REMARKS ON COMPONENT FACTORS IN GRAPHS

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Abstract. For a family of connected graphs $\mathcal{F}$, a spanning subgraph $H$ of a graph $G$ is called an $\mathcal{F}$-factor of $G$ if its each component is isomorphic to an element of $\mathcal{F}$. In particular, $H$ is called an $S_k$-factor of $G$ if $\mathcal{F} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$, where integer $k \geq 2$; $H$ is called a $P_{2,k}$-factor of $G$ if every component in $\mathcal{F}$ is a path of order at least three. As an extension of $S_k$-factors, the induced star-factor (i.e., $I\mathcal{S}_k$-factor) is a spanning subgraph each component of which is an induced subgraph isomorphic to some graph in $\mathcal{F} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$. In this paper, we firstly prove that a graph $G$ has an $S_k$-factor if and only if its isolated toughness $I(G) \geq \frac{1}{k}$. Secondly, we prove that a planar graphs $G$ has an $S_k$-factors if its minimum degree $\delta(G) \geq 3$. Thirdly, we give two sufficient conditions for graphs with $I\mathcal{S}_k$-factors by toughness and minimum degree, respectively. Additionally, we obtain three special classes of graphs admitting $P_{2,k}$-factors.

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1. Introduction

The graphs considered here are finite and simple, unless explicitly stated. Let $G = (V(G), E(G))$ be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of $v$ and the set of vertices adjacent to $v$ in $G$, respectively. For $S \subseteq V(G)$, we write $N_G(S) = \bigcup_{v \in S} N_G(v)$. A graph $G$ is called an $r$-regular graph if $d_G(v) = r$ for each $v \in V(G)$. We use $\delta(G)$ to denote the minimum degree of a graph $G$. The number of connected components and isolated vertices of a graph $G$ is denoted by $\omega(G)$ and $i(G)$, respectively. We refer to [3] for the notation and terminologies not defined here.

The complete bipartite graph $K_{1,r}$ is called the star of order $r + 1$, where $r$ is a positive integer. We use $S_k$ to denote the set $\{K_{1,1}, K_{1,2}, K_{1,3}, \ldots, K_{1,k}\}$, where integer $k \geq 2$.

Let $\mathcal{F}$ be a family of connected graphs. Then a spanning subgraph $H$ of $G$ is called an $\mathcal{F}$-factor of $G$ if each component of $H$ is isomorphic to an element of $\mathcal{F}$. In particular, for an integer $k \geq 2$, a $\{K_{1,1}, K_{1,2}, K_{1,3}, \ldots, K_{1,k}\}$-factor is briefly called an $S_k$-factor. Similarly, a $\{P_k, P_{k+1}, \ldots\}$-factor is called a $P_{2,k}$-factor.

In 1947, Tutte [10] presented a criterion for the existence of $I$-factors (perfect matchings), which is one of the classical results in graph theory. Denote by $o(G)$ the number of odd components of $G$, whose orders are odd.

Keywords. Star-factor, Induced star-factor, $P_{2,k}$-factor, Toughness, Minimum degree.

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Theorem 1.1. (Tutte [10]) A graph $G$ has a 1-factor if and only if $o(G - S) = |S|$ for any $S \subseteq V(G)$.

Since the well-known Tutte 1-factor theorem [10] was proposed, there are many results about component-factors, see [5,9,13,14], etc.

Akiyama, Avis and Era [1] demonstrated the following classical result, which is a characterization for the existence of $P_{2}$-factors in a graph.

Theorem 1.2. (Akiyama, Avis and Era [1]) A graph $G$ has a $P_{2}$-factor if and only if $i(G - S) \leq 2|S|$ for any $S \subseteq V(G)$.

Amahashi and Kano [2] and Las Vergnas [11] gave independently a characterization for graphs with $S_{k}$-factors, which is a generalization of Theorem 1.2.

Theorem 1.3. (Amahashi and Kano [2]; Las Vergnas [11]) Let $k$ be an integer with $k \geq 2$. Then a graph $G$ has an $S_{k}$-factor if and only if $i(G - S) \leq k|S|$ for any $S \subseteq V(G)$.

A connected graph is called a cactus if each block of the graph is a complete subgraph. A cactus of odd order is called an odd-cactus. As an extension of $S_{k}$-factors, the induced star factor, denoted by $\mathcal{IS}_{k}$-factor, is a spanning subgraph each component of which is an induced subgraph isomorphic to some graph in $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}\}$. Denote by $oc(G - S)$ the number of odd-cactus of $G - S$. The criterion for $\mathcal{IS}_{k}$-factors was obtained by Egawa, Kano and Kelmans as following.

Theorem 1.4. (Egawa, Kano and Kelmans [6]) Let $k \geq 2$ be an integer. A graph $G$ has an $\mathcal{IS}_{k}$-factor if and only if $oc(G - S) \leq k|S|$ for any $S \subseteq V(G)$.

The toughness of a connected graph $G$, denoted by $\tau(G)$, was first introduced by Chvátal [4] as follows. If $G$ is complete, then $\tau(G) = +\infty$; otherwise,

$$\tau(G) = \min \left\{ \frac{|S|}{\omega(G - S)} : S \subseteq V(G), \omega(G - S) \geq 2 \right\}.$$  

Kaneko [7] introduced the concept of a sun and gave a characterization for the existence of $P_{2}$-3-factors in a graph. It is perhaps the first criterion of graphs admitting path factors not including $P_{2}$. Additionally, Kano et al. [8] obtained a simpler proof for Kaneko’s result [7].

A graph $H$ is said to be a factor-critical graph if for each $v \in V(H)$, $H - \{v\}$ has a 1-factor. Let $H$ be a factor-critical graph such that $V(H) = \{v_{1}, v_{2}, \ldots, v_{n}\}$. A graph is called a sun if it is obtained from $H$ by adding new vertices $\{u_{1}, u_{2}, \ldots, u_{n}\}$ together with new edges $\{v_{i}u_{i} : 1 \leq i \leq n\}$ to $H$. Note that, according to Kaneko [7], $K_{1}$ and $K_{2}$ are also regarded as a sun, respectively. Usually, the suns other than $K_{1}$ are called big suns. We use $sun(G - X)$ to denote the number of sun components of $G - X$.

Theorem 1.5. (Kaneko [7]) A graph $G$ has a $P_{2}$-factor if and only if $sun(G - S) \leq 2|S|$ for any $S \subseteq V(G)$.

Corollary 1.6. (Kaneko [7]) A graph $G$ has a $P_{2}$-3-factor if one of the following holds: (i) $G$ is $r$-regular where $r \geq 2$; (ii) $\tau(G) = 1$; (iii) $\tau(G) = \frac{1}{2}$ and $\delta(G) \geq 2$; (iv) $G$ is 3-connected planar; (v) $G$ is claw-free with $\delta(G) \geq 2$.

This paper attempts to find more sufficient conditions for the existence of these component factors by different graphic parameters including minimum degree, toughness, isolated toughness, binding number, etc.
2. Star-factor

The isolated toughness of a connected graph $G$ denoted by $I(G)$. If $G$ is complete, then $I(G) = +\infty$; otherwise,

$$I(G) = \min \left\{ \frac{|S|}{i(G-S)} : S \subseteq V(G), i(G-S) \geq 2 \right\}.$$

**Lemma 2.1.** [15] Let $G$ be a graph and $k \geq 1$ be a real number. Then the following three statements are equivalent.

(i) $i(G-S) \leq k|S|$ for all $S \subset V(G)$.
(ii) $|U| \leq k|N_G(U)|$ for all independent set $U$ of $G$.

**Theorem 2.2.** A connected nontrivial graph $G$ has an $S_k$-factor if and only if $I(G) \geq 1/k$, where integer $k \geq 2$.

**Proof.** Sufficiency: If $G$ is complete and nontrivial, then $G$ has an $S_k$-factor obviously. Thus we may assume that $G$ is a graph of order at least two and not complete. Suppose, by way of contradiction, that $G$ has no $S_k$-factor, then by Theorem 1.3, there is a subset $S \subseteq V(G)$ such that $i(G-S) > k|S|$. Then, by the integrality of $i(G-S)$, we obtain that

$$i(G-S) \geq k|S| + 1. \quad (2.1)$$

If $|S| = 0$, then $i(G) = i(G-S) \geq k|S| + 1 = 1$, which contradicts the fact that $G$ is connected.

If $|S| = 1$, then $i(G-S) > k|S| = k$. By the definition of $I(G)$, we have that

$$I(G) \leq \frac{|S|}{i(G-S)} < \frac{1}{k},$$

a contradiction.

If $|S| \geq 2$, then by (2.1), we have

$$|S| \leq \frac{i(G-S) - 1}{k}.$$  

By the definition of $I(G)$, we have

$$I(G) \leq \frac{|S|}{i(G-S)} \leq \frac{i(G-S) - 1}{k \times i(G-S)} = \frac{1}{k} - \frac{1}{k \times i(G-S)} < \frac{1}{k},$$

a contradiction.

Necessity: Suppose that $G$ has an $S_k$-factor and $I(G) < 1/k$. Then by Theorem 1.3 and Lemma 2.1, for each independent set $U \subseteq V(G)$, we have

$$|U| \leq k|N_G(U)|. \quad (2.2)$$

Since $I(G) < 1/k$, there is a subset $S \subseteq V(G)$ such that $\frac{|S|}{i(G-S)} < \frac{1}{k}$. Let $U$ be the set of isolated vertices of $G-S$, then $N_G(U) \subseteq S$. Obviously, $U$ is independent and

$$|N_G(U)| \leq |S| < \frac{i(G-S)}{k} = \frac{|U|}{k},$$

which contradicts (2.2). \qed
Lemma 2.3. [3] Let $G$ be a simple connected planar graph of order at least three. If $G$ does not contain triangles, then $|E(G)| \leq 2|V(G)| - 4$.

Theorem 2.4. Let $G$ be a connected planar graph. If $\delta(G) \geq 3$, then $G$ has an $S_2$-factor.

Proof. Suppose that $G$ is a connected planar graph with no $S_2$-factor. By Theorem 1.3, there exists a subset $S \subseteq V(G)$ such that $i(G - S) > 2|S|$. According to the integrality of $i(G - S)$, we obtain that

$$i(G - S) \geq 2|S| + 1. \quad (2.3)$$

Claim 2.5. $S \neq \emptyset$.

Proof. Suppose $S = \emptyset$, by (2.3), $i(G - S) \geq 2|S| + 1 = 1$. On the other hand, $i(G) \leq \omega(G) = 1$ since $G$ is a connected graph. So, we obtain that $G$ is an isolated vertex, which contradicts that $\delta(G) \geq 3$. \hfill $\square$

By Claim 2.5, $S \neq \emptyset$. Set $|S| = s$. Then by (2.3), $i(G - S) \geq 2s + 1$. The set of isolated vertices in $G - S$ is denoted by $I(G - S)$. Then we construct a simple bipartite graph $H = H[X, Y]$ as follows. Let $X = S$ and $Y \subseteq I(G - S)$ such that $|Y| = 2s + 1$. For any $s \in X$ and $y \in Y$, $sy \in E(H)$ if and only if $sy \in E(G)$. Since $\delta(G) \geq 3$, it is clear that for each $y \in Y$, we have $|N_H(y)| \geq 3$. Hence, $|H| = s + (2s + 1) = 3s + 1 \geq 4$ and

$$|E(H)| \geq 3 \times (2s + 1) = 6s + 3 > 6s. \quad (2.4)$$

As $G$ is a connected planar graph, it is easy to see that $H$ is also a connected planar graph. According to the fact that a bipartite graph does not contain any odd cycles, Lemma 2.3 implies that

$$|E(H)| \leq 2|H| - 4 = 2 \times (3s + 1) - 4 = 6s - 2 < 6s,$$

which is a contradiction to (2.4). \hfill $\square$

Remark 2.6. Now, we explain that the condition of minimum degree $\delta(G) \geq 3$ in Theorem 2.4 is the best possible. Let $G = 2K_1 \lor 5K_1$ be a complete bipartite graph, where $\lor$ means “join”. We know that $G$ is a connected planar graph with $\delta(G) = 2 < 3$. Choose $X =: V(2K_1)$ with $|X| = 2$, then we have that

$$i(G - X) = 5 > 2|X| = 4.$$

In view of Theorem 1.3, $G$ has no $S_2$-factor.

3. Induced star-factor

Theorem 3.1. Let $G$ be a connected graph of order at least three. If $G$ is not an odd cactus and $\tau(G) \geq \frac{1}{k}$, then $G$ has an $IS_k$-factor.

Proof. Suppose, to the contrary, that $G$ is a connected graph with no $IS_k$-factor. If $G$ is a complete graph, then $G$ has a Hamilton cycle, denoted by $C$. Since $G$ is not an odd cactus, $C$ is an even cycle and thus $G$ has a 1-factor. Hence, $G$ has an $IS_k$-factor, a contradiction. Thus, we may assume that $G$ is not a complete graph.

By Theorem 1.4, there is a subset $S \subseteq V(G)$ such that $oc(G - S) > k|S|$. Due to the integrality, we obtain

$$oc(G - S) \geq k|S| + 1. \quad (3.1)$$
Claim 3.2. \( S \neq \emptyset \).

Proof. Suppose that \( S = \emptyset \), then by (3.1), we have \( oc(G) = oc(G - S) \geq k|S| + 1 = 1 \). Note that \( oc(G) \leq \omega(G) = 1 \) since \( G \) is connected. Thus \( G \) is an odd cactus, a contradiction. \( \blacksquare \)

By Claim 3.2, we have \( |S| \geq 1 \).

If \( |S| = 1 \), then by (3.1), we have \( oc(G - S) \geq k|S| + 1 = k + 1 \). Then due to the definition of \( \tau(G) \), we obtain that
\[
\frac{1}{k} \leq \tau(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{oc(G - S)} \leq \frac{1}{k + 1} < \frac{1}{k},
\]
a contradiction.

If \( |S| \geq 2 \), then by (3.1), we have
\[
|S| \leq \frac{oc(G - S) - 1}{k}.
\]
(3.2)
Then by (3.2) and the definition of \( \tau(G) \), we obtain that
\[
\frac{1}{k} \leq \tau(G) \leq \frac{|S|}{\omega(G - S)} \leq \frac{|S|}{oc(G - S)} \leq \frac{oc(G - S) - 1}{k \times oc(G - S)} = \frac{1}{k} - \frac{1}{k \times oc(G - S)} < \frac{1}{k},
\]
a contradiction. \( \blacksquare \)

Theorem 3.3. Let \( G \) be a connected graph of order \( n \geq 3 \) which is not an odd cactus. Then \( G \) has an \( IS_k \)-factor if \( \delta(G) \geq \max\{ \frac{n}{k+1}, \frac{4n}{3k+1} - 1 \} \).

Proof. Suppose, to the contrary, that \( G \) is a connected graph having no \( IS_k \)-factor. By Theorem 1.4, there exists \( S \subseteq V(G) \) such that \( oc(G - S) > k|S| \). Due to the integrality, we obtain
\[
oc(G - S) \geq k|S| + 1.
\]
(3.3)

Claim 3.4. \( S \neq \emptyset \).

Proof. Suppose that \( S = \emptyset \), then by (3.3), we have \( oc(G) = oc(G - S) \geq k|S| + 1 = 1 \). Note that \( oc(G) \leq \omega(G) = 1 \) since \( G \) is connected. Thus \( G \) is an odd cactus, a contradiction. \( \blacksquare \)

By Claim 3.4 and (3.3), we have that
\[
oc(G - S) \geq k|S| + 1 \geq k + 1.
\]
(3.4)

Let \( C_1, C_2, \ldots, C_m \) be the odd cactus components of \( G - S \), where \( m = oc(G - S) \). Choose an odd cactus component \( C_i \) of \( G - S \) such that \( |C_i| \) is as small as possible, where \( 1 \leq i \leq m \). Without loss of generality, we assume that \( C_1 \) is such an odd cactus component and \( |C_1| = t \).
**Case 1.** \( t = 1. \)

In this case, let \( C_1 = \{ x \} \). Since \( N_G(x) \subseteq S \), we have that

\[
|S| \geq d_G(x) \geq \delta(G) \geq \frac{n}{k+1}.
\]

It follows from (3.3) that

\[
|G| \geq |S| + \sum_{i=1}^{m} |C_i| \\
\geq |S| + (k|S| + 1) \\
= (k + 1)|S| + 1 \\
\geq (k + 1) \times \frac{n}{k+1} + 1 \\
= n + 1,
\]

a contradiction.

**Case 2.** \( t \geq 2. \)

Since \( C_1 \) is an odd cactus and \( t \geq 2 \), we find that \( |C_1| = t \geq 3 \). On the other hand, according to the minimality property, we have that

\[
t \leq \frac{|G|}{\text{oc}(G - S)} \leq \frac{n}{k|S| + 1} \leq \frac{n}{k+1} < \frac{3n}{3k+1}. \tag{3.5}
\]

Let \( u \) be the vertex with maximum degree in \( C_1 \), then \( d_{C_1}(u) \leq t - 1 \). It follows that

\[
|S| \geq d_S(u) \\
\geq \delta(G) - d_{C_1}(u) \\
\geq \frac{4n}{3k+1} - 1 - (t - 1) \\
= \frac{4n}{3k+1} - t.
\]

This together with (3.3), (3.5) and \( t \geq 3 \) implies that

\[
|G| \geq |S| + \sum_{i=1}^{m} |C_i| \\
\geq |S| + (k|S| + 1) \times t \\
> (kt + 1)|S| \\
\geq (kt + 1) \times \left( \frac{4n}{3k+1} - t \right) \\
= (kt + 1) \times \left( \frac{n}{3k+1} + \left( \frac{3n}{3k+1} - t \right) \right) \\
> (kt + 1) \times \frac{n}{kt + 1} = n,
\]

a contradiction. \( \square \)

**Remark 3.5.** Now, we explain that the condition of toughness \( \tau(G) \geq \frac{1}{2} \) in Theorem 3.1 and minimum degree \( \delta(G) \geq \max\left\{ \frac{n}{k+1}, \frac{4n}{3k+1} - 1 \right\} \) in Theorem 3.3 are all the best possible. Let \( H_1, H_2, \ldots, H_{k+1} \) be \( k + 1 \) odd complete
graphs, each of which contains exactly \( \frac{n-1}{k+1} \) vertices, where integer \( k \geq 2 \) and \( \frac{n-1}{k+1} \) is an integer. We construct a connected graph \( G = K_1 \lor (\bigcup_{i=1}^{k+1} H_i) \), the order of which is \( n \). It is obviously that \( \tau(G) = \frac{1}{k+1} < \frac{1}{k} \), and \( \delta(G) = \frac{n-1}{k+1} < \frac{n}{k+1} \). Choose \( X = V(K_1) \) with \( |X| = 1 \), then we have that
\[
ocite{oc(G - X) = k + 1 > k|X| = k.}
\]
It follows from Theorem 1.4 that \( G \) has no \( IS_k \)-factor.

4. Path-factor

In this section, we obtain some sufficient conditions for the existence of graphs admitting \( P_{\geq 3} \)-factors.

The binding number is introduced by Woodall \cite{12} and defined as
\[
\text{bind}(G) = \min \left\{ \frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G) \right\}.
\]

**Theorem 4.1.** Let \( G \) be a connected graph of order \( n \geq 3 \). Then \( G \) has a \( P_{\geq 3} \)-factor if one of the following statements holds:
(i) \( I(G) \geq \frac{3}{2} \);
(ii) \( \text{bind}(G) \geq \frac{5}{4} \);
(iii) \( n \geq 8 \) and for all three independent vertices \( u, v, w \in V(G) \),
\[
\max\{d_G(u), d_G(v), d_G(w)\} \geq \frac{n}{3}.
\]

**Proof.** By way of contradiction, suppose that \( G \) is a connected graph with no \( P_{\geq 3} \)-factor. Then by Theorem 1.5, there is a subset \( S \subseteq V(G) \) such that \( \text{sun}(G - S) > 2|S| \). Due to the integrality of \( \text{sun}(G - S) \), we obtain
\[
\text{sun}(G - S) \geq 2|S| + 1. \tag{4.1}
\]

(i) Obviously \( G \) has a \( P_{\geq 3} \)-factor if \( G \) is complete, a contradiction. Thus, we may assume that \( G \) is not complete. We shall consider two cases by the value of \( |S| \) and derive a contradiction in each case.

**Case 1.** \( |S| = 0 \).

By (4.1), we have \( \text{sun}(G) = \text{sun}(G - S) \geq 2|S| + 1 = 1 \). Note that \( \text{sun}(G) \leq \omega(G) = 1 \) since \( G \) is connected. Then, \( \text{sun}(G) = 1 \) and thus \( G \) is a big sun. Of course, \( G \) is not an isolated edge since its order at least three. Let \( R \) be the factor-critical subgraph of \( G \) and set \( U = V(R) \). It is clear that \( G - U \) is an independent set and \( |G - U| = |U| \). By the definition of \( I(G) \) and \( I(G) \geq \frac{3}{2} \), we have that
\[
\frac{3}{2} \leq I(G) \leq \frac{|U|}{i(G - U)} = 1,
\]
a contradiction.

**Case 2.** \( |S| \geq 1 \).

By (4.1), we have that
\[
|S| \leq \frac{\text{sun}(G - S) - 1}{2}. \tag{4.2}
\]
Assume that \( \text{sun}(G - S) - i(G - S) = m \), i.e., there are \( m \) big sun components of \( G - S \), denoted by \( C = \{C_1, C_2, \ldots, C_m\} \). For each \( i \in [1, m] \), let \( R_i \) be the factor-critical subgraph of \( C_i \) if \( C_i \) is not an isolated edge,
and choose vertices $c_i \in V(R_i)$. If $C_i$ is an isolated edge, then choose arbitrarily $c_i \in V(R_i)$ where $1 \leq i \leq m$. Let $S' = \{c_i : 1 \leq i \leq m\}$. Then by (4.2), we have that
\[
|S \cup S'| = |S| + \text{sun}(G - S) - i(G - S)
\leq |S| + \text{sun}(G - S)
\leq \frac{\text{sun}(G - S) - 1}{2} + \text{sun}(G - S)
= \frac{3 \times \text{sun}(G - S) - 1}{2}.
\]
By the definition of $I(G)$, it follows that
\[
\frac{3}{2} \leq I(G) \leq \frac{|S \cup S'|}{i(G - S - S')}
\leq \frac{3 \times \text{sun}(G - S) - 1}{2 \times i(G - S - S')}
= \frac{3 \times \text{sun}(G - S) - 1}{2 \times \text{sun}(G - S)} < \frac{3}{2},
\]
a contradiction.

The statement (i) in Theorem 4.1 is proved.

(ii) Let $S' = V(G - S)$. By the definition of $\text{bind}(G)$, we have that
\[
|N_G(S')| \geq \frac{5}{4}|S'|.
\] (4.3)

Case 1. $|S| \geq \frac{n}{5}$.

In this case, $|S'| = |G| - |S| \leq \frac{4n}{5}$. By (4.3),
\[
\frac{5}{4}(n - |S|) = \frac{5}{4}|S'| \leq |N_G(S')|
= |N_G(G - S)| \leq n - i(G - S).
\]
It follows immediately that
\[
i(G - S) \leq \frac{5}{4}|S| - \frac{n}{4}.
\] (4.4)
Hence, by (4.4),
\[
n \geq |S| + i(G - S) + 2 \times (\text{sun}(G - S) - i(G - S))
= |S| + 2 \times \text{sun}(G - S) - i(G - S)
> |S| + 4|S| - \left(\frac{5}{4}|S| - \frac{n}{4}\right)
= \frac{15}{4}|S| + \frac{n}{4} \geq n,
\]
a contradiction.
Case 2. $|S| \leq \frac{n}{3}$.

In this case, $|S'| = |G| - |S| > \frac{4n}{5}$. Let $S_0 \subseteq S'$ such that $|S_0| = \frac{4n}{5}$. By (4.3), we have that $|N_G(S_0)| \geq \frac{5}{4}|S_0| = n$ and so $V(G) \subseteq N_G(S')$. Consequently, there exists no singleton component of $G - S$, i.e.,

$$i(G - S) = 0.$$  \hspace{5cm} \text{(4.5)}

Consider all the sun components in $G - S$ and let $S'' = V(Sun(G - S))$. Since $sun(G - S) > 2|S|$, by (4.5), $|S''| > 2 \times sun(G - S) > 4|S|$. Hence,

$$bind(G) \leq \frac{|N_G(S'')|}{|S''|} \leq \frac{|S''| + |S|}{|S''|} = 1 + \frac{|S|}{|S''|} < 1 + \frac{1}{4} = \frac{5}{4},$$

a contradiction.

The statement (ii) in Theorem 4.1 is proved.

(iii) We first give the argument as following.

Claim 4.2. $S \neq \emptyset$.

Proof. Suppose $S = \emptyset$, by (4.1), $sun(G) = sun(G - S) \geq 1$. On the other hand, $sun(G) \leq \omega(G) = 1$. So, we obtain that $G$ is a big sun containing at least 8 vertices. It follows that there exist three vertices of degree one, denoted by $\{u, v, w\}$, which contradicts that $\max\{d_G(u), d_G(v), d_G(w)\} \geq \frac{n}{3} > 2$. \hfill $\Box$

By Claim 4.2 and (4.1), we have $sun(G - S) \geq 2|S| + 1 \geq 3$.

Case 1. $i(G - S) \geq 3$.

Let $\{x, y, z\}$ be three distinct isolated vertices of $G - S$. Since $\max\{d_G(x), d_G(y), d_G(z)\} \geq \frac{n}{3}$ and $N_G(x) \cup N_G(y) \cup N_G(z) \subseteq S$, we have that

$$|S| \geq \max\{d_G(x), d_G(y), d_G(z)\} \geq \frac{n}{3}.$$

It follows from (4.1) that

$$sun(G - S) \geq 2|S| + 1 \geq \frac{2n}{3} + 1$$

and thus

$$n \geq |S| + sun(G - S) \geq \frac{n}{3} + \frac{2n}{3} + 1 = n + 1,$$

a contradiction.

Case 2. $i(G - S) \leq 2$.

In this case, by (4.1), there exist at least three suns of $G - S$, denoted by $C_1, C_2, \ldots, C_t$ where $t \geq 3$. Then we choose $c_i \in V(C_i)$ such that $d_G(c_i) \leq 1$, where $i = 1, 2, 3$. Obviously, $\{c_1, c_2, c_3\}$ is an independent set of $G$. Then $\max\{d_G(c_1), d_G(c_2), d_G(c_3)\} \geq \frac{n}{3}$. Without loss of generality, we assume $d_G(c_1) \geq \frac{n}{3}$. Since $d_S(c_1) = d_G(c_1) - d_{C_1}(c_1) \geq \frac{n}{3} - 1$, we have that $|S| \geq d_S(c_1) \geq \frac{n}{3} - 1$. It follows from (4.1) that

$$sun(G - S) \geq 2|S| + 1 \geq \frac{2n}{3} - 1,$$

and thus

$$n \geq |S| + 2 \times sun(G - S) - i(G - S)$$

$$\geq \frac{n}{3} - 1 + 2 \times \left(\frac{2n}{3} - 1\right) - 2$$

$$= \frac{5n}{3} - 5 > n,$$

a contradiction. The statement (iii) in Theorem 4.1 is proved. \hfill $\Box$
Remark 4.3. Now, we claim that the conditions of isolated toughness $I(G) \geq \frac{2}{3}$ and binding number $bind(G) \geq \frac{5}{4}$ in Theorem 4.1 are all the best possible. Let $P_5$ be a path of order 5, the center vertex of which is denoted by $u$. We construct a connected graph $G = P_5 \cup \{v\} \cup e$, where $e = uv$. It is obvious that $I(G) = 1 < \frac{2}{3}$, and $bind(G) = 1 < \frac{5}{4}$. Choose $X = \{u\}$, then we have that $sun(G-X) = 3 > 2 = 2|X|$. It follows from Theorem 1.5 that $G$ has no $P_{2,3}$-factor.

Remark 4.4. Now, we explain that the degree condition in the statement (iii) of Theorem 4.1 is the best possible. Let $G = 2K_1 \lor 7K_1$ be a connected complete bipartite graph of order $n = 9$. We know there exists three independent vertices $\{u, v, w\} \subseteq V(7K_1)$ such that $\max\{d_G(u), d_G(v), d_G(w)\} = 2 < 3 = \frac{n}{3}$. Choose $X = V(2K_1)$ with $|X| = 2$, then we have that $sun(G-X) = 7 > 2|X| = 4$. Using Theorem 1.5, $G$ has no $P_{2,3}$-factor.

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