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REMARKS ON COMPONENT FACTORS IN GRAPHS

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Abstract. For a family of connected graphs $\mathcal{F}$, a spanning subgraph H of a graph G is called an $\mathcal{F}$-factor of G if its each component is isomorphic to an element of $\mathcal{F}$. In particular, H is called an $\mathcal{S}_k$-factor of G if $\mathcal{F} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$, where integer $k \geq 2$; H is called a $P_{\geq 3}$-factor of G if every component in $\mathcal{F}$ is a path of order at least three. As an extension of $\mathcal{S}_k$-factors, the induced star-factor (*i.e.*, $\mathcal{IS}_k$-factor) is a spanning subgraph each component of which is an induced subgraph isomorphic to some graph in $\mathcal{F} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$. In this paper, we firstly prove that a graph G has an $\mathcal{S}_k$-factor if and only if its isolated toughness $I(G) \geq \frac{1}{k}$. Secondly, we prove that a planar graphs G has an $\mathcal{S}_2$-factors if its minimum degree $\delta(G) \geq 3$. Thirdly, we give two sufficient conditions for graphs with $\mathcal{IS}_k$-factors by toughness and minimum degree, respectively. Additionally, we obtain three special classes of graphs admitting $P_{\geq 3}$-factors.

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1. Introduction

The graphs considered here are finite and simple, unless explicitly stated. Let $G = (V(G), E(G))$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of v and the set of vertices adjacent to v in G, respectively. For $S \subseteq V(G)$, we write $N_G(S) = \cup_{v \in S} N_G(v)$. A graph G is called an r-regular graph if $d_G(v) = r$ for each $v \in V(G)$. We use $\delta(G)$ to denote the minimum degree of a graph G. The number of connected components and isolated vertices of a graph G is denoted by $\omega(G)$ and $i(G)$, respectively. We refer to [3] for the notation and terminologies not defined here.

The complete bipartite graph $K_{1,r}$ is called the *star* of order $r+1$, where r is a positive integer. We use $\mathcal{S}_k$ to denote the set $\{K_{1,1}, K_{1,2}, K_{1,3}, \ldots, K_{1,k}\}$, where integer $k \geq 2$.

Let $\mathcal{F}$ be a family of connected graphs. Then a spanning subgraph H of G is called an $\mathcal{F}$-factor of G if each component of H is isomorphic to an element of $\mathcal{F}$. In particular, for an integer $k \geq 2$, a $\{K_{1,1}, K_{1,2}, K_{1,3}, \ldots, K_{1,k}\}$-factor is briefly called an $\mathcal{S}_k$-factor. Similarly, a $\{P_k, P_{k+1}, ...\}$-factor is called a $P_{\geq k}$-factor.

In 1947, Tutte [10] presented a criterion for the existence of 1-factors (perfect matchings), which is one of the classical results in graph theory. Denote by $o(G)$ the number of odd components of G, whose orders are odd.

Keywords. Star-factor, Induced star-factor, $P_{\geq 3}$-factor, Toughness, Minimum degree.

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Theorem 1.1. *(Tutte [10]) A graph G has a 1-factor if and only if $o(G-S) \le |S|$ for any $S \subseteq V(G)$.*

Since the well-known Tutte 1-factor theorem [10] was proposed, there are many results about component-factors, see [5, 9, 13, 14], etc.

Akiyama, Avis and Era [1] demonstrated the following classical result, which is a characterization for the existence of $P_{\ge 2}$-factors in a graph.

Theorem 1.2. *(Akiyama, Avis and Era [1]) A graph G has a $P_{\ge 2}$-factor if and only if $i(G-S) \le 2|S|$ for any $S \subseteq V(G)$.*

Amahashi and Kano [2] and Las Vergnas [11] gave independently a characterization for graphs with $\mathcal{S}_k$-factors, which is a generalization of Theorem 1.2.

Theorem 1.3. *(Amahashi and Kano [2]; Las Vergnas [11]) Let k be an integer with $k \ge 2$. Then a graph G has an $\mathcal{S}_k$-factor if and only if $i(G-S) \le k|S|$ for any $S \subseteq V(G)$.*

A connected graph is called a *cactus* if each block of the graph is a complete subgraph. A cactus of odd order is called an *odd-cactus*. As an extension of $\mathcal{S}_k$-factors, the induced star factor, denoted by $\mathcal{IS}_k$-factor, is a spanning subgraph each component of which is an induced subgraph isomorphic to some graph in $\{K_{1,1}, K_{1,2}, \dots, K_{1,k}\}$. Denote by $oc(G-S)$ the number of odd-cactus of $G-S$. The criterion for $\mathcal{IS}_k$-factors was obtained by Egawa, Kano and Kelmans as following.

Theorem 1.4. *(Egawa, Kano and Kelmans [6]) Let $k \ge 2$ be an integer. A graph G has an $\mathcal{IS}_k$-factor if and only if $oc(G-S) \le k|S|$ for any $S \subseteq V(G)$.*

The *toughness* of a connected graph G, denoted by $\tau(G)$, was first introduced by Chvátal [4] as follows. If G is complete, then $\tau(G) = +\infty$; otherwise,

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \ge 2\right\}.$$

Kaneko [7] introduced the concept of a *sun* and gave a characterization for the existence of $P_{\ge 3}$-factors in a graph. It is perhaps the first criterion of graphs admitting path factors not including P_2. Additionally, Kano *et al.* [8] obtained a simpler proof for Kaneko's result [7].

A graph H is said to be a factor-critical graph if for each $v \in V(H)$, $H - \{v\}$ has a 1-factor. Let H be a factor-critical graph such that $V(H) = \{v_1, v_2, \dots, v_n\}$. A graph is called a sun if it is obtained from H by adding new vertices $\{u_1, u_2, \dots, u_n\}$ together with new edges $\{v_i u_i : 1 \le i \le n\}$ to H. Note that, according to Kaneko [7], K_1 and K_2 are also regarded as a sun, respectively. Usually, the suns other than K_1 are called big suns. We use $sun(G-X)$ to denote the number of sun components of $G-X$.

Theorem 1.5. *(Kaneko [7]) A graph G has a $P_{\ge 3}$-factor if and only if $sun(G-S) \le 2|S|$ for any $S \subseteq V(G)$.*

Corollary 1.6. *(Kaneko [7]) A graph G has a $P_{\ge 3}$-factor if one of the following holds:* (i) *G is r-regular where $r \ge 2$;* (ii) *$\tau(G) = 1$;* (iii) *$\tau(G) = \frac{1}{2}$ and $\delta(G) \ge 2$;* (iv) *G is 3-connected planar;* (v) *G is claw-free with $\delta(G) \ge 2$.*

This paper attempts to find more sufficient conditions for the existence of these component factors by different graphic parameters including minimum degree, toughness, isolated toughness, binding number, etc.

2. Star-factor

The *isolated toughness* of a connected graph G denoted by $I(G)$. If G is complete, then $I(G) = +\infty$; otherwise,

$$I(G) = \min\left\{\frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2\right\}.$$

Lemma 2.1. *[15] Let G be a graph and $k \geq 1$ be a real number. Then the following three statements are equivalent.*

(i) $i(G-S) \leq k|S|$ for all $S \subset V(G)$.

(ii) $|U| \leq k|N_G(U)|$ for all independent set U of G.

Theorem 2.2. *A connected nontrivial graph G has an $\mathcal{S}_k$-factor if and only if $I(G) \geq \frac{1}{k}$, where integer $k \geq 2$.*

Proof. Sufficiency: If G is complete and nontrivial, then G has an $\mathcal{S}_k$-factor obviously. Thus we may assume that G is a graph of order at least two and not complete. Suppose, by way of contradiction, that G has no $\mathcal{S}_k$-factor, then by Theorem 1.3, there is a subset $S \subseteq V(G)$ such that $i(G-S) > k|S|$. Then, by the integrality of $i(G-S)$, we obtain that

$$i(G-S) \geq k|S| + 1. \tag{2.1}$$

If $|S| = 0$, then $i(G) = i(G-S) \geq k|S| + 1 = 1$, which contradicts the fact that G is connected.

If $|S| = 1$, then $i(G-S) > k|S| = k$. By the definition of $I(G)$, we have that

$$I(G) \leq \frac{|S|}{i(G-S)} < \frac{1}{k},$$

a contradiction.

If $|S| \geq 2$, then by (2.1), we have

$$|S| \leq \frac{i(G-S)-1}{k}.$$

By the definition of $I(G)$, we have

$$\begin{aligned} I(G) &\leq \frac{|S|}{i(G-S)} \\ &\leq \frac{i(G-S)-1}{k \times i(G-S)} \\ &= \frac{1}{k} - \frac{1}{k \times i(G-S)} \\ &< \frac{1}{k}, \end{aligned}$$

a contradiction.

Necessity: Suppose that G has an $\mathcal{S}_k$-factor and $I(G) < \frac{1}{k}$. Then by Theorem 1.3 and Lemma 2.1, for each independent set $U \subseteq V(G)$, we have

$$|U| \leq k|N_G(U)|. \tag{2.2}$$

Since $I(G) < \frac{1}{k}$, there is a subset $S \subseteq V(G)$ such that $\frac{|S|}{i(G-S)} < \frac{1}{k}$. Let U be the set of isolated vertices of $G - S$, then $N_G(U) \subseteq S$. Obviously, U is independent and

$$|N_G(U)| \leq |S| < \frac{i(G-S)}{k} = \frac{|U|}{k},$$

which contradicts (2.2). $\square$

Lemma 2.3. *[3] Let G be a simple connected planar graph of order at least three. If G does not contain triangles, then $|E(G)| \leq 2|V(G)| - 4$.*

Theorem 2.4. *Let G be a connected planar graph. If $\delta(G) \geq 3$, then G has an $\mathcal{S}_2$-factor.*

Proof. Suppose that G is a connected planar graph with no $\mathcal{S}_2$-factor. By Theorem 1.3, there exists a subset $S \subseteq V(G)$ such that $i(G-S) > 2|S|$. According to the integrality of $i(G-S)$, we obtain that

$$i(G-S) \geq 2|S| + 1. \tag{2.3}$$

Claim 2.5. *$S \neq \emptyset$.*

Proof. Suppose $S = \emptyset$, by (2.3), $i(G-S) \geq 2|S| + 1 = 1$. On the other hand, $i(G) \leq \omega(G) = 1$ since G is a connected graph. So, we obtain that G is an isolated vertex, which contradicts that $\delta(G) \geq 3$. □

By Claim 2.5, $S \neq \emptyset$. Set $|S| = s$. Then by (2.3), $i(G-S) \geq 2s + 1$. The set of isolated vertices in $G - S$ is denoted by $I(G-S)$. Then we construct a simple bipartite graph $H = H[X, Y]$ as follows. Let $X = S$ and $Y \subseteq I(G-S)$ such that $|Y| = 2s + 1$. For any $s \in X$ and $y \in Y$, $sy \in E(H)$ if and only if $sy \in E(G)$. Since $\delta(G) \geq 3$, it is clear that for each $y \in Y$, we have $|N_H(y)| \geq 3$. Hence, $|H| = s + (2s+1) = 3s + 1 \geq 4$ and

$$|E(H)| \geq 3 \times (2s+1) = 6s + 3 > 6s. \tag{2.4}$$

As G is a connected planar graph, it is easy to see that H is also a connected planar graph. According to the fact that a bipartite graph does not contain any odd cycles, Lemma 2.3 implies that

$$\begin{aligned} |E(H)| &\leq 2|H| - 4 \\ &= 2 \times (3s+1) - 4 \\ &= 6s - 2 \\ &< 6s, \end{aligned}$$

which is a contradiction to (2.4). □

Remark 2.6. Now, we explain that the condition of minimum degree $\delta(G) \geq 3$ in Theorem 2.4 is the best possible. Let $G = 2K_1 \vee 5K_1$ be a complete bipartite graph, where $\vee$ means "join". We know that G is a connected planar graph with $\delta(G) = 2 < 3$. Choose $X =: V(2K_1)$ with $|X| = 2$, then we have that

$$i(G-X) = 5 > 2|X| = 4.$$

In view of Theorem 1.3, G has no $\mathcal{S}_2$-factor.

3. Induced star-factor

Theorem 3.1. *Let G be a connected graph of order at least three. If G is not an odd cactus and $\tau(G) \geq \frac{1}{k}$, then G has an $\mathcal{IS}_k$-factor.*

Proof. Suppose, to the contrary, that G is a connected graph with no $\mathcal{IS}_k$-factor. If G is a complete graph, then G has a Hamilton cycle, denoted by C. Since G is not an odd cactus, C is an even cycle and thus G has a 1-factor. Hence, G has an $\mathcal{IS}_k$-factor, a contradiction. Thus, we may assume that G is not a complete graph.

By Theorem 1.4, there is a subset $S \subseteq V(G)$ such that $oc(G-S) > k|S|$. Due to the integrality, we obtain

$$oc(G-S) \geq k|S| + 1. \tag{3.1}$$

Claim 3.2. *$S \neq \emptyset$.*

Proof. Suppose that $S = \emptyset$, then by (3.1), we have $oc(G) = oc(G-S) \geq k|S|+1 = 1$. Note that $oc(G) \leq \omega(G) = 1$ since G is connected. Thus G is an odd cactus, a contradiction. □

By Claim 3.2, we have $|S| \geq 1$.

If $|S| = 1$, then by (3.1), we have $oc(G-S) \geq k|S|+1 = k+1$. Then due to the definition of $\tau(G)$, we obtain that

$$\frac{1}{k} \leq \tau(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{oc(G-S)} \leq \frac{1}{k+1} < \frac{1}{k},$$

a contradiction.

If $|S| \geq 2$, then by (3.1), we have

$$|S| \leq \frac{oc(G-S)-1}{k}. \tag{3.2}$$

Then by (3.2) and the definition of $\tau(G)$, we obtain that

$$\begin{aligned}\frac{1}{k} \leq \tau(G) &\leq \frac{|S|}{\omega(G-S)}\\ &\leq \frac{|S|}{oc(G-S)}\\ &\leq \frac{oc(G-S)-1}{k \times oc(G-S)}\\ &= \frac{1}{k} - \frac{1}{k \times oc(G-S)} < \frac{1}{k},\end{aligned}$$

a contradiction. □

Theorem 3.3. *Let G be a connected graph of order $n \geq 3$ which is not an odd cactus. Then G has an $\mathcal{IS}_k$-factor if $\delta(G) \geq \max\{\frac{n}{k+1}, \frac{4n}{3k+1} - 1\}$.*

Proof. Suppose, to the contrary, that G is a connected graph having no $\mathcal{IS}_k$-factor. By Theorem 1.4, there exists $S \subseteq V(G)$ such that $oc(G-S) > k|S|$. Due to the integrality, we obtain

$$oc(G-S) \geq k|S|+1. \tag{3.3}$$

Claim 3.4. *$S \neq \emptyset$.*

Proof. Suppose that $S = \emptyset$, then by (3.3), we have $oc(G) = oc(G-S) \geq k|S|+1 = 1$. Note that $oc(G) \leq \omega(G) = 1$ since G is connected. Thus G is an odd cactus, a contradiction. □

By Claim 3.4 and (3.3), we have that

$$oc(G-S) \geq k|S|+1 \geq k+1. \tag{3.4}$$

Let $C_1, C_2, \ldots, C_m$ be the odd cactus components of $G-S$, where $m = oc(G-S)$. Choose an odd cactus component C_i of $G-S$ such that $|C_i|$ is as small as possible, where $1 \leq i \leq m$. Without loss of generality, we assume that C_1 is such an odd cactus component and $|C_1| = t$.

Case 1. $t = 1$.

In this case, let $C_1 = \{x\}$. Since $N_G(x) \subseteq S$, we have that

$$|S| \geq d_G(x) \geq \delta(G) \geq \frac{n}{k+1}.$$

It follows from (3.3) that

$$\begin{aligned}
|G| &\geq |S| + \sum_{i=1}^{m} |C_i| \\
&\geq |S| + (k|S| + 1) \\
&= (k+1)|S| + 1 \\
&\geq (k+1) \times \frac{n}{k+1} + 1 \\
&= n + 1,
\end{aligned}$$

a contradiction.

Case 2. $t \geq 2$.

Since C_1 is an odd cactus and $t \geq 2$, we find that $|C_1| = t \geq 3$. On the other hand, according to the minimality property, we have that

$$t \leq \frac{|G|}{oc(G-S)} \leq \frac{n}{k|S|+1} \leq \frac{n}{k+1} < \frac{3n}{3k+1}. \tag{3.5}$$

Let u be the vertex with maximum degree in C_1, then $d_{C_1}(u) \leq t - 1$. It follows that

$$\begin{aligned}
|S| &\geq d_S(u) \\
&\geq \delta(G) - d_{C_1}(u) \\
&\geq \frac{4n}{3k+1} - 1 - (t-1) \\
&= \frac{4n}{3k+1} - t.
\end{aligned}$$

This together with (3.3), (3.5) and $t \geq 3$ implies that

$$\begin{aligned}
|G| &\geq |S| + \sum_{i=1}^{m} |C_i| \\
&\geq |S| + (k|S| + 1) \times t \\
&> (kt+1)|S| \\
&\geq (kt+1) \times \left(\frac{4n}{3k+1} - t \right) \\
&= (kt+1) \times \left(\frac{n}{3k+1} + (\frac{3n}{3k+1} - t) \right) \\
&> (kt+1) \times \frac{n}{kt+1} = n,
\end{aligned}$$

a contradiction. □

Remark 3.5. Now, we explain that the condition of toughness $\tau(G) \geq \frac{1}{k}$ in Theorem 3.1 and minimum degree $\delta(G) \geq \max\{\frac{n}{k+1}, \frac{4n}{3k+1} - 1\}$ in Theorem 3.3 are all the best possible. Let $H_1, H_2, \ldots, H_{k+1}$ be $k+1$ odd complete graphs, each of which contains exactly $\frac{n-1}{k+1}$ vertices, where integer $k \geq 2$ and $\frac{n-1}{k+1}$ is an integer. We construct a connected graph $G = K_1 \vee (\bigcup_{i=1}^{k+1} H_i)$, the order of which is n. It is obviously that $\tau(G) = \frac{1}{k+1} < \frac{1}{k}$, and $\delta(G) = \frac{n-1}{k+1} < \frac{n}{k+1}$. Choose $X =: V(K_1)$ with $|X| = 1$, then we have that

$$oc(G - X) = k + 1 > k|X| = k.$$

It follows from Theorem 1.4 that G has no $\mathcal{IS}_k$-factor.

4. Path-factor

In this section, we obtain some sufficient conditions for the existence of graphs admitting $P_{\geq 3}$-factors.

The *binding number* is introduced by Woodall [12] and defined as

$$bind(G) = \min\left\{\frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G)\right\}.$$

Theorem 4.1. *Let G be a connected graph of order $n \geq 3$. Then G has a $P_{\geq 3}$-factor if one of the following statements holds:*

(i) $I(G) \geq \frac{3}{2}$;

(ii) $bind(G) \geq \frac{5}{4}$;

(iii) $n \geq 8$ *and for all three independent vertices* $u, v, w \in V(G)$,

$$\max\{d_G(u), d_G(v), d_G(w)\} \geq \frac{n}{3}.$$

Proof. By way of contradiction, suppose that G is a connected graph with no $P_{\geq 3}$-factor. Then by Theorem 1.5, there is a subset $S \subseteq V(G)$ such that $sun(G - S) > 2|S|$. Due to the integrality of $sun(G - S)$, we obtain

$$sun(G - S) \geq 2|S| + 1. \tag{4.1}$$

(i) Obviously G has a $P_{\geq 3}$-factor if G is complete, a contradiction. Thus, we may assume that G is not complete. We shall consider two cases by the value of $|S|$ and derive a contradiction in each case.

Case 1. $|S| = 0$.

By (4.1), we have $sun(G) = sun(G - S) \geq 2|S| + 1 = 1$. Note that $sun(G) \leq \omega(G) = 1$ since G is connected. Then, $sun(G) = 1$ and thus G is a big sun. Of course, G is not an isolated edge since its order at least three. Let R be the factor-critical subgraph of G and set $U = V(R)$. It is clear that $G - U$ is an independent set and $|G - U| = |U|$. By the definition of $I(G)$ and $I(G) \geq \frac{3}{2}$, we have that

$$\frac{3}{2} \leq I(G) \leq \frac{|U|}{i(G - U)} = 1,$$

a contradiction.

Case 2. $|S| \geq 1$.

By (4.1), we have that

$$|S| \leq \frac{sun(G - S) - 1}{2}. \tag{4.2}$$

Assume that $sun(G - S) - i(G - S) = m$, *i.e.*, there are m big sun components of $G - S$, denoted by $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$. For each $i \in [1, m]$, let R_i be the factor-critical subgraph of C_i if C_i is not an isolated edge, and choose vertices $c_i \in V(R_i)$. If C_i is an isolated edge, then choose arbitrarily $c_i \in V(R_i)$ where $1 \le i \le m$. Let $S' = \{c_i : 1 \le i \le m\}$. Then by (4.2), we have that

$$\begin{aligned}|S \cup S'| &= |S| + sun(G-S) - i(G-S)\\ &\le |S| + sun(G-S)\\ &\le \frac{sun(G-S)-1}{2} + sun(G-S)\\ &= \frac{3 \times sun(G-S)-1}{2}\cdot\end{aligned}$$

By the definition of $I(G)$, it follows that

$$\begin{aligned}\frac{3}{2} \le I(G) &\le \frac{|S \cup S'|}{i(G-S-S')}\\ &\le \frac{3 \times sun(G-S)-1}{2 \times i(G-S-S')}\\ &= \frac{3 \times sun(G-S)-1}{2 \times sun(G-S)} < \frac{3}{2},\end{aligned}$$

a contradiction.

The statement (i) in Theorem 4.1 is proved.

(ii) Let $S' = V(G-S)$. By the definition of $bind(G)$, we have that

$$|N_G(S')| \ge \frac{5}{4}|S'|. \tag{4.3}$$

Case 1. $|S| \ge \frac{n}{5}$.

In this case, $|S'| = |G| - |S| \le \frac{4n}{5}$. By (4.3),

$$\begin{aligned}\frac{5}{4}(n-|S|) = \frac{5}{4}|S'| &\le |N_G(S')|\\ &= |N_G(G-S)| \le n - i(G-S).\end{aligned}$$

It follows immediately that

$$i(G-S) \le \frac{5}{4}|S| - \frac{n}{4}\cdot \tag{4.4}$$

Hence, by (4.4),

$$\begin{aligned}n &\ge |S| + i(G-S) + 2 \times (sun(G-S) - i(G-S))\\ &= |S| + 2 \times sun(G-S) - i(G-S)\\ &> |S| + 4|S| - (\frac{5}{4}|S| - \frac{n}{4})\\ &= \frac{15}{4}|S| + \frac{n}{4} \ge n,\end{aligned}$$

a contradiction.

Case 2. $|S| < \frac{n}{5}$.

In this case, $|S'| = |G| - |S| > \frac{4n}{5}$. Let $S_0 \subseteq S'$ such that $|S_0| = \frac{4n}{5}$. By (4.3), we have that $|N_G(S_0)| \geq \frac{5}{4}|S_0| = n$ and so $V(G) \subseteq N_G(S')$. Consequently, there exists no singleton component of $G - S$, *i.e.*,

$$i(G - S) = 0. \tag{4.5}$$

Consider all the sun components in $G - S$ and let $S'' = V(Sun(G - S))$. Since $sun(G - S) > 2|S|$, by (4.5), $|S''| > 2 \times sun(G - S) > 4|S|$. Hence,

$$\begin{aligned} bind(G) &\leq \frac{|N_G(S'')|}{|S''|} \leq \frac{|S''| + |S|}{|S''|} \\ &= 1 + \frac{|S|}{|S''|} < 1 + \frac{1}{4} = \frac{5}{4}, \end{aligned}$$

a contradiction.

The statement (ii) in Theorem 4.1 is proved.

(iii) We first give the argument as following.

Claim 4.2. *$S \neq \emptyset$.*

Proof. Suppose $S = \emptyset$, by (4.1), $sun(G) = sun(G - S) \geq 1$. On the other hand, $sun(G) \leq \omega(G) = 1$. So, we obtain that G is a big sun containing at least 8 vertices. It follows that there exist three vertices of degree one, denoted by $\{u, v, w\}$, which contradicts that $\max\{d_G(u), d_G(v), d_G(w)\} \geq \frac{n}{3} > 2$. □

By Claim 4.2 and (4.1), we have $sun(G - S) \geq 2|S| + 1 \geq 3$.

Case 1. $i(G - S) \geq 3$.

Let $\{x, y, z\}$ be three distinct isolated vertices of $G - S$. Since $\max\{d_G(x), d_G(y), d_G(z)\} \geq \frac{n}{3}$ and $N_G(x) \cup N_G(y) \cup N_G(z) \subseteq S$, we have that

$$|S| \geq \max\{d_G(x), d_G(y), d_G(z)\} \geq \frac{n}{3}.$$

It follows from (4.1) that $sun(G - S) \geq 2|S| + 1 \geq \frac{2n}{3} + 1$ and thus

$$n \geq |S| + sun(G - S) \geq \frac{n}{3} + \frac{2n}{3} + 1 = n + 1,$$

a contradiction.

Case 2. $i(G - S) \leq 2$.

In this case, by (4.1), there exist at least three suns of $G - S$, denoted by $C_1, C_2, \ldots, C_t$ where $t \geq 3$. Then we choose $c_i \in V(C_i)$ such that $d_{C_i}(c_i) \leq 1$, where $i = 1, 2, 3$. Obviously, $\{c_1, c_2, c_3\}$ is an independent set of G. Then $\max\{d_G(c_1), d_G(c_2), d_G(c_3)\} \geq \frac{n}{3}$. Without loss of generality, we assume $d_G(c_1) \geq \frac{n}{3}$. Since $d_S(c_1) = d_G(c_1) - d_{C_1}(c_1) \geq \frac{n}{3} - 1$, we have that $|S| \geq d_S(c_1) \geq \frac{n}{3} - 1$. It follows from (4.1) that

$$sun(G - S) \geq 2|S| + 1 \geq \frac{2n}{3} - 1,$$

and thus

$$\begin{aligned} n &\geq |S| + 2 \times sun(G - S) - i(G - S) \\ &\geq \frac{n}{3} - 1 + 2 \times \left(\frac{2n}{3} - 1\right) - 2 \\ &= \frac{5n}{3} - 5 > n, \end{aligned}$$

a contradiction. The statement (iii) in Theorem 4.1 is proved. □

Remark 4.3. Now, we claim that the conditions of isolated toughness $I(G) \geq \frac{2}{3}$ and binding number $bind(G) \geq \frac{5}{4}$ in Theorem 4.1 are all the best possible. Let P_5 be a path of order 5, the center vertex of which is denoted by u. We construct a connected graph $G = P_5 \cup \{v\} \cup e$, where $e = uv$. It is obvious that $I(G) = 1 < \frac{3}{2}$, and $bind(G) = 1 < \frac{5}{4}$. Choose $X =: \{u\}$, then we have that $sun(G-X) = 3 > 2 = 2|X|$. It follows from Theorem 1.5 that G has no $P_{\geq 3}$-factor.

Remark 4.4. Now, we explain that the degree condition in the statement (iii) of Theorem 4.1 is the best possible. Let $G = 2K_1 \vee 7K_1$ be a connected complete bipartite graph of order $n = 9$. We know there exists three independent vertices $\{u, v, w\} \subseteq V(7K_1)$ such that $\max\{d_G(u), d_G(v), d_G(w)\} = 2 < 3 = \frac{n}{3}$. Choose $X =: V(2K_1)$ with $|X| = 2$, then we have that $sun(G - X) = 7 > 2|X| = 4$. Using Theorem 1.5, G has no $P_{\geq 3}$-factor.

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