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GENERALIZED PERIODIC REPLACEMENT POLICIES FOR REPAIRABLE SYSTEMS SUBJECT TO TWO TYPES OF FAILURES

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Abstract. The purpose of this current research is to schedule generalized periodic replacement policies for a single unit system executing random working jobs. The system is subject to two types of failures when it has failed, including a minor failure (Type I failure), which can be thoroughly removed by the minimal repair and a catastrophic failure (Type II failure), which should be rectified by the corrective replacement. To be specific, four distinct periodic replacement models including a periodic replacement first model (Model A1), a modified periodic replacement first model (Model A2), a periodic replacement last model (Model B1), and a modified periodic replacement last model (Model B2) are investigated. The long run average cost rate (ACR) over an infinite time span under different situations is obtained theoretically and optimal periodic replacement interval for each condition is derived analytically. Numerical examples are exhibited to verify the derived results.

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1. Introduction

Maintenance activities, especially replacement behaviors are widely arranged in advance to avoid disastrous systems failures and to decrease economic losses in various industrial scenes. Generally, replacement operations where the procedure time is arranged before system failure and after system failure are called as preventive replacement and corrective replacement, respectively [18]. Replacing a system either too frequently or too less is not advisable as it increases unnecessary maintenance costs or reduces system availability. Therefore, scheduling the optimal replacement cycle and numbers according to some criteria, such as long-run average cost rate (ACR), expected long-run profit rate, or system availability is attracting more and more attention in maintenance activity [3].

Age replacement (AR) models and periodic replacement (PR) models are two fundamental replacement policies in preventive maintenance theory and they have been extensively studied in the past few decades [9,13]. Barlow and Hunter [4] investigated the AR policy for a single unit system, in which the original system is replaced with an auxiliary system at a constant age T after its operation or at system failure, whichever occurs first. Different from AR policies, PR models are more practical since they do not need to keep records of usage time, where the system is periodically replaced at kT $(k = 1, 2, \ldots)$ and only minimal repair at system failure is addressed such that system failure rate is undisturbed by any repair for failures between two proximate

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replacements [8, 18]. Various replacement models and their theoretical computations are sufficiently analyzed [1, 7, 12, 14].

In the late 1990s, power companies in Brazil would be penalized stupendously for non-scheduled repairs when a major overhaul of the electrical power sector happened. Therefore, the company wanted to adopt a preventive maintenance policy, as opposed to repair actions adopted after failures and periodic visual inspections were arranged for the power switch disconnectors. While some potential failures are detected and fixed at inspection time, some other failures cannot be predicted through periodic visual inspection but only by a preventive maintenance. It is evident that removing potential failures at inspection is less expensive than that the failures have happened.

To deal with the above question in theory, Colosimo *et al.* [11] treated these two kinds of failures as events governed by two non-homogeneous Poisson processes (NHPPs) for the random occurrence of failures, while the concept of two categories of failures was initially put forward in 1980s. In 1983, the definition of two kinds of failures (*i.e.*, a minor failure and a major failure) was explicated by Brown and Proschan [6], in which the minor failure is non-fatal and can be thoroughly rectified by a minimal repair, and the major failure is fatal and should be removed by a corrective replacement. Later, Sheu *et al.* [30] studied a system with age-dependent failures and random working missions, and developed three generalized age maintenance policies, where the system is also subject to two failure types (Type I failure and Type II failure). The system is replaced ahead at a planned age T or at the completion of the Nth working mission, or correctively replaced at the first occurrence of Type II failure. Wang *et al.* [27] extended the generalized age replacement policies for a single-unit system into a series system and a parallel system with n non-identical components, where two types of failures for both systems were considered as well. Other maintenance policies on systems incorporated with two types of failures can be found [2, 10, 25, 26]. The NHPP serves as an effective way to deal with different kinds of failures when system failure can be categorized into distinct types.

When considering multiple type conditions for maintenance policies, a classical assumption is "whichever occurs first" [20, 24]. Such an assumption is more reasonable for situations where failures may bring catastrophic production interruptions. It should be noted that maintenance actions of "whichever triggering event occurs first" may be more frequent when several combined policies are scheduled [17, 23]. In addition, it would be inappropriate to arrange a strict replacement on time at a planned age T or at periodic cycles kT $(k = 1, 2, \dots)$ especially when the system needs to finish some successive working missions because any interruption of working periods may incur tremendous losses of production to different degrees. Therefore, it would be not wise to replace the system until the job is completed even though the scheduled maintenance time has arrived [19, 22]. By considering the above aspects, the concept of "whichever occurs last" is developed and has been investigated considerably [29, 32].

To the best of our knowledge, more generalized periodic replacement policies for a system subject to two types of failures (namely, Type I failure and Type II failure used in previous researches) with random working periods have not been completely addressed yet, while their models are analytically investigated in this paper. It is assumed that the system needs to execute several random working periods $Y_1, Y_2, \dots, Y_n$ during its operation and is subject to two types of failures when it has failed, where Type I failure is called as a minor failure and can be removed by a minimal repair and Type II failure is a catastrophic failure which requires a corrective replacement or an overhaul. Type I failure occurs with a probability q $(0 \le q \le 1)$ and Type II failure occurs with another probability $p \equiv 1 - q$. In the first generalized periodic replacement model, the system is replaced at age T $(0 < T \le \infty)$, or at the completion of n working missions, or at the first occurrence time of a Type II failure, whichever comes first. While in the second generalized periodic replacement model, the system is replaced at age T $(0 < T \le \infty)$, or at the completion of n working times, or at the first occurrence time of Type II failure, whichever comes last. Except for the two above replacement models, their respective extended models are developed as well. The ACR function is minimized to seek the optimal replacement cycle in each model.

The remainder of this paper is organized as follows. Notations and some assumptions are offered in Section 2. Sections 3 and 4 investigate the generalized periodic replacement policies under the assumption of "whichever occurs first" and "whichever occurs last", respectively. In both models, theoretical computations are derived and numerical examples are given to verify the results. Finally, some conclusions are summarized in Section 5.

2. Notations and assumptions

2.1. Notations

For the case of exposition, the notations used in this paper are firstly presented.

T	Replacement interval
Y_i	The ith $(i = 1, 2, \ldots, n)$ random working period
$G_i(t)$	Distribution of Y_i
Y_F	Minimum of $Y_1, Y_2, \ldots, Y_n$
$G_F(t)$	Distribution of Y_F
Y_L	Maximum of $Y_1, Y_2, \ldots, Y_n$
$G_L(t)$	Distribution of Y_L
X	System failure time
$F(t)$	Distribution of X
$\lambda(t)$	System failure rate function
$\Lambda(t)$	Cumulative hazard rate function
q	Occurrence probability of Type I failure
p	Occurrence probability of Type II failure
Z	Waiting time until the first occurrence of Type II failure
$\overline{F}_p(t)$	Survival function of Z
c_T	Replacement cost at periodic cycles kT $(k = 1, 2, \ldots)$
c_Y	Replacement cost at the completion of Y_i
c_F	Replacement cost at the first Type II failure
c_M	Replacement cost for each minimal repair
$C(T)$	ACR
$C_F(T)$	ACR for Model A1
$\widetilde{C}_F(T)$	ACR for Model A2
$C_L(T)$	ACR for Model B1
$\widetilde{C}_L(T)$	ACR for Model B2

2.2. Some assumptions

Assume that a system has to operate for finite working jobs $Y_1, Y_2, \ldots, Y_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed (*i.i.d*). More specifically, Y_i $(i = 1, 2, \ldots, n)$ are assumed to be exponentially distributed with a parameter θ, *i.e.*, the distribution of Y_i is $G_i(t) = \Pr\{Y_i \leq t\} = 1 - \mathrm{e}^{-\theta t}$ $(0 < \theta < \infty)$. The system deteriorates with the operating time and has a lifetime X according to a general distribution $F(t) = \Pr\{X \leq t\}$, where system failure time X is statistically independent with $Y_1, Y_2, \ldots, Y_n$. Let $\lambda(t) \equiv f(t)/\overline{F}(t)$ be the failure rate of X, where $f(t)$ is the density function of $F(t)$, *i.e.*, $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$. $\lambda(t)$ is assumed to increase strictly with t from $\lambda(0) = 0$ to $\lambda(\infty)$. $\Lambda(t) \equiv \int_0^t \lambda(x)\mathrm{d}x$ is the cumulative hazard rate and $\Phi(t) \equiv 1 - \overline{\Phi}(t)$ holds for any function $\Phi(t)$ in the whole contents.

Two types of failures are introduced for the deterioration system when it has failed at t. Type I failure (minor failure) is occurred with a probability q $(0 \leq q \leq 1)$ and it can be removed by a minimal repair, where minimal repair means that system failure rate $\lambda(t)$ remains undisturbed by any maintenance [16, 31]. Whereas Type II failure (catastrophic failure) is formed with another probability $p \equiv 1 - q$, resulting in a total breakdown and needing a corrective replacement to rectify it [33].

The preventive replacement costs at periodic times kT $(k = 1, 2, \ldots)$ and at the completion of random working periods $Y_1, Y_2, \ldots, Y_n$ are c_T and c_Y, respectively. The corrective replacement cost at the first Type II failure is c_F, and the maintenance cost for each minimal repair is c_M. It is set that $c_F > c_Y > c_T$. In addition, the preparation time for every maintenance activity including the replacement and the minimal repair is negligible.

3. Periodic replacement first policies

According to the assumptions, system failure is subject to events following an NHPP with intensity $\lambda(t)$, increasing strictly with respect to t from $\lambda(0) = 0$ to $\lambda(\infty)$. Denote $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ as the respective counting numbers of Type I failures and Type II failures in $[0, t]$. Then, the processes $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are two independent NHPPs with intensities $q\lambda(t)$ and $p\lambda(t)$, respectively [28]. Let Z be the waiting time until the first occurrence of Type II failure in time interval $[0, t]$, *i.e.*,

$$Z = \inf\{t \geq 0 : \ N_2(t) = 1\}. \tag{3.1}$$

The survival function of Z is

$$\overline{F}_p(t) = \Pr\{Z > t\} = \Pr\{N_2(t) = 0\} = \exp\left[-p\int_0^t \lambda(x)\mathrm{d}x\right]. \tag{3.2}$$

The mean number of Type I failures in time interval $[0, t]$ is

$$E[N_1(t)] = q\Lambda(t) = q\int_0^t \lambda(x)\mathrm{d}x. \tag{3.3}$$

Let U_i be the length of the ith $(i = 1, 2, \ldots)$ replacement cycle and V_i be the cost over the replacement cycle U_i. Then, $\{U_i, V_i\}$ constitutes a renewal reward process. Defining $D(t)$ as the expected cost of the operating system over the time interval $[0, t]$, according to the renewal reward theorem [5, 15], we have

$$C(T) = \lim_{t\to\infty}\frac{D(t)}{t} = \frac{E[V_1]}{E[U_1]}. \tag{3.4}$$

3.1. Periodic replacement first policy (Model A1)

For Model A1, we consider the following replacement situations in a renewal cycle and derive the corresponding probabilities.

(1) The probability that the system is preventively replaced at periodic times kT $(k = 1, 2, \ldots)$ is

$$\Pr\{T < Y_F, T < Z\} = \overline{F}_p(T)\overline{G}_F(T), \tag{3.5}$$

in which $Y_F = \min\{Y_1, Y_2, \ldots, Y_n\}$ and

$$G_F(t) = \Pr\{Y_F \leq t\} = 1 - \prod_{i=1}^n \Pr\{Y_i > t\} = 1 - \prod_{i=1}^n \overline{G}_i(t). \tag{3.6}$$

Thus, (3.5) becomes

$$\Pr\{T < Y_F, T < Z\} = \overline{F}_p(T)\prod_{i=1}^n \overline{G}_i(T). \tag{3.7}$$

(2) The probability that the system is preventively replaced at the completion of random working jobs is

$$\Pr\{Y_F < T, Y_F < Z\} = \int_0^T \overline{F}_p(t)\mathrm{d}\left(1 - \prod_{i=1}^n \overline{G}_i(t)\right). \tag{3.8}$$

(3) The probability that the system is correctively replaced at the first occurrence of Type II failure is

$$\Pr\{Z \leq T, Z \leq Y_F\} = \int_0^T \prod_{i=1}^n \overline{G}_i(t)\mathrm{d}F_p(t), \tag{3.9}$$

where should note that $\Pr\{T < Z, T < Y_F\} + \Pr\{Y_F < T, Y_F < Z\} + \Pr\{Z \leq T, Z \leq Y_F\} \equiv 1$.

It is clear that each replacement time for the deterioration system is a regeneration point, and therefore, the expected length of the first renewal cycle U_1 is

$$\begin{aligned} E[U_1] &= T\overline{F}_p(T)\prod_{i=1}^{n}\overline{G}_i(T) + \int_0^T t\overline{F}_p(t)\mathrm{d}\left(1-\prod_{i=1}^{n}\overline{G}_i(t)\right) + \int_0^T t\prod_{i=1}^{n}\overline{G}_i(t)\mathrm{d}F_p(t) \\ &= \int_0^T \overline{F}_p(t)\prod_{i=1}^{n}\overline{G}_i(t)\mathrm{d}t. \end{aligned} \tag{3.10}$$

The total mean number of Type I failures before replacement is

$$\begin{aligned} &qT\overline{F}_p(T)\prod_{i=1}^{n}\overline{G}_i(T)\int_0^T \lambda(t)\mathrm{d}t + q\int_0^T\int_0^t \lambda(x)\overline{F}_p(t)\mathrm{d}x\mathrm{d}\left(1-\prod_{i=1}^{n}\overline{G}_i(t)\right) \\ &\quad + q\int_0^T\int_0^t \lambda(x)\prod_{i=1}^{n}\overline{G}_i(t)\mathrm{d}x\mathrm{d}F_p(t) \\ &= q\int_0^T\prod_{i=1}^{n}\overline{G}_i(t)\lambda(t)\overline{F}_p(t)\mathrm{d}t. \end{aligned} \tag{3.11}$$

The expected maintenance cost of the first renewal cycle is

$$\begin{aligned} E[V_1] &= c_T\overline{F}_p(T)\prod_{i=1}^{n}\overline{G}_i(T) + c_Y\int_0^T \overline{F}_p(t)\mathrm{d}\left(1-\prod_{i=1}^{n}\overline{G}_i(t)\right) \\ &\quad + c_F\int_0^T\prod_{i=1}^{n}\overline{G}_i(t)\mathrm{d}F_p(t) + c_M q\int_0^T\prod_{i=1}^{n}\overline{G}_i(t)\lambda(t)\overline{F}_p(t)\mathrm{d}t \\ &= c_T + (c_Y - c_T)\int_0^T \overline{F}_p(t)\mathrm{d}\left(1-\prod_{i=1}^{n}\overline{G}_i(t)\right) \\ &\quad + (c_F - c_T)\int_0^T\prod_{i=1}^{n}\overline{G}_i(t)\mathrm{d}F_p(t) + c_M q\int_0^T\prod_{i=1}^{n}\overline{G}_i(t)\lambda(t)\overline{F}_p(t)\mathrm{d}t. \end{aligned} \tag{3.12}$$

According to (3.4), the ACR for Model A1 is

$$C_F(T) = \frac{\begin{array}{l} c_T + (c_Y - c_T)\int_0^T \bar{F}_p(t)\mathrm{d}\left(1-\prod_{i=1}^{n}\bar{G}_i(t)\right) \\ +(c_F - c_T)\int_0^T\left(\prod_{i=1}^{n}\bar{G}_i(t)\right)\mathrm{d}F_p(t) + c_M q\int_0^T\prod_{i=1}^{n}\bar{G}_i(t)\lambda(t)\bar{F}_p(t)\mathrm{d}t \end{array}}{\int_0^T \bar{F}_p(t)\prod_{i=1}^{n}\bar{G}_i(t)\mathrm{d}t}. \tag{3.13}$$

In order to find the optimal T_F^* minimizing $C_F(T)$ in (3.13) for an infinite time horizon, we differentiate $C_F(T)$ with respect to T and set it equal to zero. From $\mathrm{d}C_F(T)/\mathrm{d}T = 0$, T_F^* satisfies

$$Q_F(T) = c_T, \tag{3.14}$$

where

$$Q_F(T) = \varphi_F(T)\int_0^T \overline{F}_p(t)\prod_{i=1}^{n}\overline{G}_i(t)\mathrm{d}t - (c_Y - c_T)\int_0^T \overline{F}_p(t)\mathrm{d}\left(1-\prod_{i=1}^{n}\overline{G}_i(t)\right)$$

$$
\begin{aligned}
&- (c_F - c_T) \int_0^T \prod_{i=1}^n \overline{G}_i(t) \mathrm{d}F_p(t) - c_M q \int_0^T \prod_{i=1}^n \overline{G}_i(t)\lambda(t)\overline{F}_p(t)\mathrm{d}t \\
&= \int_0^T \overline{F}_p(t) \mathrm{e}^{-n\theta t} \left[\varphi_F(T) - \varphi_F(t)\right]\mathrm{d}t,
\end{aligned}
$$

with

$$
\begin{aligned}
\varphi_F(t) &= (c_Y - c_T) \frac{\mathrm{d}\left(1 - \prod_{i=1}^n \overline{G}_i(t)\right)/\mathrm{d}t}{\prod_{i=1}^n \overline{G}_i(t)} + (c_F - c_T)\frac{\mathrm{d}F_p(t)/\mathrm{d}t}{\overline{F}_p(t)} + c_M q \lambda(t) \\
&= (c_Y - c_T)n\theta + \left[(c_F - c_T)p + c_M q\right]\lambda(t).
\end{aligned}
$$

Then, the optimal T_F^* is obtained according to the following theorem.

Theorem 3.1. *If $Q_F(\infty) > c_T$, there exists an optimal T_F^* $(0 < T_F^* < \infty)$ which satisfies (3.14), and the optimal replacement cost rate is $C_F(T_F^*) = \varphi_F(T_F^*)$. Otherwise, $T_F^* = \infty$.*

Proof. Differentiating $Q_F(T)$ with respect to T, we have

$$
\begin{aligned}
\frac{\mathrm{d}Q_F(T)}{\mathrm{d}T} &= \frac{\mathrm{d}\varphi_F(T)}{\mathrm{d}T} \int_0^T \overline{F}_p(t) \prod_{i=1}^n \overline{G}_i(t)\mathrm{d}t + \varphi_F(T)\overline{F}_p(T)\prod_{i=1}^n \overline{G}_i(T) \\
&\quad - (c_Y - c_T)\overline{F}_p(T) \frac{\mathrm{d}\left(1 - \prod_{i=1}^n \overline{G}_i(T)\right)}{\mathrm{d}T} - (c_F - c_T)\prod_{i=1}^n \overline{G}_i(T)\frac{\mathrm{d}F_p(T)}{\mathrm{d}T} \\
&\quad - c_M q \prod_{i=1}^n \overline{G}_i(T)\lambda(T)\overline{F}_p(T) \\
&= \frac{\mathrm{d}\varphi_F(T)}{\mathrm{d}T} \int_0^T \overline{F}_p(t) \prod_{i=1}^n \overline{G}_i(t)\mathrm{d}t \\
&= \left[(c_F - c_T)p + c_M q\right] \frac{\mathrm{d}\lambda(T)}{\mathrm{d}T} \int_0^T \overline{F}_p(t)\mathrm{e}^{-n\theta t}\mathrm{d}t.
\end{aligned}
$$

We judge that $\mathrm{d}Q_F(T)/\mathrm{d}T > 0$ given the condition that $\lambda(t)$ increases strictly with t from $\lambda(0) = 0$ to $\lambda(\infty)$, illustrating that $Q_F(T)$ also increases strictly from $Q_F(0) = 0$ to $Q_F(\infty) = \lim_{T\to\infty} Q_F(T)$ with respect to T, and

$$
Q_F(\infty) = \lim_{T\to\infty} Q_F(T) = \int_0^\infty \overline{F}_p(t)\mathrm{e}^{-n\theta t}\left[\varphi_F(\infty) - \varphi_F(t)\right]\mathrm{d}t.
$$

Thus, a finite and unique T_F^* $(0 < T_F^* < \infty)$ exists when $Q_F(\infty) > c_T$, otherwise $T_F^* = \infty$ when $Q_F(\infty) \le c_T$, which completes the proof process of Theorem 3.1. □

Remark 3.2. (1) When $q = 1$, *i.e.*, the system undergoes only minimal repair at failure, $C_F(T)$ in (3.13) becomes

$$
C_F(T) = \frac{c_T \prod_{i=1}^n \overline{G}_i(T) + c_Y\left(1 - \prod_{i=1}^n \overline{G}_i(T)\right) + c_M \int_0^T \prod_{i=1}^n \overline{G}_i(t)\lambda(t)\mathrm{d}t}{\int_0^T \prod_{i=1}^n \overline{G}_i(t)\mathrm{d}t}, \tag{3.15}
$$

which consists with the result in Wang *et al.* [27].

(2) When $T \to \infty$, *i.e.*, the system is replaced at the completion of random working jobs, or at the first occurrence of Type II failure, whichever occurs first. $C_F(T)$ in (3.13) becomes

$$C_F(T) = \frac{c_Y \int_0^\infty \overline{F}_p(t)\mathrm{d}\left(1 - \prod_{i=1}^n \overline{G}_i(t)\right) + c_F \int_0^\infty \prod_{i=1}^n \overline{G}_i(t)\mathrm{d}F_p(t) + c_M q \int_0^\infty \prod_{i=1}^n \overline{G}_i(t)\lambda(t)\overline{F}_p(t)\mathrm{d}t}{\int_0^\infty \overline{F}_p(t) \prod_{i=1}^n \overline{G}_i(t)\mathrm{d}t}. \tag{3.16}$$

(3) When $q = 1$, $Y_F \to \infty$, *i.e.*, the system undergoes only minimal repair at failure and no random working times are considered, then $C_F(T)$ in (3.13) becomes

$$C_F(T) = \frac{c_T + c_M \int_0^T \lambda(t)\mathrm{d}t}{T}, \tag{3.17}$$

which is the classical periodic replacement policy.

3.2. Modified periodic replacement first policy (Model A2)

In this section we develop a modified periodic replacement first policy (Model A2) based on Section 3.1. Suppose that the system is preventively replaced at the periodic time points kT ($k = 1, 2, \ldots$), or at when at least one of the n random working times is longer than T, or correctively replaced at the first time of Type II failure, whichever occurs first. Replacing $G_F(t) = 1 - \prod_{i=1}^n \overline{G}_i(t)$ with $G_L(t) = \prod_{i=1}^n G_i(t)$ in (3.13), we have the following ACR for Model A2 as

$$\widetilde{C}_F(T) = \frac{c_T + (c_Y - c_T)\int_0^T \overline{F}_p(t)\mathrm{d}\left(\prod_{i=1}^n G_i(t)\right) + (c_F - c_T)\int_0^T \left(1 - \prod_{i=1}^n G_i(t)\right)\mathrm{d}F_p(t) + c_M q \int_0^T \left(1 - \prod_{i=1}^n G_i(t)\right)\lambda(t)\overline{F}_p(t)\mathrm{d}t}{\int_0^T \overline{F}_p(t)\left(1 - \prod_{i=1}^n G_i(t)\right)\mathrm{d}t}. \tag{3.18}$$

In order to find the optimal $\widetilde{T}_F^*$ which minimizes $\widetilde{C}_F(T)$ in (3.18) for an infinite time horizon, we differentiate $\widetilde{C}_F(T)$ with respect to T and set it equal to zero. From $\mathrm{d}\widetilde{C}_F(T)/\mathrm{d}T = 0$, we have

$$\widetilde{Q}_F(T) = c_T, \tag{3.19}$$

where

$$\begin{aligned}
\widetilde{Q}_F(T) = {} & \widetilde{\varphi}_F(T)\int_0^T \overline{F}_p(t)\left(1 - \prod_{i=1}^n G_i(t)\right)\mathrm{d}t - (c_Y - c_T)\int_0^T \overline{F}_p(t)\mathrm{d}\left(\prod_{i=1}^n G_i(t)\right) \\
& - (c_F - c_T)\int_0^T \left(1 - \prod_{i=1}^n G_i(t)\right)\mathrm{d}F_p(t) \\
& - c_M q \int_0^T \left(1 - \prod_{i=1}^n G_i(t)\right)\lambda(t)\overline{F}_p(t)\mathrm{d}t \\
= {} & \int_0^T \overline{F}_p(t)\left[1 - \left(1 - \mathrm{e}^{-\theta t}\right)^n\right][\widetilde{\varphi}_F(T) - \widetilde{\varphi}_F(t)]\mathrm{d}t,
\end{aligned}$$

with

$$\widetilde{\varphi}_F(t) = (c_Y - c_T)\frac{\mathrm{d}\left(\prod_{i=1}^{n} G_i(t)\right)/\mathrm{d}t}{1 - \prod_{i=1}^{n} G_i(t)} + (c_F - c_T)\frac{\mathrm{d}F_p(t)/\mathrm{d}t}{\overline{F}_p(t)} + c_M q\lambda(t)$$
$$= (c_Y - c_T)\frac{n\theta \mathrm{e}^{-\theta t}(1-\mathrm{e}^{-\theta t})^{n-1}}{1-(1-\mathrm{e}^{-\theta t})^n} + \left[(c_F - c_T)p + c_M q\right]\lambda(t).$$

Then, the optimal $\widetilde{T}_F^*$ is obtained according to the following theorem.

Theorem 3.3. *If $\widetilde{Q}_F(\infty) > c_T$, there exists an optimal $\widetilde{T}_F^*$ $(0 < \widetilde{T}_F^* < \infty)$ which satisfies (3.19), and the optimal replacement cost rate is $\widetilde{C}_F(\widetilde{T}_F^*) = \widetilde{\varphi}_F(\widetilde{T}_F^*)$. Otherwise, $\widetilde{T}_F^* = \infty$.*

Proof. Differentiating $\widetilde{Q}_F(T)$ with respect to T, we have

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{Q}_F(T)}{\mathrm{d}T} &= \frac{\mathrm{d}\widetilde{\varphi}_F(T)}{\mathrm{d}T}\int_0^T \overline{F}_p(t)\left(1 - \prod_{i=1}^{n} G_i(t)\right)\mathrm{d}t + \widetilde{\varphi}_F(T)\overline{F}_p(T)\left(1 - \prod_{i=1}^{n} G_i(T)\right) \\
&\quad - (c_Y - c_T)\overline{F}_p(T)\frac{\mathrm{d}\left(\prod_{i=1}^{n} G_i(T)\right)}{\mathrm{d}T} \\
&\quad - (c_F - c_T)\left(1 - \prod_{i=1}^{n} G_i(T)\right)\frac{\mathrm{d}F_p(T)}{\mathrm{d}T} - c_M q\lambda(T)\left(1 - \prod_{i=1}^{n} G_i(T)\right)\overline{F}_p(T) \\
&= \frac{\mathrm{d}\widetilde{\varphi}_F(T)}{\mathrm{d}T}\int_0^T \overline{F}_p(t)\left(1 - \prod_{i=1}^{n} G_i(t)\right)\mathrm{d}t \\
&= \Bigg\{(c_Y - c_T)\frac{n(n-1)\theta^2\mathrm{e}^{-\theta T}(1-\mathrm{e}^{-\theta T})^{n-2}}{1-(1-\mathrm{e}^{-\theta T})^n} + (c_Y - c_T)\left[\frac{n\theta\mathrm{e}^{-\theta T}(1-\mathrm{e}^{-\theta T})^{n-1}}{1-(1-\mathrm{e}^{-\theta T})^n}\right]^2 \\
&\quad + \left[(c_F - c_T)p + c_M q\right]\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\Bigg\}\int_0^T \overline{F}_p(t)\left[1-(1-\mathrm{e}^{-\theta t})^n\right]\mathrm{d}t.
\end{aligned}$$

If $\lambda(t)$ increases strictly with t from $\lambda(0) = 0$ to $\lambda(\infty)$, it is clear that $\mathrm{d}\widetilde{Q}_F(T)/\mathrm{d}T > 0$, then $\widetilde{Q}_F(T)$ increases strictly from $\widetilde{Q}_F(0) = 0$ to $\widetilde{Q}_F(\infty) = \lim_{T\to\infty}\widetilde{Q}_F(T)$ with respect to T, and

$$\widetilde{Q}_F(\infty) = \lim_{T\to\infty}\widetilde{Q}_F(T) = \int_0^\infty \overline{F}_p(t)\left[1-(1-\mathrm{e}^{-\theta t})^n\right]\left[\widetilde{\varphi}_F(\infty) - \widetilde{\varphi}_F(t)\right]\mathrm{d}t.$$

Thus, a finite and unique $\widetilde{T}_F^*$ $(0 < \widetilde{T}_F^* < \infty)$ exists when $\widetilde{Q}_F(\infty) > c_T$, otherwise $\widetilde{T}_F^* = \infty$ when $\widetilde{Q}_F(\infty) \le c_T$, which completes the proof process of Theorem 3.3. □

3.3. Numerical examples

In this section, numerical examples are given to verify the theoretical results obtained. Assume that system failure time follows a Weibull distribution, *i.e.*, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$. The ith $(i = 1, 2, \ldots, n)$ working time is exponentially distributed with $G_i(t) = 1 - \mathrm{e}^{-0.1t}$. For convenient computation, the following costs are introduced: $c_T = 500$, $c_Y = 750$, $c_F = 1000$, and $c_M = 100$. Tables 1 and 2 show the optimal replacement cycles T_F^* and $\widetilde{T}_F^*$, and the corresponding minimized maintenance cost rates $C_F(T_F^*)$ and $\widetilde{C}_F(\widetilde{T}_F^*)$ for Model A1 and Model A2, respectively.

TABLE 1. Optimal T_F^* and $C_F(T_F^*)$ for different q and n ($c_T = 500$, $c_Y = 750$, $c_F = 1000$, $c_M = 100$, $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$).

q	$n = 1$		$n = 2$		$n = 3$	
	T_F^*	$C_F(T_F^*)$	T_F^*	$C_F(T_F^*)$	T_F^*	$C_F(T_F^*)$
1.0	34.69	94.38	54.99	160.01	78.33	231.67
0.9	28.31	104.27	41.95	167.45	57.91	237.16
0.8	24.32	112.54	34.47	174.09	46.47	242.29
0.7	21.55	119.82	29.58	180.15	39.12	247.13
0.6	19.50	126.41	26.11	185.77	33.99	251.74
0.5	17.91	132.47	23.51	191.05	30.19	256.14
0.4	16.63	138.12	21.48	196.04	27.26	260.37
0.3	15.58	143.43	19.84	200.79	24.93	264.44
0.2	14.70	148.45	18.49	205.33	23.02	268.37
0.1	13.94	153.24	17.36	209.68	21.43	272.18
0	13.28	157.82	16.39	213.88	20.09	275.88

TABLE 2. Optimal $\widetilde{T}_F^*$ and $\widetilde{C}_F(\widetilde{T}_F^*)$ for different q and n ($c_T = 500$, $c_Y = 750$, $c_F = 1000$, $c_M = 100$, $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$).

q	$n = 1$		$n = 2$		$n = 3$	
	$\widetilde{T}_F^*$	$\widetilde{C}_F(\widetilde{T}_F^*)$	$\widetilde{T}_F^*$	$\widetilde{C}_F(\widetilde{T}_F^*)$	$\widetilde{T}_F^*$	$\widetilde{C}_F(\widetilde{T}_F^*)$
1.0	34.69	94.38	23.50	70.74	20.29	62.14
0.9	28.31	104.27	19.89	78.87	17.66	69.92
0.8	24.32	112.54	17.58	85.93	15.90	76.81
0.7	21.55	119.82	15.93	92.29	14.61	83.06
0.6	19.50	126.41	14.69	98.12	13.61	88.84
0.5	17.91	132.47	13.70	103.55	12.80	94.25
0.4	16.63	138.12	12.89	108.66	12.13	99.36
0.3	15.58	143.43	12.21	113.49	11.56	104.20
0.2	14.70	148.45	11.63	118.10	11.07	108.83
0.1	13.94	153.24	11.13	122.51	10.63	113.26
0	13.28	158.82	10.69	126.74	10.25	117.53

Tables 1 and 2 illustrate that T_F^* and $\widetilde{T}_F^*$ increase with q for a fixed n, while $C_F(T_F^*)$ and $\widetilde{C}_F(\widetilde{T}_F^*)$ decrease with q. When q is given, both T_F^* and $C_F(T_F^*)$ increase with n, whereas on the other aspect, both $\widetilde{T}_F^*$ and $\widetilde{C}_F(\widetilde{T}_F^*)$ decrease with n. Figure 1 shows the average cost rate $C_F(T)$ for different n in terms of $q = 1$ for Model A1 and Figure 2 shows the average cost rate $\widetilde{C}_F(T)$ for different n in terms of $q = 1$ for Model A2, where $q = 1$ illustrates that the failure rate of the system is undisturbed by any shocks. From Figures 1 and 2, it is clear that the finite and unique replacement intervals T_F^* and $\widetilde{T}_F^*$ exist when $q = 1$, *i.e.*, $0 < T_F^* < \infty$ and $0 < \widetilde{T}_F^* < \infty$.

4. Periodic replacement last policies

Implementing replacement first policies may lead to too frequent unnecessary replacement, as well as interrupting random working jobs. In this case, we develop generalized periodic replacement last models. The system is preventively replaced at periodic cycles kT ($k = 1, 2, \ldots$) before Type II failure, or at the completion of random working times, whichever occurs last. Corrective replacement is arranged immediately at the first occurrence of Type II failure.

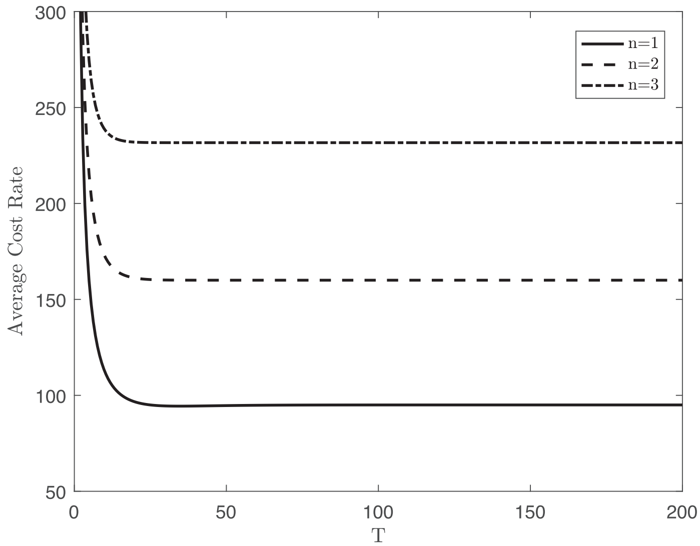

FIGURE 1. $C_F(T)$ for different n in terms of $q = 1$ ($c_T = 500$, $c_Y = 750$, $c_F = 1000$, $c_M = 100$, $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$).

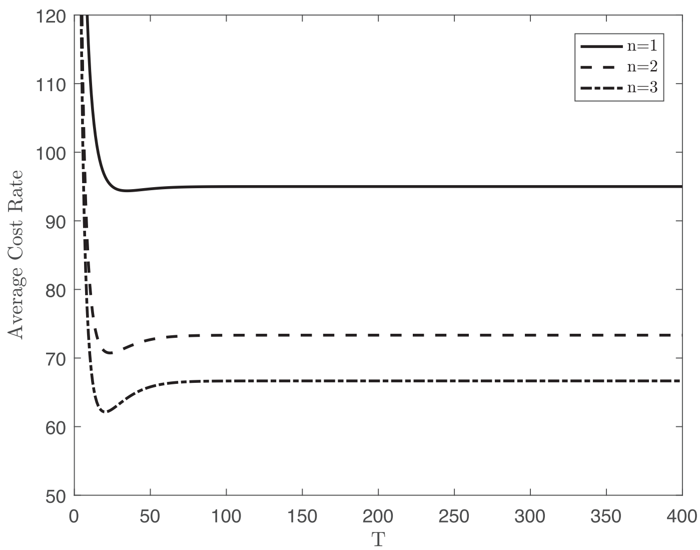

FIGURE 2. $\widetilde{C}_F(T)$ for different n in terms of $q = 1$ ($c_T = 500$, $c_Y = 750$, $c_F = 1000$, $c_M = 100$, $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$).

4.1. Periodic replacement last policy (Model B1)

For Model B1, the following three distinct situations are considered and their corresponding probabilities are derived.

(1) The probability that the system is preventively replaced at periodic times kT $(k = 1, 2, \ldots)$ is

$$\Pr\{T < Z, Y_L \le T\} = \overline{F}_p(T) \prod_{i=1}^{n} G_i(T). \tag{4.1}$$

(2) The probability that the system is preventively replaced at the completion of n random working jobs is

$$\Pr\{Y_L > T, Y_L \le Z\} = \int_T^{\infty} \overline{F}_p(t) \mathrm{d}\left(\prod_{i=1}^{n} G_i(t)\right). \tag{4.2}$$

(3) The probability that the system is correctively replaced at the first occurrence of Type II failure is

$$\Pr\{Z \le T\} + \Pr\{T < Z, Z < Y_L\} = F_p(T) + \int_T^{\infty} \left(1 - \prod_{i=1}^{n} G_i(t)\right) \mathrm{d}F_p(t), \tag{4.3}$$

where should note that $\Pr\{T < Z, Y_L \le T\} + \Pr\{Y_L > T, Y_L \le Z\} + \Pr\{Z \le T\} + \Pr\{T < Z, Z < Y_L\} \equiv 1$.

The expected length of the first renewal cycle is

$$\begin{aligned} E[U_1] &= T\overline{F}_p(T) \prod_{i=1}^{n} G_i(T) + \int_T^{\infty} t\overline{F}_p(t) \mathrm{d}\left(\prod_{i=1}^{n} G_i(t)\right) + \int_0^T t \mathrm{d}F_p(t) \\ &\quad + \int_T^{\infty} t \left(1 - \prod_{i=1}^{n} G_i(t)\right) \mathrm{d}F_p(t) \\ &= \int_0^T \overline{F}_p(t) \mathrm{d}t + \int_T^{\infty} \overline{F}_p(t) \left(1 - \prod_{i=1}^{n} G_i(t)\right) \mathrm{d}t. \end{aligned} \tag{4.4}$$

The total mean number of Type I failures before replacement is

$$\begin{aligned} &q\overline{F}_p(T) \prod_{i=1}^{n} G_i(T) \int_0^T \lambda(t) \mathrm{d}t + q \int_T^{\infty} \int_0^t \lambda(x) \overline{F}_p(t) \mathrm{d}x \mathrm{d}\left(\prod_{i=1}^{n} G_i(t)\right) \\ &\quad + q \int_0^T \int_0^t \lambda(x) \mathrm{d}x \mathrm{d}F_p(t) + q \int_T^{\infty} \int_0^t \lambda(x) \left(1 - \prod_{i=1}^{n} G_i(t)\right) \mathrm{d}x \mathrm{d}F_p(t) \\ &= q\left[\int_0^T \overline{F}_p(t) \lambda(t) \mathrm{d}t + \int_T^{\infty} \overline{F}_p(t) \left(1 - \prod_{i=1}^{n} G_i(t)\right) \lambda(t) \mathrm{d}t\right]. \end{aligned} \tag{4.5}$$

The expected maintenance cost in a renewal cycle is

$$\begin{aligned} E[V_1] &= c_T \overline{F}_p(T) \prod_{i=1}^{n} G_i(T) + c_Y \int_T^{\infty} \overline{F}_p(t) \mathrm{d}\left(\prod_{i=1}^{n} G_i(t)\right) + c_F \left[\int_T^{\infty} \left(1 - \prod_{i=1}^{n} G_i(t)\right) \mathrm{d}F_p(t)\right. \\ &\quad \left. + F_p(T)\right] + c_M \left[q \int_0^T \overline{F}_p(t) \lambda(t) \mathrm{d}t + q \int_T^{\infty} \overline{F}_p(t) \left(1 - \prod_{i=1}^{n} G_i(t)\right) \lambda(t) \mathrm{d}t\right] \\ &= c_T + (c_Y - c_T) \int_T^{\infty} \overline{F}_p(t) \mathrm{d}\left(\prod_{i=1}^{n} G_i(t)\right) + (c_F - c_T) \left[\int_T^{\infty} \left(1 - \prod_{i=1}^{n} G_i(t)\right) \mathrm{d}F_p(t)\right. \\ &\quad \left. + F_p(T)\right] + c_M q \left[\int_0^T \overline{F}_p(t) \lambda(t) \mathrm{d}t + \int_T^{\infty} \overline{F}_p(t) \left(1 - \prod_{i=1}^{n} G_i(t)\right) \lambda(t) \mathrm{d}t\right]. \end{aligned} \tag{4.6}$$

According to (3.4), the ACR for Model B1 is

$$C_L(T) = \frac{c_T + (c_Y - c_T)\int_T^{\infty} \overline{F}_p(t)\mathrm{d}\left(\prod_{i=1}^n G_i(t)\right) + (c_F - c_T)\left[\int_T^{\infty}\left(1 - \prod_{i=1}^n G_i(t)\right)\mathrm{d}F_p(t) + F_p(T)\right] + c_M q\left[\int_0^T \overline{F}_p(t)\lambda(t)\mathrm{d}t + \int_T^{\infty}\overline{F}_p(t)\left(1-\prod_{i=1}^n G_i(t)\right)\lambda(t)\mathrm{d}t\right]}{\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^{\infty}\overline{F}_p(t)\left(1-\prod_{i=1}^n G_i(t)\right)\mathrm{d}t}. \tag{4.7}$$

In order to find the optimal T_L^* which minimizes $C_L(T)$ in (4.7) for an infinite time horizon, we differentiate $C_L(T)$ with respect to T and set it equal to zero. From $\mathrm{d}C_L(T)/\mathrm{d}T = 0$, we have

$$Q_L(T) = c_T, \tag{4.8}$$

where

$$\begin{aligned}
Q_L(T) = {} & \varphi_L(T)\left[\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^{\infty}\overline{F}_p(t)\left(1-\prod_{i=1}^n G_i(t)\right)\mathrm{d}t\right] - (c_Y - c_T)\int_T^{\infty}\overline{F}_p(t)\mathrm{d}\left(\prod_{i=1}^n G_i(t)\right) \\
& - (c_F - c_T)\left[\int_T^{\infty}\left(1-\prod_{i=1}^n G_i(t)\right)\mathrm{d}F_p(t) + F_p(T)\right] \\
& - c_M\left[q\int_0^T \overline{F}_p(t)\lambda(t)\mathrm{d}t + q\int_T^{\infty}\overline{F}_p(t)\left(1-\prod_{i=1}^n G_i(t)\right)\lambda(t)\mathrm{d}t\right] \\
= {} & \int_0^T \overline{F}_p(t)\left[\varphi_L(T) - \varphi_L(t)\right]\mathrm{d}t + \int_T^{\infty}\overline{F}_p(t)\left[1-\left(1-\mathrm{e}^{-\theta t}\right)^n\right]\left[\varphi_L(T)-\varphi_L(t)\right]\mathrm{d}t,
\end{aligned}$$

with

$$\begin{aligned}
\varphi_L(t) &= -(c_Y - c_T)\frac{\mathrm{d}\left(\prod_{i=1}^n G_i(t)\right)/\mathrm{d}t}{\prod_{i=1}^n G_i(t)} + (c_F - c_T)\frac{\mathrm{d}F_p(t)/\mathrm{d}t}{\overline{F}_p(t)} + c_M q\lambda(t) \\
&= -(c_Y - c_T)\frac{n\theta\mathrm{e}^{-\theta t}}{1-\mathrm{e}^{-\theta t}} + \left[(c_F - c_T)p + c_M q\right]\lambda(t).
\end{aligned}$$

Then, the optimal T_L^* is obtained according to the following theorem.

Theorem 4.1. *If $Q_L(\infty) > c_T$, there exists an optimal T_L^* $(0 < T_L^* < \infty)$ which satisfies (4.8), and the optimal replacement cost rate is $C_L(T_L^*) = \varphi_L(T_L^*)$. Otherwise, $T_L^* = \infty$.*

Proof. Differentiating $Q_L(T)$ with respect to T, we have

$$\begin{aligned}
\frac{\mathrm{d}Q_L(T)}{\mathrm{d}T} = {} & \frac{\mathrm{d}\varphi_L(T)}{\mathrm{d}T}\left[\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^{\infty}\overline{F}_p(t)\left(1-\prod_{i=1}^n G_i(t)\right)\mathrm{d}t\right] \\
& - \varphi_L(T)\overline{F}_p(T)\prod_{i=1}^n G_i(T) + (c_Y - c_T)\overline{F}_p(T)\frac{\mathrm{d}\left(\prod_{i=1}^n G_i(T)\right)}{\mathrm{d}T} \\
& - (c_F - c_T)\prod_{i=1}^n G_i(T)\frac{\mathrm{d}F_p(T)}{\mathrm{d}T} - c_M q\lambda(T)\overline{F}_p(T)\prod_{i=1}^n \overline{G}_i(T)
\end{aligned}$$

$$= \frac{\mathrm{d}\varphi_L(T)}{\mathrm{d}T}\left[\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^\infty \overline{F}_p(t)\left(1-\prod_{i=1}^n G_i(t)\right)\mathrm{d}t\right]$$

$$= \left\{(c_Y - c_T)\frac{n\theta^2 \mathrm{e}^{-\theta T}}{(1-\mathrm{e}^{-\theta T})^2} + [(c_F - c_T)p + c_M q]\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\right\}\left\{\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^\infty \overline{F}_p(t)\left[1-(1-\mathrm{e}^{-\theta t})^n\right]\mathrm{d}t\right\}.$$

If $\lambda(t)$ increases strictly with t from $\lambda(0)=0$ to $\lambda(\infty)$, it is clear that $\mathrm{d}Q_L(T)/\mathrm{d}T > 0$, then $Q_L(T)$ increases strictly from $Q_L(0)$ to $Q_L(\infty) = \lim_{T\to\infty} Q_L(T)$ with respect to T, and

$$Q_L(\infty) = \int_0^\infty \overline{F}_p(t)\left[\varphi_L(\infty) - \varphi_L(t)\right]\mathrm{d}t.$$

Thus, a finite and unique T_L^* $(0 < T_L^* < \infty)$ exists when $Q_L(\infty) > c_T$, otherwise $T_L^* = \infty$ when $Q_L(\infty) \le c_T$, which completes the proof process of Theorem 4.1. □

Remark 4.2. (1) When $q = 1$, *i.e.*, the system undergoes only minimal repair at failure, $C_L(T)$ in (4.7) becomes

$$C_L(T) = \frac{c_T \prod_{i=1}^n G_i(T) + c_Y\left(1-\prod_{i=1}^n G_i(T)\right) + c_M\left[q\int_0^T \lambda(t)\mathrm{d}t + q\int_T^\infty \left(1-\prod_{i=1}^n G_i(t)\right)\lambda(t)\mathrm{d}t\right]}{T + \int_T^\infty \left(1-\prod_{i=1}^n G_i(t)\right)\mathrm{d}t}\cdot \tag{4.9}$$

(2) When $T \to \infty$, *i.e.*, the system is replaced at the completion of random working times, or at the first occurrence of Type II failure, whichever occurs last. $C_L(T)$ in (4.7) becomes

$$C_L(T) = \frac{c_F + c_M q\int_0^\infty \overline{F}_p(t)\lambda(t)\mathrm{d}t}{\int_0^\infty \overline{F}_p(t)\mathrm{d}t}\cdot \tag{4.10}$$

4.2. Modified periodic replacement last policy (Model B2)

In this section we develop a modified periodic replacement last policy (Model B2) based on Section 4.1. Suppose that the system is preventively replaced at the first completion of Y_1 among n random working jobs after the periodic cycle T, or at periodic cycles when at least one of the n random working jobs is less than T, or correctively replaced at the occurrence of Type II failure, whichever occurs last. Replacing $G_L(t) = \prod_{i=1}^n G_i(t)$ with $G_F(t) = 1 - \prod_{i=1}^n \overline{G}_i(t)$ in (4.7), we have the ACR for Model B2 as

$$\widetilde{C}_L(T) = \frac{c_T + (c_Y - c_T)\int_T^\infty \overline{F}_p(t)\mathrm{d}\left(1-\prod_{i=1}^n \overline{G}_i(t)\right) + (c_F - c_T)\left[\int_T^\infty \left(\prod_{i=1}^n \overline{G}_i(t)\right)\mathrm{d}F_p(t) + F_p(T)\right] + c_M\left[q\int_0^T \overline{F}_p(t)\lambda(t)\mathrm{d}t + q\int_T^\infty \overline{F}_p(t)\prod_{i=1}^n \overline{G}_i(t)\lambda(t)\mathrm{d}t\right]}{\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^\infty \overline{F}_p(t)\prod_{i=1}^n \overline{G}_i(t)\mathrm{d}t}\cdot \tag{4.11}$$

We differentiate $\widetilde{C}_L(T)$ with respect to T and set it equal to zero, finding the optimal $\widetilde{T}_L^*$ which minimizes $\widetilde{C}_L(T)$ in (4.11) and having

$$\widetilde{Q}_L(T) = c_T, \tag{4.12}$$

where

$$\widetilde{Q}_L(T) = \widetilde{\varphi}_L(T)\left[\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^\infty \overline{F}_p(t)\prod_{i=1}^n \overline{G}_i(t)\mathrm{d}t\right] - (c_Y - c_T)\int_T^\infty \overline{F}_p(t)\mathrm{d}\left(1-\prod_{i=1}^n \overline{G}_i(t)\right)$$
$$- (c_F - c_T)\left[\int_T^\infty \left(\prod_{i=1}^n \overline{G}_i(t)\right)\mathrm{d}F_p(t + F_p(T)\right]$$
$$- c_M\left[q\int_0^T \overline{F}_p(t)\lambda(t)\mathrm{d}t + q\int_T^\infty \overline{F}_p(t)\prod_{i=1}^n \overline{G}_i(t)\lambda(t)\mathrm{d}t\right]$$
$$= \int_0^T \overline{F}_p(t)\left[\widetilde{\varphi}_L(T) - \widetilde{\varphi}_L(t)\right]\mathrm{d}t + \int_T^\infty \overline{F}_p(t)\mathrm{e}^{-n\theta t}\left[\widetilde{\varphi}_L(T) - \widetilde{\varphi}_L(t)\right]\mathrm{d}t,$$

with

$$\widetilde{\varphi}_L(t) = -(c_Y - c_T)\frac{\mathrm{d}\left(1-\prod_{i=1}^n \overline{G}_i(t)\right)/\mathrm{d}t}{1-\prod_{i=1}^n \overline{G}_i(t)} + (c_F - c_T)\frac{\mathrm{d}F_p(t)/\mathrm{d}t}{\overline{F}_p(t)} + c_M q\lambda(t)$$
$$= -(c_Y - c_T)\frac{n\theta\mathrm{e}^{-n\theta t}}{1-\mathrm{e}^{-n\theta t}} + \left[(c_F - c_T)p + c_M q\right]\lambda(t).$$

Then, the optimal $\widetilde{T}_L^*$ is obtained according to the following theorem.

Theorem 4.3. *If $\widetilde{Q}_L(\infty) > c_T$, there exists an optimal $\widetilde{T}_L^*$ $(0 < \widetilde{T}_L^* < \infty)$ which satisfies (4.12), and the optimal maintenance cost rate is $\widetilde{C}_L(\widetilde{T}_L^*) = \widetilde{\varphi}_L(\widetilde{T}_L^*)$. Otherwise, $\widetilde{T}_L^* = \infty$.*

Proof. Differentiating $\widetilde{Q}_L(T)$ with respect to T, we have

$$\frac{\mathrm{d}\widetilde{Q}_L(T)}{\mathrm{d}T} = \frac{\mathrm{d}\widetilde{\varphi}_L(T)}{\mathrm{d}T}\left[\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^\infty \overline{F}_p(t)\prod_{i=1}^n \overline{G}_i(t)\mathrm{d}t\right]$$
$$+ \widetilde{\varphi}_L(T)\overline{F}_p(T)\left(1-\prod_{i=1}^n \overline{G}_i(T)\right) + (c_Y - c_T)\overline{F}_p(T)\frac{\mathrm{d}\left(1-\prod_{i=1}^n \overline{G}_i(T)\right)}{\mathrm{d}T}$$
$$- (c_F - c_T)\left(1-\prod_{i=1}^n \overline{G}_i(T)\right)\frac{\mathrm{d}F_p(T)}{\mathrm{d}T} - c_M q\lambda(T)\overline{F}_p(T)\left(1-\prod_{i=1}^n \overline{G}_i(T)\right)$$
$$= \frac{\mathrm{d}\widetilde{\varphi}_L(T)}{\mathrm{d}T}\left[\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^\infty \overline{F}_p(t)\prod_{i=1}^n \overline{G}_i(t)\mathrm{d}t\right]$$
$$= \left\{(c_Y - c_T)\frac{(n\theta)^2\mathrm{e}^{-n\theta T}}{(1-\mathrm{e}^{-n\theta T})^2} + \left[(c_F - c_T)p + c_M q\right]\frac{\mathrm{d}\lambda(T)}{\mathrm{d}T}\right\}\left\{\int_0^T \overline{F}_p(t)\mathrm{d}t + \int_T^\infty \overline{F}_p(t)\mathrm{e}^{-n\theta t}\mathrm{d}t\right\}.$$

$\mathrm{d}\widetilde{Q}_L(T)/\mathrm{d}T > 0$ if $\lambda(t)$ increases strictly with t from $\lambda(0) = 0$ to $\lambda(\infty)$, then $\widetilde{Q}_L(T)$ also increases strictly from $\widetilde{Q}_F(0) = 0$ to $\widetilde{Q}_L(\infty) = \lim_{T\to\infty} \widetilde{Q}_L(T)$ with respect to T, and

$$\widetilde{Q}_L(\infty) = \int_0^\infty \overline{F}_p(t)\left[\widetilde{\varphi}_L(\infty) - \widetilde{\varphi}_L(t)\right]\mathrm{d}t.$$

TABLE 3. Optimal T_L^* and $C_L(T_L^*)$ for different q and n ($c_T = 500$, $c_Y = 750$, $c_F = 1000$, $c_M = 100$, $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$).

q	$n=1$		$n=2$		$n=3$	
	T_L^*	$C_L(T_L^*)$	T_L^*	$C_L(T_L^*)$	T_L^*	$C_L(T_L^*)$
1.0	24.48	46.58	25.87	47.96	27.25	49.08
0.9	21.25	55.91	22.63	56.88	23.09	57.64
0.8	18.84	64.19	20.32	65.04	21.25	65.67
0.7	17.55	71.58	18.94	72.39	19.86	72.96
0.6	16.17	78.32	17.55	79.10	18.48	79.63
0.5	15.24	84.54	16.17	85.30	17.09	85.81
0.4	14.32	90.34	17.09	92.21	16.17	91.59
0.3	13.86	95.80	14.78	96.56	15.24	97.03
0.2	12.82	101.01	14.32	101.71	14.78	102.21
0.1	12.47	105.90	13.86	106.72	13.86	107.11
0	11.78	110.61	12.93	111.40	12.93	111.83

TABLE 4. Optimal $\widetilde{T}_L^*$ and $\widetilde{C}_L(\widetilde{T}_L^*)$ for different q and n ($c_T = 500$, $c_Y = 750$, $c_F = 1000$, $c_M = 100$, $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$).

q	$n=1$		$n=2$		$n=3$	
	$\widetilde{T}_L^*$	$\widetilde{C}_L(\widetilde{T}_L^*)$	$\widetilde{T}_L^*$	$\widetilde{C}_L(\widetilde{T}_L^*)$	$\widetilde{T}_L^*$	$\widetilde{C}_L(\widetilde{T}_L^*)$
1.0	24.48	46.58	22.63	44.87	22.17	44.74
0.9	21.25	55.91	19.86	54.73	19.52	54.57
0.8	18.84	64.19	18.01	63.15	17.55	62.96
0.7	17.55	71.58	16.63	70.62	16.63	70.41
0.6	16.17	78.32	15.24	77.38	14.78	77.14
0.5	15.24	84.54	14.32	83.62	14.32	83.35
0.4	14.32	90.34	13.39	89.43	13.39	89.14
0.3	13.86	95.80	12.93	94.89	12.93	94.58
0.2	12.82	101.01	12.47	100.11	12.01	99.73
0.1	12.47	105.90	12.01	105.04	11.55	104.62
0	11.78	110.61	11.55	109.73	11.09	109.33

Thus, a finite and unique $\widetilde{T}_L^*$ ($0 < \widetilde{T}_L^* < \infty$) exists when $\widetilde{Q}_L(\infty) > c_T$, otherwise $\widetilde{T}_L^* = \infty$ when $\widetilde{Q}_L(\infty) \le c_T$, which completes the proof of Theorem 4.3. □

4.3. Numerical examples

In this section, we use the same parameters with them in Section 4.3 to verify the theoretical results for the generalized periodic replacement last models, *i.e.*, system failure distribution is $F(t) = 1 - \mathrm{e}^{-0.01t^2}$, the distribution of the ith ($i = 1, 2, \ldots, n$) working time is $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, and the replacement costs are $c_T = 500$, $c_Y = 750$, $c_F = 1000$, and $c_M = 100$. Tables 3 and 4 show the optimal replacement cycles T_L^* and $\widetilde{T}_L^*$, and the corresponding minimized maintenance cost rates $C_L\,(T_L^*)$ and $\widetilde{C}_L(\widetilde{T}_L^*)$, respectively.

Tables 3 and 4 illustrate that T_L^* and $\widetilde{T}_L^*$ increase with q for a fixed n, while $C_L(T_L^*)$ and $\widetilde{C}_L(\widetilde{T}_L^*)$ decrease with q. When q is fixed, T_L^* and $C_L(T_L^*)$ increase with n while $\widetilde{T}_L^*$ and $\widetilde{C}_L(\widetilde{T}_L^*)$ decrease with n. Figure 3 shows the average cost rate $C_L(T)$ for different n in terms of $q = 1$ for Model B1 and Figure 4 shows the average cost rate $\widetilde{C}_L(T)$ for different n in terms of $q = 1$ for Model B2. From Figures 3 and 4, it is clear that the finite and unique replacement intervals T_L^* and $\widetilde{T}_L^*$ exist when $q = 1$, *i.e.*, $0 < T_L^* < \infty$ and $0 < \widetilde{T}_L^* < \infty$.

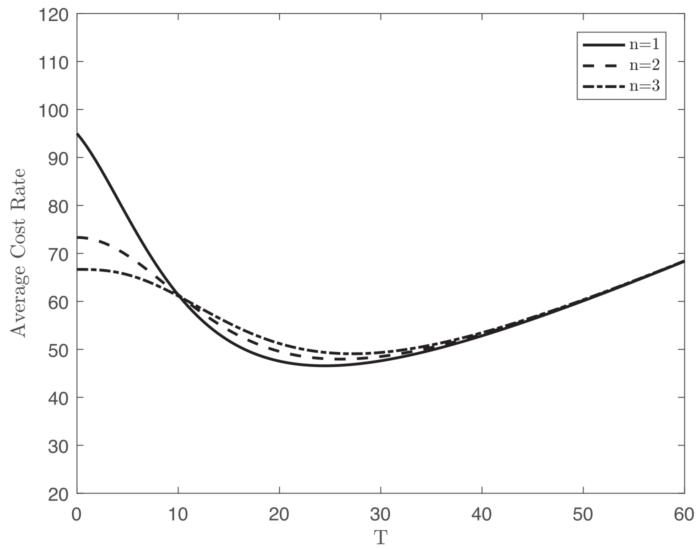

FIGURE 3. $C_L(T)$ for different n in terms of $q = 1$ ($c_T = 500$, $c_Y = 750$, $c_F = 1000$, $c_M = 100$, $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$).

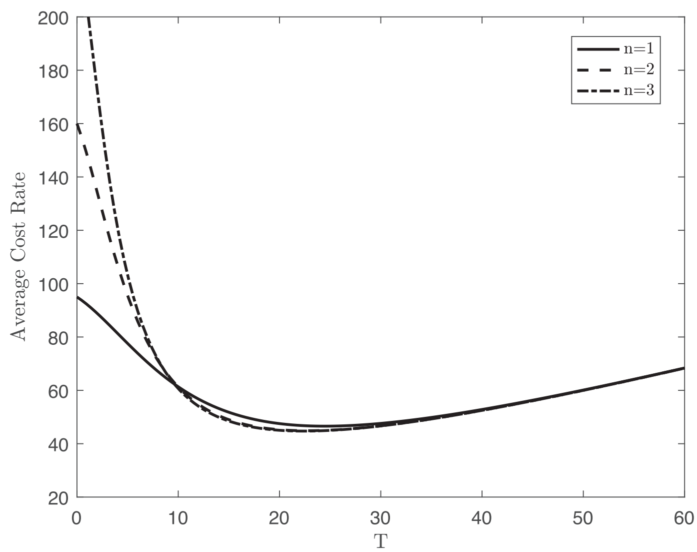

FIGURE 4. $\widetilde{C}_L(T)$ for different n in terms of $q = 1$ ($c_T = 500$, $c_Y = 750$, $c_F = 1000$, $c_M = 100$, $G_i(t) = 1 - \mathrm{e}^{-0.1t}$, $F(t) = 1 - \mathrm{e}^{-0.01t^2}$).

5. Conclusions

We have investigated preventive replacement policies in this paper and constructed four models, *i.e.*, a periodic replacement first model (Model A1), a modified periodic replacement first model (Model A2), a periodic replacement last model (Model B1), and a modified periodic replacement last model (Model B2). In each modeling framework, the infinite time span is considered and average replacement cost rate is minimized to seek the optimal replacement interval. All discussions have been conducted analytically and examined numerically. For the generalized periodic replacement first policy and generalized periodic replacement last policy, both the optimal replacement intervals T_F^* and T_L^* increase with the number of random working periods n, while on the contrary, both $\widetilde{T}_F^*$ and $\widetilde{T}_L^*$ for the modified generalized periodic replacement first policy and modified generalized periodic replacement last policy decrease with n. The developed maintenance models in this paper have potential applications in practical products such as the unmanned aerial vehicle (UAV), micro-electro-mechanical system (MEMS), and gyroscope in the inertial navigation system (INS) as soon as their operating conditions satisfy the assumptions in Section 2.

For future research, we should consider the condition that times for repair and replacement are not neglected. In addition, more complex maintenance models should be developed on reliability for deterioration systems as they are capable of describing the sophisticated degrading behaviors of engineering systems.

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