mathrm{d}t} \times \frac{\|T\|+1}{\|B\|+1} \\ &= \lim_{t\to 0} \frac{\frac{1}{s}\sum_{r=1}^s \delta_r \frac{\mathrm{d}\beta(t)}{\mathrm{d}t}}{\frac{1}{m}\sum_{i=1}^m \omega_i} \times \frac{\|T\|+1}{\|B\|+1} = \lim_{t\to 0} \frac{\frac{1}{s}\sum_{r=1}^s \delta_r}{\frac{1}{m}\sum_{i=1}^m \omega_i} \times \frac{\frac{1}{m}\sum_{i=1}^m \omega_i t + 1}{\frac{1}{s}\sum_{r=1}^s \delta_r \beta(t) + 1} \times \frac{\mathrm{d}\beta(t)}{\mathrm{d}t}\cdot\end{aligned} \tag{2.8}$$

At DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$, where $t = 0$ and $\beta(t) = 0$, the directional SE derived from formula (2.8) can be represented as follows:

$$\varepsilon\left(\mathbf{X}_0^*, \mathbf{Y}_0^*\right) = \frac{\frac{1}{s}\sum_{r=1}^s \delta_r}{\frac{1}{m}\sum_{i=1}^m \omega_i} \times \frac{\mathrm{d}\beta(t)}{\mathrm{d}t}\cdot \tag{2.9}$$

Further, if we let $\sum_{i=1}^m \omega_i = p$ and $\sum_{r=1}^s \delta_r = q$ (where p and q are arbitrary positive numbers) be the completely generalized direction setting (as in [2]), then the proposed directional SE can be rewritten as

$$\varepsilon\left(\mathbf{X}_0^*, \mathbf{Y}_0^*\right) = \frac{ps}{qm} \times \frac{\mathrm{d}\beta(t)}{\mathrm{d}t}\cdot \tag{2.10}$$

By comparing the proposed directional SE and the classic SE (*i.e.*, $\mathrm{d}\beta(t)/\mathrm{d}t$), we find that the proposed directional SE is ps/qm time of the classic SE. To maintain consistency with the classic SE in the form, we consider the normalization condition of $p = m$ and $q = s$ for standardizing our directional vector. In this setting, the proposed and classic SEs can be uniformly expressed as

$$\varepsilon\left(\mathbf{X}_0^*, \mathbf{Y}_0^*\right) = \frac{\mathrm{d}\beta(t)}{\mathrm{d}t}\cdot \tag{2.11}$$

Now, the rationale of the formulae (2.11) is that a marginal increase in the inputs by a scale factor t (where $t \to 0$) in the direction $\boldsymbol{\omega}$, corresponds to the maximum increase of the outputs by a scale factor $t \times \varepsilon(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ in the direction $\boldsymbol{\delta}$ for the DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$. Therefore, the directional SE measure in formula (2.11) has "reciprocal" interpretation with the classic case, *i.e.*, the value greater than 1 indicates about increasing directional RTS, equal to 1 indicates about constant directional RTS, and less than 1 indicates decreasing directional RTS.

Since the piecewise linear frontier constructed by DEA is not smooth, it is not differentiable everywhere, in particular at some corner points. According to Podinovski *et al.* [19], the right-hand and left-hand derivatives of $\beta'(0)$ always exist. Here, we use one-sided derivatives $\beta'_+(0)$, $\beta'_-(0)$ instead of $\beta'(0)$ to get the definitions of the right-hand and left-hand directional SE:

$$\varepsilon_+\left(\mathbf{X}_0^*, \mathbf{Y}_0^*\right) = \lim_{t\downarrow 0} \frac{\beta(t)-\beta(0)}{t-0} = \beta'_+(0) \tag{2.12}$$

$$\varepsilon_-\left(\mathbf{X}_0^*, \mathbf{Y}_0^*\right) = \lim_{t\uparrow 0} \frac{\beta(t)-\beta(0)}{t-0} = \beta'_-(0). \tag{2.13}$$

Then, we can define the directional RTS as follows:

(1) The directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$

(i) if $\varepsilon_+(\mathbf{X}_0^*, \mathbf{Y}_0^*) > 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is said increasing in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$;

(ii) if $\varepsilon_+(\mathbf{X}_0^*, \mathbf{Y}_0^*) = 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is said constant in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$;

(iii) if $\varepsilon_+(\mathbf{X}_0^*, \mathbf{Y}_0^*) < 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is said decreasing in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$.

(2) The directional RTS to the "left" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$

(i) if $\varepsilon_-(\mathbf{X}_0^*, \mathbf{Y}_0^*) > 1$ holds, then directional RTS to the "left" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is said increasing in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$;

(ii) if $\varepsilon_-(\mathbf{X}_0^*, \mathbf{Y}_0^*) = 1$ holds, then directional RTS to the "left" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is said constant in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$;

(iii) if $\varepsilon_-(\mathbf{X}_0^*, \mathbf{Y}_0^*) < 1$ holds, then directional RTS to the "left" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is said decreasing in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$.

3. Measurement of directional SE

In Section 3, we propose a DEA-based approach to derive the formulae for the directional SE for the target DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$. In view of both the definitions of traditional SE and directional SE, we assume henceforth that DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is strongly efficient with respect to T. If not, we project it onto the strongly frontier of T using model (2.2) (or model (2.3)), and the directional SE can be measured through its strongly efficient projection ((2.4) or (2.5)).

Based on the definition (2.6) for the strongly efficient DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$, the function $\beta(t)$ is determined by solving the following linear program:

$$\begin{aligned} \beta(t) = \max\ & \beta \\ \text{s.t.}\ & \sum_{j=1}^{n} \lambda_j \mathbf{X}_j^* \leq (1 + t\boldsymbol{\omega}) \circ \mathbf{X}_0^*, \\ & -\sum_{j=1}^{n} \lambda_j \mathbf{Y}_j^* + (1 + \beta\,\boldsymbol{\delta}) \circ \mathbf{Y}_0^* \leq \mathbf{0}, \\ & \sum_{j=1}^{n} \lambda_j = 1,\ \ \lambda_j \geq 0,\ \ j = 1, \ldots n, \\ & \beta \text{ sign free} \end{aligned} \tag{3.1}$$

where the vectors $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)^T(\omega_i \geq 0, i = 1, \ldots, m)$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_s)^T(\delta_r \geq 0, r = 1, \ldots, s)$ represent the directions of input changes and output changes, and satisfy $\sum_{i=1}^{m} \omega_i = m$, $\sum_{r=1}^{s} \delta_r = s$, where t and β are input and output scaling factors, respectively. Our goal is to obtain $\beta'_+(0)$ and $\beta'_-(0)$, which can be applied to calculate the numerical values of the left- and right directional SE of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ by formulae (2.12) and (2.13).

Theorem 3.1. *Given the direction of input changes $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)^T(\omega_i \geq 0, i = 1, \ldots, m,\ \sum_{i=1}^{m} \omega_i = m)$ and the direction of output changes $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_s)^T(\delta_r \geq 0, r = 1, \ldots, s,\ \sum_{r=1}^{s} \delta_r = s)$, let $\mathbf{v} = (v_1, \ldots, v_m)$ represent the dual variables of the input constraint corresponding to the program (3.1) with $t = 0$. Then the value of $\beta'_+(0)$ and $\beta'_-(0)$ can be determined as follow*

$$\beta'_+(0) = \min_{\mathbf{v}_0} \mathbf{v}\,(\boldsymbol{\omega} \circ \mathbf{X}_0^*)$$
$$\beta'_-(0) = \max_{\mathbf{v}_0} \mathbf{v}\,(\boldsymbol{\omega} \circ \mathbf{X}_0^*)\,.$$

To prove Theorem 3.1, we use the concept of the directional derivatives of the optimal value function in linear programming problems. Podinovski *et al.* [19] formulated this in the form of Proposition 3.2.

Proposition 3.2. *Let function $\varphi(\mathbf{b})$ of vector $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2)$ be defined as the optimal value in the following linear program formulated in terms of the vector $\mathbf{X} \in R^n$ (assuming it has an optimal solution), where J is some subset of $\{1, \ldots, n\}$:*

$$\varphi(\mathbf{b}) = \max\left\{\mathbf{cX} \,|\mathbf{A}_1\mathbf{X} \leq \mathbf{b}_1, \mathbf{A}_2\mathbf{X} = \mathbf{b}_2, x_j \geq 0, j \in J\right\}. \tag{3.2}$$

Assume a non-zero vector $\mathbf{d}$ is a feasible direction at $\mathbf{b}$. (This means (3.2) as an optimal solution with vector $\mathbf{b}' = \mathbf{b} + \theta\mathbf{d}$ on its right-hand side for all sufficiently small $\theta > 0$.) Then, the directional derivative $\varphi'(\mathbf{b}; \mathbf{d})$ of φ at $\mathbf{b}$ in the direction $\mathbf{d}$ exists and is equal to

$$\varphi'\left(\mathbf{b}; \mathbf{d}\right) = \min\left\{\mathbf{wd}\,|\mathbf{w} \text{ optimal in the dual to (3.2)}\right\}. \tag{3.3}$$

Proof of Theorem 3.1. Program (3.1) is a special case of (3.2) and its optimal value can be represented as $\beta(t) = \varphi((1+t\boldsymbol{\omega}) \circ \mathbf{X}_0^*, -\mathbf{Y}_0^*, 1)$, then $\beta(0) = \varphi(\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1)$ with $t = 0$, particularly. According to Proposition 3.2, the right-hand side of $\beta(0)$, that is, $\beta'_+(0)$, is equal to the directional derivative of the optimal value function $\varphi(\mathbf{b})$ with $\mathbf{b} = (\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1)$ in the direction of $\mathbf{d} = (\boldsymbol{\omega} \circ \mathbf{X}_0^*, \mathbf{0}, 0)$:

$$\begin{aligned}\beta'_+(0) &= \lim_{t\downarrow 0}\frac{\beta(t)-\beta(0)}{t-0} = \lim_{t\downarrow 0}\frac{\varphi(\mathbf{X}_0^* + t\boldsymbol{\omega}\circ\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1) - \varphi(\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1)}{t-0}\\ &= \lim_{\sigma\downarrow 0}\frac{\varphi(\mathbf{b}+\sigma\mathbf{d})-\varphi(\mathbf{b})}{\sigma} = \varphi'\left(\mathbf{b};\mathbf{d}\right) = \varphi'\left((\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1); (\boldsymbol{\omega}\circ\mathbf{X}_0^*, \mathbf{0}, 0)\right).\end{aligned} \tag{3.4}$$

Similarly, $\beta'_-(0)$ is equal to the directional derivative of the corresponding optimal value function $\varphi(\mathbf{b})$ with $\mathbf{b} = (\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1)$ in the direction of $-\mathbf{d} = (-\boldsymbol{\omega} \circ \mathbf{X}_0^*, \mathbf{0}, 0)$:

$$\begin{aligned}\beta'_-(0) &= \lim_{t\uparrow 0}\frac{\beta(t)-\beta(0)}{t} = \lim_{t\uparrow 0}\frac{\varphi(\mathbf{X}_0^* + t\boldsymbol{\omega}\circ\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1) - \varphi(\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1)}{t}\\ &= \lim_{\sigma\downarrow 0}\frac{\varphi(\mathbf{b}+\sigma(-\mathbf{d}))-\varphi(\mathbf{b})}{-\sigma} = -\varphi'\left(\mathbf{b};-\mathbf{d}\right) = -\varphi'\left((\mathbf{X}_0^*, -\mathbf{Y}_0^*, 1); (-\boldsymbol{\omega}\circ\mathbf{X}_0^*, \mathbf{0}, 0)\right).\end{aligned} \tag{3.5}$$

Then, consider the dual model of program (3.1) with $t = 0$ is as follows:

$$\begin{aligned}&\beta(0) = \min \mathbf{vX}_0^* - \mathbf{uY}_0^* + u_0\\ &\text{s.t. } \mathbf{vX}_j^* - \mathbf{uY}_j^* + u_0\mathbf{e} \geq 0, \qquad j = 1, \ldots, n,\\ &\quad\ \mathbf{u}(\boldsymbol{\delta} \circ \mathbf{Y}_0^*) = 1\\ &\quad\ \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}, u_0 \text{ sign free}\end{aligned} \tag{3.6}$$

where $\mathbf{e}$ is a row vector with all its elements being equal to one. Let $\boldsymbol{\Psi}$ represent the set of all the optimal solutions of program (3.6). We follow with interest the directional RTS measurement of strongly efficient DMUs and the strongly efficient projections of inefficient DMUs, and it's easy to get $\beta(0) = 0$, for each optimal solution from $\boldsymbol{\Psi}$, and we have $\mathbf{vX}_0^* - \mathbf{uY}_0^* + u_0 = 0$. From the program (3.6), the formulae (3.4) and (3.5) can be expressed by:

$$\begin{aligned}\beta'_+(0) &= \min_{(\mathbf{v},\mathbf{u},u_0)\in\boldsymbol{\Psi}}\left\langle(\mathbf{v},\mathbf{u},u_0), (\boldsymbol{\omega}\circ\mathbf{X}_0^*, \mathbf{0}, 0)\right\rangle = \min_{(\mathbf{v},\mathbf{u},u_0)\in\Omega}\mathbf{v}\left(\boldsymbol{\omega}\circ\mathbf{X}_0^*\right)\\ \beta'_-(0) &= -\min_{(\mathbf{v},\mathbf{u},u_0)\in\boldsymbol{\Psi}}\left\langle(\mathbf{v},\mathbf{u},u_0), (-\boldsymbol{\omega}\circ\mathbf{X}_0^*, \mathbf{0}, 0)\right\rangle = \max_{(\mathbf{v},\mathbf{u},u_0)\in\Omega}\mathbf{v}\left(\boldsymbol{\omega}\circ\mathbf{X}_0^*\right).\end{aligned}$$

□

Intuitively, Theorem 3.1 provides the formulae for directional SE under VRS assumption. Based on Theorem 3.1, we further construct the two auxiliary linear programs (3.7) and (3.8) to calculate the minimum and maximum values of $\mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*)$:

$$\begin{aligned} \min\ & \mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*) \\ \text{s.t. } & \mathbf{v}\mathbf{X}_0^* - \mathbf{u}\mathbf{Y}_0^* + u_0 = 0, \\ & \mathbf{v}\mathbf{X}_j^* - \mathbf{u}\mathbf{Y}_j^* + u_0\mathbf{e} \geq 0, \qquad j = 1, \ldots, n, \\ & \mathbf{u}(\boldsymbol{\delta} \circ \mathbf{Y}_0^*) = 1 \\ & \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}, u_0 \text{ sign free} \end{aligned} \tag{3.7}$$

$$\begin{aligned} \max\ & \mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*) \\ \text{s.t. } & \mathbf{v}\mathbf{X}_0^* - \mathbf{u}\mathbf{Y}_0^* + u_0 = 0, \\ & \mathbf{v}\mathbf{X}_j^* - \mathbf{u}\mathbf{Y}_j^* + u_0\mathbf{e} \geq 0, \qquad j = 1, \ldots, n, \\ & \mathbf{u}(\boldsymbol{\delta} \circ \mathbf{Y}_0^*) = 1 \\ & \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}, u_0 \text{ sign free.} \end{aligned} \tag{3.8}$$

From the objective function of programs (3.7) and (3.8) we can obtain the numerical values of the right- and left-directional SE in the direction of $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)^T$ (inputs) and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_s)^T$ (outputs). Accordingly, the directional RTS to the "right" and the "left" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ can be determined as follows:

(1) The directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$
  - (i) if $\min\{\mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*)\} > 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is increasing in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$;
  - (ii) if $\min\{\mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*)\} = 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is constant in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$;
  - (iii) if $\min\{\mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*)\} < 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is decreasing in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$.

(2) The directional RTS to the "left" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$
  - (i) if $\max\{\mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*)\} > 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is increasing in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$;
  - (ii) if $\max\{\mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*)\} = 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is constant in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$;
  - (iii) if $\max\{\mathbf{v}(\boldsymbol{\omega} \circ \mathbf{X}_0^*)\} < 1$ holds, then directional RTS to the "right" of DMU$(\mathbf{X}_0^*, \mathbf{Y}_0^*)$ is decreasing in the direction of $(\omega_1, \ldots, \omega_m)^T$ and $(\delta_1, \ldots, \delta_s)^T$.

It is worth emphasizing that our formulae of the directional SE should be applied to the strongly efficient unit with respect to the DEA frontier. For the inefficient DMU, one might not be able to distinguish whether the output increases resulting from the input increases is due to the improvement of technology inefficiency or the economies of scale, so it is necessary to analyze the directional SE and directional RTS for the target DMU under the premise of technology efficiency.

4. Numerical example

4.1. Directional SE and directional RTS analysis

In this section, to illustrate our proposed method, we analyze an empirical dataset consisting of 16 basic research institutes in the CAS in 2016. The data set consists of two inputs (research staff and research expenditures) and three outputs (external research funding, high SCI publications and granted patents). Table 1 lists the input and output datasets of these 16 basic research institutes in 2016.

As discussed previously, the analyses of directional SE and directional RTS are aimed at a strongly efficient DMU of T. We start by identifying the scale states with respect to strongly efficient projections under the input

TABLE 1. Input-output data of 16 basic CAS research institutes in 2016.

Institutes	Inputs		Outputs		
	Staff (FTE)	Res. Expen. (RMB million)	Exter. Fund. (RMB million)	High Pub. (Number)	Granted patents (Number)
DMU 1	327	296.6066	67.1469	183	10
DMU 2	442	253.1420	295.7381	112	37
DMU 3	2589	1485.7362	922.1845	432	336
DMU 4	1472	1218.8277	424.3740	298	60
DMU 5	1338	780.1315	193.3859	204	49
DMU 6	449	365.3578	77.5895	90	66
DMU 7	609	629.1216	306.1235	783	236
DMU 8	321	376.2365	324.9000	428	153
DMU 9	1105	741.7895	534.8300	253	48
DMU 10	276	257.3831	41.1500	67	2
DMU 11	793	498.1555	141.8561	303	109
DMU 12	327	365.9673	152.7000	74	12
DMU 13	63	58.1003	12.4700	71	0
DMU 14	473	676.5251	967.1305	429	75
DMU 15	476	239.0912	5.5200	4	13
DMU 16	919	559.3781	108.3900	66	38

orientation of these 16 DMUs computed by model (2.2). The results of the inefficiency scores are reported in the second column of Table 2, and there are six institutes are efficient. As shown in Table 2, the corresponding projections are reported in columns 3–7.

Then, we apply the auxiliary models (3.7) and (3.8) to calculate the directional SE to the "left" and "right" for the input-oriented projections of these DMUs. Here, for comparison purposes, we set three different directions of output changes, which respectively represent different management preferences of decision-makers:

Case 1: $\delta_1 = 0.75,\ \delta_2 = 0.75,\ \delta_2 = 1.5$;
Case 2: $\delta_1 = 1,\ \delta_2 = 1,\ \delta_2 = 1$;
Case 3: $\delta_1 = 1.25,\ \delta_2 = 1.25,\ \delta_2 = 0.5$.

The above three directions can be expressed as different management preferences of decision-makers with respect to the three outputs: external research funding, high SCI publications, and granted patents. In Case 1, the given direction of output changes represents the management preference that the decision-maker prefers to prioritize increasing patent quantity compared to the other two outputs; the given direction in Case 2 can be interpreted that the decision-maker has an equal priority on the increase of all three outputs. Contrary to the management preference assumed in Case 1, the direction given by Case 3 indicates that the decision-maker prioritizes the increases in external research funding and high SCI publications over granted patents. Based on the three sets of output directions, using Theorem 3.1 proposed in this paper, we can calculate the values of right-hand and left-hand directional SE for 16 institutes. Here, we take the results for the input-oriented projections of DMU 1 and DMU 2 as an example to conduct an in-depth explanation.

Tables 3 and 4 display the directional RTS and directional SE to the "right" and "left" for the input-oriented projections of DMU 1 and DMU 2 in the three cases, respectively. To present these results intuitively, we offer Figures 1–6 to show the RTS status for the input-oriented projections of DMU 1 and DMU 2 in different input change directions under these cases. Figures 1–3 show the results regarding the input-oriented projection of DMU 1, and Figures 4–6 show the results with respect to the input-oriented projection of DMU 2.

From the results in Figures 1–6 and Tables 3, 4, for the input-oriented projections of DMU 1 and DMU 2, we can find that the directional SE and directional RTS have obvious differences due to the different management

Table 2. Efficiency and projections under the input orientation for 16 basic research institutes.

Institutes	Inefficiency scores (model 2)	Projections under input orientation				
		Staff (FTE)	Res. Expen. (RMB million)	Exter. Fund. (RMB million)	High Pub. (Number)	Granted patents (Number)
DMU 1	0.5042	148.0779	149.5583	67.1469	183	38.9042
DMU 2	1	442	253.1420	295.7381	112	37
DMU 3	1	2589	1485.7362	922.1845	432	336
DMU 4	0.2953	277.3486	359.9351	424.3740	298	71.2090
DMU 5	0.2650	174.3362	206.7410	193.3859	204	49.1101
DMU 6	0.5346	174.2941	195.3355	147.2437	225	66
DMU 7	1	609	629.1216	306.1235	783	236
DMU 8	1	321	376.2365	324.9000	428	153
DMU 9	0.5423	364.5208	402.2505	534.8300	253	48
DMU 10	0.2979	75.3173	76.6791	41.1500	81.7551	2.2532
DMU 11	0.5716	246.8039	284.7464	235.0508	325.3333	109
DMU 12	0.4092	133.8116	149.7575	152.7000	121.7450	12
DMU 13	1	63	58.1003	12.4700	71	0
DMU 14	1	473	676.5251	967.1305	429	75
DMU 15	0.3561	84.9216	85.1315	39.0163	101.3333	13
DMU 16	0.2637	155.0074	147.5331	108.3900	155.6677	38

preferences of decision-makers. Several observations can be made regarding the above results on these input-oriented projections.

Table 3 and Figures 1–3 report the directional SE and directional RTS for the input-oriented projection of DMU 1. The following conclusions emerge. From the results of the right directional RTS in Figure 1a, it can be observed intuitively for the input-oriented projection of DMU 1, if the proportion of the increases in Staff and Res. Expen. locates in Region 2 (including radial proportion), increasing directional RTS prevails. On the contrary, if the proportion of the inputs' increases locates in Region 3, decreasing directional RTS prevails. As mentioned earlier, the management preference indicated in Case 1 is that the decision-maker prefers to increasing patent quantity *versus* external research funding and high SCI publications. Considering such management preference, the DMU 1 which has been realized strongly efficient in PPS under the input orientation is suggested to increase the two inputs in these proportions that are located in Region 2. From the results of left directional RTS in Figure 1b, It indicates that increasing directional RTS prevails at the input-oriented projection of DMU 1 for any proportion of the decreases in Staff and Res. Expen. Similar to Case 1, the results with respect to Cases 2 and 3 can be observed intuitively in Figures 2 and 3, which are not discussed in detail here.

Comparing the results of Figures 1, 2 and 3, we can easily find that the SE and RTS for the input-oriented projection of DMU 1 are different if the decision-maker have different management preference. Cases 1, 2 and 3 represent three different management preferences of decision-makers with respect to the three outputs: external research funding, high SCI publications, and granted patents, respectively. Here, as an example of the results of increasing inputs in equal proportions (*i.e.*, $\omega_1 = \omega_2 = 1$), the calculation results of the right directional SE for the input-oriented projection of DMU 1 in the three cases are 1.23, 1.03 and 0.82, and the corresponding directional RTS status are increasing, increasing, and decreasing, respectively (see column 4 and column 6 of Tab. 3). Therefore, the SE and RTS need to consider different management preferences of decision-makers in practice.

In the meanwhile, looking at the results of Case 1 in Table 3, we can find that the best SE for the input-oriented projection of DMU 1 is achieved as the proportion of increased in Staff and Res. Expen. is 0.9:1.1 ($\omega_1 = 0.9$ and $\omega_2 = 1.1$), not as the equal-proportion ($\omega_1 = 1.0$ and $\omega_2 = 1.0$) of increases in the two inputs.

Table 3. Directional SE and directional RTS for the input-oriented projection of DMU 1 in different directions.

	ω_1	ω_2	$\beta'_+(0)$ (right)	$\beta'_-(0)$ (left)	Directional RTS (right)	Directional RTS (left)
	0.1	1.9	0.97	2.61	Decreasing	Increasing
	0.2	1.8	1.02	2.47	Increasing	Increasing
	0.3	1.7	1.07	2.34	Increasing	Increasing
	0.4	1.6	1.11	2.20	Increasing	Increasing
	0.5	1.5	1.16	2.06	Increasing	Increasing
	0.6	1.4	1.21	1.92	Increasing	Increasing
	0.7	1.3	1.25	1.79	Increasing	Increasing
	0.8	1.2	1.30	1.65	Increasing	Increasing
	0.9	1.1	1.34	1.51	Increasing	Increasing
Case 1	1	1	1.23	1.39	Increasing	Increasing
	1.1	0.9	1.10	1.44	Increasing	Increasing
	1.2	0.8	0.98	1.48	Decreasing	Increasing
	1.3	0.7	0.86	1.53	Decreasing	Increasing
	1.4	0.6	0.74	1.58	Decreasing	Increasing
	1.5	0.5	0.61	1.62	Decreasing	Increasing
	1.6	0.4	0.49	1.67	Decreasing	Increasing
	1.7	0.3	0.37	1.71	Decreasing	Increasing
	1.8	0.2	0.25	1.76	Decreasing	Increasing
	1.9	0.1	0.12	1.81	Decreasing	Increasing
	0.1	1.9	0.73	2.24	Decreasing	Increasing
	0.2	1.8	0.76	2.12	Decreasing	Increasing
	0.3	1.7	0.80	2.01	Decreasing	Increasing
	0.4	1.6	0.83	1.89	Decreasing	Increasing
	0.5	1.5	0.87	1.77	Decreasing	Increasing
	0.6	1.4	0.90	1.65	Decreasing	Increasing
	0.7	1.3	0.94	1.53	Decreasing	Increasing
	0.8	1.2	0.97	1.42	Decreasing	Increasing
	0.9	1.1	1.01	1.30	Increasing	Increasing
Case 2	1	1	1.03	1.18	Increasing	Increasing
	1.1	0.9	0.93	1.08	Decreasing	Increasing
	1.2	0.8	0.82	1.11	Decreasing	Increasing
	1.3	0.7	0.72	1.15	Decreasing	Increasing
	1.4	0.6	0.62	1.18	Decreasing	Increasing
	1.5	0.5	0.52	1.22	Decreasing	Increasing
	1.6	0.4	0.41	1.25	Decreasing	Increasing
	1.7	0.3	0.31	1.29	Decreasing	Increasing
	1.8	0.2	0.21	1.32	Decreasing	Increasing
	1.9	0.1	0.10	1.36	Decreasing	Increasing
	0.1	1.9	0.58	2.16	Decreasing	Increasing
	0.2	1.8	0.61	2.04	Decreasing	Increasing
	0.3	1.7	0.64	1.93	Decreasing	Increasing
	0.4	1.6	0.67	1.82	Decreasing	Increasing
	0.5	1.5	0.70	1.70	Decreasing	Increasing
	0.6	1.4	0.72	1.59	Decreasing	Increasing
	0.7	1.3	0.75	1.48	Decreasing	Increasing
	0.8	1.2	0.78	1.36	Decreasing	Increasing
	0.9	1.1	0.81	1.25	Decreasing	Increasing
Case 3	1	1	0.82	1.14	Decreasing	Increasing
	1.1	0.9	0.74	1.02	Decreasing	Increasing
	1.2	0.8	0.66	0.91	Decreasing	Decreasing
	1.3	0.7	0.58	0.92	Decreasing	Decreasing
	1.4	0.6	0.49	0.92	Decreasing	Decreasing
	1.5	0.5	0.41	0.97	Decreasing	Decreasing
	1.6	0.4	0.33	1.00	Decreasing	Constant
	1.7	0.3	0.25	1.03	Decreasing	Increasing
	1.8	0.2	0.16	1.06	Decreasing	Increasing
	1.9	0.1	0.11	0.14	Decreasing	Decreasing

Table 4. Directional SE and directional RTS for the input-oriented projection of DMU 2 in different directions.

	ω_1	ω_2	$\beta'_+(0)$ (right)	$\beta'_-(0)$ (left)	Directional RTS (right)	Directional RTS (left)
	0.1	1.9	2.68	2.77	Increasing	Increasing
	0.2	1.8	2.54	2.63	Increasing	Increasing
	0.3	1.7	2.40	2.48	Increasing	Increasing
	0.4	1.6	2.26	2.33	Increasing	Increasing
	0.5	1.5	2.12	2.19	Increasing	Increasing
	0.6	1.4	1.98	2.04	Increasing	Increasing
	0.7	1.3	1.84	1.90	Increasing	Increasing
	0.8	1.2	1.70	1.75	Increasing	Increasing
	0.9	1.1	1.55	1.61	Increasing	Increasing
Case 1	1	1	1.41	1.46	Increasing	Increasing
	1.1	0.9	1.27	1.31	Increasing	Increasing
	1.2	0.8	1.13	1.17	Increasing	Increasing
	1.3	0.7	0.99	1.02	Decreasing	Increasing
	1.4	0.6	0.85	0.88	Decreasing	Decreasing
	1.5	0.5	0.71	0.73	Decreasing	Decreasing
	1.6	0.4	0.57	0.59	Decreasing	Decreasing
	1.7	0.3	0.42	0.45	Decreasing	Decreasing
	1.8	0.2	0.28	0.31	Decreasing	Decreasing
	1.9	0.1	0.14	0.18	Decreasing	Decreasing
	0.1	1.9	2.35	2.38	Increasing	Increasing
	0.2	1.8	2.22	2.25	Increasing	Increasing
	0.3	1.7	2.10	2.13	Increasing	Increasing
	0.4	1.6	1.98	2.00	Increasing	Increasing
	0.5	1.5	1.85	1.88	Increasing	Increasing
	0.6	1.4	1.73	1.75	Increasing	Increasing
	0.7	1.3	1.60	1.63	Increasing	Increasing
	0.8	1.2	1.48	1.50	Increasing	Increasing
	0.9	1.1	1.36	1.38	Increasing	Increasing
Case 2	1	1	1.23	1.25	Increasing	Increasing
	1.1	0.9	1.11	1.13	Increasing	Increasing
	1.2	0.8	0.99	1.01	Decreasing	Increasing
	1.3	0.7	0.86	0.88	Decreasing	Decreasing
	1.4	0.6	0.74	0.76	Decreasing	Decreasing
	1.5	0.5	0.62	0.64	Decreasing	Decreasing
	1.6	0.4	0.49	0.52	Decreasing	Decreasing
	1.7	0.3	0.37	0.40	Decreasing	Decreasing
	1.8	0.2	0.25	0.27	Decreasing	Decreasing
	1.9	0.1	0.12	0.15	Decreasing	Decreasing
	0.1	1.9	2.08	2.12	Increasing	Increasing
	0.2	1.8	1.97	2.01	Increasing	Increasing
	0.3	1.7	1.86	1.89	Increasing	Increasing
	0.4	1.6	1.75	1.78	Increasing	Increasing
	0.5	1.5	1.64	1.67	Increasing	Increasing
	0.6	1.4	1.53	1.56	Increasing	Increasing
	0.7	1.3	1.42	1.45	Increasing	Increasing
	0.8	1.2	1.31	1.34	Increasing	Increasing
	0.9	1.1	1.20	1.23	Increasing	Increasing
Case 3	1	1	1.09	1.11	Increasing	Increasing
	1.1	0.9	0.99	1.00	Decreasing	Constant
	1.2	0.8	0.88	0.89	Decreasing	Decreasing
	1.3	0.7	0.77	0.78	Decreasing	Decreasing
	1.4	0.6	0.66	0.68	Decreasing	Decreasing
	1.5	0.5	0.55	0.57	Decreasing	Decreasing
	1.6	0.4	0.44	0.46	Decreasing	Decreasing
	1.7	0.3	0.33	0.35	Decreasing	Decreasing
	1.8	0.2	0.22	0.24	Decreasing	Decreasing
	1.9	0.1	0.11	0.14	Decreasing	Decreasing

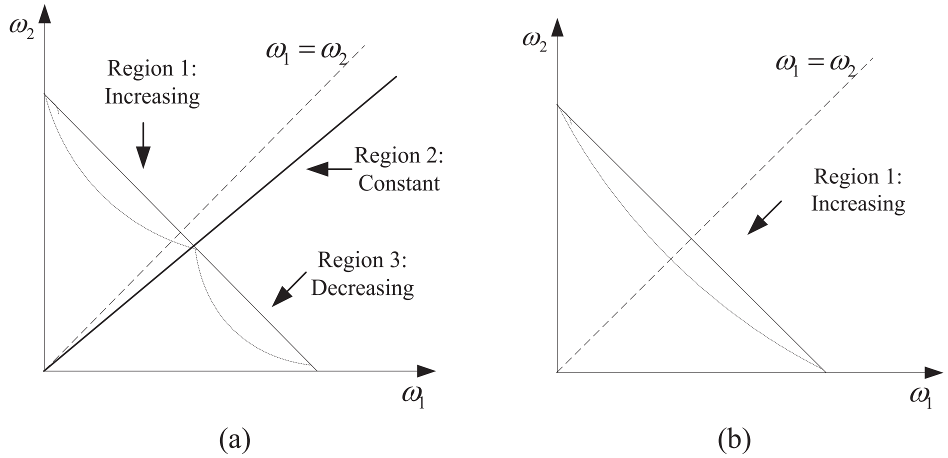

FIGURE 1. *Right* and *left* directional RTS for the input-oriented projection of DMU 1 in Case 1.

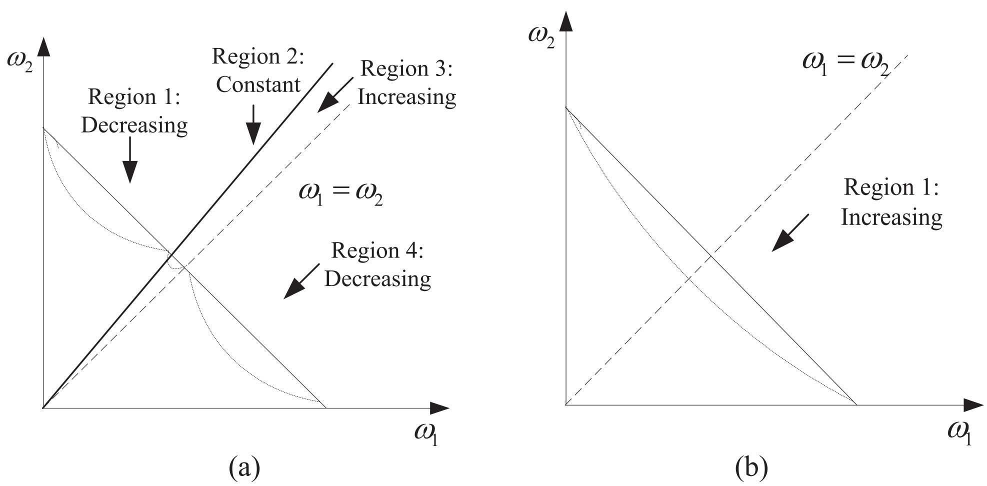

FIGURE 2. *Right* and *left* directional RTS for the input-oriented projection of DMU 1 in Case 2.

This result confirms our supposition above whereby an equal-proportional change in all inputs may not be the best strategy for practical management. It is possible that an unequal-proportional increase in inputs may contribute to a lager SE for a strongly efficient DMU, especially concerning the management preferences of decision-makers.

For DMU 2, since it is a strongly efficient DMU of PPS, that is, we can directly analyze the RTS status in terms of its current inputs and outputs. According to the results of the right-hand and left-hand directional RTS (see Figs. 4–6), we can observe that the directional RTS to the "right" and "left" of DMU 2 in the give directions of inputs and outputs are approximately equal. Thus, it can be summarized briefly that on the basis of existing inputs, if the proportion of Staff and Res. Expen. increases (or decreases) of DMU 2 locates in Region 1 of Figures 4, 5 and 6 (corresponding to Cases 1, 2, and 3), increasing directional RTS prevails. If the proportion of increases (or decreases) in the two inputs locates in Region 2, constant directional RTS prevails. If this proportion locates in Region 3, constant directional RTS prevails. In addition, under the three cases, we note that the feasible ranges for the proportion of increases (or decreases) in Staff and Res. Expen. corresponding to

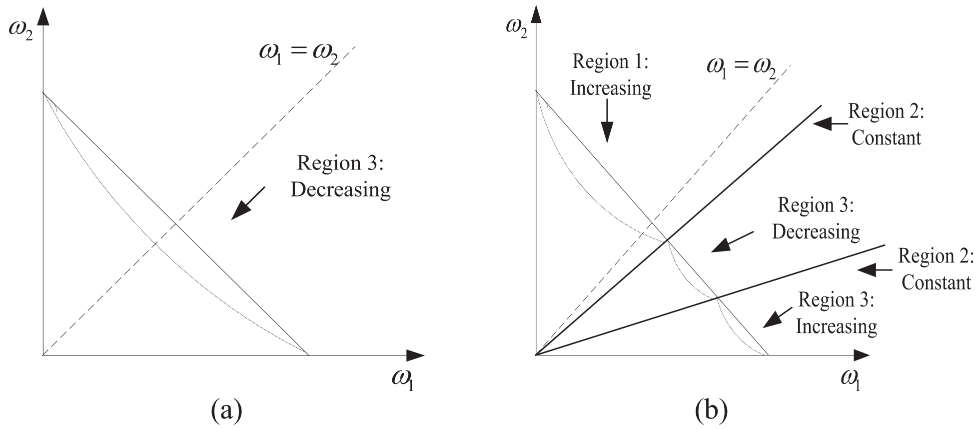

FIGURE 3. *Right* and *left* directional RTS for the input-oriented projection of DMU 1 in Case 3.

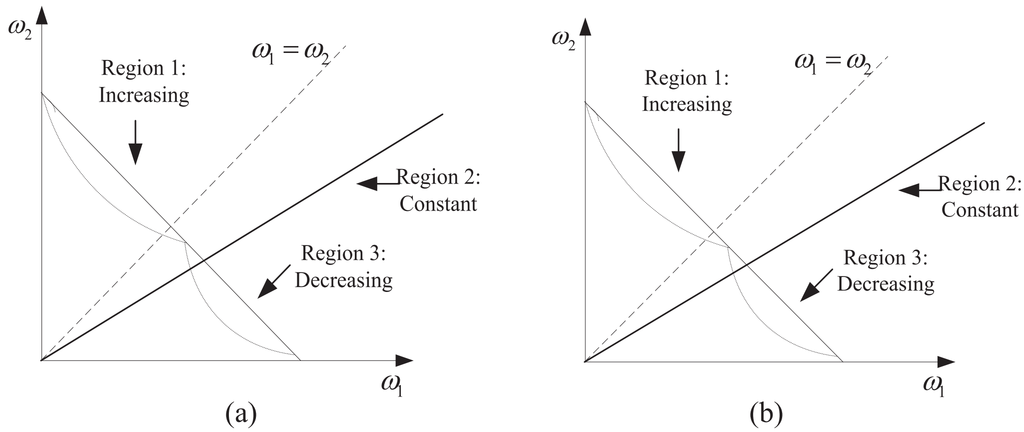

FIGURE 4. *Right* and *left* directional RTS for the input-oriented projection of DMU 2 in Case 1.

three directional RTS statuses are different. Combined with Table 4, for example, we can find that in Case 1, if the proportion of increases in the two inputs is 1.2:0.8 ($\omega_1 = 1.2$ and $\omega_2 = 0.8$), increasing directional RTS prevails on DMU 2. However, in Case 3, if the two inputs increase in this proportion, decreasing directional RTS prevails. The results further verify our supposition above that the SE and RTS are different under different management preference of the decision-makers.

In the following, let us analyze the scale issue of the strongly efficient projections of 16 DMUs under the output orientation. Table 5 reports the output-oriented inefficiency scores and projections computed by model (2.3).

Looking at Table 5, it is clear that the six strongly efficient DMUs under the input orientation remain strongly efficient under the output orientation. While for the inefficient DMUs, a significant difference can be found in their strongly efficient projections under these two orientations. As we discussed previously, the directional SE and directional RTS are measured at the strongly efficient DMU or the strongly efficient projections (*i.e.*, benchmarks) of inefficient DMUs in this contribution.

By employing models (3.7) and (3.8), the "left" and "right" directional SE and directional RTS are calculated similarly based on the output-oriented projections of the 16 DMUs. Given that DMU 2 is strongly efficient under

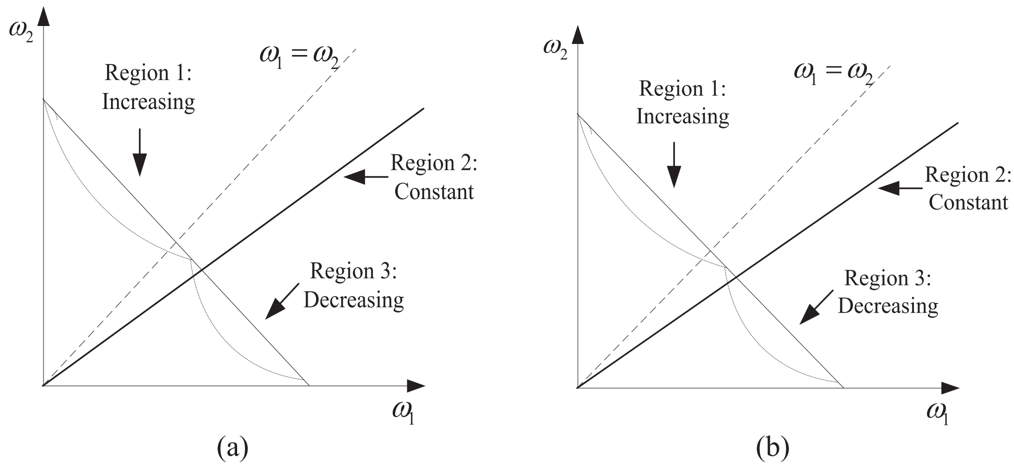

FIGURE 5. *Right* and *left* directional RTS for the input-oriented projection of DMU 2 in Case 2.

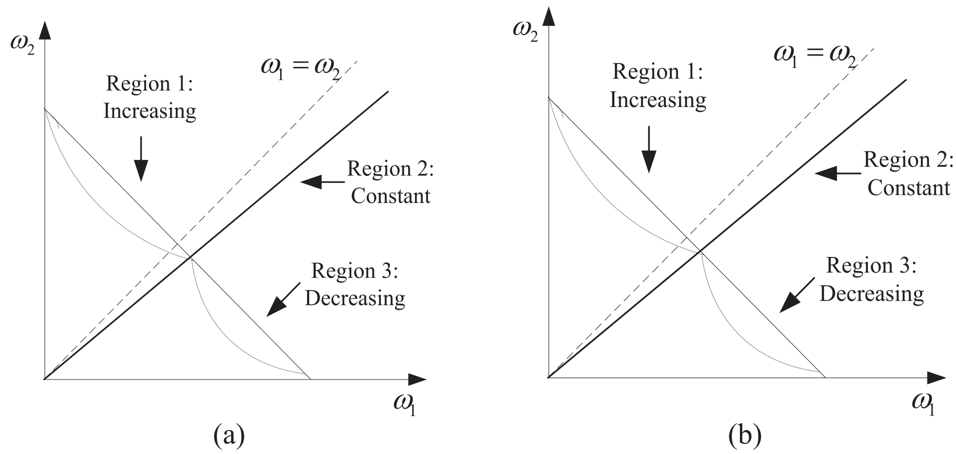

FIGURE 6. *Right* and *left* directional RTS for the input-oriented projection of DMU 2 in Case 3.

both input and output orientations, the same results on the directional SE and directional RTS (as in Tab. 4) are not repeatedly presented. Also taking the output-oriented projection of DMU 1 as example for a detailed explanation, the results are shown in Table 6.

According to results in Table 6, we can find that in these given directions of input change and output change, the values of the directional SE (see columns 4 and 5 of Tab. 6) at the output-oriented projection of DMU 1 are little different from the results at its input-oriented projection (see columns 4 and 5 of Tab. 3), and naturally the resulting directional RTS at these two projections are similar. Therefore, we make no repeat explanation.

Consequently, according to the analyses of directional SE and RTS proposed in this paper, research institutions can identify the status of increasing, constant and decreasing directional RTS with respect to their own management preference, and then adjust the proportion of inputs based on this. From a management point of view, the directional SE and directional RTS parameter is of particular importance to develop a feasible strategy for input adjustment and effectively avoid blind investment and resource waste.

Table 5. Efficiency and projections under the output orientation for 16 basic research institutes.

Institutes	Inefficiency scores (model (3))	Projections under output orientation				
		Staff (FTE)	Res. Expen. (RMB million)	Exter. Fund. (RMB million)	High Pub. (Number)	Granted patents (Number)
DMU 1	0.4968	291.0421	296.6066	135.1669	368.3795	98.5795
DMU 2	1	442	253.142	295.7381	112	37
DMU3	1	2589	1485.7362	922.1845	432	336
DMU4	0.5547	514.5779	662.0329	765.0477	537.2247	124.2209
DMU5	0.3248	549.4823	649.8668	595.4001	628.0789	165.5416
DMU6	0.4466	312.1777	365.3578	314.2164	415.7923	147.7681
DMU7	1	609	629.1216	306.1235	783	236
DMU8	1	321	376.2365	324.9	428	153
DMU9	0.5697	498.3625	682.0083	938.6812	444.0408	84.2449
DMU 10	0.2227	231.8745	257.3831	184.7848	300.8647	92.4582
DMU 11	0.5647	459.8483	498.1555	315.8476	599.1498	193.0153
DMU 12	0.3139	283.8904	365.9673	486.5258	244.3881	38.2339
DMU 13	1	63	58.1003	12.47	71	0
DMU 14	1	473	676.5251	967.1305	429	75
DMU 15	0.1493	209.7788	239.0912	190.2146	274.1009	87.0432
DMU 16	0.2258	481.3389	559.3781	480.0108	578.3269	168.2850

4.2. Comparison analysis with Yang's method

Furthermore, we have also provided a comparison analysis on the directional SE and RTS computed by the proposed method and Yang's method (see [25]). As introduced previously, the underlying idea of our method is to derive the formulate of the directional SE based on the directional derivatives of the optimal value function in linear programming programs, and present two auxiliary linear programs (models (3.7) and (3.8)) to obtain the exact values of the right- and left-hand directional SE. Yang's method mainly adopts the basic idea of the finite difference method (FDM) to provide an approximate estimate for the directional SE by testing the ratio of the amount of change of outputs on the efficient frontier in a specified direction caused by a change in a small enough amount t_0 of inputs in a specified direction.

For comparison, we use Yang's method to estimate the directional SE and directional RTS for the input-oriented and output-oriented projections of DMU 1, respectively. Following Yang and Liu [25], this small enough amount for the input change is set to $t_0 = 1E^{-6}$. Table 7 lists these estimated directional SE and directional RTS for the input-oriented projections of DMU 1, and Table 8 reports the results regarding the output-oriented projections of DMU 1.

As we can see from Tables 7 and 8, we notice that Yang's method used for estimating the directional SE suffers from the infeasibility problems in some directions, and clearly the corresponding RTS status in these directions cannot be identified using their method. Specifically, for the input-oriented projection of DMU 1, the incidences of the infeasibility problem are 21.05% and 29.82% when using Yang's method to estimate the left- and right-hand directional SE, respectively. For the output-oriented projection of DMU 1, the incidences of the infeasibility problem for estimating the left- and right-hand SE are 15.79% and 5.26%, respectively. Compared to the results computed by our method (see the results listed Tabs. 3 and 6), the exact values of the directional SE are obtained explicitly in all the above directions without any infeasibility problem.

Furthermore, the proposed method used for calculating the directional SE is not subject to any simplifying assumptions, such as the setting on the small enough quantity of the input change in FDM methods, and actually the setting on this quantity may perturb the estimations of directional SE and directional RTS. Finally, the proposed method provides a straightforward derivation for the directional scale elasticity, where all calculations

TABLE 6. Directional SE and directional RTS for the output-oriented projection of DMU 1 in different directions.

	ω_1	ω_2	$\beta'_+(0)$ (right)	$\beta'_-(0)$ (left)	Directional RTS (right)	Directional RTS (left)
	0.1	1.9	0.96	2.57	Decreasing	Increasing
	0.2	1.8	1.00	2.44	Constant	Increasing
	0.3	1.7	1.05	2.30	Increasing	Increasing
	0.4	1.6	1.09	2.17	Increasing	Increasing
	0.5	1.5	1.14	2.03	Increasing	Increasing
	0.6	1.4	1.18	1.89	Increasing	Increasing
	0.7	1.3	1.23	1.76	Increasing	Increasing
	0.8	1.2	1.27	1.62	Increasing	Increasing
	0.9	1.1	1.19	1.49	Increasing	Increasing
Case 1	1	1	1.09	1.36	Increasing	Increasing
	1.1	0.9	0.98	1.41	Decreasing	Increasing
	1.2	0.8	0.87	1.45	Decreasing	Increasing
	1.3	0.7	0.76	1.50	Decreasing	Increasing
	1.4	0.6	0.65	1.54	Decreasing	Increasing
	1.5	0.5	0.54	1.59	Decreasing	Increasing
	1.6	0.4	0.43	1.63	Decreasing	Increasing
	1.7	0.3	0.33	1.68	Decreasing	Increasing
	1.8	0.2	0.22	1.72	Decreasing	Increasing
	1.9	0.1	0.11	1.76	Decreasing	Increasing
	0.1	1.9	0.72	2.06	Decreasing	Increasing
	0.2	1.8	0.75	1.95	Decreasing	Increasing
	0.3	1.7	0.79	1.84	Decreasing	Increasing
	0.4	1.6	0.82	1.73	Decreasing	Increasing
	0.5	1.5	0.85	1.62	Decreasing	Increasing
	0.6	1.4	0.89	1.52	Decreasing	Increasing
	0.7	1.3	0.92	1.41	Decreasing	Increasing
	0.8	1.2	0.95	1.30	Decreasing	Increasing
	0.9	1.1	0.99	1.19	Decreasing	Increasing
Case 2	1	1	1.02	1.08	Increasing	Increasing
	1.1	0.9	0.91	1.05	Decreasing	Increasing
	1.2	0.8	0.81	1.09	Decreasing	Increasing
	1.3	0.7	0.71	1.12	Decreasing	Increasing
	1.4	0.6	0.61	1.16	Decreasing	Increasing
	1.5	0.5	0.51	1.19	Decreasing	Increasing
	1.6	0.4	0.41	1.22	Decreasing	Increasing
	1.7	0.3	0.30	1.26	Decreasing	Increasing
	1.8	0.2	0.20	1.29	Decreasing	Increasing
	1.9	0.1	0.10	1.32	Decreasing	Increasing
	0.1	1.9	0.58	2.05	Decreasing	Increasing
	0.2	1.8	0.60	1.95	Decreasing	Increasing
	0.3	1.7	0.63	1.84	Decreasing	Increasing
	0.4	1.6	0.66	1.73	Decreasing	Increasing
	0.5	1.5	0.68	1.62	Decreasing	Increasing
	0.6	1.4	0.71	1.51	Decreasing	Increasing
	0.7	1.3	0.74	1.41	Decreasing	Increasing
	0.8	1.2	0.76	1.30	Decreasing	Increasing
	0.9	1.1	0.79	1.19	Decreasing	Increasing
Case 3	1	1	0.81	1.08	Decreasing	Increasing
	1.1	0.9	0.73	0.97	Decreasing	Decreasing
	1.2	0.8	0.65	0.87	Decreasing	Decreasing
	1.3	0.7	0.57	0.90	Decreasing	Decreasing
	1.4	0.6	0.49	0.92	Decreasing	Decreasing
	1.5	0.5	0.41	0.95	Decreasing	Decreasing
	1.6	0.4	0.32	0.98	Decreasing	Decreasing
	1.7	0.3	0.24	1.01	Decreasing	Increasing
	1.8	0.2	0.16	1.03	Decreasing	Increasing
	1.9	0.1	0.08	1.06	Decreasing	Increasing

Table 7. Directional SE and directional RTS for the input-oriented projection of DMU 1 in different directions estimated by Yang's method: Infeasibilities (Inf).

	ω_1	ω_2	$\beta'_+(0)$ (right)	$\beta'_-(0)$ (left)	Directional RTS (right)	Directional RTS (left)
Case 1	0.1	1.9	13.17	0.39	Inf	Inf
	0.2	1.8	1.30	2.10	Increasing	Inf
	0.3	1.7	1.38	2.33	Increasing	Increasing
	0.4	1.6	1.37	2.67	Increasing	Inf
	0.5	1.5	1.37	4.18	Increasing	Inf
	0.6	1.4	1.36	1.93	Increasing	Increasing
	0.7	1.3	3.08	1.80	Inf	Increasing
	0.8	1.2	1.33	4.73	Increasing	Inf
	0.9	1.1	1.39	1.33	Increasing	Increasing
	1	1	1.39	1.48	Increasing	Increasing
	1.1	0.9	1.36	1.37	Inf	Increasing
	1.2	0.8	0.98	1.35	Decreasing	Increasing
	1.3	0.7	0.86	1.34	Decreasing	Increasing
	1.4	0.6	0.74	1.33	Decreasing	Increasing
	1.5	0.5	0.62	−1.92	Decreasing	Inf
	1.6	0.4	0.57	1.34	Decreasing	Increasing
	1.7	0.3	0.40	1.30	Decreasing	Increasing
	1.8	0.2	−1.07	1.30	Inf	Increasing
	1.9	0.1	0.15	−1.48	Decreasing	Inf
Case 2	0.1	1.9	1.07	1.19	Increasing	Inf
	0.2	1.8	1.12	3.73	Increasing	Inf
	0.3	1.7	12.66	2.00	Inf	Increasing
	0.4	1.6	1.10	1.88	Increasing	Increasing
	0.5	1.5	1.05	1.77	Increasing	Increasing
	0.6	1.4	1.01	2.74	Increasing	Inf
	0.7	1.3	1.09	0.82	Increasing	Inf
	0.8	1.2	1.05	2.10	Increasing	Inf
	0.9	1.1	1.44	1.13	Inf	Increasing
	1	1	1.06	1.25	Increasing	Increasing
	1.1	0.9	1.02	0.96	Increasing	Inf
	1.2	0.8	−5.19	1.12	Inf	Increasing
	1.3	0.7	0.72	1.16	Decreasing	Increasing
	1.4	0.6	1.45	1.10	Inf	Increasing
	1.5	0.5	0.52	1.41	Decreasing	Increasing
	1.6	0.4	2.52	0.96	Inf	Decreasing
	1.7	0.3	0.75	−9.37	Inf	Inf
	1.8	0.2	0.19	1.88	Decreasing	Inf
	1.9	0.1	0.12	0.94	Decreasing	Decreasing
Case 3	0.1	1.9	0.94	2.22	Decreasing	Increasing
	0.2	1.8	0.93	2.04	Decreasing	Increasing
	0.3	1.7	0.00	1.96	Inf	Increasing
	0.4	1.6	0.88	1.28	Decreasing	Increasing
	0.5	1.5	0.93	1.74	Decreasing	Increasing
	0.6	1.4	0.83	0.61	Decreasing	Inf
	0.7	1.3	0.86	1.53	Decreasing	Increasing
	0.8	1.2	0.83	1.36	Decreasing	Increasing
	0.9	1.1	0.84	2.65	Decreasing	Inf
	1	1	0.81	1.01	Decreasing	Increasing
	1.1	0.9	0.83	1.01	Decreasing	Increasing
	1.2	0.8	0.71	0.92	Inf	Decreasing
	1.3	0.7	0.58	0.76	Decreasing	Decreasing
	1.4	0.6	0.46	0.97	Decreasing	Decreasing
	1.5	0.5	0.35	0.91	Decreasing	Decreasing
	1.6	0.4	0.37	0.89	Decreasing	Decreasing
	1.7	0.3	0.27	0.91	Decreasing	Decreasing
	1.8	0.2	0.13	0.86	Decreasing	Decreasing
	1.9	0.1	0.04	0.87	Decreasing	Decreasing

Table 8. Directional SE and directional RTS for the output-oriented projection of DMU 1 in different directions estimated by Yang's method: Infeasibilities (Inf).

	ω_1	ω_2	$\beta'_+(0)$ (right)	$\beta'_-(0)$ (left)	Directional RTS (right)	Directional RTS (left)
	0.1	1.9	0.96	2.62	Decreasing	Increasing
	0.2	1.8	1.00	2.40	Constant	Increasing
	0.3	1.7	1.04	2.30	Increasing	Increasing
	0.4	1.6	1.08	2.17	Increasing	Increasing
	0.5	1.5	1.14	2.09	Increasing	Increasing
	0.6	1.4	4.77	1.90	Inf	Increasing
	0.7	1.3	1.32	3.04	Increasing	Inf
	0.8	1.2	1.23	1.69	Increasing	Increasing
	0.9	1.1	1.18	1.56	Increasing	Increasing
Case 1	1	1	1.08	1.37	Increasing	Increasing
	1.1	0.9	0.97	1.41	Decreasing	Increasing
	1.2	0.8	0.91	1.45	Decreasing	Increasing
	1.3	0.7	1.63	1.50	Inf	Increasing
	1.4	0.6	0.65	1.54	Decreasing	Increasing
	1.5	0.5	−1.52	1.60	Inf	Increasing
	1.6	0.4	−2.09	1.61	Inf	Increasing
	1.7	0.3	0.32	1.67	Decreasing	Increasing
	1.8	0.2	0.22	1.72	Decreasing	Increasing
	1.9	0.1	0.10	1.73	Decreasing	Increasing
	0.1	1.9	0.77	2.11	Decreasing	Increasing
	0.2	1.8	−9.54	1.95	Inf	Increasing
	0.3	1.7	0.79	1.93	Decreasing	Increasing
	0.4	1.6	0.81	1.60	Decreasing	Increasing
	0.5	1.5	0.87	1.59	Decreasing	Increasing
	0.6	1.4	0.89	1.40	Decreasing	Increasing
	0.7	1.3	0.00	1.30	Inf	Increasing
	0.8	1.2	0.98	1.20	Decreasing	Increasing
	0.9	1.1	1.01	1.11	Increasing	Increasing
Case 2	1	1	1.01	1.00	Increasing	Constant
	1.1	0.9	0.91	1.08	Decreasing	Increasing
	1.2	0.8	0.14	1.17	Decreasing	Increasing
	1.3	0.7	0.69	1.14	Decreasing	Increasing
	1.4	0.6	0.63	1.15	Decreasing	Increasing
	1.5	0.5	−1.14	1.20	Inf	Increasing
	1.6	0.4	0.46	−1.57	Decreasing	Inf
	1.7	0.3	0.28	1.26	Decreasing	Increasing
	1.8	0.2	0.20	1.29	Decreasing	Increasing
	1.9	0.1	0.10	1.32	Decreasing	Increasing
	0.1	1.9	0.55	2.05	Decreasing	Increasing
	0.2	1.8	0.87	1.91	Inf	Increasing
	0.3	1.7	0.63	1.89	Decreasing	Increasing
	0.4	1.6	0.64	1.72	Decreasing	Increasing
	0.5	1.5	0.78	1.61	Decreasing	Increasing
	0.6	1.4	0.73	1.40	Decreasing	Increasing
	0.7	1.3	0.74	1.39	Decreasing	Increasing
	0.8	1.2	0.76	1.48	Decreasing	Inf
	0.9	1.1	0.77	1.18	Decreasing	Increasing
Case 3	1	1	0.80	1.05	Inf	Increasing
	1.1	0.9	0.74	1.02	Decreasing	Increasing
	1.2	0.8	0.65	0.92	Decreasing	Decreasing
	1.3	0.7	0.56	0.82	Decreasing	Decreasing
	1.4	0.6	0.41	0.92	Decreasing	Decreasing
	1.5	0.5	0.39	0.94	Decreasing	Decreasing
	1.6	0.4	0.31	0.98	Decreasing	Decreasing
	1.7	0.3	0.21	1.00	Decreasing	Constant
	1.8	0.2	0.16	1.03	Decreasing	Increasing
	1.9	0.1	0.08	1.06	Decreasing	Increasing

are based on the simple linear programming. Hence, this method can be easily extended to a more generalized case and has a low computational cost.

5. Concluding remarks

The most existing approaches for quantitative analysis of RTS follow the concepts of the traditional RTS and SE in economics, *i.e.*, the radial change in outputs resulting from the radial changes of all inputs. In actual multiple-input and multiple-output activities, due to the constraints of actual management needs and objective conditions, the goals of expanding inputs are not only to obtain increases in outputs, but also to expect the proportions of such increases consistent with the management preference of the decision-makers, which is usually not radial changes in outputs.

This paper introduces the direction of output changes to describe the management preference of decision-makers, and extends the concepts of directional SE and directional RTS in multi-output activities. Then, we propose a general DEA-based approach to numerically calculate the directional SE at any point on the DEA frontier. In this approach, the formula derivation of the directional SE requires no simplifying assumptions and can be easily extended to more general cases. Through the proposed definitions and measures of directional SE, one can identify the status of RTS in any given directions of input changes and output changes. Finally, we employ the data from 16 basic research institutions in CAS as the empirical example to illustrate the advantages of the analyses of directional SE and directional RTS. The results show that the proposed directional SE and directional RTS provide more precise information on how to adjust inputs to maximize output increases consistent with the decision maker's management preference.

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