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WEIGHT RESTRICTIONS FOR THE DEA MODEL IN THE PRESENCE OF DUAL-ROLE FACTORS: AN APPLICATION TO THE IRANIAN BANKING SECTOR

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Abstract. A production process transforming multiple inputs to different outputs is considered in conventional data envelopment analysis (DEA) models. In various settings, however, there are factors that simultaneously play the roles of both input and output called dual-role factors. In some situations, additional information is available to impose on a DEA model with dual-role factors, or the decision maker is forced to impose some restrictions regarding the importance of dual-role factors on the model. Toward this end, the current research employs two different weighting methods to introduce various weighted DEA models in the presence of dual-role factors. To strengthen the accuracy of the new models, their properties are discussed. Then, each new model is illustrated in details by a numerical example. Moreover, to show that the new models are applicable, they are applied to the Iranian banking sector. To do this, 20 bank branches which have dual-role factors are assessed. At last, to show the outcome of weight restrictions, the results obtained by each new model are compared with those from Cook and Zhu's model [*Eur. J. Oper. Res.* **180** (2007) 692–699].

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1. Introduction

Preliminary data envelopment analysis (DEA) proposed by Charnes *et al.* [8] is a non-parametric frontier estimation methodology on the basis of linear programming problems. It has been widely applied to measure the relative efficiency for a group of homogeneous decision making units (DMUs) with multiple input and output factors. All in all, DEA includes a collection of linear and non-linear models which are extensions of the original work of Charnes *et al.* [8] for the evaluation of DMUs. It has been developed over the last years and emerged as a body of methodologies and concepts. Also, its popularity is reflected by a multitude of successful applications.

In the DEA methodology, outputs show what a DMU generates whereas inputs are resources that have resulted in creating those outputs. Typical DEA studies assume that a production process transforms multiple inputs to various outputs. In some cases, however, as well as those having a clear input or output role, there exist some factors which simultaneously play both input and output roles; they are known as dual-role factors. The dual-role factor was first noted by Beasly [6,7]. In the evaluation of universities, he finds that the research funding

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is a resource, an input factor, and is a performance criterion which strengthens the university's performance, an output factor. Another example for factors which simultaneously play both input and output roles is trainees in organizations, like nurses and doctoral students in hospitals. Trainees are staff in a study for the evaluation of hospitals, hence, they are inputs; on the other hand, they obviously are output factors in evaluations.

Assessing DMUs with dual-role factors, and consequently, specifying the role of dual-role factors were conducted from different perspective. A substantial body of previous DEA research indicates that there exist a considerable number of papers published in this area; some of which are briefly reviewed as follows. Cook *et al.* [14] developed a linear programming problem to evaluate DMUs having a dual-role factor barring input and output factors. Also, they determined the dual-role factor status, whenever possible. In modelling, they assumed that the dual-role factor behaves like a non-discretionary input. Cook and Zhu [12] suggested two mixed integer linear programming problems to acquire the efficiency score of DMUs having L dual-role factors ($L \geqslant 1$), and to characterize the status of each dual-role factor. Drawing upon the DEA model proposed by Cook *et al.* [14], Farzipoor Saen [16] presented the supplier selection process through a DEA model in the presence of dual-role factors allowing the incorporation of decision maker's preferences on input/output factors. In contrast to the previous DEA models mentioned based on the multiplier form, Amirteimoori and Emrouznejad [2] and Amirteimoori *et al.* [3] suggested mixed integer linear programming problems, in the envelopment form, to calculate the efficiency score of DMUs having dual-role factors, and to specify the status of dual-role factors. In 2014, Chen [11] expanded the work of Cook *et al.* [14] to incorporate dual-role factors in DEA models. To do so, he [11] considered two individual production processes acting together by summarizing the intuitive thinking. Toloo *et al.* [33] developed mixed integer linear programming problems in the presence of imprecise dual-role factors. Also, they proposed a structure to calculate an optimal reallocation model for dual-role factors among all DMUs. Su and Sun [31] suggested a network DEA model which includes undesirable outputs and dual-role factors. They calculated the optimistic and pessimistic efficiency scores, and ranked DMUs by computing the overall performance measures. Noveiri *et al.* [19] developed an alternative DEA approach to incorporate dual-role factors in the production process. In estimating the efficiency of each DMU by their model, a dual-role factor can simultaneously take both input and output roles, but in the previous DEA models, after the evaluation, a dual-role factor can only take one role. Toloo *et al.* [35] presented DEA models in the presence of dual-role factors for the interval data. The formulated models are a pair of mixed binary linear programming problems to calculate the relative efficiencies in the interval forms. Also, they suggested a stepwise procedure to specify the status of each dual-role factor. Moreover, Toloo *et al.* [34] indicated that the epsilon-free classifier models in the presence of dual-role factors may lead to unacceptable results. Hence, they presented a pair of multiplier and envelopment epsilon-based classifier models. In addition, an approach was developed to find a suitable value for the epsilon in the proposed models.

In the multiplier form of conventional DEA models, in some cases, there exists additional information regarding the structure of factors leading to adding conditions to the model as weight constraints, apart from non-negativity weights. On the other hand, a popular way to prevent uncommon weights in the conventional DEA models is to imposing weight restrictions [1, 22]. Weight restrictions represent value judgements incorporated in the form of additional constraints on input and output weights within conventional DEA models. These constraints reduce the flexibility of weights and typically improve the discrimination of models [1, 13, 20, 26, 29]. Several reasons motivating the application of weight restrictions in the DEA methodology were mentioned in the work of Allen *et al.* [1]. From the technology perspective, the incorporation of additional weight restrictions in multiplier DEA models results in the expansion of the production possibility set (PPS) [17, 23, 27, 28]. It is obvious that the shift of the production frontier away from DMUs fails to increase the efficiency scores which results in failure to recognize efficient DMUs. In the literature of the DEA methodology, there are different types of inserting additional homogeneous and non-homogeneous weight restrictions. Often, applying weight restrictions to models may result in infeasibility and may not calculate the relative efficiency score of DMUs; they are well-known drawbacks in restricting weights for DEA models [21, 24, 25, 32].

There are a wide range of methods for weighting the conventional DEA models. In 1997, Allen *et al.* [1] classified most of the weighting methods in the three categories, direct restrictions on the weights, adjusting the observed input-output levels to capture value judgements, and restricting the virtual inputs and outputs. Two weighting methods used in the current study are "virtual weight restrictions" and "cone-ratio"; they belong to the third and second categories. In what follows, these two methods are briefly reviewed. In the virtual weight restrictions method, except the restriction of actual DEA weights, non-negativity weights, the proportion of the total virtual inputs/outputs of a DMU devoted to a virtual input/output is restricted in a distance. Also, the decision maker sets appropriate bounds in the range between 0 and 1. The virtual weight restrictions can be formed for all DMUs, the assessed DMU, or the average unit. A preliminary work on the use of restricting virtual input/output weights within the conventional DEA model was undertaken by Wong and Beasley [36]. In addition to the virtual weight restrictions method, the cone-ratio method lends itself to a variety of additional uses in restricting weights. This weighting method is based on applying polyhedral cones for weights which results in artificial data. Using the cone-ratio method, Charnes *et al.* [9] ($\mathrm{C^2WH}$) provided definitions of efficiency over the conventional DEA model whose data are insufficient to capture restrictions which should be involved. Furthermore, Charnes *et al.* [10] applied this restricting method to assess commercial banks. It should be noted that these two weighting methods are different, and have various aims in weighting.

The model with the dual-role factors proposed by Cook and Zhu [12] overestimates the relative efficiency of DMUs because it obtains an optimal set of weights for input, output, and dual-role factors to represent the assessed DMU in the best light in comparison to all the other DMUs. Now, the difficulty for flexibility of weights is resolved by weight restrictions. In recent studies, weight restrictions represent value judgements on only input and output weights in a multiplier model; however, there does not exist any additional weight constraint on a DEA model as for restricting weights of dual-role factors, even the DEA model weighted by Farzipoor Saen [16]. To fill this gap, the current paper establishes how the virtual weight restrictions and cone-ratio methods are separately employed for restricting input, output, and dual-role factor weights within the DEA model presented by Cook and Zhu [12]. Consequently, two different weighted DEA models are formed for assessing DMUs having dual-role factors apart from input and output factors. Then, for confirming the newly proposed models, their properties are discussed and illustrated by simple examples. It should be noted that the two newly weighted DEA models are not comparable, they are non-equivalent with different aims. Given that the importance of the banking industry in the economy of each country, an application to a data set of 20 bank branches is offered. In these bank branches, in addition to input and output factors, there exist factors playing simultaneously both input and output roles. Hence, we employ the new weighting models to control the flexibility of weights for input, output, and dual-role factors.

The current paper proceeds as follows. In Section 2, a brief review on the previous DEA models used in this study is accomplished. Section 3 proposes two different weighted DEA models in the presence of dual-role factors. They are constructed based on the two weighting DEA methods, virtual weight restrictions and cone-ratio. Section 4 illustrates the proposed models by the two numerical examples. Then, the applicability of the new models is discussed by a real data set of 20 Iranian bank branches. At last, conclusion and future research directions are discussed in Section 5.

2. Preliminaries

In this section, the preliminary DEA models applied within Section 3 are reviewed in two subsections. First, DEA models with weight restrictions presented by Wong and Beasley [36] and Charnes *et al.* [9] ($\mathrm{C^2WH}$) are summarized. Then, the DEA model developed by Cook and Zhu [12] for the evaluation of DMUs in the presence of dual-role factors is briefly presented.

2.1. The weighted DEA models

Assume there exist n DMUs each of which consumes m inputs to produce s outputs. The ith input and the rth output of DMU_j $(j = 1, \ldots, n)$ are respectively denoted as x_{ij} $(i = 1, \ldots, m)$ and y_{rj} $(r = 1, \ldots, s)$. Without loss of generality, let DMU_o $(o \in \{1, \ldots, n\})$ be the DMU assessed.

Wong and Beasley [36] suggested the application of restricting virtual inputs and outputs on the conventional CCR DEA model in the multiplier form presented by Charnes *et al.* [8]. To do this, except restricting the actual DEA weights, non-negativity weights, the proportion of the total virtual outputs of a unit devoted to the rth virtual output is restricted to the range between $[\underline{U}_r, \overline{U}_r]$ $(r = 1, \ldots, s)$. The decision maker sets $\underline{U}_r$ and $\overline{U}_r$ as suitable lower and upper bounds for the importance of the virtual output r normalized $\left(0 \leqslant \underline{U}_r \leqslant \overline{U}_r \leqslant 1\right)$. In the mathematical term, the restriction on the rth virtual output takes the form $\underline{U}_r \leqslant \frac{u_r y_{ra}}{\sum_{r=1}^{s} u_r y_{ra}} \leqslant \overline{U}_r$ where $u_r y_{ra}$ and $\sum_{r=1}^{s} u_r y_{ra}$ respectively represent the virtual output r and the total virtual outputs for the average unit $(x_{1a}, \ldots, x_{ma}, y_{1a}, \ldots, y_{sa}) = \left(\sum_{j=1}^{n} \frac{x_{1j}}{n}, \ldots, \sum_{j=1}^{n} \frac{x_{mj}}{n}, \sum_{j=1}^{n} \frac{y_{1j}}{n}, \ldots, \sum_{j=1}^{n} \frac{y_{sj}}{n}\right)$. A similar restriction can be imposed on the ith input as well. By considering the above-mentioned virtual weight restrictions to the average unit, $m+s$ additional weighting constraints can be added to the conventional CCR DEA model in the multiplier form. Hence, the DEA model constructed by restricting the virtual weights is as follows [1, 36]:

$$
\begin{aligned}
\text{Max} \quad & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} & & \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1, & & j = 1, \ldots, n, \\
& \underline{V}_i \leqslant \frac{v_i x_{ia}}{\sum_{i=1}^{m} v_i x_{ia}} \leqslant \overline{V}_i, & & i = 1, \ldots, m, \\
& \underline{U}_r \leqslant \frac{u_r y_{ra}}{\sum_{r=1}^{s} u_r y_{ra}} \leqslant \overline{U}_r & & r = 1, \ldots, s, \\
& v_i \geqslant 0, & & i = 1, \ldots, m, \\
& u_r \geqslant 0, & & r = 1, \ldots, s.
\end{aligned} \tag{2.1}
$$

Also, in the virtual weight restrictions method, the under evaluation DMU or DMU_j $(j = 1, \ldots, n)$ can be used rather than the average unit. For computational reasons, the average unit is generally used in this weighting method [36].

For distributing more general conditions imposing restrictions on the weights of input and output factors, the conventional CCR DEA model was generalized by Charnes *et al.* [9] (C^2WH) known as the cone-ratio CCR DEA model. For formulating the preliminary cone-ratio CCR DEA model, the feasible regions of decision space for the input and output weights are supposed within the polyhedral convex cones V and U spanned by $k1$ and $k2$ admissible non-negative direction vectors $a_h \in \Re^m$ $(h = 1, \ldots, k1)$ and $b_p \in \Re^s$ $(p = 1, \ldots, k2)$, respectively. By adding these cones to the conventional CCR DEA model and after doing some transformations, the cone-ratio CCR DEA model is acquired as follows [9, 10]:

$$
\begin{aligned}
\text{Max} \quad & \sum_{p=1}^{k2} \beta_p \overline{y}_{po} & & \\
\text{s.t.} \quad & \sum_{h=1}^{k1} \alpha_h \overline{x}_{ho} = 1, & & \\
& \sum_{p=1}^{k2} \beta_p \overline{y}_{pj} - \sum_{h=1}^{k1} \alpha_h \overline{x}_{hj} \leqslant 0, & & j = 1, \ldots, n, \\
& \alpha_h \geqslant 0, & & h = 1, \ldots, k1,
\end{aligned}
$$

$$\beta_p \geqslant 0, \qquad\qquad p = 1, \ldots, k2, \tag{2.2}$$

where $\overline{x}_{hj} = \sum_{i=1}^{m} a_{hi} x_{ij}$ $(h = 1, \ldots, k1)$ and $\overline{y}_{pj} = \sum_{r=1}^{s} b_{pr} y_{rj}$ $(p = 1, \ldots, k2)$ are respectively the artificial input and output data sets for DMU_j $(j = 1, \ldots, n)$ obtained by the $k1$ and $k2$ admissible non-negative direction vectors. Also, α_h and β_p are respectively the importance of the hth and the pth artificial input and output generated by the cone-ratio method. Essentially, the process of constructing the weighted DEA model by the cone-ratio method generates artificial input and output data sets in $\Re^{k1}$ and $\Re^{k2}$.

2.2. The DEA model in the presence of dual-role factors

In traditional DEA models, each factor explicitly plays one of the roles of input or output. Still, some factors in several applications can simultaneously play both input and output roles; they are known as dual-role factors. In addition to the notations mentioned in the previous subsection, here, consider L dual-role factors w_{lj} $(l = 1, \ldots, L,\ j = 1, \ldots, n)$. To evaluate DMU_o having dual-role factors apart from input and output factors, Cook and Zhu [12] developed the following mixed integer linear programming problem,

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} \delta_l w_{lo} & \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} \gamma_l w_{lo} - \sum_{l=1}^{L} \delta_l w_{lo} = 1, & \\
& \sum_{r=1}^{s} u_r y_{rj} + 2\sum_{l=1}^{L} \delta_l w_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} \gamma_l w_{lj} \leqslant 0, \qquad & j = 1, \ldots, n, \\
& 0 \leqslant \delta_l \leqslant M d_l, & l = 1, \ldots, L, \\
& \delta_l \leqslant \gamma_l \leqslant \delta_l + M(1 - d_l), & l = 1, \ldots, L, \\
& d_l \in \{0, 1\}, & l = 1, \ldots, L, \\
& \delta_l, \gamma_l \geqslant 0, & l = 1, \ldots, L, \\
& v_i \geqslant 0, & i = 1, \ldots, m, \\
& u_r \geqslant 0, & r = 1, \ldots, s.
\end{aligned} \tag{2.3}$$

Model (2.3) is solved for each DMU to acquire the efficiency score and to detect the status of L dual-role factors. The binary variable d_l $(l \in \{1, \ldots, L\})$ takes the value 0 or 1 in which $d_l = 0/d_l = 1$ determines that the lth dual-role factor behaves as an input/output factor. After assessing all DMUs, L sets including n optimal values 0 and/or 1 are characterized. In the lth set $(l \in \{1, \ldots, L\})$, providing that the number of 0 is more than the number of 1, all DMUs accept the input role for the lth dual-role factor, otherwise, the output role. Finally, the L sets gained by Model (2.3) specify the status of all the dual-role factors. In the case that the numbers of obtained 0 and 1 for the lth set are equal, Model (2.3) cannot specify the status of the lth dual-role factor.

3. The proposed DEA models

Consider each DMU_j consumes m inputs x_{ij} $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ to produce s outputs y_{rj} $(r = 1, \ldots, s,\ j = 1, \ldots, n)$, and has L dual-role factors w_{lj} $(l = 1, \ldots, L,\ j = 1, \ldots, n)$ which simultaneously play both input and output roles. Furthermore, DMU_o is assumed as the under evaluation DMU.

It is well-known that the model proposed by Cook and Zhu [12] in the presence of dual-role factors fails to have any additional weight restriction. On the other hand, there exist many weighted models in the DEA methodology but none of them restricts dual-role factor weights, even the one proposed by Farzipoor Saen [16] restricted only weights of input/output factors in evaluating DMUs having dual-role factors. Hence, his work [16] does not have any contribution to controlling weights of dual-role factors. Previous studies in the literature have reported that most of developments for the evaluation of DMUs in the presence of dual-role factors were followed by an extension of applications without challenging the fundamental basis of the DEA methodology like the flexibility in selecting weights of dual-role factors. To fill this gap, the current study makes noteworthy contributions to restrict additional weighting constraints to the all factors, even dual-role, by the two different methods, virtual weight restrictions and cone-ratio. Consequently, these weighting methods lead to construct two different weighted DEA models in the presence of dual-role factors.

3.1. The virtual weighted DEA model in the presence of dual-role factors

This subsection proposes an extension of the virtual weight restrictions method for weighting the DEA model with dual-role factors developed by Cook and Zhu [12]. By an inspiration of the additional weight restrictions in Section 2.1, here, the virtual weighting constraints of the average unit, apart from m inputs and s outputs, for L dual-role factors are newly constructed. The level of ith input, rth output, and lth dual-role factor for the average unit are respectively considered as $x_{ia} = \sum_{j=1}^{n} \frac{x_{ij}}{n}$, $y_{ra} = \sum_{j=1}^{n} \frac{y_{rj}}{n}$, and $w_{la} = \sum_{j=1}^{n} \frac{w_{lj}}{n}$. Since each dual-role factor plays two roles, input and output, two weighting constraints are formed for each dual-role factor. The process of constructing them is as follows. For the average unit, the ratio of the lth virtual dual-role factor playing the input role $((1-d_l)\gamma_l w_{la})$ against the total virtual inputs $\left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L}(1-d_l)\gamma_l w_{la}\right)$ should be between the lower bound $\underline{V}_{1l}$ and the upper bound $\overline{V}_{1l}$ $(l \in \{1,\ldots,L\})$. In a mathematical term, a virtual weighting constraint for the lth dual-role factor playing the input role is $\underline{V}_{1l}(1-d_l) \leqslant \frac{(1-d_l)\gamma_l w_{la}}{\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L}(1-d_l)\gamma_l w_{la}} \leqslant \overline{V}_{1l}$ $(l \in \{1,\ldots,L\})$. Similarly, a virtual weighting constraint for the lth dual-role factor whenever it plays the output role is formed as $\underline{U}_{1l} d_l \leqslant \frac{d_l \gamma_l w_{la}}{\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L} d_l \gamma_l w_{la}} \leqslant \overline{U}_{1l}$ $(l \in \{1,\ldots,L\})$. In addition to imposing the restrictions for weights of dual-role factors, virtual weighting constraints for weights of other factors, inputs and outputs, can be added to the new model constructed. Toward this end, the weighting constraints for the weights of input and output factors are respectively constructed as $\underline{V}_{2i} \leqslant \frac{v_i x_{ia}}{\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L}(1-d_l)\gamma_l w_{la}} \leqslant \overline{V}_{2i}$ $(i = 1,\ldots,m)$ and $\underline{U}_{2r} \leqslant \frac{u_r y_{ra}}{\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L} d_l \gamma_l w_{la}} \leqslant \overline{U}_{2r}$ $(r = 1,\ldots,s)$. As regards the structure of these fractional weighting constraints, the lower bounds must be greater than or equal to zero and upper bounds must be less than or equal to unity. After all, these bounds are considered by the decision maker. These weighting constraints which are homogeneous can be added to the fractional DEA model presented by Cook and Zhu [12]. Hence, the fractional virtual weighted DEA model in the presence of dual-role factors is constructed as follows:

$$
\begin{aligned}
\text{Max} \quad & \frac{\sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} d_l \gamma_l w_{lo}}{\sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L}(1-d_l)\gamma_l w_{lo}} & & \\
\text{s.t.} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{l=1}^{L} d_l \gamma_l w_{lj}}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{l=1}^{L}(1-d_l)\gamma_l w_{lj}} \leqslant 1, & & j = 1,\ldots,n \\
& \underline{V}_{1l}(1-d_l) \leqslant \frac{(1-d_l)\gamma_l w_{la}}{\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L}(1-d_l)\gamma_l w_{la}} \leqslant \overline{V}_{1l}, & & l = 1,\ldots,L, \quad (*_1) \\
& \underline{U}_{1l} d_l \leqslant \frac{d_l \gamma_l w_{la}}{\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L} d_l \gamma_l w_{la}} \leqslant \overline{U}_{1l}, & & l = 1,\ldots,L, \quad (*_2) \\
& \underline{V}_{2i} \leqslant \frac{v_i x_{ia}}{\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L}(1-d_l)\gamma_l w_{la}} \leqslant \overline{V}_{2i}, & & i = 1,\ldots,m, \\
& \underline{U}_{2r} \leqslant \frac{u_r y_{ra}}{\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L} d_l \gamma_l w_{la}} \leqslant \overline{U}_{2r}, & & r = 1,\ldots,s, \\
& d_l \in \{0,1\}, & & l = 1,\ldots,L, \\
& v_i \geqslant 0, & & i = 1,\ldots,m, \\
& u_r \geqslant 0, & & r = 1,\ldots,s,
\end{aligned}
$$

$$\gamma_l \geqslant 0, \qquad l = 1, \ldots, L. \tag{3.1}$$

Controlling the contribution of each factor in the efficiency is the aim of this weighted DEA model. The average unit applied to restricting weights, all DMUs in the peer set comply with the virtual weight constraints. The reason is that the weights are not dependent on the observed input, output, and dual-role factor levels.

In Model (3.1), the binary variable d_l ($l \in \{1, \ldots, L\}$) implies that the lth dual-role factor takes either the input or output role. Its status is determined so that, taking into account the expert preferences, the relative efficiency score of the given DMU increases as much as possible. In the fractional virtual weighted DEA model (3.1), if $d_l = 0$, the lth virtual weighting constraint in $(*_1)$ will be active and the corresponded constraint in $(*_2)$ will be redundant because in this case the lth dual-role factor plays the input role. Furthermore, if $d_l = 1$, the lth virtual weighting constraint in $(*_2)$ will be active and the corresponded one in $(*_1)$ will be redundant because in this case the lth dual-role factor plays the output role.

It is obvious that Model (3.1) is non-linear. Some of its constraints can be linearized by the changing variables $\delta_l = d_l\gamma_l$ ($l = 1, \ldots, L$) and imposing the following constraints [12]:

$$\begin{aligned} &0 \leqslant \delta_l \leqslant M d_l, && l = 1, \ldots, L,\\ &\delta_l \leqslant \gamma_l \leqslant \delta_l + M(1 - d_l), && l = 1, \ldots, L. \end{aligned} \tag{3.2}$$

Yet, it is continuously non-linear. Now, the changing variables $g_{li} = d_l v_i$ ($l = 1, \ldots, L,\ i = 1, \ldots, m$), $f_{lr} = d_l u_r$ ($l = 1, \ldots, L,\ r = 1, \ldots, s$), $q_{ll'} = d_l\delta_{l'}$, and $p_{ll'} = d_l\gamma_{l'}$ ($l = 1, \ldots, L,\ l' = 1, \ldots, L$) by imposing the constraints

$$\begin{aligned} &0 \leqslant g_{li} \leqslant M d_l, && l = 1, \ldots, L,\ i = 1, \ldots, m,\\ &g_{li} \leqslant\ v_i \leqslant g_{li} + M(1 - d_l), && l = 1, \ldots, L,\ i = 1, \ldots, m, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &0 \leqslant f_{lr} \leqslant M d_l, && l = 1, \ldots, L,\ r = 1, \ldots, s,\\ &f_{lr} \leqslant\ u_r \leqslant f_{lr} + M(1 - d_l), && l = 1, \ldots, L,\ r = 1, \ldots, s, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &0 \leqslant q_{ll'} \leqslant M d_l, && l = 1, \ldots, L,\ l' = 1, \ldots, L,\\ &q_{ll'} \leqslant \delta_{l'} \leqslant q_{ll'} + M(1 - d_l), && l = 1, \ldots, L,\ l' = 1, \ldots, L, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &0 \leqslant p_{ll'} \leqslant M d_l, && l = 1, \ldots, L,\ l' = 1, \ldots, L,\\ &p_{ll'} \leqslant \gamma_{l'} \leqslant p_{ll'} + M(1 - d_l), && l = 1, \ldots, L,\ l' = 1, \ldots, L, \end{aligned} \tag{3.6}$$

are utilized to the linearization. Given the above-mentioned changing variables and the variable transformation method ([30], pp. 340-341), Model (3.1) is converted to the following mixed integer linear programming problem,

$$\begin{aligned} \text{Max}\quad & \sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} \delta_l w_{lo}\\ \text{s.t.}\quad & \sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} \gamma_l w_{lo} - \sum_{l=1}^{L} \delta_l w_{lo} = 1,\\ & \sum_{r=1}^{s} u_r y_{rj} + 2\sum_{l=1}^{L} \delta_l w_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} \gamma_l w_{lj} \leqslant 0, && j = 1, \ldots, n,\\ & \underline{V}_{1l}\left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L} \gamma_l w_{la} - \sum_{l=1}^{L} \delta_l w_{la}\right. \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{m} g_{li}x_{ia} - \sum_{l'=1}^{L} p_{ll'}w_{l'a} + \sum_{l'=1}^{L} q_{ll'}w_{l'a}\Bigg) \\
&\leqslant \gamma_l w_{la} - \delta_l w_{la} \leqslant \overline{V}_{1l}\left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L}\gamma_l w_{la} - \sum_{l=1}^{L}\delta_l w_{la}\right), && l=1,\ldots,L,\\
&\underline{U}_{1l}\left(\sum_{r=1}^{s} f_{lr}y_{ra} + \sum_{l'=1}^{L} q_{ll'}w_{l'a}\right) \leqslant \delta_l w_{la} \leqslant \overline{U}_{1l}\left(\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L}\delta_l w_{la}\right), && l=1,\ldots,L,\\
&\underline{V}_{2i}\left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L}\gamma_l w_{la} - \sum_{l=1}^{L}\delta_l w_{la}\right) \leqslant v_i x_{ia} \\
&\leqslant \overline{V}_{2i}\left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L}\gamma_l w_{la} - \sum_{l=1}^{L}\delta_l w_{la}\right), && i=1,\ldots,m,\\
&\underline{U}_{2r}\left(\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L}\delta_l w_{la}\right) \leqslant u_r y_{ra} \leqslant \overline{U}_{2r}\left(\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L}\delta_l w_{la}\right), && r=1,\ldots,s,\\
&0 \leqslant \delta_l \leqslant M d_l, && l=1,\ldots,L,\\
&\delta_l \leqslant \gamma_l \leqslant \delta_l + M(1-d_l), && l=1,\ldots,L,\\
&0 \leqslant g_{li} \leqslant M d_l, && l=1,\ldots,L,\ i=1,\ldots,m,\\
&g_{li} \leqslant\ v_i \leqslant g_{li} + M(1-d_l), && l=1,\ldots,L,\ i=1,\ldots,m,\\
&0 \leqslant f_{lr} \leqslant M d_l, && l=1,\ldots,L,\ r=1,\ldots,s,\\
&f_{lr} \leqslant\ u_r \leqslant f_{lr} + M(1-d_l), && l=1,\ldots,L,\ r=1,\ldots,s,\\
&0 \leqslant q_{ll'} \leqslant M d_l, && l=1,\ldots,L,\ l'=1,\ldots,L,\\
&q_{ll'} \leqslant \delta_{l'} \leqslant q_{ll'} + M(1-d_l), && l=1,\ldots,L,\ l'=1,\ldots,L,\\
&0 \leqslant p_{ll'} \leqslant M d_l, && l=1,\ldots,L,\ l'=1,\ldots,L,\\
&p_{ll'} \leqslant \gamma_{l'} \leqslant p_{ll'} + M(1-d_l), && l=1,\ldots,L,\ l'=1,\ldots,L,\\
&d_l \in \{0,1\}, && l=1,\ldots,L. \qquad (3.7)
\end{aligned}
$$

In assessing DMU$_o$ by Model (3.7), 2^L expanded PPSs are constructed, there being L dual-role factors which play different roles, inputs and/or outputs. Then, one of them which gives the maximum efficiency score for DMU$_o$ is selected. The virtual weight restrictions are unlinked weight constraints as $-P_{t_1}^T v \leqslant 0$ ($t_1 = 1,\ldots,T_1$) and $Q_{t_2}^T u \leqslant 0$ ($t_2 = 1,\ldots,T_2$) where $T_1 \geqslant 2m$, $T_2 \geqslant 2s$, and $T_1 + T_2 = 2(m+s+L)$ [27]. The axiom for the feasibility of the trade-offs $(-P_{t_1}, 0)$ ($t_1 = 1,\ldots,T_1$) and $(0, Q_{t_2})$ ($t_2 = 1,\ldots,T_2$) implies that in the expanded PPS with the CRS technology ($\overline{T}_C$), each unlinked weight constraint is a virtual DMU as $A_{t_1} = (-P_{t_1}, 0) \in \Re^{m+s+L}$ or $B_{t_2} = (0, Q_{t_2}) \in \Re^{m+s+L}$ which has zero outputs or inputs [23]. It is obvious that A_{t_1} ($t_1 \in \{1,\ldots,T_1\}$) has zero outputs and B_{t_2} ($t_2 \in \{1,\ldots,T_2\}$) has at least one negative output. Hence, not all of the new virtual DMUs dominate any observed DMU or any virtual DMU gotten by the axiom of the unbounded ray for an observed DMU in $\overline{T}_C$, with outputs for the observed DMUs non-negative. On the other hand, there exists at least one negative component in each positive combination of $B_{t_2} \in \overline{T}_C$ ($t_2 \in \{1,\ldots,T_2\}$). Therefore, each non-negative combination of the trade-offs corresponding to the new virtual DMUs fails to dominate any observed DMU. As a result, after adding the virtual weighting constraints to Cook and Zhu's model, some of the observed DMUs still remain efficient in the technologies expanded by the production trade-offs of Model (3.7).

Adding unlinked weight constraints to a feasible DEA model may lead to infeasibility because the weighting constraints may be in conflict with each other. Thus, the virtual weighting constraints may create the empty feasible region of decision space for Model (3.7). In fact, on condition that the lower and upper bounds for the virtual weighting constraints are selected inappropriate, they lead to the infeasibility of Model (3.7). To overcome this difficulty, the bounds should be appropriately chosen by the decision maker so that there should be freedom in selecting weights. In most cases, however, determining these bounds by the decision maker is a hard task. In the current research, the authors are argued that, to ensuring the feasibility of Model (3.7), the bounds for the virtual weighting constraints can be considered as goals of the decision maker. Thereby, Model (3.7) is revised by the goal programming approach ([30], pp. 282–301). After adding deviation variables to the virtual weighting constraints, Model (3.7) is modified in the following form,

$$
\begin{aligned}
\text{Max}\quad & \sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} \delta_l w_{lo} - M \sum_{l=1}^{L} \big(\underline{A}_{1l} + \overline{A}_{1l} + \underline{B}_{1l} + \overline{B}_{1l}\big) \\
& - M \sum_{i=1}^{m} \big(\underline{A}_{2i} + \overline{A}_{2i}\big) - M \sum_{r=1}^{s} \big(\underline{B}_{2r} + \overline{B}_{2r}\big) \\
\text{s.t.}\quad & \sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} \gamma_l w_{lo} - \sum_{l=1}^{L} \delta_l w_{lo} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} + 2\sum_{l=1}^{L} \delta_l w_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} \gamma_l w_{lj} \leqslant 0, && j = 1,\ldots,n, \\
& \underline{V}_{1l} \left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L} \gamma_l w_{la} - \sum_{l=1}^{L} \delta_l w_{la} - \sum_{i=1}^{m} g_{li} x_{ia} - \sum_{l'=1}^{L} p_{ll'} w_{l'a} \right. \\
& \left. + \sum_{l'=1}^{L} q_{ll'} w_{l'a} \right) - \gamma_l w_{la} + \delta_l w_{la} - \underline{A}_{1l} \leqslant 0, && l = 1,\ldots,L, \\
& \gamma_l w_{la} - \delta_l w_{la} - \overline{V}_{1l} \left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L} \gamma_l w_{la} - \sum_{l=1}^{L} \delta_l w_{la} \right) - \overline{A}_{1l} \leqslant 0, && l = 1,\ldots,L, \\
& \underline{U}_{1l} \left(\sum_{r=1}^{s} f_{lr} y_{ra} + \sum_{l'=1}^{L} q_{ll'} w_{l'a} \right) - \delta_l w_{la} - \underline{B}_{1l} \leqslant 0, && l = 1,\ldots,L, \\
& \delta_l w_{la} - \overline{U}_{1l} \left(\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L} \delta_l w_{la} \right) - \overline{B}_{1l} \leqslant 0, && l = 1,\ldots,L, \\
& \underline{V}_{2i} \left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L} \gamma_l w_{la} - \sum_{l=1}^{L} \delta_l w_{la} \right) - v_i x_{ia} - \underline{A}_{2i} \leqslant 0, && l = 1,\ldots,L, \\
& v_i x_{ia} - \overline{V}_{2i} \left(\sum_{i=1}^{m} v_i x_{ia} + \sum_{l=1}^{L} \gamma_l w_{la} - \sum_{l=1}^{L} \delta_l w_{la} \right) - \overline{A}_{2i} \leqslant 0, && i = 1,\ldots,m, \\
& \underline{U}_{2r} \left(\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L} \delta_l w_{la} \right) - u_r y_{ra} - \underline{B}_{2r} \leqslant 0, && r = 1,\ldots,s, \\
& u_r y_{ra} - \overline{U}_{2r} \left(\sum_{r=1}^{s} u_r y_{ra} + \sum_{l=1}^{L} \delta_l w_{la} \right) - \overline{B}_{2r} \leqslant 0, && r = 1,\ldots,s, \\
& 0 \leqslant \delta_l \leqslant M d_l, && l = 1,\ldots,L, \\
& \delta_l \leqslant \gamma_l \leqslant \delta_l + M(1 - d_l), && l = 1,\ldots,L, \\
& 0 \leqslant g_{li} \leqslant M d_l, && l = 1,\ldots,L,\ i = 1,\ldots,m,
\end{aligned}
$$

$$
\begin{aligned}
& g_{li} \leqslant\ v_i \leqslant g_{li} + M(1-d_l), && l=1,\ldots,L,\ i=1,\ldots,m,\\
& 0 \leqslant f_{lr} \leqslant M d_l, && l=1,\ldots,L,\ r=1,\ldots,s,\\
& f_{lr} \leqslant\ u_r \leqslant f_{lr} + M(1-d_l), && l=1,\ldots,L,\ r=1,\ldots,s,\\
& 0 \leqslant q_{ll'} \leqslant M d_l, && l=1,\ldots,L,\ l'=1,\ldots,L,\\
& q_{ll'} \leqslant \delta_{l'} \leqslant q_{ll'} + M(1-d_l), && l=1,\ldots,L,\ l'=1,\ldots,L,\\
& 0 \leqslant p_{ll'} \leqslant M d_l, && l=1,\ldots,L,\ l'=1,\ldots,L,\\
& p_{ll'} \leqslant \gamma_{l'} \leqslant p_{ll'} + M(1-d_l), && l=1,\ldots,L,\ l'=1,\ldots,L,\\
& d_l \in \{0,1\}, && l=1,\ldots,L,\\
& \underline{A}_{1l} \geqslant 0,\ \overline{A}_{1l} \geqslant 0,\ \underline{B}_{1l} \geqslant 0,\ \overline{B}_{1l} \geqslant 0, && l=1,\ldots,L,\\
& \underline{A}_{2i} \geqslant 0,\ \overline{A}_{2i} \geqslant 0,\ \underline{B}_{2r} \geqslant 0,\ \overline{B}_{2r} \geqslant 0, && i=1,\ldots,m,\ r=1,\ldots,s, \quad (3.8)
\end{aligned}
$$

where M is a large positive number. Model (3.8) is the finally virtual weighted DEA model for the evaluation of DMU_o having dual-role factors except to inputs and outputs. As mentioned in Section 2.2, after assessing n DMUs, the role of all dual-role factors are specified by optimal values of d_l $(l = 1,\ldots,L)$. Regarding the results of Model (3.8), if the number of cases that a dual-role factor is considered as an input is greater (smaller) than the number of cases of output, then the dual-role factor is considered as an input (output) factor.

In the following, Theorem 3.1 shows that there exists a relationship between the feasibility of Models (3.7) and (3.8).

Theorem 3.1. *If Model (3.7) is feasible, then, in the optimal solution of Model (3.8), the value of all deviation variables are zero.*

Proof. By contradiction, suppose that in the optimal solution $\chi^* = \Big(v^*, u^*, \gamma^*, \delta^*, g^*, f^*, q^*, p^*, d^*, \underline{A}_1^*, \overline{A}_1^*, \underline{B}_1^*, \overline{B}_1^*, \underline{A}_2^*, \overline{A}_2^*, \underline{B}_2^*, \overline{B}_2^*\Big)$ for Model (3.8), the value for not all derivation variables are zero. Without loss of generality, assume that $\underline{A}_1^* \neq 0$. Therefore, there is at least one index l $(l \in \{1,\ldots,L\})$ such that $\underline{A}_{1l}^* \neq 0$. With Model (3.7) feasible, there exists a solution $\Big(\widehat{v}, \widehat{u}, \widehat{\gamma}, \widehat{\delta}, \widehat{g}, \widehat{f}, \widehat{q}, \widehat{p}, \widehat{d}\Big)$ for this model which satisfies in the all of its constraints. It is clear that $\widehat{\chi} = (v = \widehat{v}, u = \widehat{u}, \gamma = \widehat{\gamma}, \delta = \widehat{\delta}, g = \widehat{g}, f = \widehat{f}, q = \widehat{q}, p = \widehat{p}, d = \widehat{d}, \underline{A}_1 = 0, \overline{A}_1 = 0, \underline{B}_1 = 0, \overline{B}_1 = 0, \underline{A}_2 = 0, \overline{A}_2 = 0, \underline{B}_2 = 0, \overline{B}_2 = 0)$ is a feasible solution for Model (3.8). Hence, in Model (3.8), the results of objective function for the optimal solution χ^* and the feasible solution $\widehat{\chi}$ are respectively as $E_o^* = \sum_{r=1}^{s} u_r^* y_{ro} + \sum_{l=1}^{L} \delta_l^* w_{lo} - M\sum_{l=1}^{L}\Big(\underline{A}_{1l}^* + \overline{A}_{1l}^* + \underline{B}_{1l}^* + \overline{B}_{1l}^*\Big) - M\sum_{i=1}^{m}\Big(\underline{A}_{2i}^* + \overline{A}_{2i}^*\Big) - M\sum_{r=1}^{s}\Big(\underline{B}_{2r}^* + \overline{B}_{2r}^*\Big)$ and $\widehat{E}_o = \sum_{r=1}^{s} \widehat{u}_r y_{ro} + \sum_{l=1}^{L} \widehat{\delta}_l w_{lo}$. Thus, it is a contradiction, M being a large positive number and $\underline{A}_1^* \geqslant 0$, $\underline{A}_1^* \neq 0$. $\square$

Now, to strengthen the accuracy of the virtual weighted DEA model (3.8), the following theorem is proven.

Theorem 3.2. *The relative efficiency score obtained by Model (3.8) is not greater than the one obtained by Model (2.3).*

Proof. Since $(m+s+L) \geqslant 1$, the number of constraints in the new model is more than the number of constraints in the model proposed by Cook and Zhu [12]. It is remembered that by adding constraints to a model its feasible region of decision space fails to become greater than before adding any new constraint. As a consequence, the relative efficiency score of DMU_o obtained by the virtual weighted DEA model will be less than or equal to the before score obtained by Cook and Zhu's model, which completes the proof. $\square$

3.2. The cone-ratio weighted DEA model in the presence of dual-role factors

This subsection employs another weighting method, cone-ratio, to introduce a new weighted DEA model for the evaluation of DMUs having dual-role factors aside from input and output factors. Similar to the process of constructing the virtual weighted DEA model, the model proposed by Cook and Zhu [12] is also employed to form the newly weighted DEA model in the current subsection. To construct the model, suppose that the feasible regions of decision space for the weights of input, output, and dual-role factors are restricted within polyhedral convex cones V, U, and Γ spanned by the $k1$, $k2$ and $k3$ admissible non-negative direction vectors $a_h \in \Re^m$ $(h = 1, \ldots, k1)$, $b_p \in \Re^s$ $(p = 1, \ldots, k2)$, and $c_f \in \Re^L$ $(f = 1, \ldots, k3)$, respectively. Accordingly, the feasible input, output, and dual-role factor weights v, u, and γ can be respectively expressed as

$$\begin{aligned} v \in V &= \sum_{h=1}^{k1} \alpha_h \quad a_h = A^T \alpha, \\ u \in U &= \sum_{p=1}^{k2} \beta_p \quad b_p = B^T \beta, \\ \gamma \in \Gamma &= \sum_{f=1}^{k3} \lambda_f \quad c_f = C^T \delta, \end{aligned} \tag{3.9}$$

where $A^T = (a_1, \ldots, a_{k1}) \in R^{m \times k1}$, $B^T = (b_1, \ldots, b_{k2}) \in R^{s \times k2}$, $C^T = (c_1, \ldots, c_{k3}) \in R^{L \times k3}$, $\alpha^T = (\alpha_1, \ldots, \alpha_{k1}) \in \Re^{k1}$, $\beta^T = (\beta_1, \ldots, \beta_{k2}) \in \Re^{k2}$, and $\lambda^T = (\lambda_1, \ldots, \lambda_{k3}) \in \Re^{k3}$. There exist several ways to select the admissible direction vectors ([15], p. 189). In the current research, the knowledge of experts is applied for characterizing these vectors.

As the cone-ratio method results in homogeneous weighting constraints, there is no problem in applying this method to a linear form of DEA models [21]. Imposing the polyhedral convex cones V, U and Γ on the weights of input, output, and dual-role factors in the mixed integer linear programming problem proposed by Cook and Zhu [12] leads to a cone-ratio DEA model with dual-role factors,

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^{s} u_r y_{ro} + \sum_{l=1}^{L} d_l \gamma_l w_{lo} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} + \sum_{l=1}^{L} (1 - d_l) \gamma_l w_{lo} = 1, \\ & \sum_{r=1}^{s} u_r y_{rj} + \sum_{l=1}^{L} d_l \gamma_l w_{lj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{l=1}^{L} (1 - d_l) \gamma_l w_{lj} \leqslant 0, \qquad j = 1, \ldots, n, \\ & d_l \in \{0, 1\}, \qquad l = 1, \ldots, L, \\ & (v_1, \ldots, v_m) \in V, \\ & (u_1, \ldots, u_s) \in U, \\ & (\gamma_1, \ldots, \gamma_L) \in \Gamma. \end{aligned} \tag{3.10}$$

The cone-ratio weighted DEA model (3.10) controls the importance of each factor in the efficiency with adding a cone to the feasible region of decision space for Model (2.3). By assuming $V = \Re^m \geqslant 0$, $U = \Re^s \geqslant 0$, and $\Gamma = \Re^L \geqslant 0$ within Model (3.10), Models (3.10) and (2.3) become coincident. With respect to the three polyhedral convex cones defined in (3.9), Model (3.10) is converted to,

$$\text{Max} \quad \sum_{r=1}^{s} \left(\sum_{p=1}^{k2} \beta_p b_{pr} \right) y_{ro} + \sum_{l=1}^{L} d_l \left(\sum_{f=1}^{k3} \lambda_f c_{fl} \right) w_{lo}$$

$$
\begin{aligned}
\text{s.t.}\quad & \sum_{i=1}^{m}\left(\sum_{h=1}^{k1}\alpha_h a_{hi}\right)x_{io}+\sum_{l=1}^{L}(1-d_l)\left(\sum_{f=1}^{k3}\lambda_f c_{fl}\right)w_{lo}=1, && \\
& \sum_{r=1}^{s}\left(\sum_{p=1}^{k2}\beta_p b_{pr}\right)y_{rj}+\sum_{l=1}^{L}d_l\left(\sum_{f=1}^{k3}\lambda_f c_{fl}\right)w_{lj}-\sum_{i=1}^{m}\left(\sum_{h=1}^{k1}\alpha_h a_{hi}\right)x_{ij} && \\
& -\sum_{l=1}^{L}(1-d_l)\left(\sum_{f=1}^{k3}\lambda_f c_{fl}\right)w_{lj}\leqslant 0, && j=1,\dots,n,\\
& d_l\in\{0,1\}, && l=1,\dots,L,\\
& \alpha_h\geqslant 0, && h=1,\dots,k1,\\
& \beta_p\geqslant 0, && p=1,\dots,k2,\\
& \lambda_f\geqslant 0, && f=1,\dots,k3. \qquad (3.11)
\end{aligned}
$$

If Model (3.11) obtains $d_l=0$ $(l\in\{1,\dots,L\})$, then, $d_l\left(\sum_{f=1}^{k3}\lambda_f c_{lf}\right)w_{lo}$ will be removed from the objective function and the second constraint set for all j; this means that the lth dual-role factor behaves as an input factor in the evaluation of DMU$_o$. Moreover, if Model (3.11) obtains $d_l=1$ $(l\in\{1,\dots,L\})$, then, $(1-d_l)\left(\sum_{f=1}^{k3}\lambda_f c_{lf}\right)w_{lo}$ will be eliminated from the first constraint and the second constraint set for all j; this means that the lth dual-role factor plays an output role within assessing DMU$_o$.

With a precise perspective, Model (3.11) deals with the artificial non-negative data sets $\overline{x}_{hj}=\sum_{i=1}^{m}a_{hi}x_{ij}(h=1,\dots,k1)$, $\overline{y}_{pj}=\sum_{r=1}^{s}b_{pr}y_{rj}(p=1,\dots,k2)$, and $\overline{w}_{fj}=\sum_{l=1}^{L}c_{fl}w_{lj}$ $(f=1,\dots,k3)$ belonging to DMU$_j(j=1,\dots,n)$. In the following, the changed data for input, output, and dual-role factors is presented in the matrix forms,

$$
\begin{aligned}
AX &= \bar{X}\in R^{k1\times n},\\
BY &= \bar{Y}\in R^{k2\times n},\\
CW &= \bar{W}\in R^{k3\times n},
\end{aligned}
\tag{3.12}
$$

where A, B, and C are the transformation matrices for the original data of input, output, and dual-role factors shown by the matrices $X\in\Re^{m\times n}$, $Y\in\Re^{s\times n}$, and $W\in\Re^{L\times n}$. Furthermore, the transformation matrices constructed by the admissible non-negative direction vectors $\{a_1,\dots,a_{k1}\}$, $\{b_1,\dots,b_{k2}\}$, and $\{c_1,\dots,c_{k3}\}$ are as follows:

$$
A=\begin{bmatrix} a_{11} & \dots & a_{1m}\\ a_{21} & \dots & a_{2m}\\ \vdots & \vdots & \vdots\\ a_{(k1)1} & \dots & a_{(k1)m}\end{bmatrix}_{k1\times m},\quad
B=\begin{bmatrix} b_{11} & \dots & b_{1s}\\ b_{21} & \dots & b_{2s}\\ \vdots & \vdots & \vdots\\ b_{(k2)1} & \dots & b_{(k2)s}\end{bmatrix}_{k2\times s},\quad
C=\begin{bmatrix} c_{11} & \dots & c_{1L}\\ c_{21} & \dots & c_{2L}\\ \vdots & \vdots & \vdots\\ c_{(k3)1} & \dots & c_{(k3)L}\end{bmatrix}_{k3\times L}.
\tag{3.13}
$$

It is easy to verify that Model (3.11) is non-linear. It can be linearized by the changing variables $t_{lf}=d_l\lambda_f$ $(l=1,\dots,L,\ f=1,\dots,k3)$ and imposing the following constraints:

$$
\begin{aligned}
0\leqslant t_{lf} &\leqslant Md_l, && l=1,\dots,L,\ f=1,\dots,k3,\\
t_{lf}\leqslant \lambda_f &\leqslant t_{lf}+M(1-d_l), && l=1,\dots,L,\ f=1,\dots,k3.
\end{aligned}
\tag{3.14}
$$

Owing to the artificial data sets and the above-mentioned changing variables, Model (3.11) is converted to the following mixed integer linear programming problem,

$$
\begin{aligned}
\text{Max}\quad & \sum_{p=1}^{k2}\beta_p\bar{y}_{po}+\sum_{l=1}^{L}\sum_{f=1}^{k3}t_{lf}\bar{w}_{fo} & \\
\text{s.t.}\quad & \sum_{h=1}^{k1}\alpha_h\bar{x}_{ho}+\sum_{f=1}^{k3}\lambda_f\bar{w}_{fo}-\sum_{l=1}^{L}\sum_{f=1}^{k3}t_{lf}\bar{w}_{fo}=1, & \\
& \sum_{p=1}^{k2}\beta_p\bar{y}_{pj}+2\sum_{l=1}^{L}\sum_{f=1}^{k3}t_{lf}\bar{w}_{fj}-\sum_{h=1}^{k1}\alpha_h\bar{x}_{hj}-\sum_{f=1}^{k3}\lambda_f\bar{w}_{fj}\leqslant 0, & j=1,\dots,n,\\
& 0\leqslant t_{lf}\leqslant Md_l, & l=1,\dots,L,\ f=1,\dots,k3,\\
& t_{lf}\leqslant\lambda_f\leqslant t_{lf}+M(1-d_l), & l=1,\dots,L,\ f=1,\dots,k3,\\
& d_l\in\{0,1\}, & l=1,\dots,L,\\
& \alpha_h\geqslant 0, & h=1,\dots,k1,\\
& \beta_p\geqslant 0, & p=1,\dots,k2.
\end{aligned}
\tag{3.15}
$$

Model (3.15) is the finally weighted DEA model based on the cone-ratio method to evaluate DMU_o having dual-role factors. After assessing n DMUs by Model (3.15), the role of all dual-role factors are specified using the optimal values of d_l $(l=1,\dots,L)$. If the number of cases that a dual-role factor is considered as an input is greater (smaller) than the number of cases of output, then the dual-role factor is considered as an input (output) factor.

Now, some properties of the cone-ratio weighted DEA model (3.15) are proven to verify its accuracy. In doing so, Theorems 3.3 and 3.4 respectively indicate that Model (3.15) is feasible and its optimal objective function value gives the relative efficiency score of the DMU assessed.

Theorem 3.3. *Model (3.15) is always feasible.*

Proof. $d_l=0$ $(l=1,\dots,L)$, $\alpha_1=\frac{1}{\bar{x}_{1o}}>0$, $\alpha_h=0$ $(h=2,\dots,k1)$, $\beta_p=0$ $(p=1,\dots,k2)$, $\lambda_f=0$ $(f=1,\dots,k3)$, and $t_{lf}=0$ $(l=1,\dots,L,\ f=1,\dots,k3)$ is a desired feasible solution for Model (3.15), which completes the proof. □

Theorem 3.4. *Model (3.15) gives the relative efficiency score of* DMU_o.

Proof. Model (3.15) similar to Model (2.3) with the artificial data, the proof is clear. □

The following theorem shows that results of the cone-ratio weighted DEA model have more power to distinguish inefficient DMUs in comparison with Cook and Zhu's model.

Theorem 3.5. *The relative efficiency score obtained by the newly proposed model (3.15) is less than or equal to the one obtained by Model (2.3).*

Proof. Model (3.15) was obtained by Model (3.10). It suffices to show that the optimal objective function value obtained by Model (3.10) is not greater than the one obtained by Model (2.3). As mentioned before, Model (3.10) is similar to Model (2.3) providing that $V=\Re^m\geqslant 0$, $U=\Re^s\geqslant 0$, and $\Gamma=\Re^L\geqslant 0$. With $V\subseteq\Re^m\geqslant 0$, $U\subseteq\Re^s\geqslant 0$, and $\Gamma\subseteq\Re^L\geqslant 0$ assumed in Model (3.10), the feasible region of decision space for this model is not greater than the one for Model (2.3). Hence, the optimal objective function value for Model (3.10) is not greater than the one obtained by Model (2.3), which completes the proof. □

What is interesting in this subsection is that the weighted DEA model (3.15) constructed by the cone-ratio method fails to have any problem with decision maker's preferences.

TABLE 1. Data for the three DMUs in Example 1.

DMU	Input	Output	Dual-role factor
A	1	1	2
B	2	1	1
C	1	1	3

4. EXAMPLES

The current section gives two numerical examples and an application to illustrate the new models proposed in the previous section.

4.1. Numerical examples

In this subsection, the effect of weight restrictions based on the virtual weight and cone-ratio methods on the DEA model with dual-role factors is illustrated by the two numerical examples.

Example 1

This example makes clear results for the virtual weighted DEA model in the presence of dual-role factors. Consider a simple example of three DMUs, A, B, and C, where each of them has one input (x), one output (y), and one dual-role factor (w). Table 1 shows a set of hypothetical data for these three DMUs.

Our illustrations can be limited only to the two-dimensional spaces depicted by Figures 1 and 2. In Figure 1, the dual-role factor plays the input role; however, it has the output role in Figure 2.

The shaded areas in these figures represent the original PPSs used for evaluating the three DMUs by Cook and Zhu's model. The results of assessing indicate that all of the DMUs are efficient. B selects only the original PPS in Figure 1 because considering the input role for the dual-role factor gives the maximum efficiency score. Yet, other DMUs can choose the input or output role for the dual-role factor because Figures 1 and 2 display A and C are efficient in both the original PPSs.

Now, the virtual weight restrictions for the weight of dual-role factor are added as $0.1(1-d) \leqslant \frac{(1-d)\gamma w_a}{vx_a+(1-d)\gamma w_a} \leqslant 0.2$ and $0.1d \leqslant \frac{d\gamma w_a}{uy_a+d\gamma w_a} \leqslant 0.2$. The first weighting constraint results in the judgements $0.13v - 1.8\gamma \leqslant 0$ and $-0.26v + 1.6\gamma \leqslant 0$. Also, they can be expressed as the trade-offs $P_1 = (-0.13, 1.8)$, $Q_1 = (0)$, $P_2 = (0.26, -1.6)$, and $Q_2 = (0)$ in Figure 1. These trade-offs expand the original PPS and add the lines AE and AF to the technology. In essence, the light dotted areas are added to the original PPS to construct the expanded PPS in Figure 1. Moreover, drawing on the second weighting constraint, the judgements $0.1u - 1.8\gamma \leqslant 0$ and $-0.2u + 1.6\gamma \leqslant 0$ lead to the trade-offs $P_1 = (0)$, $Q_1 = (0.1, -1.8)$, $P_2 = (0)$, and $Q_2 = (-0.2, 1.6)$ in Figure 2. These trade-offs insert the lines CG and CH and the light dotted areas to the original PPS. It is obvious that the expanded PPSs are larger than the original ones.

Table 2 reports the results of efficiency score in evaluating the three DMUs by Model (3.8).

The results show that A and B select the expanded PPS in Figure 1 but C selects the expanded PPS in Figure 2. In the case that the dual-role factor is considered as input, the efficiency score for C is 0.941. Hence, by considering the above-mentioned weighting constraints, the best situation for the dual-role factor in the evaluation of C is that it plays the output role. In assessing A and B, if the dual-role factor is considered as the output, their efficiency scores are 0.952 and 0.452, respectively. That is why, these DMUs selected the input role for the dual-role factor.

Example 2

The current numerical example shows results of the cone-ratio weighted DEA model in the presence of dual-role factors. Consider a hypothetical data set exhibited in Table 3.

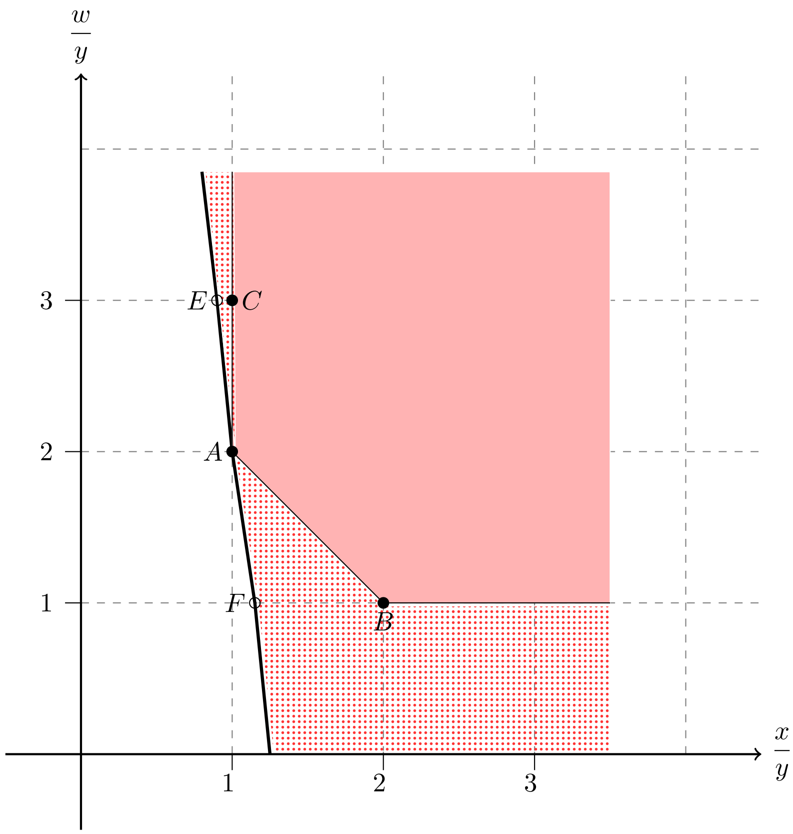

FIGURE 1. The original PPS and the trade-offs when the dual-role factor plays the input role.

There exist five DMUs labelled as A, B, C, D, and E. Each DMU has two inputs, one output, and two dual-role factors.

Assessing these five DMUs by Model (2.3) reports that all of them are efficient. Also, the first dual-role factor has the input role and the second one plays as the output.

Drawing upon the notations in Section 3.2, the decision maker assumes $k1 = 3$, $k2 = 1$, and $k3 = 2$ which are the numbers of admissible non-negative direction vectors for the input, output, and dual-role factor weight spaces to construct the polyhedral convex cones V, U, and Γ. The transformation matrices considered by the decision maker for the hypothetical data set are

$$A = \begin{bmatrix} 0.2 & 0.8 \\ 0.41 & 0.59 \\ 0.39 & 0.61 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.35 & 0.65 \\ 0.42 & 0.58 \end{bmatrix}. \tag{4.1}$$

Consequently, the matrices for the artificial data are obtained as follows:

$$\overline{X} = \begin{bmatrix} 6.6 & 3.6 & 5.8 & 6.8 & 8.6 \\ 7.23 & 3.18 & 6.64 & 6.59 & 8.18 \\ 7.17 & 3.22 & 6.56 & 6.61 & 8.22 \end{bmatrix}, \quad \overline{Y} = \begin{bmatrix} 4 & 5 & 1 & 4 & 2 \end{bmatrix},$$
$$\overline{W} = \begin{bmatrix} 6.55 & 1.7 & 4.95 & 3.6 & 6.6 \\ 6.06 & 1.84 & 4.74 & 3.32 & 6.32 \end{bmatrix}. \tag{4.2}$$

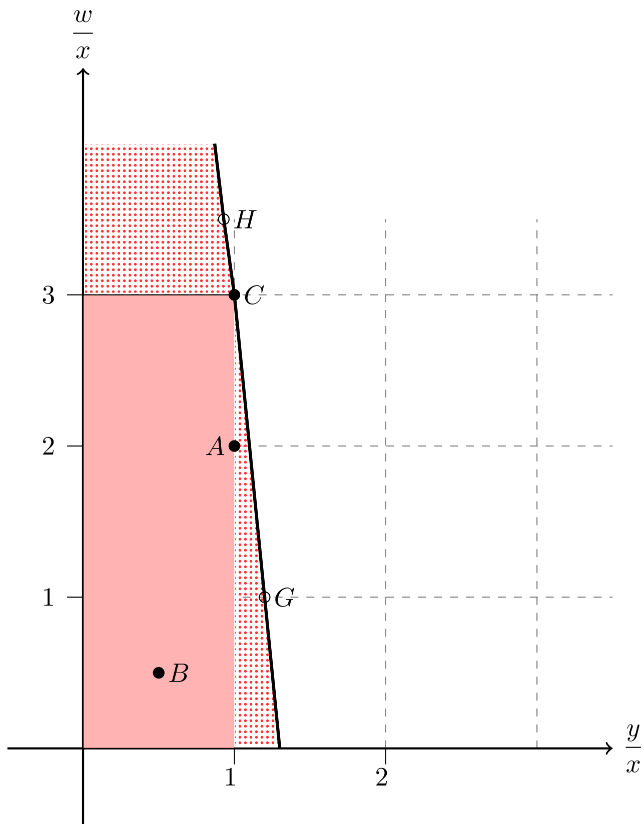

FIGURE 2. The original PPS and the trade-offs when the dual-role factor plays the output role.

TABLE 2. Results of assessing DMUs in Example 1.

DMU	A	B	C
Efficiency scores	1.000	0.613	1.000

With respect to the artificial data in (4.2), the efficiency scores for the five DMUs gotten by Model (3.15) are reported in Table 4.

Also, the results show that the first dual-role factor behaves as the input factor and the second one has the output role.

4.2. An application to the Iranian banking sector

This subsection clarify the application of the theoretical results obtained in Section 3. To this end, first, the structure of the case study is introduced. Then, computational results obtained by Models (2.3), (3.8), and (3.15) for this case study are reported.

Iran is an Asian country, located in the Middle East. It has vast agriculture lands, various mines, and other reserves of underground resources like oil and natural gas. Unfortunately, notwithstanding rich underground wealth, the economy of this country has many problems and challenges. On the other hand, Iran's economy has been hit hard because of the US economic sanctions that came to effect in mid 2018. Meanwhile, on account

TABLE 3. Data for the five DMUs in Example 2.

DMU	Input 1	Input 2	Output	Dual-role factor 1	Dual-role factor 2
A	9	6	4	2	9
B	2	4	5	3	1
C	9	5	1	3	6
D	6	7	4	1	5
E	7	9	2	4	8

TABLE 4. Results of assessing DMUs in Example 2.

DMU	A	B	C	D	E
Efficiency scores	1.000	1.000	0.890	0.833	0.922

of this sanctions, Iran's national currency, the rial, has lost its value with a high percentage against the US dollar within a few months in this year. It has followed that most Iranians have had a low livelihood level. It is well-known that commercial banks play a critical role in the enhancement and the economic expansion of each country. Consequently, given the problems mentioned, the Iranian banking sector is selected for the case study of this research.

The evaluation of banks or bank branches has different aims such as profitability, provision of services, and curbing liquidity. The current study considers the view point of profitability for the evaluation. It should be noted that assessing banks is different from assessing bank branches. The difference between evaluating banks and bank branches and also the purposes mentioned for evaluating them lead to choosing different factors as inputs and outputs within various case studies in the banking industry [18]. For instance, the automated teller machine (ATM) is not considered as a factor whenever bank branches are assessed because each ATM gives services for all customers of the considered bank belonging to the all branches. Still, it is supposed as an input factor in most studies of evaluating banks [4, 5].

The data of this case study were obtained from the central management of Mellat bank branches located in a region of Tehran. There are 20 branches in this region, each of which consumes 3 inputs to produce 2 outputs. As well as these factors, there exist 2 other ones playing simultaneously both input and output roles, known dual-role factors. The input factors are staff privilege (x_1), interest paid (x_2), and demand arrear (x_3). The output factors include interest received (y_1) and charge received (y_2). Regarding the aim of evaluating these bank branches, loan (w_1) and deposit (w_2) are considered as dual-role factors. A period of time considered for this case study is a month. Now, the above-mentioned factors applied for the application are briefly described.

– The staff privilege is an input factor including the quantitative and qualitative indexes of manpower in a branch. These indexes are the number of staff, experience, degree of education, and training rate. Using the weighted sum approach, they are combined, and then, considered as a factor called the staff privilege.
– Each bank branch is forced to pay interests for some client deposits. For Gharzolhasaneh deposits, popular in Iranian banks, branches do not pay any interest. Yet, they pay various interests for short-term and long-term deposits. The sum of all interests that a branch should pay, in a period of time, to some deposits is called the interest paid.
– To grant a loan is one of the activities for commercial banks to earn income. After the loan has been received by a customer, it should be returned by monthly instalments to the granter bank branch. In some cases, however, customers delay to pay or never pay their instalments, even though branches attain guarantees

TABLE 5. Descriptive statistics of factors for the 20 bank branches.

Factor	Mean	Minimum	Maximum	Coefficient of variation
Staff privilege (x_1)	15.361	5.11	31.09	0.483
Interest paid (x_2)	3 294 079 885.200	373 144 412	11 461 622 508	0.744
Demand arrear (x_3)	6 293 320 459.250	206 149 410	21 208 699 780	1.011
Interest received (y_1)	5 264 044 354.500	1 674 570 195	15 460 389 283	0.606
Charge received (y_2)	163 501 789.300	60 916 564	338 207 770	0.407
Loan (w_1)	78 592 945 448.950	24 526 627 277	266 434 914 435	0.752
Deposit (w_2)	58 409 279 730.250	20 439 226 024	128 044 781 417	0.512

from clients due to granting loans. The amount of non-payment for loan instalments by clients, in a period of time, is called the demand arrear.

- Bank branches on loans earn interests. The sum of all interests of a branch received in a period of time for all previous loans granted is called the interest received. The interest received is a percentage of loans granted. It should be noted that different loans have various interests.
- Bank branches give services like selling bonds and bank guarantees and remitting money. For doing these services, they receive charges from customers. Total received charges of a branch for the sake of all services performed, in a period of time, is called the bank charge received.
- The amount of total loans which a branch grants to authentic or legal clients, in a period of time, is considered as a factor called the loan. After all, in return for granting loans, a branch receives valid legal guarantees from clients.
- Total deposits of clients in different sectors, Gharzolhasaneh, short-term, and long-term, is assumed to be one of the factors used in this case study, the deposit. In essence, the deposit shows the mean of deposits balanced within all days for a period of time. Bank branches can apply the deposit to investments and granting loans.

In what follows, the reasons for considering the loan and the deposit as dual-role factors are discussed. It is worthwhile to know that how much the loan is granted to customers because it results in the interest received. Therefore, the loan is an input factor to produce the interest received. It is good that more interests are earned in return for granting fewer loans. On the other hand, granting the loan is a service leading to business boom, job creation, etc. As such, granting the loan is an output factor for bank branches, from the service perspective. As mentioned before, curbing liquidity is one of the bank purposes. Thus, collecting deposits of customers is an output factor. Actually, increasing the deposit in a branch results in decreasing the cash among people. That is, liquidity is inhibited in the society. From another point of view, the deposit results in granting the loan. Furthermore, the loan receives the interest. Consequently, the deposit is considered an input factor which leads to producing the interest received. As the aim of assessing the bank branches in this research is profitability, whether is not recognized that the increase or decrease in granting the loan and receiving the deposit are good for a branch. Hence, they are considered as dual-role factors. Yet, with respect to different purposes, in some case studies, the loan and the deposit may be confidently supposed as input and/or output factors [18].

The summary statistics for all of the factors appear in Table 5. It is worth mentioning that the unit for all factors except the staff privilege is 1 000 000 Rials.

Now, the computational results of applying Models (2.3), (3.8), and (3.15) for the 20 bank branches are presented. Note that comparing the computational results gotten by Models (3.8) and (3.15) is not correct, they being based on the two weighting methods with different aims in the DEA methodology. GAMS (http:

Table 6. Results of efficiency scores for the 20 bank branches.

No. of branch	Efficiency score of Model (2.3)	Efficiency score of Model (3.8)	Efficiency score of Model (3.15)
1	0.944	0.777	0.895
2	1.000	0.831	0.796
3	1.000	0.809	0.813
4	0.998	0.753	0.669
5	1.000	1.000	1.000
6	1.000	0.963	1.000
7	1.000	0.890	0.680
8	0.987	0.843	0.882
9	1.000	0.622	0.910
10	1.000	0.938	1.000
11	0.912	0.632	0.672
12	1.000	0.837	0.755
13	1.000	0.768	0.837
14	0.917	0.742	0.901
15	1.000	0.975	0.961
16	0.919	0.704	0.734
17	0.895	0.752	0.788
18	1.000	1.000	1.000
19	1.000	1.000	1.000
20	1.000	1.000	0.856
Mean	0.979	0.842	0.857

//www.gams.com) package is used for the software code of these models. The original data are used within coding these three models. Table 6 reports the results.

First, the 20 bank branches are assessed by Model (2.3). The second column of Table 6 shows the results of this model. There exist 13 efficient branches. The mean of efficiency scores for all the branches obtained by this model is 0.979.

Owing to the structure and the data set for the 20 bank branches, the decision maker considers a set of bounds for the virtual weighting constraints in Model (3.8), *e.g.* $\underline{V}_{11} = 0.2$, $\overline{V}_{11} = 0.65$, $\underline{U}_{21} = 0.35$, and $\overline{U}_{21} = 0.8$. The results of efficiency scores acquired by Model (3.8) are shown in the third column of Table 6. There are 4 efficient branches. The mean of efficiency scores for all the branches gotten by this model is 0.842. Comparing the second and third columns shows that the efficiency score of each branch taken by Model (3.8) is not greater than the one obtained by Cook and Zhu's model. Therefore, the new constraints reduce the flexibility of weights and improve the discrimination of the model (3.8) than Cook and Zhu's model. The computational results obtained by Model (3.8) confirm the properties discussed in Section 3.1.

The polyhedral convex cones spanned by the $k1 = 2$, $k2 = 3$, and $k3 = 1$ admissible non-negative direction vectors are considered in $\Re^3$, $\Re^2$, and $\Re^2$ by the decision maker. For instance, the direction vector considered by the decision maker for Γ in the normalized form is $\{(0.747, 0.253)\}$. The results of the cone-ratio weighted DEA model (3.15) are listed in the last column of Table 6. It shows that there exist 5 efficient branches. Comparing the second and forth columns of Table 6 indicates that the results obtained by Model (3.15) have better discrimination power for DMUs than Cook and Zhu's model. Also, the last row of this table confirms this issue. That is, the mean of the 20 efficiency scores is decreased whenever Model (3.15) is used instead of Model (2.3). By changing the parameters of three polyhedral convex cones V, U, and Γ, the results are changed. All of the results obtained by Model (3.15), in this case study, confirm the theorems proven in Section 3.2.

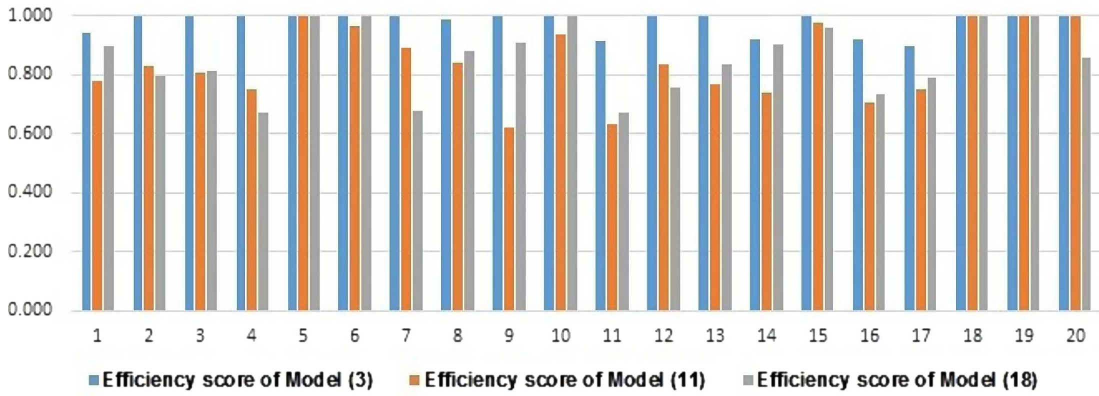

FIGURE 3. The efficiency scores obtained by Models (2.3), (3.8), and (3.15).

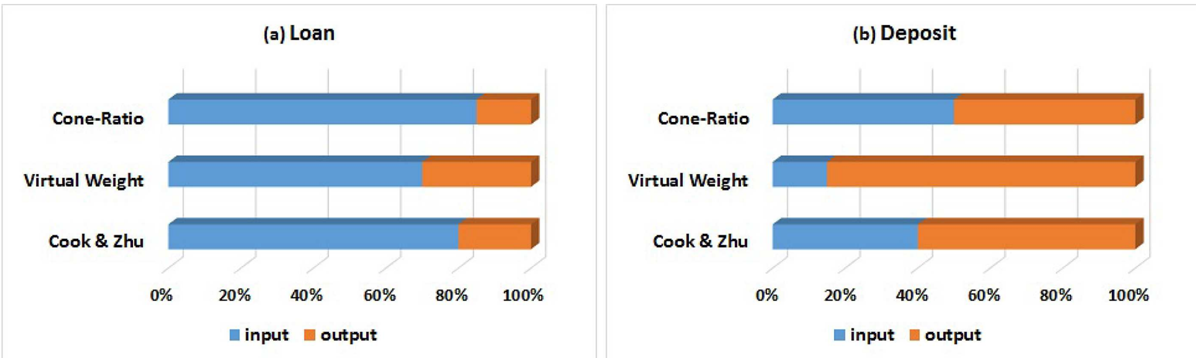

FIGURE 4. The status of dual-role factors.

From a geometrical perspective, Figure 3 displays the results of efficiency scores obtained by the three models (2.3), (3.8), and (3.15).

Figure 3 is geometrically indicated the correctness of the notes stated above.

Now, the results for the status of the two dual-role factors, loan and deposit, are discussed. Models (2.3), (3.8), and (3.15) imply that the loan is an input factor with the contributions 80%, 70%, and 85%, respectively. Also, the results show that the second dual-role factor, deposit, is an output factor with the contributions 60%, 85%, and 50%. Figure 4 shows the bar charts of the percentages for each dual-role factor obtained by Models (2.3), (3.8), and (3.15).

To assume the loan and the deposit as the dual-role factors is the best way to increase the system performance as much as possible. However, in the case that they should have explicit roles to improve the system performance, the results obtained show that the loan should be considered as an input factor and the deposit as an output factor. The following results confirm our claim. First, suppose the correct roles for the two dual-role factors; the means of the efficiency scores for the 20 bank branches taken by the CCR DEA models corresponding to Models (2.3) and (3.8) are 0.970 and 0.815, respectively. Next, consider incorrect roles for them; the means of the efficiency scores are 0.907 and 0.691 generated respectively by the CCR DEA models corresponding to Models (2.3) and (3.8).

Note that the loan and the deposit fail to have always the input and output roles, respectively. Regarding the data set for this case study, they play these roles but by changing the data set they may have other roles. It is evident that by changing the parameters of Models (3.8) and (3.15), in addition to changing efficiency scores, the results for the status of the dual-role factors will be changed.

5. Conclusion

Conventional DEA models consider the assumption of an explicit designation for each factor whether it is input or output. Still, there exist applications where this assumption is not realistic for some factors, known dual-role factors. In fact, the status of dual-role factors is not clear and they simultaneously play two roles. To date, there exist a good number of papers published for the evaluation of DMUs having dual-role factors apart from input and output factors. However, they are optimistic *i.e.* acquire the highest efficiency score for DMUs. Restricting weights is one of the approaches to overcome this weakness. In the literature of the DEA methodology, there does not exist any model whose contribution is the restriction of dual-role factor weights, even the weighted DEA model in the presence of dual-role factors presented by Farzipoor Saen [16]. To fill this gap, the current paper addressed this issue. To this end, Cook and Zhu's model [12] to evaluate DMUs with dual-role factors was improved using weighting methods. In this study, imposing additional weight restrictions was performed using two different methods, virtual weight restrictions and cone-ratio. The newly weighted DEA models not only do not overestimate the efficiency of the DMUs but also make relatively better discrimination for DMUs. We believe that the models presented could be employed in the real world applications. To prove the claim, the models were separately applied to the Iranian banking sector. Providing the number of cases that a dual-role factor can be considered as an input is equal to the number of cases of output, then some appropriate criteria can be developed. Discussion about these suitable criteria can be considered as a further research topic.

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