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COMBINING EFFICIENCY AND SCALING EFFECTS IN ACTIVITY ANALYSIS: TOWARDS AN IMPROVED BEST PRACTICE CRITERION

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Abstract. Efficiency is the main issue in any data envelopment analysis. Realizing output by a minimum of input or reaching a maximum of output by a given input is the credo, scale effects often are only a sort of accessory. In modern economics scale effects play a prominent role, however. What is the right size of a decision making unit (DMU) and how to proceed there. Returns to scale inform a DMU about its hitherto disregard of scale effects and show the way towards its ideal activity size. Combining efficiency aspects and scaling effects leads to a new DEA-best practice criterion of DMUs and gives them a profound orientation of their current position. This combination turns out to be the relation of weighted outputs to weighted inputs – in optimal prices under variable returns to scale (VRS). It is the VRS-productivity. For DMUs with increasing returns to scale the recommended growth path is in accordance with economic rationales, for decreasing returns to scale the recommended shrinking path uncovers severe flaws in VRS-models and needs adjustment. All theoretical considerations are illustrated by little numerical examples. A real world application of 37 Brazilian banks demonstrates the benefits of the new concept.

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1. Introduction

Enterprises transform goods into goods for the purpose of humans' need satisfaction: They transform inputs into outputs. Such transformation processes over a certain time period are called activities, see [17]. The intrinsic desire of men to compare and evaluate things makes economists evaluate such transformation processes, either for each isolated activity or comparing similar activities. Here, efficiency comes into play: In a set of activities a selected activity is efficient if its outputs cannot be produced by less inputs, or *vice versa* its inputs do not permit more outputs, *cf.* again [17].

For multiple inputs and multiple outputs, from an economic point of view, comparability of activities needs prices of goods, be them market or virtual prices. In a descriptive model of economic activities, prices might be the result of model calculations and hence are called virtual. Such a descriptive model is data envelopment analysis (DEA). It permits measurement of efficiency and productivity of activities. In the model with constant

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returns to scale (CRS-model), efficiency and productivity coincide. For variable returns to scale (VRS-model), we calculate the efficiencies of activities and their relation to productivity: scale efficiency. For more details on DEA in general, *cf. e.g.* [4,5,10,11,13,19].

Many researchers in the field of DEA regret weaknesses of the classical CRS-efficiency and VRS-efficiency measures as they do not have any discriminative power beyond efficiency, as discussed *e.g.* by Despotis [8]. All CRS-efficient acitivities show equal assessment and so do all VRS-efficient activities, in their respective models. There are some attempts to overcome this problem by modifying the original activity set and thus assigning super-efficiencies to some activities, see *e.g.* [14,18]. A little satisfying approach, we feel, because it distorts the technology contradicting fundamental axioms of DEA. As to scale efficiency, this measure improves the results of mere CRS-efficiency and/or VRS-efficiency, joining them both. So it shows the difference between global and technical aspects and in parts is a good indicator for an activity's closeness to the point of most productive scale size. Unfortunately, it is unable to discriminate between different activities with equal CRS-projection and VRS-projection, however. So there is a need for a more sophisticated assessment criterion than the classical ones.

The VRS-model suffers from a lack of autonomy. What is VRS-productivity and how does this link efficiency to scale effects? Is there an economically reliable assessment, combining these parameters? Returns to scale (RTS) offer valuable clues to answer these questions. As long as an activity still has great increasing returns to scale (IRS) and hence unused scale effects, its assessment should be downgraded. An activity which has already exploited its scale effects should not be punished. Such reasoning applies for decreasing returns to scale (DRS) as well.

Combining efficiency and RTS yields a concept which conjoins static and dynamic aspects. We call it improved efficiency measure IEM. And for IRS it meets the rationale of economics. For DRS, Dellnitz and Rödder [6] detected misbehaviour of the RTS-concept. Such misbehaviour contaminates IEM and needs amendments which will be presented in this contribution, too.

The paper is organized as follows. After this introduction, we give preliminaries of DEA. In Section 2.1 we sum up the concepts of efficiency, productivity, and (virtual) prices. In Section 2.2 follow returns to scale RTS, scale efficiency SE, and most productive scale size MPSS. Section 2.3 eventually sketches the misbehaviour of RTS in the case of DRS. Section 3 we dedicate to IEM. In Section 3.1 shortcomings of classical efficiency measures are brought up, in Section 3.2 the new IEM is presented and its nature is exemplified by a little example. In Section 4 VRS-productivity is defined (Sect. 4.1) and its characteristics are developed (Sect. 4.2). As a result of severe flaws in the VRS-model, we propose an NDRS-model instead and develop a respective modified NDRS-IEM. Section 5 shows rankings of DMUs for all efficiency measures reported on so far. Section 6 applies the new findings to 37 Brazilian banks and Section 7 is a resume and a prospect of future research.

2. Preliminaries

2.1. Efficiency, productivity, and virtual prices in DEA

The activities of J decision making units (DMUs) are $(\mathbf{x}_j, \mathbf{y}_j), j = 1, \ldots, J$; being $\mathbf{x}_j$ and $\mathbf{y}_j$ the vectors of M inputs and S outputs, respectively. As is well known, these activities – by means of the axioms completeness, convexity, monotonicity, minimal extrapolation, and expansion – build the possibility set or technology T. Expansion has four different forms:

- radial unboundedness U,
- radial dilatation D,
- radial reduction R,
- radial boundedness B.

These are classical DEA-models, *cf.* [4,5,16]. There are numerous other types of models, like non-convex technologies, pollution-generating technologies, the consideration of dual role factors, etc.; *cf. e.g.* [1,2,9].

In the so-called envelopment form, the input-oriented efficiency for each DMU $k, k \in \{1, \ldots, J\}$, is calculated by the linear programs

$$\begin{aligned} &\min h_k \\ \text{s.t. } & h_k \mathbf{x}_k - \sum_{j=1}^{J} \lambda_{kj} \mathbf{x}_j \geq 0 \qquad (2.1)\\ & \sum_{j=1}^{J} \lambda_{kj} \mathbf{y}_j \geq \mathbf{y}_k \\ & \lambda_{kj} \geq 0 \, \forall j, h_k \text{ free} \\ \text{and } & - \text{ for U} \qquad (2.1\text{U})\\ & \sum_{j=1}^{J} \lambda_{kj} \geq 1 \text{ for D} \qquad (2.1\text{D})\\ & \sum_{j=1}^{J} \lambda_{kj} \leq 1 \text{ for R} \qquad (2.1\text{R})\\ & \sum_{j=1}^{J} \lambda_{kj} = 1 \text{ for B.} \qquad (2.1\text{B}) \end{aligned}$$

The first one often is named CCR after their creators and the last one BCC for the same reason. To get Pareto-Koopmans-efficiency, consider slack-variables in (2.1). Any textbook contains respective considerations. The duals of (2.1U)–(2.1B) are (2.2U)–(2.2B).

$$\begin{aligned} & \max g_k = \mathbf{u}_k^{\mathsf{T}} \mathbf{y}_k + u_k \\ \text{s.t. } & \mathbf{v}_k^{\mathsf{T}} \mathbf{x}_k = 1 \qquad (2.2)\\ & \mathbf{u}_k^{\mathsf{T}} \mathbf{y}_j + u_k - \mathbf{v}_k^{\mathsf{T}} \mathbf{x}_j \leq 0 \, \forall j \\ & \mathbf{u}_k, \mathbf{v}_k \geq 0 \\ \text{and } & u_k = 0 \text{ for U} \qquad (2.2\text{U})\\ & u_k \geq 0 \text{ for D} \qquad (2.2\text{D})\\ & u_k \leq 0 \text{ for R} \qquad (2.2\text{R})\\ & u_k \text{ free for B.} \qquad (2.2\text{B}) \end{aligned}$$

The vectors of multipliers $\mathbf{v}_k$ and $\mathbf{u}_k$ are virtual prices of inputs and outputs. They result from the axioms and from the geometrical positions of activities in T. Even if not market-based, in DEA they

- permit the determination of efficiencies,
- weigh inputs and outputs of transformation processes and hence
- inform whether these processes are economically balanced or not.

Even if the prices are virtual rather than market-based, they must be in line with basic economic principles. Outputs in virtual prices must not exceed inputs in virtual prices; otherwise the transformation generates "free lunch".

u_k is the scale variable, it is of great importance in the remainder of this contribution. Optimal solutions of

$$\begin{aligned} &(2.2\text{B}) \text{ we name } g_k^*, \mathbf{u}_k^*, \mathbf{v}_k^*, u_k^*, \\ &(2.2\text{U}) \text{ we name } g_k^{**}, \mathbf{u}_k^{**}, \mathbf{v}_k^{**}, -, \end{aligned}$$

$$\text{(2.2D) we name } \bar{g}_k, \bar{\mathbf{u}}_k, \bar{\mathbf{v}}_k, \bar{u}_k,$$

optimal solutions of (2.2R) are of less interest here.

In DEA, ${}^{\mathbf{u}_k^{**\mathsf{T}}\mathbf{y}_k}/_{\mathbf{v}_k^{**\mathsf{T}}\mathbf{x}_k}$ is the CRS-productivity of DMU k, the counterpart ${}^{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}/_{\mathbf{v}_k^{*\mathsf{T}}\mathbf{x}_k}$ is not to be found in relevant DEA literature.

2.2. Returns to scale RTS, scale efficiency SE, MPSS

After solving (2.2B) for DMU k, we have $g_k^*, \mathbf{u}_k^*, \mathbf{v}_k^*, u_k^*$. Investigation of the DMU's improvement potential by scale effects is based on RTS. It is the radial change rate of outputs $\epsilon(\delta)$ as a function of the radial change rate δ of inputs – under constant VRS-efficiency g_k^*, as discussed by Førsund and Hjalmarsson [12] as well as Dellnitz *et al.* [7]. RTS informs about change of $\mathbf{y}_k$ to $(1+\epsilon_k)\mathbf{y}_k$ when $\mathbf{x}_k$ changes to $(1+\delta)\mathbf{x}_k$. For the purpose of conserving efficiency, such alterations must satisfy

$$\mathbf{u}_k^{*\mathsf{T}}(1+\epsilon_k)\mathbf{y}_k + u_k^* - g_k^*\mathbf{v}_k^{*\mathsf{T}}(1+\delta)\mathbf{x}_k = 0 \tag{2.3}$$

and after reorganisation of terms

$$\epsilon_k = \delta\frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k + u_k^*}{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}\cdot \tag{2.4}$$

That is the RTS-equation and ${}^{\epsilon_k}/_{\delta}$ is the scale elasticity, *cf.* [11, 13, 19].

$$(\mathbf{x},\ \mathbf{y}) = \left((1+\delta)\mathbf{x}_k, \left(1+\delta\frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k + u_k^*}{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}\right)\mathbf{y}_k\right) \tag{2.5}$$

are the loci of altered activities, as long as optimal $g_k^*, \mathbf{u}_k^*, \mathbf{v}_k^*, u_k^*$ remain valid. For this stability issue see [7].

For such virtual activities $(\mathbf{x},\ \mathbf{y})$ the respective RTS-equation reads

$$\frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y} + u_k^*}{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}} = 1 + \frac{u_k^*}{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}}\cdot \tag{2.6}$$

Obviously,

- for $u_k^* > 0$ this function decreases with a radial increase of $\mathbf{y}$,
- for $u_k^* < 0$ this function increases with a radial increase of $\mathbf{y}$.

Alternative optima might be present when solving (2.2B) and consequently influence scale elasticities. Equation (2.7) determines the variation of RTS for alternative optima.

$$\begin{aligned} &\inf\ \frac{\mathbf{u}_k^{\mathsf{T}}\mathbf{y}_k + u_k}{\mathbf{u}_k^{\mathsf{T}}\mathbf{y}_k} \text{ and } \sup\ \frac{\mathbf{u}_k^{\mathsf{T}}\mathbf{y}_k + u_k}{\mathbf{u}_k^{\mathsf{T}}\mathbf{y}_k} \\ \text{s.t.}\ & \mathbf{v}_k^{\mathsf{T}}\mathbf{x}_k = 1 \\ & \mathbf{u}_k^{\mathsf{T}}\mathbf{y}_k + u_k = g_k^* \\ & \mathbf{u}_k^{\mathsf{T}}\mathbf{y}_j + u_k - \mathbf{v}_k^{\mathsf{T}}\mathbf{x}_j \leqq 0 \quad \forall j \\ & \mathbf{u}_k, \mathbf{v}_k \geqq \mathbf{0} \text{ and } u_k \text{ free.} \end{aligned} \tag{2.7}$$

Because of

$$\frac{\mathbf{u}_k^{\mathsf{T}}\mathbf{y}_k + u_k}{\mathbf{u}_k^{\mathsf{T}}\mathbf{y}_k} = \frac{g_k^*}{g_k^* - u_k},$$

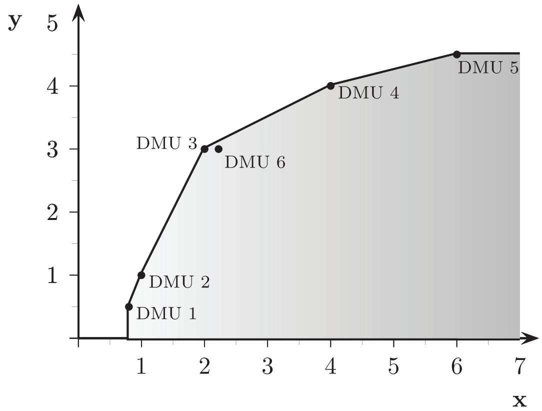

FIGURE 1. VRS-technology for 6 DMUs.

the objective function in (2.7) can be substituted by this equation, depending only on the decision variable u_k. The solutions of (2.7) we name $\mathbf{u}_k^-, \mathbf{v}_k^-, u_k^-, \mathbf{u}_k^+, \mathbf{v}_k^+, u_k^+$. They justify the DEA-parlance

– $u_k^+ \geq u_k^- \geq 0 \longrightarrow$ non-decreasing returns to scale (NDRS),
  $u_k^+ \geq u_k^- > 0 \longrightarrow$ increasing returns to scale (IRS),

– $u_k^- \leq u_k^+ \leq 0 \longrightarrow$ non-increasing returns to scale (NIRS),
  $u_k^- \leq u_k^+ < 0 \longrightarrow$ decreasing returns to scale (DRS),

– $\left.\begin{array}{r} u_k^+ = u_k^- = 0 \\ u_k^- < 0 < u_k^+ \end{array}\right\} \longrightarrow$ constant returns to scale (CRS).

As is well-known,

$$\mathrm{SE}_k = \frac{g_k^{**}}{g_k^*} \tag{2.8}$$

is the scale (in)efficiency of DMU k, see [4]. It informs to what extent the DMU realized scaling effects to improve productivity. For $\mathrm{SE}_k = 1$ it is scale-efficient and cannot exploit further scale effects. For $g_k^{**} = g_k^* = 1$ it is scale-efficient as well as VRS-efficient and CRS-efficient; it has most productive scale size (MPSS), *cf.* [3].

2.3. Misbehaviour of RTS in VRS-technologies

Dellnitz and Rödder [6] show the behaviour of RTS on the efficient boundary of a VRS-technology. We sketch their results as they affect further developments in this paper.

Example 2.1. Figure 1 shows VRS-technology for 6 DMUs. Table 1 contains all results for (2.1B) and (2.7).

For arbitrary activities $(\mathbf{x},\ \mathbf{y})$ on the technology's boundary, respective elasticities read

$$\frac{\epsilon(\mathbf{y})}{\delta} = \frac{\mathbf{u}^{*\mathsf{T}}(\mathbf{y}) \cdot \mathbf{y} + u^*(\mathbf{y})}{\mathbf{u}^{*\mathsf{T}}(\mathbf{y}) \cdot \mathbf{y}}. \tag{2.9}$$

Running through boundary activities and calculating $\mathbf{u}^*(\mathbf{y})$ und $u^*(\mathbf{y})$ at a time by means of (2.2B) and (2.7), respectively, will yield elasticities like in Figure 2.

Please verify:

– The vertical lines for fix outputs $\mathbf{y}_k$ are the elasticities for all $u_k^- \leq u_k \leq u_k^+$ like in (2.7).

TABLE 1. Inputs, outputs, and solutions for 6 DMUs.

DMU k	$(\mathbf{x}_k, \mathbf{y}_k)$	g_k^*	$(\mathbf{u}_k^+, \mathbf{v}_k^+, u_k^+)$	$(\mathbf{u}_k^-, \mathbf{v}_k^-, u_k^-)$	ϵ_k^+/δ	ϵ_k^-/δ
DMU 1	(0.8, 0.5)	1	$\left(0, \frac{5}{4}, 1\right)$	$\left(\frac{1}{2}, \frac{5}{4}, \frac{3}{4}\right)$	∞	4
DMU 2	(1, 1)	1	$\left(\frac{2}{5}, 1, \frac{3}{5}\right)$	$\left(\frac{1}{2}, 1, \frac{1}{2}\right)$	2.5	2
DMU 3	(2, 3)	1	$\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$	$\left(1, \frac{1}{2}, -2\right)$	$1\frac{1}{3}$	$\frac{1}{3}$
DMU 4	(4, 4)	1	$\left(\frac{1}{2}, \frac{1}{4}, -1\right)$	$\left(1, \frac{1}{4}, -3\right)$	$\frac{1}{2}$	$\frac{1}{4}$
DMU 5	(6, 4.5)	1	$\left(\frac{4}{6}, \frac{1}{6}, -2\right)$	$\left(+\infty, \frac{1}{6}, -\infty\right)$	$\frac{1}{3}$	0
DMU 6	$\left(2\frac{2}{9}, 3\right)$	$\frac{9}{10}$	$\left(\frac{9}{40}, \frac{9}{20}, \frac{9}{40}\right)$	$\left(\frac{9}{10}, \frac{9}{20}, -1\frac{4}{5}\right)$	$1\frac{1}{3}$	$\frac{1}{3}$

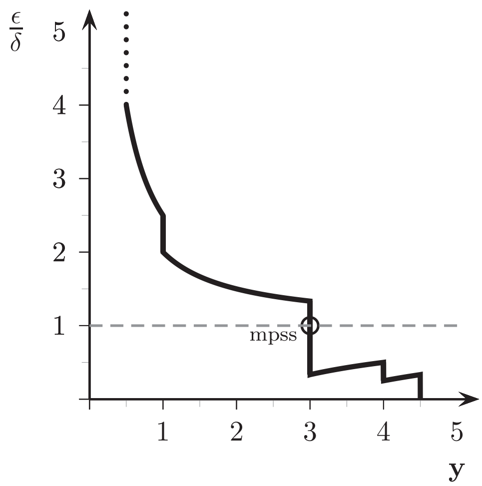

FIGURE 2. Elasticities under varying $\mathbf{y}$.

- For IRS elasticity $\epsilon(\mathbf{y})/\delta$ is greater than 1 and falls with increasing $\mathbf{y}$ till MPSS-activity of DMU 3. The potential for radially improving output/input lessens, in accordance with economic rationales.
- For DRS the elasticities are less than 1, increase on facets of T's boundary and collapse for alternative optima, against economic rationale. We would expect an increase of $\epsilon(\mathbf{y})/\delta$ towards 1 – the elasticity of MPSS – for decreasing $\mathbf{y}$.

Even joining input and output orientations, like in Førsund [11] and the therein studied beam variation equations, does not cure the model's weakness.

After these preliminaries, in Section 3.1 we justify the creation of a new DEA-best practice criterion in VRS-models, beyond VRS-efficiency. In Section 3.2 the new index will be presented and in Section 3.3 it will be analyzed.

3. The improved efficiency measure IEM

3.1. Shortcomings of classical efficiency measures

For measuring efficiency relevant DEA literature cites

(i) CRS-efficiency,

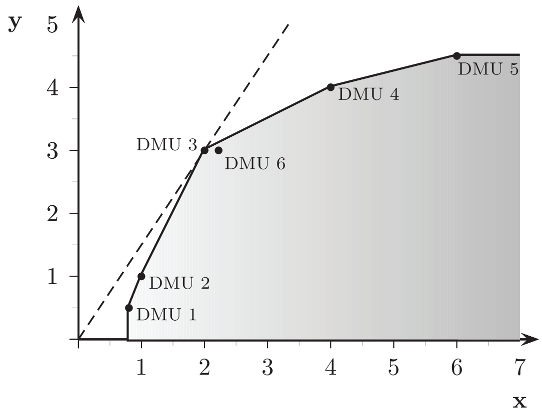

FIGURE 3. VRS-technology and CRS-technology for 6 DMUs.

(ii) VRS-efficiency,
(iii) scale efficiency.

To highlight their advantages and disadvantages, we again study the 6 DMUs of Example 2.1.

Example 3.1. Figure 3 in addition to the 6 DMUs shows the CRS-technology.

(Ad i) CRS-efficiencies for DMUs 1, 2, 3, 4, 5, 6 are 0.417, 0.667, 1.0, 0.667, 0.5, 0.9. The CRS-model shows current productivity but hides possible former and potential future scale effects. This is painful for "little" DMUs: They must compete with "big" DMUs and have little hope to get there.
(Ad ii) The VRS-model permits the calculation of scale effects but does not make use therefrom. VRS-efficiencies 1 of DMUs 1, 2, 3, 4, 5 and 0.9 of DMU 6 hide scale effects. Mind the fact that DMU 1 is scored equivalent to DMU 3, even if far away from MPSS. Here, VRS-efficiency is a bad indicator for a DMU's performance.
(Ad iii) Scale efficiency does not heal the problems. This measure rates DMUs 3 and 6 equally but does not identify different VRS-efficiencies nor CRS-efficiencies.

The following section is an attempt to cure these deficits.

3.2. The improved efficiency measure IEM

Combining elasticity – as a measure of not yet realized scaling potential – and efficiency is a promising attempt to improve the assessment of a DMU.

Definition 3.2. For a DMU k with activity $\mathbf{x}_k, \mathbf{y}_k$, VRS-efficiency g_k^* and elasticity ${}^{\epsilon_k}/_{\delta}$, its IEM is the fraction

$$\mathrm{IEM}_k = \frac{g_k^*}{{}^{\epsilon_k}/_{\delta}}. \tag{3.1}$$

Equation (3.1) evaluates a DMU the better the greater efficiency and the lower elasticity. A great elasticity indicates "not yet realized scaling potential" – see above.

Picking up again Example 2.1 from Section 2.3 and using the values from Table 1, we get the following results:

- DMU 1 has VRS-efficiency 1 and IEM varies in $[0.000, 0.250]$,
- DMU 2 has VRS-efficiency 1 and IEM varies in $[0.400, 0.500]$,
- DMU 3 has VRS-efficiency 1 and IEM varies in $[0.750, 3.000]$,
- DMU 6 has VRS-efficiency 0.9 and IEM varies in $[0.625, 2.700]$,
- DMU 4 has VRS-efficiency 1 and IEM varies in $[2.000, 4.000]$,

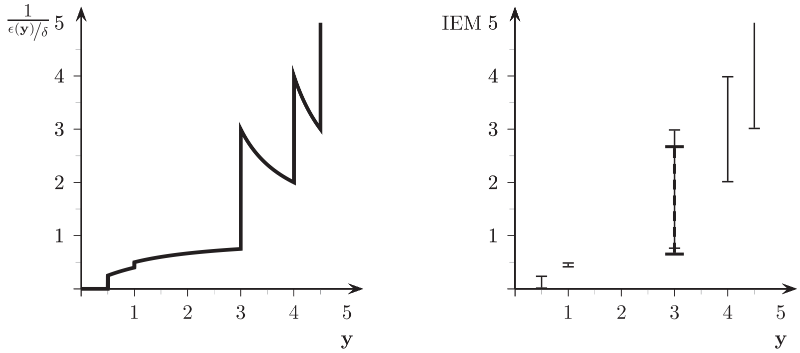

Figure 4. IEMs (*left*) and IEM intervals for 6 DMUs (*right*).

– DMU 5 has VRS-efficiency 1 and IEM varies in $[2.000, \infty]$.

The intervals of IEM result from alternative optima and hence from ambiguity of RTS, as was outlined in Section 2.2. Furthermore, the values reassure intended effects. DMU 1 has lower IEM than DMU 2 due to greater RTS. DMU 2 has a lower index than DMU 3. DMU 6 loses because of its VRS-efficiency 0.9. High IEMs for DMU 4 and 5 result from elasticities less than 1. They even outperform DMU 3. This irritating fact needs further research and will be picked up later.

Visualization of IEMs might be helpful and so we show it in the next section.

3.3. IEM for IRS and DRS

Figure 2 in Section 2.3 showed all elasticities on the boundary of VRS-technology T. Due to efficiency 1, IEMs are just reciprocal to these values, see Figure 4. Figure 4 (left) shows all IEMs and Figure 4 (right) the IEM-intervals for respective DMUs, including the dashed line for DMU 6.

We observe the following behaviour of IEM:

– For IRS the new IEM $\frac{1}{\epsilon(\delta)/\delta}$ is less than 1 and increases when approximating MPSS-activity of DMU 3. Realizing scale effects improves IEM.
– For DRS, IEM is greater than 1, falls on the boundary's facets and shows an abrupt increase at points of alternative optima. This behaviour is economically irrational. For DRS, IEM should be ≤ 1 and increase when approximating MPSS.

Intervals of IEM need getting used to. To overcome this problem, we make them point estimates.

Definition 3.3. For a DMU k with VRS-efficiency g_k^* and

– CRS make $\text{PIEM}_k = \frac{g_k^*}{1}$,
– IRS make $\text{PIEM}_k = \frac{g_k^*}{\epsilon_k^-/\delta}$; see (2.7),
– DRS make $\text{PIEM}_k = \frac{g_k^*}{\epsilon_k^+/\delta}$; see again (2.7).

Definition 3.3 yields benevolent point estimates. In other words: DMU k with IRS gets a high and with DRS gets a low IEM – close to the ideal $g_k^*/1$.

The next section introduces VRS-productivity. This concept helps illuminating economical irregularities for DRS in VRS-models.

4. VRS-productivity

4.1. From IEM to VRS-productivity

From Definitions 3.2 and 3.3, IEM is the relation between VRS-efficiency and elasticity, combining a static and a dynamic aspect in DEA. Interesting enough, this idea can be put further towards VRS-productivity.

Dividing

$$g_k^* = \frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k + u_k^*}{\mathbf{v}_k^{*\mathsf{T}}\mathbf{x}_k} \tag{4.1}$$

by

$$\frac{\epsilon_k}{\delta} = \frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k + u_k^*}{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k} \tag{4.2}$$

yields such productivity. We put this result as a theorem.

Theorem 4.1. $\frac{g_k^*}{\epsilon_k/\delta} = \frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}{\mathbf{v}_k^{*\mathsf{T}}\mathbf{x}_k}$.

Proof. Trivial. □

$\frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}{\mathbf{v}_k^{*\mathsf{T}}\mathbf{x}_k}$ is productivity in optimal VRS-prices. Productivity as an economic basic concept should fulfil basic principles, see again our observations in Section 2.1: It should be ≤ 1, avoiding "free lunch". It should be in line with quantity-based productivity. The next section is dedicated to such questions.

4.2. Properties of VRS-productivity

Reordering terms in (4.1) yields

$$\frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}{\mathbf{v}_k^{*\mathsf{T}}\mathbf{x}_k} + \frac{u_k^*}{\mathbf{v}_k^{*\mathsf{T}}\mathbf{x}_k} = g_k^*. \tag{4.3}$$

First put $g_k^* = 1$ and study (4.3) under NDRS and DRS.

(1a) NDRS, $u_k^* \geq 0$. Here, VRS-productivity $\frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}{\mathbf{v}_k^{*\mathsf{T}}\mathbf{x}_k}$ is ≤ 1, meeting economic rationale.

Example 4.2. Consider DMU 2 from Example 2.1. Applying values of Table 1 its benevolent VRS-productivity $\frac{1}{\epsilon_2^-/\delta} = \frac{1}{2}$ is in accordance with economic theory.

(1b) DRS, $u_k^* < 0$. Here, VRS-productivity $\frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}{\mathbf{v}_k^{*\mathsf{T}}\mathbf{x}_k}$ exceeds 1. In VRS-prices output is greater than input which makes the activity enjoy free lunch.

Example 4.3. Consider DMU 4 from Example 2.1. Applying values of Table 1 its benevolent VRS-productivity $\frac{1}{\epsilon_2^+/\delta} = \frac{1}{1/2} = 2$ is not in accordance with economic theory.

Second put $g_k^* < 1$ and again study (4.3) under NDRS and DRS.

Dividing (4.3) by g_k^* yields

$$\frac{\mathbf{u}_k^{*\mathsf{T}}\mathbf{y}_k}{\mathbf{v}_k^{*\mathsf{T}}(g_k^*\mathbf{x}_k)} + \frac{u_k^*}{\mathbf{v}_k^{*\mathsf{T}}(g_k^*\mathbf{x}_k)} = 1. \tag{4.4}$$

(2a) NDRS, $u_k^* \geq 0$. After input-projection the

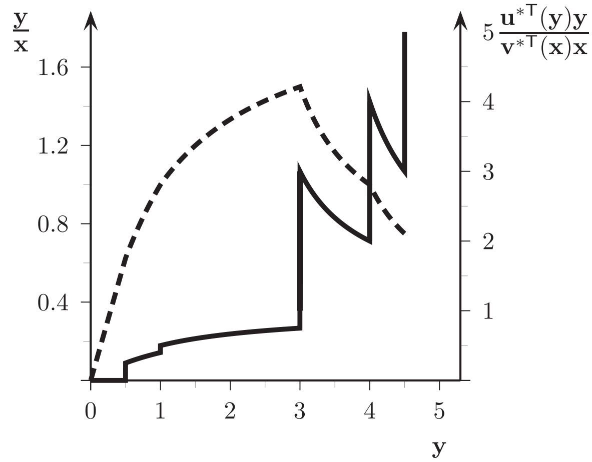

FIGURE 5. Quantity-based *vs.* VRS-productivity.

– activity becomes VRS-efficient, of course,
– VRS-productivity remains ≤ 1.

This meets economic rationale.

(2b) DRS, $u_k^* < 0$. For a non-efficient DMU k ($g_k^* < 1$), DEA-theory recommends input-projection. And this even whilst VRS-productivity exceeds 1. The inadequate free lunch grows.

Example 4.4. Consider activity $(\mathbf{x}, \mathbf{y}) = (4.5, 4)$ in Example 2.1. VRS-efficiency is $\frac{4}{4.5} = 0.\bar{8} < 1$ and benevolent VRS-productivity amounts to $\frac{0.\bar{8}}{1/2} = 1.\bar{7} > 1$. Even in this economically irrational situation DEA recommends input-projection and consequently an increase to productivity = 2.

(1b) and (2b) vividly demonstrate misleading recommendations of DEA-theory in VRS-models. This severe flaw is reinforced when studying quantity-based *vs.* VRS-productivity. Figure 5 shows quantity-based productivities $\mathbf{y}/\mathbf{x}$ – dashed line – and VRS-productivities – solid line. As expected, $\mathbf{y}/\mathbf{x}$ increases until activity of DMU 3 and then decreases. First output increment exceeds input increment and then falls below. The solid line is known from Figure 4. Its zigzaging form foils quantity structure. It increases from $\mathbf{y} = 4 - d$ to $\mathbf{y} = 4 + d$ for a sufficiently small $d > 0$, in spite of decreasing quantity-based productivity. VRS-productivity does not follow economic rationale.

Resuming:

(3a) NDRS, $u^* \geq 0$. VRS-productivity is in accordance with quantity-based productivity.
(3b) DRS, $u^* < 0$. VRS-productivity is inconsistent with quantity-based productivity.

All observed economical irritations in VRS-model are due to negative values of the scale variable u. They should be eliminated to make DEA a coherent instrument for evaluating DMUs. This is what the next section is about.

4.3. IEM in NDRS-models

To overcome problems related on in the last section, we consider the NDRS-model, which is briefly repeated here.

$$\max g_k = \mathbf{u}_k^{\mathsf{T}} \mathbf{y}_k + u_k$$

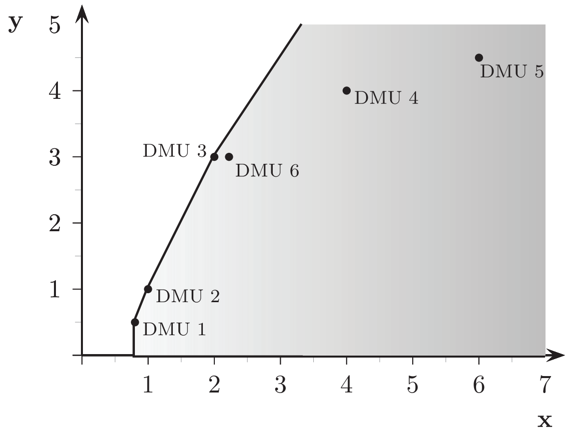

FIGURE 6. 6 DMUs in NDRS-technology.

$$\text{s.t. } \mathbf{v}_k^{\mathsf{T}}\mathbf{x}_k = 1 \tag{2.2}$$
$$\mathbf{u}_k^{\mathsf{T}}\mathbf{y}_j + u_k - \mathbf{v}_k^{\mathsf{T}}\mathbf{x}_j \leq 0\,\forall j$$
$$\mathbf{u}_k, \mathbf{v}_k \geq 0$$
$$\text{and } u_k \geq 0. \tag{2.2D}$$

Let the optimal solution be $\bar{g}_k, \bar{\mathbf{u}}_k, \bar{\mathbf{v}}_k, \bar{u}_k$, see also Section 2.1.

We notice:

- If for DMU k, $u_k^* \geq 0$ in the VRS-model (2.2B), then $\mathbf{u}_k^* = \bar{\mathbf{u}}_k, \mathbf{v}_k^* = \bar{\mathbf{v}}_k, u_k^* = \bar{u}_k$. Furthermore, IEMs and productivities coincide.
- If for DMU k, $u_k^* < 0$ in the VRS-model (2.2B), then we have $\bar{\mathbf{u}}_k, \bar{\mathbf{v}}_k, 0$ in (2.D). Furthermore, IEM and productivity are equal $\bar{g}_k/1$ and we have $\bar{g}_k = g_k^{**}, \bar{\mathbf{u}}_k = \mathbf{u}_k^{**}, \bar{\mathbf{v}}_k = \mathbf{v}_k^{**}, \bar{u}_k = u_k^{**} = 0$.

In other words: For DMUs with NDRS and optimal dual prices, IEMs and productivities remain valid. For DMUs with DRS such prices become those of the CRS-model. IEMs equal CRS-efficiencies and consequently equal CRS-productivities.

Definition 4.5. For a DMU k with activity $\mathbf{x}_k, \mathbf{y}_k$, NDRS-efficiency $\bar{g}_k$ and NDRS-elasticity ϵ_k/δ, its NDRS-IEM reads

$$\text{NDRS-IEM}_k = \frac{\bar{g}_k}{\epsilon_k/\delta}. \tag{4.5}$$

There also is a counterpart to Definition 4.5 avoiding ambiguity.

Definition 4.6. For a DMU k with NDRS-efficiency $\bar{g}_k$ and

- CRS, NDRS-IEM is equal $\frac{\bar{g}_k}{1}$,
- IRS, NDRS-IEM is equal $\frac{\bar{g}_k}{\epsilon_k^-/\delta}$.

For the calculation of ϵ_k^-/δ solve again (2.7), but replace g_k^* by $\bar{g}_k$ and u_k free by $u_k \geq 0$.

Example 4.7. Consider 6 DMUs in NDRS-technology like in Figure 6. Table 2 provides efficiencies $\bar{g}_k$, (benevolent) NDRS-IEMs, and productivities, respectively.

Obviously, DMUs 1, 2, 3 maintain knowledge about elasticities, IEMs, and productivities. DMU 1 and 2, for instance, know their benevolent $\epsilon_1^-/\delta = 4$ and $\epsilon_2^-/\delta = 2$ and consequently their improvement potential. DMU 3

TABLE 2. 6 DMUs in NDRS-technology.

DMU k	$(\mathbf{x}_k, \mathbf{y}_k)$	$\bar{g}_k$	ϵ_k^+/δ	ϵ_k^-/δ	NDRS-IEM k/ Productivity
DMU 1	(0.8, 0.5)	1	∞	4	1/4
DMU 2	(1, 1)	1	2.5	2	1/2
DMU 3	(2, 3)	1	1 1/3	1/3	1
DMU 4	(4, 4)	2/3	1	1	2/3
DMU 5	(6, 4.5)	1/2	1	1	1/2
DMU 6	$(2.2\bar{2}, 3)$	9/10	1 1/3	1/3	9/10

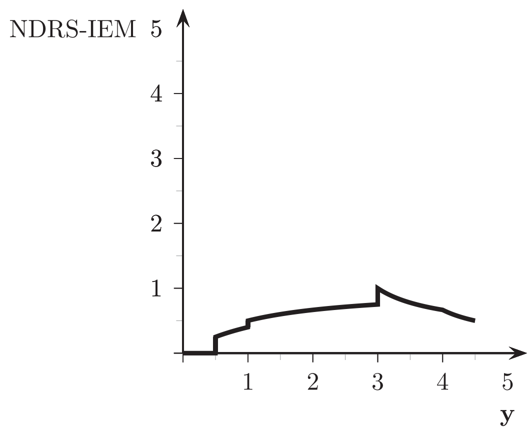

FIGURE 7. IEMs/productivities in the NDRS-model.

and 6 have benevolent elasticity 1. DMUs 4, 5 receive new assessment. Because of $\bar{u}_4 = \bar{u}_5 = 0$, elasticities equal 1 and hence IEMs and productivities are $\bar{g}_4/1 = 2/3$ and $\bar{g}_5/1 = 1/2$, respectively.

Running through all activities on the efficient boundary of *former* VRS-technology we get NDRS-IEMs like in Figure 7.

The course of IEMs now is in line with quantity-based productivity, see again our remarks on this issue in Section 4.2.

5. Rankings of 6 DMUs by different efficiencies

Previous sections presented classical efficiencies like g_k^*, g_k^{**}, scale efficiency SE_k, improved efficiency IEM $= \frac{g^*}{\epsilon/\delta}$, NDRS-efficiency $\bar{g}_k$, and finally NDRS-IEM $= \frac{\bar{g}}{\epsilon/\delta}$. All these indices allow rankings of DMUs. Table 3 shows respective results complemented by the following comments:

- g_k^* have little explanatory power, only DMU 6 is inefficient. Scale effects are not perceptible.
- Scale efficiencies rank DMU 3 on par with DMU 6, an obvious flaw of this criterion.
- $\bar{g}_k$ hide improvement potentials for all DMUs.
- IEM_k for IRS outperform $\bar{g}_k$ but for DRS are inapt.
- NDRS-IEM show all desired effects. For "little" DMUs they conjoin VRS-efficiencies and improvement potentials and for "big" DMUs they cure deficits of VRS-IEM.

All theoretical results will be applied to a set of 37 Brazilian banks in Section 6.

TABLE 3. Efficiencies and rankings of 6 DMUs.

DMU k	g_k^{**}	g_k^*	SE_k	$\bar{g}_k$	$\underset{\epsilon/\delta}{g_k^*}$	$\underset{\epsilon/\delta}{\bar{g}_k}$
DMU 1	0.417	1.000	0.417	1.000	0.250	0.250
DMU 2	0.667	1.000	0.667	1.000	0.500	0.500
DMU 3	1.000	1.000	1.000	1.000	1.000	1.000
DMU 4	0.667	1.000	0.667	0.667	2.000	0.667
DMU 5	0.500	1.000	0.500	0.500	2.000	0.500
DMU 6	0.900	0.900	1.000	0.900	0.900	0.900
			Rankings			
DMU 1	6	1	6	1	6	6
DMU 2	3	1	3	1	5	4
DMU 3	1	1	1	1	3	1
DMU 4	3	1	3	5	1	3
DMU 5	5	1	5	6	1	4
DMU 6	2	6	1	4	4	2

6. Assessment of Brazilian banks

Efficiency and productivity assessments are of great importance, especially in the banking sector due to the difficult interest rate environment. In many applications of DEA – like in banking and finance – divisability of goods (loans, financial instruments) are evident. Scaling effects are present in the VRS-model, of course, and such an effect is the driving force for consolidation processes in the banking sector in order to improve a bank's efficiency or productivity.

In Henriques *et al.* [15] the authors relate on a DEA-study of Brazilian banks and focus on the so-called intermediation approach. They investigate the banks' efficiencies in the period from 2012 to 2016, using three inputs (fixed assets, total deposits, and personell expenses) and one output (total loans). Table 4 shows data of the fiscal year 2016. Table 5 then provides classical efficiencies and RTS-classification. 4 DMUs have MPSS, 10 IRS, and a predominant number of 23 DMUs are oversized showing DRS. The latter being a typical phenomenon in almost all countries' banking systems; *cf.* [20]. The CRS-efficiencies range between 0.027 and 1, VRS-efficiencies range between 0.09 and 1. This highlights the differences between the banks' intermediation strenghts. Scale efficiencies can be seen as "distances" from MPSS.

Table 6 compares classical with new indices developed in this paper. Eye-catching are some entries > 1 for VRS-IEM. They confirm the flaw of VRS-productivity for DRS, see Section 4.2. This defect is cured by NDRS-IEM. Here, (former) IRS-DMUs are downgraded due to unrealized scale effects, former MPSS-activities remain MPSS, and finally former DRS-banks lose efficiency. Table 7 shows all rankings. Determining best practice DMUs is one of the main objectives in DEA. Respective rankings are based on classical indices, such as CRS-efficiency, VRS-efficiency and scale efficiency. As demonstrated in this paper, they all suffer from certain flaws, resulting in erroneous best practice DMUs. To overcome these flaws, we have defined new indices combining efficiency and scale effects. These are the two components a responsible management should take into account in order to successfully pilot the decision making unit.

To support managers and policymakers, we show a path that is economically sound from a bank's reality, adapting the procedure of Rödder *et al.* [21]. For this purpose, we pick up the bank Semear (inefficient and operating under IRS) and show its improvements *via* this management tool. The iterative algorithm consists of two improvement vehicles:

(1) RTS-based activity scalings like in equation (2.5).
(2) A further input reduction in each planning period (iteration), confirmed as practicable by the management.

Table 4. Inputs and outputs of 37 Brazilian banks.

Bank k	No.	Fixed assets	Total deposits	Personnel expenses	Total loans
Alfa	1	323 417	40 626	309 151	6 748 462
Bonsucesso	2	301 688	8779	939 761	308 364
Semear	3	1689	3023	567 958	400 229
Topázio	4	4005	3197	266 165	147 256
Banestes	5	276 560	79 967	9 310 156	3 473 396
Banif	6	6696	6029	490 918	73 687
Banrisul	7	956 272	401 681	37 793 700	29 808 188
BB	8	31 221 063	5 246 319	455 560 520	667 786 191
Arbi	9	8558	1544	70 717	46 319
Capital	10	354	657	5478	3079
Cooperativo Sicredi	11	151 596	26 463	10 362 623	14 442 009
Banco da Amazónia	12	278 514	130 794	2 909 788	3 873 265
Banco da China Brasil	13	6451	4777	294 503	484 293
Banese	14	82 376	39 238	2 895 553	2 050 738
Banpará	15	114 978	67 197	3 884 973	3 431 025
BNB	16	236 206	426 027	10 352 508	12 678 428
Fibra	17	78 659	23 233	2 173 689	2 479 147
Ficsa	18	1074	972	79 236	6116
La Nacion Argentina	19	16 351	1251	4433	29 052
Luso Brasileiro	20	12 463	5876	639 616	697 948
Rep Oriental Uruguay BCE	21	2294	553	1272	14 248
Ribeirão Preto	22	1575	1698	67 483	373 867
BMG	23	1 873 997	46 798	5 200 705	8 087 786
Bradesco	24	51 076 723	3 209 178	189 864 277	317 809 283
BRB	25	418 334	214 699	9 157 803	9 522 840
CEF	26	13 153 796	5 018 876	451 018 737	672 513 474
Citibank	27	619 525	296 551	14 677 936	16 009 264
HSBC	28	3 099 668	894 990	55 709 668	55 630 103
Intermedium	29	6627	14 391	1 220 503	2 187 713
Itaú	30	84 219 449	3 641 920	297 347 284	396 500 032
Mercantil do Brasil	31	235 083	87 432	7 825 089	7 646 678
Original	32	728 170	35 671	1 466 660	2 587 370
Panamericano	33	840 450	87 330	12 960 426	16 230 243
Rendimento	34	38 449	29 799	583 234	318 071
Safra	35	3 099 710	440 788	9 228 824	38 610 052
Santander	36	16 448 887	1 736 403	137 822 766	212 243 750
Sofisa	37	83 495	16 278	2 885 708	1 738 000

We opted for five iterations including RTS-based activity scaling and a further input reduction of five percent in each period. For the bank Semear, the results of this procedure are:

– VRS-efficiency (g_3^*) improves from 0.913 to 0.961.
– scale efficiency (SE_3) increases from 0.872 to 0.999.
– NDRS-IEM_3 consequently upgrades from 0.707 to approx. 0.961.

We notice that already after five planning periods, the bank Semear reaches an economically valuable position. IEM is an appropriate instrument to control a DMU's activity design.

TABLE 5. CRS-efficiency, VRS-efficiency and RTS-classification.

Bank k	g_k^{**}	g_k^{*}	u_k^{-}	u_k^{+}	RTS-class
Alfa	1.000	1.000	−5.111	0.010	CRS
Bonsucesso	0.124	0.146	0.045	0.045	IRS
Semear	0.796	0.913	0.206	0.206	IRS
Topázio	0.190	0.313	0.185	0.185	IRS
Banestes	0.149	0.191	−0.012	−0.012	DRS
Banif	0.053	0.134	0.100	0.100	IRS
Banrisul	0.274	0.527	−0.026	−0.026	DRS
BB	0.483	0.981	−0.088	−0.088	DRS
Arbi	0.133	0.395	0.342	0.342	IRS
Capital	0.091	1.000	0.917	1.000	IRS
Cooperativo Sicredi	1.000	1.000	−3.137	0.020	CRS
Banco da Amazónia	0.199	0.390	−0.097	−0.097	DRS
Banco da China Brasil	0.422	0.457	−0.352	−0.352	DRS
Banese	0.208	0.315	−0.036	−0.036	DRS
Banpará	0.216	0.433	−0.126	−0.126	DRS
BNB	0.226	0.787	−0.051	−0.051	DRS
Fibra	0.395	0.520	−0.048	−0.048	DRS
Ficsa	0.027	0.659	0.644	0.644	IRS
La Nacion Argentina	0.300	0.508	0.359	0.359	IRS
Luso Brasileiro	0.418	0.445	−0.193	−0.193	DRS
Rep Oriental Uruguay BCE	0.513	1.000	0.488	1.000	IRS
Ribeirão Preto	1.000	1.000	−1.874	0.363	CRS
BMG	0.604	0.643	−0.003	−0.003	DRS
Bradesco	0.416	1.000	−0.222	−0.060	DRS
BRB	0.199	0.513	−0.045	−0.045	DRS
CEF	0.503	1.000	$-\infty$	−0.002	DRS
Citibank	0.236	0.589	−0.055	−0.055	DRS
HSBC	0.258	0.553	−0.042	−0.042	DRS
Intermedium	1.000	1.000	−2.905	0.052	CRS
Itaú	0.420	0.857	−0.041	−0.041	DRS
Mercantil do Brasil	0.329	0.553	−0.062	−0.062	DRS
Original	0.328	0.360	−0.006	−0.006	DRS
Panamericano	0.579	0.748	−0.246	−0.246	DRS
Rendimento	0.088	0.090	0.007	0.007	IRS
Safra	0.465	1.000	−1.706	−0.114	DRS
Santander	0.476	0.959	−0.021	−0.021	DRS
Sofisa	0.307	0.314	−0.007	−0.007	DRS

7. Resume and prospect of further research

Scale effects have a cardinal impact on modern economies. Marketers seek growth to increase productivity: more output per input. Sometimes this ambition causes overdimensionality and consequently causes a drawback due to, *e.g.* inefficient control, transports, and personnel structure. Here, economic rationale would command reduction of activities. Data envelopment analysis evaluates activities of marketers and – especially in the VRS-model – recommends each decision making unit to proceed against its most productive scale size. Returns to scale or production elasticity are respective indicators which might help the units to find the right size.

Data envelopment analysis provides such indicators but unfortunately, they play an ancillary role in the theory. So in the present paper, we show how to combine two objectives: the right way towards efficiency and the way to the right scale size. To this end, classical efficiency becomes improved efficiency. Interestingly

Table 6. Classical measures of efficiency and IEMs.

Bank k	g_k^{**}	g_k^{*}	SE_k	$\bar{g}_k$	VRS-IEM_k	NDRS-IEM_k
Alfa	1.000	1.000	1.000	1.000	1.000	1.000
Bonsucesso	0.124	0.146	0.849	0.146	0.101	0.101
Semear	0.796	0.913	0.872	0.913	0.707	0.707
Topázio	0.190	0.313	0.607	0.313	0.128	0.128
Banestes	0.149	0.191	0.780	0.149	0.203	0.149
Banif	0.053	0.134	0.396	0.134	0.034	0.034
Banrisul	0.274	0.527	0.520	0.274	0.553	0.274
BB	0.483	0.981	0.492	0.483	1.069	0.483
Arbi	0.133	0.395	0.337	0.395	0.054	0.054
Capital	0.091	1.000	0.091	1.000	0.083	0.083
Cooperativo Sicredi	1.000	1.000	1.000	1.000	1.000	1.000
Banco da Amazónia	0.199	0.390	0.510	0.199	0.487	0.199
Banco da China Brasil	0.422	0.457	0.923	0.422	0.809	0.422
Banese	0.208	0.315	0.660	0.208	0.351	0.208
Banpará	0.216	0.433	0.499	0.216	0.559	0.216
BNB	0.226	0.787	0.287	0.226	0.838	0.226
Fibra	0.395	0.520	0.760	0.395	0.568	0.395
Ficsa	0.027	0.659	0.041	0.659	0.015	0.015
La Nacion Argentina	0.300	0.508	0.591	0.508	0.149	0.149
Luso Brasileiro	0.418	0.445	0.939	0.418	0.638	0.418
Rep Oriental Uruguay BCE	0.513	1.000	0.513	1.000	0.512	0.512
Ribeirão Preto	1.000	1.000	1.000	1.000	1.000	1.000
BMG	0.604	0.643	0.939	0.604	0.646	0.604
Bradesco	0.416	1.000	0.416	0.416	1.060	0.416
BRB	0.199	0.513	0.388	0.199	0.558	0.199
CEF	0.503	1.000	0.503	0.503	1.002	0.503
Citibank	0.236	0.589	0.401	0.236	0.644	0.236
HSBC	0.258	0.553	0.467	0.258	0.595	0.258
Intermedium	1.000	1.000	1.000	1.000	1.000	1.000
Itaú	0.420	0.857	0.490	0.420	0.898	0.420
Mercantil do Brasil	0.329	0.553	0.595	0.329	0.615	0.329
Original	0.328	0.360	0.911	0.328	0.366	0.328
Panamericano	0.579	0.748	0.774	0.579	0.994	0.579
Rendimento	0.088	0.090	0.978	0.090	0.083	0.083
Safra	0.465	1.000	0.465	0.465	1.114	0.465
Santander	0.476	0.959	0.496	0.476	0.980	0.476
Sofisa	0.307	0.314	0.978	0.307	0.321	0.307

enough, this new measure equals productivity in VRS-prices: weighted output per weighted input. Double-checking the new productivity reveals serious flaws of the VRS-model. Productivity can exceed 1 and hence permits free lunch; inefficient productivity, besides being greater than 1, demands even more free lunch. And VRS-productivity is not in line with quantity-based productivity. Less output per input leads to more weighted output per weighted input. To sum up, the VRS-model shows severe flaws. The NDRS-model provides a loophole out of such inconsistencies. All new findings are illustrated by little numerical examples and a study of Brazilian banks.

Is NDRS in accordance with market behaviour of real-world decision making units? A future empirical investigation might clarify such questions. And if NDRS is not adequate, is there another implied approach to market behaviour? This is the road ahead for future research.

Table 7. Rankings of 37 Brazilian banks.

	Ranks					
Bank k	g_k^{**}	g_k^{*}	SE_k	$\bar{g}_k$	VRS-IEM$_k$	NDRS-IEM$_k$
Alfa	1	1	1	1	5	1
Bonsucesso	33	35	12	35	32	32
Semear	5	12	11	7	14	5
Topázio	30	33	17	24	31	31
Banestes	31	34	13	34	29	29
Banif	36	36	32	36	36	36
Banrisul	22	21	20	26	23	21
BB	10	10	26	13	2	10
Arbi	32	28	34	20	35	35
Capital	34	1	36	1	33	33
Cooperativo Sicredi	1	1	1	1	5	1
Banco da Amazónia	28	29	22	32	25	27
Banco da China Brasil	13	25	9	16	13	13
Banese	27	31	16	31	27	26
Banpará	26	27	24	30	21	25
BNB	25	14	35	29	12	24
Fibra	17	22	15	20	20	17
Ficsa	37	16	37	8	37	37
La Nacion Argentina	21	24	19	11	30	30
Luso Brasileiro	15	26	8	18	17	15
Rep Oriental Uruguay BCE	8	1	21	1	24	8
Ribeirão Preto	1	1	1	1	5	1
BMG	6	17	7	9	15	6
Bradesco	16	1	30	19	3	16
BRB	28	23	33	32	22	27
CEF	9	1	23	12	4	9
Citibank	24	18	31	28	16	23
HSBC	23	19	28	27	19	22
Intermedium	1	1	1	1	5	1
Itaú	14	13	27	17	11	14
Mercantil do Brasil	18	19	18	22	18	18
Original	19	30	10	23	26	19
Panamericano	7	15	14	10	9	7
Rendimento	35	37	5	37	34	34
Safra	12	1	29	15	1	12
Santander	11	11	25	14	10	11
Sofisa	20	32	6	25	28	20

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