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PERFORMANCE ANALYSIS OF DISCRETE-TIME $Geo^X/G/1$ RETRIAL QUEUE WITH VARIOUS VACATION POLICIES AND IMPATIENT CUSTOMERS

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Abstract. This paper studies the behavior of a batch arrival single server retrial queueing model under three different vacation policies. Three types of vacation policies, single vacation, multiple vacations, and atmost J-vacations with impatient customers in general retrial times are considered. The probability generating function and marginal generating function of orbit size are obtained in a steady state. The stability condition for each vacation model is derived. Performance measures such as mean orbit size, mean system size, mean waiting time of a customer, and the probabilities of the server being in different states have also been determined. Based on performance characteristics, a comparative analysis is performed among the three vacations. Numerical illustrations are displayed to establish the consistency of the theory developed.

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1. Introduction

Waiting in queues is inevitable in human lives. Classical queues and retrial queues are becoming essential tools in manufacturing systems, communication systems, and computer networks. Modern communication systems work digitally rather than through analogue mode. In computer communication systems, telephone switching systems, the machine cycle time of a processor, and the bit or byte duration of signals on a transmission line are allowed to occur only at regularly spaced time points. The production system of factories operates on a discrete-time basis, where the events can occur only at regularly spaced epochs. Alfa [3] discussed applications of discrete-time queues and modeled single node telecommunication systems using the discrete-time queueing model in [4]. Anupam *et al.* [5] obtained optimal power consumption of DRX mechanism in LTE-A network which is studied in continuous-time but discrete-time queueing model is more appropriate than the continuous-time. Queueing models pay attention mainly to two groups of people; namely the customers (users) of the systems and the servers (service providers). In classical queues, customers must stand before the queue and servers are always ready to serve round the clock which is not practically possible. In many situations, customers who do not wait in front of the service centre, may book their slot and wait in a virtual queue. It is also not possible for the

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server to be present in the service station always. Further, the server may not be available for a certain duration due to many reasons. The service provider wishes that the customers are reasonably satisfied. To address this issue, discrete-time retrial queuing model with vacations would be an appropriate solution. Single, multiple, and atmost J-vacations are the most common vacation policies. In designing queueing models, the selection of suitable vacation and analyzing its performance characteristics would be the biggest challenges to the designer. Literature shows that most contributions are either based on any one of the vacation models or queueing models designed for one particular application. In multiple working or production environments, assigning the server with a suitable vacation based on the demand is one of the most important tasks. The goal of the proposed model is to provide a simple tool for allocating a server with appropriate vacation time.

1.1. Motivation

A single vacation policy can be seen in the flexible manufacturing system such that the machine (server) takes up maintenance activity (single vacation) after completing service, and when no items remain in the queue waiting for being processed. In the same flexible manufacturing system, consider a machine that is mainly used for producing customized products. If there is no arrival of customer's orders, the facility may be used to produce the items in stock. The machine will not back to process until the system receives the orders. In this model, "producing the customized products" is a service process and "producing the items in stock" (the secondary job of server) can be modeled as servers multiple vacations. In the production system, the machine (server) is kept in an idle state (vacation) after completing the service and no items are in the queue to get processed. The machine cannot remain in an idle state and as one of the measure of preventive maintenance, the machine should be brought to the service mode after completing maximum vacations (J-vacations).

Primary Health Care (PHC) is a vital component of the health-care system in which a doctor is available and the patient will be given consultation. Sometimes, the doctors may be availing the physical break but diagnoze the patients in online mode (working vacation) or the doctors may be working in some other patients (multiple vacations). Some of the patients (impatient customers) who are unable to wait due to many reasons will leave the PHC. This motivated the authors to study the different vacation policy queueing models and impatient customers.

The objective of the proposed model is to develop a simple tool for the allocation of servers with suitable vacation. As far as the author's knowledge is concerned, the literature reveals reported work on a discrete-time bulk arrival retrial queue with different vacation policies and comparative analysis of them. This paper will help the designers to select the appropriate model suitable for their work and determine performances of the system, particularly the mean orbit size. This motivated the authors to research $Geo^X/G/1$ retrial queue with various vacation policies.

The paper is structured as follows. Model description and preliminary notations used are given in Section 2. In Sections 3–5, the probability generating function for orbit size under three types of vacations with impatient customers are analyzed. In steady-state, performance characteristics such as mean orbit size, mean waiting time, and orbit size probabilities are obtained. In Section 6, a comparative analysis among vacations is done. In Section 7, numerical illustrations have been presented for the consistency of the theory developed. The conclusion of the research work with future direction is given in Section 8.

1.2. Related literature study

In the recent years, discrete-time retrial queues have grown rapidly due to their potential applications in digital communication systems. Performances of queueing models with the various patterns of arrivals, different kinds of services along with vacation policies have been explored by many researchers. Falin and Templeton [13] contributed a classical retrial queue and Yang and Li [30] emerged with the first work in discrete-time retrial queue. A detailed discussion on retrial can be viewed in Artalejo [6], Atencia and Morena [8], Gomez-Corral [15], Rein Nobel [22], and Ioannis Dimitriou [11]. The type of arrival is an important behavior in both classical and retrial queueing models. Queueing models differ based on different types of arriving customers such as balking, preemptive, feedback, positive, negative, recurrent, etc.. In many applications, customers arrive in bulk, for instance, a random size of packets is transmitted in digital communication systems. Takahashi *et al.* [28] considered a $Geo^X/G/1$ retrial queue with non-preemptive priority. Artalejo *et al.* [7] investigated a batch arrival $Geo^X/G/1$ retrial queue with control of admission using the maximum entropy method. Aboul Hassan *et al.* [1, 2] focus on the effect of balking customers and batch arrivals in $Geo/G/1$ retrial queueing model. Choudhury *et al.* [10] derived the distribution of orbit size of a batch arrival retrial queueing system with two phases of service and server interruption. Non-persistent customers will occur due to the long waiting time in the queue or orbit. Queues with non-persistent behavior tend to arise in many situations, particularly in the telecommunication industry where impatient customers tend to hang up their calls even before getting a response from a service station. Liu and Song [20] derived the recursive formulae for a steady-state distribution of $Geo/Geo/1$ retrial queueing model with vacation interruption and non-persistent retrial customers. Tao Jiang [19] investigated the polling system by modeling a bulk service retrial queue with impatient customers and obtained the joint probability distribution using a matrix geometric approach.

The interest in studying the queueing model with vacations has increased inorder to the utilize the idle time of the server and to optimize the total average cost. In a queuing system, when the queue length or orbit size is empty, the server will switch over to some other job – this period is referred to as the vacation period. In the last few decades, performances of queueing models with various vacation policies have been explored. Queues with various types of vacations were developed by Hunter [17], Doshi [12] and Takagi [27]. Servi and Finn [24] initiated a $M/M/1$ queue with a single working vacation in which the server gives a smaller service rate in the vacation period. $M/G/1$ queueing system with multiple working vacations was analyzed by Wu and Takagi [26]. After the completion of the service period, the server avails a vacation if no customer is waiting in the orbit and returns to the idle state – such a model is known as a single vacation policy. Zhang *et al.* [33] studied a discrete-time $Geo/G/1$ retrial queue with single vacation and starting failure. Shweta Upadhyaya and Geetika Malik [21] obtained the expected orbit size for a single-vacation retrial queueing model with preferred and impatient customers. In the J-vacations model, the server avails the first vacation if no customer is in the orbit after the service is completed. The server continues the vacation until it finds a customer for service or completes J-vacations following which the server returns to an idle state. Chang and Ke [9] discussed the characteristics of the batch arrival retrial queue with J-vacations in continuous time. Performance analysis of retrial model with J-vacations in discrete-time contributed by Yue and Zhang [31]. Randomized J-vacations in the discrete-time $Geo/G/1$ queue was modeled by Wang *et al.* [29] and the authors also obtained the waiting time and optimum cost analysis. In multiple vacations policy, if there is no customer waiting in the orbit at the time of server's vacation completion epoch, the server avails of another vacation and this state of vacation continues until the server finds another customer or a new customer to start the service.

Rosenberg and Yechiali [23] studied both single and multiple vacations. Jinting and Gao [14] focused on-orbit size distribution and sojourn time for a discrete-time bulk arrival retrial model with a working vacation policy. Arumuganathan *et al.* [16, 18] derived queue size probability generating function for batch arrival retrial queue and bulk state-dependent arrival $M^X/G/1$ retrial queue with multiple vacations using the supplementary variable technique. Zhang and Zhu [32] contributed a steady-state analysis of a queueing model with two different types of vacations out which one is a regular vacation and the other, a non-exhaustive urgent vacation. Sudhesh *et al.* [25] obtained time-dependent system size for queue under single and multiple working vacations with impatient customers.

2. Model description

The proposed model consists of batch arrival, single server retrial queue with impatient customers, and three types of vacations. Primary customers arrive in batches at the service station according to a geometric process with the parameter α, where α is the probability that a bulk of customers will arrive in the slot (m, m^+). An arrival of batch of L customers find the server to be free, one of the customers from the arrival batch gets its service immediately, while the remaining $L-1$ customers join the orbit with probability β. On the other

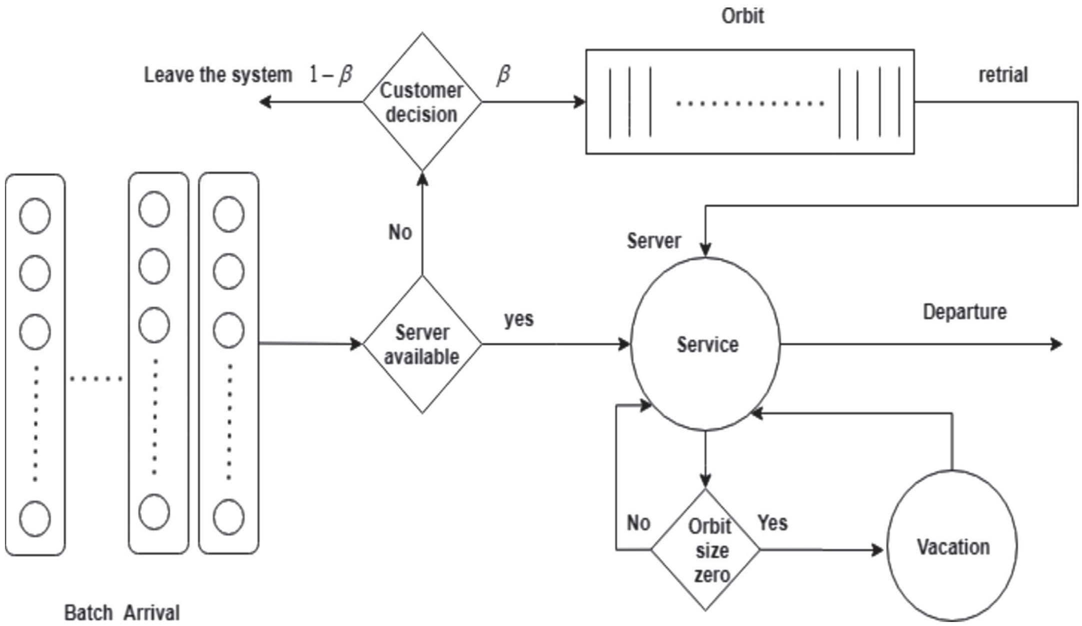

FIGURE 1. Schematic diagram – Model.

hand, if the server is not available at the time of batch arrival, the batch decides whether to join the orbit with probability β or to abandon the system with the probability $1 - \beta = \bar{\beta}$. The customers who leave the system never return later. It is assumed that customers in the orbit are usually persistent. At the time of service completion, the server avails of a vacation if the orbit size is zero. This paper aims to investigate batch arrival with impatient customers single server retrial queue with three policies namely single vacation, multiple vacations, and J-vacations policy.

Case (i). In a single vacation policy, the server avails of a single vacation when the orbit size is zero. After completing the vacation, if there is no arrival at the service station, the server returns to an idle state. Otherwise, the server starts a service when the server finds at least one arrival in the system.

Case (ii). The server avails of the first vacation when there are no customers in the orbit. Once the server is completed vacation, the server avails another vacation and continues to be on vacation untill there is a new arrival to the service station at each vacation completion time. The server returns to the service state only when the server finds at least one arrival to the system. This is called multiple vacations.

Case (iii). Under the J-vacations policy, the server avails the first vacation if the orbit size is zero at the time of service completion. The server either returns to the service state when the server finds at least one arrival to the system or the server is permitted to avail at most J number of vacations if there is no arrival of the customer at the end of each vacation completion period.

The batch size L is identically distributed random variable with the probability mass function $b_i = P(L = i), i \geq 1$, with probability generating function $B(X) = \sum_{h=1}^{\infty} b_h x^h$. It is assumed that inter-arrival times, service times, retrial times, and vacation times follow the general distribution and are independent of one another. Let $S(x) = \sum_{h=1}^{\infty} s_h x^h$, and $V(x) = \sum_{h=1}^{\infty} v_h x^h$ be the probability generating function of regular service time and vacation time, respectively. Let B_i, S_i and V_i are ith order factorial moment of batch size, service time and vacation time, respectively. The inter-retrial time is considered to be general with the probability distribution function $R(x) = \sum_{h=1}^{\infty} r_h x^h$. The schematic diagram of the model and vacation types are shown in Figures 1 and 2, respectively.

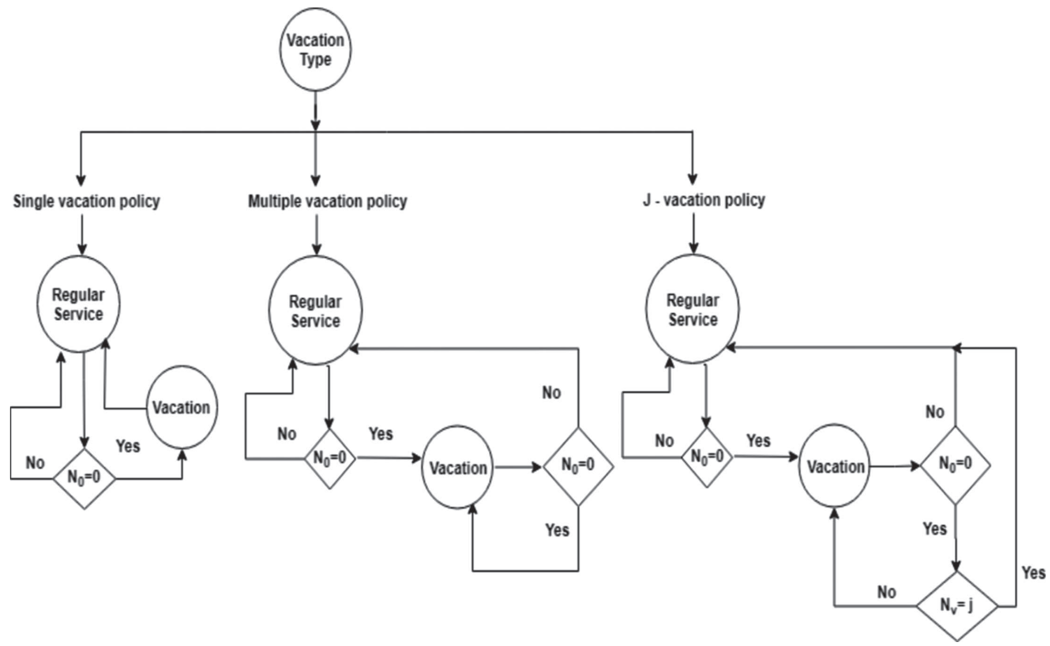

FIGURE 2. Schematic diagram – Different types of vacations.

3. Single vacation model (SVM)

In this section, a $Geo^X/G/1$ retrial queue with impatient customers and a single vacation is studied. A geometric arrival of bulk customers with rate α is considered. An arriving batch of L customers, if find the server free, one of the customers from the batch gets its service immediately and the remaining $L-1$ customers join the orbit with probability β. On the other hand, if the server is not free, the decision is made by the batch, with probability β joins the orbit or with probability, $\bar{\beta}$ the entire batch of L customers abandon from the system and never returns later. It is assumed that customers in the orbit are persistent. If the orbit is empty at the service completion epoch, the server avails a single vacation and resumes to an idle state immediately. The schematic diagram of the single vacation is shown in Figure 2. It is considered that service time, vacation time, and the inter-arrival time follow the general distribution and it is assumed that all are independent of one another.

At the time m^+, the state of the server is denoted as follows:

$$S_m^{(1)} = \begin{cases} 0 & \text{if the server is idle;} \\ 1 & \text{if the server is busy;} \\ 2 & \text{if the server is on vacation.} \end{cases}$$

Let $N_o^{(1)}$ denote the number of customers in the orbit, $R_{0,m}^{(1)}$ denote the remaining time for retrial, $R_{1,m}^{(1)}$ denote the remaining service time of the customer, $R_{2,m}^{(1)}$ denote the remaining vacation time of the server. The system is modeled by the stochastic process $\Gamma_m^{(1)} = \left(S_m^{(1)}, R_{k,m}^{(1)}, N_o^{(1)}\right)$. $\Gamma_m^{(1)}$ is a Markov chain and state space is

$$\zeta_m^{(1)} = \{(0,0) \cup (0,i,k) \cup (1,i,k) \cup (2,i,k); i \geq 1, k \geq 0\}.$$

To find the stationary distribution, the system steady-state probabilities are defined as follows

$$\Pi_{0,0}^{(1)} = \lim_{m \to \infty} P\left\{S_m^{(1)} = 0, N_o^{(1)} = 0\right\};$$

$$\Pi^{(1)}_{0,i,k} = \lim_{m\to\infty} P\Big\{S^{(1)}_m = 0, R^{(1)}_{0,m} = i, N^{(1)}_o = k\Big\}; \quad i \geq 1,\ k \geq 1,$$
$$\Pi^{(1)}_{1,i,k} = \lim_{m\to\infty} P\Big\{S^{(1)}_m = 1, R^{(1)}_{1,m} = i, N^{(1)}_o = k\Big\}; \quad i \geq 1,\ k \geq 0,$$
$$\Pi^{(1)}_{2,i,k} = \lim_{m\to\infty} P\Big\{S^{(1)}_m = 2, R^{(1)}_{2,m} = i, N^{(1)}_o = k\Big\}; \quad i \geq 1,\ k \geq 0.$$

The forward Kolmogorov equations are as follows:

$$\Pi^{(1)}_{0,0} = \bar{\alpha}\Pi^{(1)}_{0,0} + \bar{\alpha}\Pi^{(1)}_{2,1,0}, \tag{3.1}$$

$$\Pi^{(1)}_{0,i,k} = \bar{\alpha}\Pi^{(1)}_{0,i+1,k} + \bar{\alpha}r_i\Pi^{(1)}_{1,1,k} + \bar{\alpha}r_i\Pi^{(1)}_{2,1,k}; \qquad i \geq 1, k \geq 1, \tag{3.2}$$

$$\begin{aligned}\Pi^{(1)}_{1,i,k} = {} & \delta_{0,k}\alpha\beta b_{k+1}s_i\Pi^{(1)}_{0,0} + (1-\delta_{0,k})\alpha\beta s_i\sum_{l=1}^{k} b_l \sum_{j=1}^{\infty}\Pi^{(1)}_{0,j,k-l+1} + \bar{\alpha}s_i\Pi^{(1)}_{0,1,k+1} \\ & + \alpha\bar{\beta}\sum_{l=1}^{\infty} b_l\Pi^{(1)}_{1,i+1,k} + \bar{\alpha}\Pi^{(1)}_{1,i+1,k} + (1-\delta_{0,k})\alpha\beta\sum_{l=1}^{k} b_l\Pi^{(1)}_{1,i+1,k-l} \\ & + \alpha\beta s_i\sum_{l=1}^{k} b_l\Pi^{(1)}_{1,1,k-l+1} + \bar{\alpha}r_0 s_i\Pi^{(1)}_{1,1,k+1} \\ & + \bar{\alpha}r_0 s_i\Pi^{(1)}_{2,1,k+1} + (1-\delta_{0,k})\alpha\beta s_i\sum_{l=1}^{k} b_l\Pi^{(1)}_{2,1,k-l+1}; \qquad i \geq 1, k \geq 0,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\Pi^{(1)}_{2,i,k} = {} & \bar{\alpha}v_i\Pi^{(1)}_{1,1,0} + \alpha\bar{\beta}\sum_{l=1}^{\infty} b_l v_i\Pi^{(1)}_{1,1,0} + \bar{\alpha}\Pi^{(1)}_{2,i+1,k} \\ & + (1-\delta_{0,k})\alpha\beta\sum_{l=1}^{k} b_l\Pi^{(1)}_{2,i+1,k-l} + \alpha\bar{\beta}\sum_{l=1}^{\infty} b_l\Pi^{(1)}_{2,i+1,k}; \qquad i \geq 1, k \geq 0,\end{aligned} \tag{3.4}$$

where $\bar{\alpha} = 1-\alpha$, $\bar{\beta} = 1-\beta$ and $\delta_{i,j}$ denotes the Kronecker delta and the condition is

$$\Pi^{(1)}_{0,0} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\Pi^{(1)}_{0,j,k} + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\Pi^{(1)}_{1,j,k} + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\Pi^{(1)}_{2,j,k} = 1. \tag{3.5}$$

3.1. Steady-state orbit size probabilities

To determine the performance characteristics of the underlying model, the auxiliary generating functions and probability generating functions are considered as follows:

$$\Phi^{(1)}_{0,i}(u) = \sum_{k=1}^{\infty}\Pi^{(1)}_{0,i,k}u^k; \quad i \geq 1,$$
$$\Phi^{(1)}_{h,i}(u) = \sum_{k=0}^{\infty}\Pi^{(1)}_{h,i,k}u^k; \quad \text{where } h = 1,2, \text{ and,}$$
$$\Phi^{(1)}_{h}(x,u) = \sum_{i=1}^{\infty}\Phi^{(1)}_{h,i}(u)x^i; \quad \text{where } h = 0,1,2.$$

This part describes how steady-state orbit size probability generating functions (PGF) are derived. The following theorem provides the result to find the PGF of the number of customers in the buffer under the single vacation model.

Theorem 3.1. *The steady-state probability distribution of the Markov chain* $\Gamma_m^{(1)} = (S_m^{(1)}, R_{k,m}^{(1)}, N_o^{(1)})$ *at an arbitrary slot has the following probability generating functions:*

$$\Phi_0^{(1)}(x,u) = \frac{R(x)-R(\bar{\alpha})}{x-\bar{\alpha}} \frac{\alpha x[(u\nu(u)-\bar{\alpha}\beta B(u)S(\nu(u))V(\tau)+u(\nu(u)-\tau V(\nu(u)))]}{V(\tau)\Omega(u)}\Pi_{0,0}^{(1)}, \tag{3.6}$$

$$\Phi_1^{(1)}(x,u) = \frac{S(x)-S(\nu(u))}{x-\nu(u)} \frac{\alpha x[\nu(u)(\bar{\alpha}(1-\beta B(u))R(\bar{\alpha})+\alpha\beta B(u))V(\tau) +(\nu(u)-\tau V(\nu(u)))(\bar{\alpha}(1-\beta B(u))R(\bar{\alpha})+\beta B(u))]}{\bar{\alpha}V(\tau)\Omega(u)}\Pi_{0,0}^{(1)}, \tag{3.7}$$

$$\Phi_2^{(1)}(x,u) = \frac{V(x)-V(\nu(u))}{x-\nu(u)} \frac{\alpha x\tau}{\bar{\alpha}V(\tau)}\Pi_{0,0}^{(1)}, \tag{3.8}$$

where

$$\Pi_{0,0}^{(1)} = \frac{\bar{\alpha}\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))V(\tau)}{(1+V(\tau))T_1 - T_2 + T_3V(\tau)}, \tag{3.9}$$

$$\tau = 1-\alpha\beta, \tag{3.10}$$

$$\nu(u) = \bar{\alpha}+\alpha\bar{\beta}+\alpha\beta B(u), \tag{3.11}$$

$$\Omega(u) = \bar{\alpha}S(\nu(u))(1-\beta B(u))R(\bar{\alpha}) + \beta B(u)S(\nu(u)) - u\nu(u), \tag{3.12}$$

$$T_1 = \beta\big[\bar{\alpha}(1-R(\bar{\alpha}) + \alpha^2\beta B_1 S_1\big], \tag{3.13}$$

$$T_2 = \alpha\beta\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))B_1(\alpha\beta S_1+\tau V_1)), \tag{3.14}$$

$$T_3 = \bar{\alpha}\beta\bar{\beta}(1-\alpha\beta B_1 S_1), \tag{3.15}$$

$$T_4 = \beta\bar{\beta}[\bar{\alpha}R(\bar{\alpha})(1-\alpha\beta B_1\tau(S_1-V_1)) - \alpha\beta B_1(\alpha\beta B_1(\alpha\beta S_1+\tau V_1))]. \tag{3.16}$$

Proof of the Theorem 3.1. From the Kolmogorov equation (3.1), the condition $\Pi_{2,1,0}^{(1)} = \frac{\alpha}{\bar{\alpha}}\Pi_{0,0}^{(1)}$ is obtained. Multiplying equations (3.2)–(3.4) by u^k, and summing up k, the set of equations in terms of auxiliary generating functions has been determined as

$$\Phi_{0,i}^{(1)}(u) = \bar{\alpha}\Phi_{0,i+1}^{(1)}(u) + \bar{\alpha}r_i\Big[\Phi_{1,1}^{(1)}(u) - \Pi_{1,1,0}^{(1)}\Big] + \bar{\alpha}r_i\Big[\Phi_{2,1}^{(1)}(u) - \Pi_{2,1,0}^{(1)}\Big]; \qquad i \geq 1, \tag{3.17}$$

$$\begin{aligned}\Phi_{1,i}^{(1)}(u) = {} & \alpha\beta\frac{B(u)}{u}s_i\Pi_{0,0}^{(1)} + \alpha\beta\frac{B(u)}{u}s_i\Phi_0^{(1)}(1,u) + \bar{\alpha}s_i\frac{\Phi_{0,1}^{(1)}(u)}{u} + \nu(u)\Phi_{1,i+1}^{(1)}(u) \\ & + \left(\frac{\alpha\beta B(u)+\bar{\alpha}r_0}{u}\right)s_i\Big[\Phi_{1,1}^{(1)}(u) + \Phi_{2,1}^{(1)}(u)\Big] \\ & - \left(\frac{\alpha\beta B(u)+\bar{\alpha}r_0}{u}\right)s_i\Big(\Pi_{1,1,0}^{(1)} + \Pi_{2,1,0}^{(1)}\Big); \qquad i \geq 1,\end{aligned} \tag{3.18}$$

$$\Phi_{2,i}^{(1)}(u) = \tau v_i\Pi_{1,1,0}^{(1)}) + \nu(u)\Phi_{2,i+1}^{(1)}(u); \qquad i \geq 1, \tag{3.19}$$

where $\tau = \bar{\alpha}+\alpha\bar{\beta} = 1-\alpha\beta, \nu(u) = \bar{\alpha}+\alpha\bar{\beta}+\alpha\beta B(u)$. Multiplying equation (3.19) by x^i, summing over i,

$$\frac{(x-\nu(u))\Phi_2^{(1)}(x,u)}{x} = \tau V(x)\Pi_{1,1,0}^{(1)} - \nu(u)\Phi_{2,1}^{(1)}(u). \tag{3.20}$$

Substituting $x=\nu(u)$ in (3.20)

$$\Phi_{2,1}^{(1)}(u) = \frac{(\tau)V(\nu(u))}{\nu(u)}\Pi_{1,1,0}^{(1)}. \tag{3.21}$$

Substituting (3.21) in (3.20)

$$\Phi_2^{(1)}(x,u) = \frac{\big(V(x)-V(\nu(u))\big)}{x-\nu(u)}x\tau\Pi_{1,1,0}^{(1)}. \tag{3.22}$$

Differentiating (3.22) w.r.t x and taking $x=u=0$, we get $\Pi^{(1)}_{2,1,0}=V(\tau)\Pi^{(1)}_{1,1,0}$. □

Inserting the condition $\Pi^{(1)}_{2,1,0}=\frac{\alpha}{\bar{\alpha}}\Pi^{(1)}_{0,0}$, it is obtained $\Pi^{(1)}_{1,1,0}=\frac{\alpha}{\bar{\alpha}V(\tau)}\Pi^{(1)}_{0,0}$, and equation (3.21) reduced as $\Phi^{(1)}_{2,1}(u)=\frac{\alpha}{\bar{\alpha}}\frac{\tau}{V(\tau)}\frac{V(\nu(u))}{\nu(u)}\Pi^{(1)}_{0,0}$.

The PGF of orbit size when the server is in vacation is derived as

$$\Phi^{(1)}_{2}(x,u)=\frac{V(x)-V(\nu(u))}{x-\nu(u)}\frac{\alpha x\tau}{\bar{\alpha}V(\tau)}\Pi^{(1)}_{0,0}. \tag{3.23}$$

To complete the theorem, multiplying the equation (3.17) by x^i, summing over i from 1 to ∞ and applying the results of $\Pi^{(1)}_{1,1,0}$ and $\Pi^{(1)}_{2,1,0}$

$$\begin{aligned}\left(\frac{x-\bar{\alpha}}{x}\right)\Phi^{(1)}_{0}(x,u)&=\bar{\alpha}(R(x)-r_0)\Big[\Phi^{(1)}_{1,1}(u)+\Phi^{(1)}_{2,1}(u)\Big]-\bar{\alpha}\Phi^{(1)}_{0,1}(u)\\&\quad-\alpha(R(x)-a_0)\frac{1+V(\tau)}{V(\tau)}\Pi^{(1)}_{0,0}.\end{aligned} \tag{3.24}$$

Multiplying equation (3.18) by x^i, summing up i from 1 to ∞ and using all boundary conditions. $\Phi^{(1)}_0(1,u)$ can be obtained as follows from (3.24), by substituting $x=1$,

$$\begin{aligned}\left(\frac{x-\nu(u)}{x}\right)\Phi^{(1)}_{1}(x,u)&=\left(\frac{1}{u}-\frac{\beta B(u)}{u}\right)\bar{\alpha}S_(x)\Phi^{(1)}_{0,1}(u)\\&\quad+\left[\left(\frac{\beta(1-\bar{\alpha}a_0)B(u)}{u}+\frac{\bar{\alpha}r_0}{u}\right)S(x)-\nu(u)\right]\Phi^{(1)}_{1,1}(u)\\&\quad+\left[\left(\frac{\beta(1-\bar{\alpha}a_0)B(u)}{u}+\frac{\bar{\alpha}r_0}{u}\right)S(x)\right]\Phi^{(1)}_{2,1}(u)\\&\quad-\left[\left(\frac{\alpha\beta(1-a_0)B(u)}{u}+\frac{\alpha\beta B(u)+(1-\bar{\alpha}a_0)}{u}\right)\frac{1+V(\tau)}{V(\tau)}-\left(\frac{\alpha\beta B(u)}{u}\right)\right]S(x)\Pi^{(1)}_{0,0}.\end{aligned} \tag{3.25}$$

Taking $x=\bar{\alpha},x=\nu(u)$ in (3.24) and (3.25) respectively, $\Phi^{(1)}_{0,1}(u)$ and $\Phi^{(1)}_{1,1}(u)$ has been obtained as follows

$$\Phi^{(1)}_{0,1}(u)=\frac{(R(\bar{\alpha}-r_0)\alpha[(u\nu(u)-\bar{\alpha}\beta B(u)S(\nu(u))V(\tau)+u(\nu(u)-\tau V(\nu(u)))]}{V(\tau)\Omega(u)}\Pi^{(1)}_{0,0}, \tag{3.26}$$

$$\Phi^{(1)}_{1,1}(u)=\frac{\begin{array}{c}[\nu(u)(\bar{\alpha}(1-\beta B(u))R(\bar{\alpha})+\alpha\beta B(u))V(\tau)\\+(\nu(u)-\tau V(\nu(u)))(\bar{\alpha}(1-\beta B(u))R(\bar{\alpha})+\beta B(u))]\end{array}}{\bar{\alpha}V(\tau)\Omega(u)}\Pi^{(1)}_{0,0}, \tag{3.27}$$

where $\Omega(u)=\bar{\alpha}S(\nu(u))(1-\beta B(u))R(\bar{\alpha})+\beta B(u)S(\nu(u))-u\nu(u)$.

Substituting the above results in (3.24) and (3.25), the PGFs of number of customers in the orbit are concluded as

$$\Phi^{(1)}_{0}(x,u)=\frac{R(x)-R(\bar{\alpha})}{x-\bar{\alpha}}\frac{\alpha x[(u\nu(u)-\bar{\alpha}\beta B(u)S(\nu(u))V(\tau)+u(\nu(u)-\tau V(\nu(u)))]}{V(\tau)\Omega(u)}\Pi^{(1)}_{0,0}, \tag{3.28}$$

$$\Phi^{(1)}_{1}(x,u)=\frac{S(x)-S(\nu(u))}{x-\nu(u)}\frac{\begin{array}{c}\alpha x[\nu(u)(\bar{\alpha}(1-\beta B(u))R(\bar{\alpha})+\alpha\beta B(u))V(\tau)\\+(\nu(u)-\tau V(\nu(u)))(\bar{\alpha}(1-\beta B(u))R(\bar{\alpha})+\beta B(u))]\end{array}}{\bar{\alpha}V(\tau)\Omega(u)}\Pi^{(1)}_{0,0}. \tag{3.29}$$

The unknown constant $\Pi_{0,0}^{(1)}$ can be obtained from the steady-state condition (3.5). The equation (3.5) is given in terms of PGFs as

$$\Pi_{0,0}^{(1)} + \Phi_0^{(1)}(1,1) + \Phi_1^{(1)}(1,1) + \Phi_2^{(1)}(1,1) = 1. \tag{3.30}$$

$\Phi_0^{(1)}(1,1), \Phi_1^{(1)}(1,1)$, and $\Phi_2^{(1)}(1,1)$ are obtained from (3.28), (3.29), and (3.23) taking $x = u = 1$, it gives

$$\Pi_{0,0}^{(1)} = \frac{\bar{\alpha}\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))V(\tau)}{(1+V(\tau))T_1 - T_2 + T_3V(\tau)}. \tag{3.31}$$

The values of $T_i, i = 1,2,3$, and 4 are as follows

$$\begin{aligned}
T_1 &= \beta\big[\bar{\alpha}(1-R(\bar{\alpha}) + \alpha^2\beta B_1 S_1\big],\\
T_2 &= \alpha\beta\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))B_1(\alpha\beta S_1 + \tau V_1),\\
T_3 &= \bar{\alpha}\beta\bar{\beta}(1-\alpha\beta B_1 S_1),\\
T_4 &= \beta\bar{\beta}[\bar{\alpha}R(\bar{\alpha})(1-\alpha\beta B_1\tau(S_1 - V_1)) - \alpha\beta B_1(\alpha\beta B_1(\alpha\beta S_1 + \tau V_1))].
\end{aligned}$$

3.2. Stability condition

Equation (3.5) holds if the value of the constant $\Pi_{0,0}^{(1)}$ in (3.9) is greater than zero. $\Pi_{0,0}^{(1)} = \frac{\bar{\alpha}\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))V(\tau)}{(1+V(\tau))T_1-T_2+T_3V(\tau)} > 0$, which implies $\bar{\alpha}\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))V(\tau) > 0$. Since $V(\tau)$ and $R(\bar{\alpha}))$ are probabilities which are always greater than 0. The stability condition of the Markov chain $\Gamma_m^{(1)}$ is $\bar{\alpha}R(\bar{\alpha}) < 1$. It can be observed from the condition that the system is stable for any set of parameters.

3.3. Marginal generating functions

Taking limit $x \to 1$, from (3.6) to (3.8) in Theorem 3.1, under the stability condition, the marginal generating functions of the orbit size when the server in different states have been obtained as follows:

(1) The marginal generating function of the orbit size when the server is in idle state is given by

$$\Pi_{0,0}^{(1)} + \Phi_0^{(1)}(1,u) = \frac{\begin{array}{r}[((\bar{\alpha}S(\nu(u)) - u\nu(u))R(\bar{\alpha}) + \alpha\beta B(u)S(\nu(u)))V(\tau)\\ +u(1-R(\bar{\alpha})(\nu(u) - \tau V(\nu(u))))]\end{array}}{V(\tau)\Omega(u)}\Pi_{0,0}^{(1)}.$$

(2) The following result gives the marginal generating function of the orbit size when the server is in busy state with regular service is obtained as

$$\Phi_1^{(1)}(1,u) = \frac{1-S(\nu(u))}{1-\nu(u)}\frac{\begin{array}{c}\alpha[\nu(u)(\bar{\alpha}(1-\beta B(u))R(\bar{\alpha}) + \alpha\beta B(u))V(\tau)\\ +(\nu(u) - \tau V(\nu(u)))(\bar{\alpha}(1-\beta B(u))R(\bar{\alpha}) + \beta B(u))]\end{array}}{\bar{\alpha}V(\tau)\Omega(u)}\Pi_{0,0}^{(1)}.$$

(3) When the server is under vacation, the marginal generating function of the orbit size is given by

$$\Phi_2^{(1)}(1,u) = \frac{1-V(\nu(u))}{1-\nu(u)}\frac{\alpha\tau}{\bar{\alpha}V(\tau)}\Pi_{0,0}^{(1)}.$$

(4) The probability generating function of the mean orbit size is derived by

$$\Psi^{(1)}(u) = \Phi_0^{(1)}(1,u) + \Phi_1^{(1)}(1,u) + \Phi_2^{(1)}(1,u) \tag{3.32}$$

$$= \frac{N_1(u) - \xi_1(u)}{D_1(u)}\Pi_{0,0}^{(1)}. \tag{3.33}$$

where the values of $N_1(u), \xi_1(u)$ and $D_1(u)$ are

$$\begin{aligned}
N_1(u) &= A_1u(B(u))^2 - A_2uB(u)V(\nu(u)) + A_3uB(u) + A_4u(1-V(\nu(u))) + \alpha A_5(B(u))^2S(\nu(u)) \\
&\quad + A_6(B(u))^2 + A_7B(u)S(\nu(u)) + A_8B(u)V(\nu(u)) - A_9B(u) + A_{11}(1-V(\nu(u))),\\
\xi_1(u) &= \bar{\alpha}\alpha^2\beta^2R(\bar{\alpha})V(\tau)u(B(u))^2 + \alpha\bar{\alpha}\beta(1-2\alpha\beta)R(\bar{\alpha})V(\tau)uB(u) + \alpha\bar{\alpha}\beta(\tau)R(\bar{\alpha})V(\tau))u\\
&\quad + \alpha^2\beta^2(1-\bar{\alpha}R(\bar{\alpha}))(B(u))^2S(\nu(u)) + \alpha^2\beta(\alpha - \bar{\alpha}R(\bar{\alpha})V(\tau))(B(u))^2\\
&\quad + \alpha^2\beta(1-\beta(1-\bar{\alpha}R(\bar{\alpha})))(B(u))S(\nu(u)) + \alpha\beta(\alpha^2 + \alpha\beta(1+\beta))R(\bar{\alpha})V(\tau) - \alpha\bar{\alpha}\beta)B(u)\\
&\quad + A_{10}S(\nu(u)) + \alpha\bar{\alpha}(1-\alpha\beta)R(\bar{\alpha})V(\tau),\\
D_1(u) &= \bar{\alpha}(1-\nu(u))V(\tau)\Omega(u),
\end{aligned}$$

$A_1 = \alpha^2\beta^2(1-\bar{\alpha}R(\bar{\alpha})); A_2 = \alpha\beta(1-\alpha\beta)(1-\bar{\alpha}R(\bar{\alpha})); A3 = \alpha\beta((1-\bar{\alpha}) - \alpha\beta(\alpha - 2\bar{\alpha}R(\bar{\alpha})); A_4 = \alpha(1-\alpha\beta)(\bar{\beta}+\bar{\alpha}\beta R(\bar{\alpha})); A_5 = \alpha^2\beta^2(1-\bar{\alpha}R(\bar{\alpha})); A_6 = \alpha^2\beta^2(\alpha(1-V(\tau))-(1-\bar{\alpha}R(\bar{\alpha}))); A_7 = \alpha^2\beta(1-\bar{\alpha}R(\bar{\alpha})(\beta(1-V(\tau)-1)+V(\tau)); A_8 = \alpha\beta(1-\alpha\beta)(1-\bar{\alpha}R(\bar{\alpha})); A_9 = \alpha\beta(\alpha\beta(\bar{\alpha}(1-(R(\bar{\alpha}))+\alpha V(\tau)+\alpha^2R(\bar{\alpha})+(1-2R(\bar{\alpha})-V(\tau))\alpha+R(\bar{\alpha})-1))); A_{10} = \alpha\bar{\alpha}\bar{\beta}(1-R(\bar{\alpha}))V(\tau)); A_{11} = \alpha(1-\alpha\beta)\bar{\alpha}R(\bar{\alpha})$.

(5) The probability generating function of the mean system size is obtained by

$$\Phi^{(1)}(u) = \Pi^{(1)}_{0,0} + \Phi^{(1)}_0(1,u) + u\Phi^{(1)}_1(1,u) + \Phi^{(1)}_2(1,u).$$

3.4. Performance characteristics

Using the steady solution derived in Theorem 3.1, performance measures such as system state probabilities, mean orbit size, mean system size, and waiting time in the orbit are obtained as follows:

(1) Probability that the server is in idle state is given by

$$\Pi^{(1)}_{0,0} + \Phi^{(1)}_0(1,1) = \frac{\alpha((R(\bar{\alpha})-\beta)V(\tau) - \beta(1-R(\bar{\alpha})))}{\Omega(1)V(\tau)}\Pi^{(1)}_{0,0}. \tag{3.34}$$

(2) Probability that the server is in busy state is given by

$$\Phi^{(1)}_1(1,1) = \frac{\left[\left(\bar{\alpha}\bar{\beta}R(\bar{\alpha}) + \alpha\beta\right)V(\tau) + \left(\bar{\alpha}\bar{\beta}R(\bar{\alpha}) + \beta\right)\alpha\beta\right]\alpha^2\beta B_1S_1}{\bar{\alpha}\Omega(1)V(\tau)}\Pi^{(1)}_{0,0}. \tag{3.35}$$

(3) Probability that the server is under vacation is given by

$$\Phi^{(1)}_2(1,1) = \frac{\alpha^2\beta\tau B_1V_1}{\bar{\alpha}V(\tau)}\Pi^{(1)}_{0,0}. \tag{3.36}$$

(4) The rate of arrival of customers to the orbit is given by

$$P_{\text{orbit}} = \alpha\beta\Phi^{(1)}_1(1,1), \tag{3.37}$$

$$P_{\text{orbit}} = \alpha\beta\frac{\alpha S_1\left[(\bar{\alpha}\bar{\beta}R(\bar{\alpha}) + \alpha\beta)V(\tau) + (\bar{\alpha}\bar{\beta}R(\bar{\alpha}) + \beta)\alpha\beta\right]}{\bar{\alpha}\Omega(1)V(\tau)}\Pi^{(1)}_{0,0}. \tag{3.38}$$

(5) The mean number of customers in the orbit is given by $E(N) = \Psi'(1)$.

(6) Mean waiting time of a customer in the orbit is given by Little's formula

$$W_{\text{orbit}} = \frac{E(N)}{P_{\text{orbit}}}. \tag{3.39}$$

3.5. Particular cases

Assume that single arrival pattern, all the customers will join the orbit if server is busy, and no vacation. When $B(u) = u, \beta = 1$, and $V(\nu(u)) = 1$, the proposed model is become $Geo/G/1$ with general retrial times. The equations (3.6) and (3.7) imply the following PGFs which coincide to Atencia and Moreno [8].

$$\Phi_0^{(1)}(x,u) = \frac{[R(x) - R(\bar{\alpha})]\alpha x u[\nu(u) - S(\nu(u))]\Pi_{0,0}^{(1)}}{[x - \bar{\alpha}][(\bar{\alpha}R(\bar{\alpha}))(1-u)S(\nu(u)) - u(\nu(u) - S(\nu(u)))]}, \tag{3.40}$$

$$\Phi_1^{(1)}(x,u) = \frac{[S(x) - S(\nu(u))]\alpha x R(\bar{\alpha}))(1-u)\Big(\nu(u)\Pi_{0,0}^{(1)}\Big)}{[x - \nu(u)][(\bar{\alpha}R(\bar{\alpha}))(1-u)S(\nu(u)) - u(\nu(u) - S(\nu(u))]}, \tag{3.41}$$

where $\Pi_{0,0}^{(1)} = \frac{\alpha + \bar{\alpha}R(\bar{\alpha}) - \alpha S_1}{R(\bar{\alpha})}$ and $\nu(u) = \bar{\alpha} + \alpha u$.

4. Multiple vacations model (MVM)

This section considers a $Geo^X/G/1$ retrial queue with impatient customers and multiple vacations. A geometric arrival of bulk customers with rate α is considered. If an arriving batch of L customers find the server free, one of the customers from the arrival batch gets its service immediately and the remaining $L-1$ customers join the orbit with probability β. If the server is not available at the time of batch arrival, the decision is made by the batch whether to join the orbit with probability β or to abandon from the system with the probability $1 - \beta = \bar{\beta}$. The customers who leave the system never return later. However, it is assumed that customers in the orbit are persistent. If the orbit size is zero at the service completion epoch, the server avails the first vacation. The server takes another vacation if there is no customer in the system at the end of the vacation completion period and continues for remain in vacation mode. The server returns to its regular service state when the server finds at least one arrival to the system. The schematic diagram of multiple vacations is shown in Figure 2. Service time, vacation time, retrial time, and inter-arrival times are considered to follow general probability distribution and is assumed to be independent of one another.

At the time m^+, the state of the server is denoted by,

$$S_m^{(2)} = \begin{cases} 0 & \text{if the server is idle;} \\ 1 & \text{if the server is busy;} \\ 2 & \text{if the server is on vacation.} \end{cases}$$

Let $N_o^{(2)}$ denote the number of customers in the orbit, $N_v^{(2)}$, the number of vacations, $R_{0,m}^{(2)}$, the remaining time for retrial, $R_{1,m}^{(2)}$, the remaining service time of the customer, and $R_{2,m}^{(2)}$, the remaining vacation time of the server. The system has been modeled by the stochastic process $\Gamma_m^{(2)} = \Big(S_m^{(2)}, R_{k,m}^{(2)}, N_o^{(2)}\Big)$.

$\Gamma_m^{(2)}$ is a Markov chain and its state space is

$$\zeta_m^{(2)} = \{(0,0) \cup (0,i,k) \cup (1,i,k) \cup (2,i,k); i \geq 1, k \geq 0\}.$$

To find the stationary distribution, the steady-state probabilities are defined as follows

$$\Pi_{0,i,k}^{(2)} = \lim_{m\to\infty} P\Big\{S_m^{(2)} = 0, R_{0,m}^{(2)} = i, N_o^{(2)} = k\Big\}; \qquad i \geq 1,\ k \geq 1,$$

$$\Pi_{1,i,k}^{(2)} = \lim_{m\to\infty} P\Big\{S_m^{(2)} = 1, R_{1,m}^{(2)} = i, N_o^{(2)} = k\Big\}; \qquad i \geq 1,\ k \geq 0,$$

$$w_{n,i,k}^{(1)} = \lim_{m\to\infty} P\Big\{S_m^{(2)} = 2, N_v^{(2)} = n, R_{2,m}^{(2)} = i, N_o^{(2)} = k\Big\}; \quad i \geq 1,\ k \geq 0.$$

The balance equations of the underlying system are obtained as follows:

$$\Pi^{(2)}_{0,i,k} = \bar{\alpha}\Pi^{(2)}_{0,i+1,k} + \bar{\alpha}r_i\Pi^{(2)}_{1,1,k} + \bar{\alpha}r_i\sum_{n=1}^{\infty} w^{(1)}_{n,1,k}; \qquad i \geq 1, k \geq 1, \tag{4.1}$$

$$\begin{aligned}\Pi^{(2)}_{1,i,k} &= (1-\delta_{0,k})\alpha\beta s_i\sum_{l=1}^{k} b_l\sum_{j=1}^{\infty}\Pi^{(2)}_{0,j,k-l+1} + \bar{\alpha}s_i\Pi^{(2)}_{0,1,k+1} + \alpha\beta b_{k+1}s_i\Pi^{(2)}_{1,1,0}\\ &+ \bar{\alpha}\Pi^{(2)}_{1,i+1,k} + \alpha\bar{\beta}\sum_{l=1}^{\infty} b_l\Pi^{(2)}_{1,i+1,k} + (1-\delta_{0,k})\alpha\beta\sum_{l=1}^{k} b_l\Pi^{(2)}_{1,i+1,k-l}\\ &+ \bar{\alpha}r_0 s_i\Pi^{(2)}_{1,1,k+1} + \alpha\beta\sum_{l=1}^{k} b_l s_i\Pi^{(2)}_{1,1,k-l+1}\\ &+ \bar{\alpha}r_0 s_i\sum_{n=1}^{\infty} w^{(1)}_{n,1,k+1} + \alpha\beta s_i\sum_{l=1}^{k} b_l\sum_{n=1}^{\infty} w^{(1)}_{n,1,k-l+1}; \qquad i \geq 1, k \geq 0,\end{aligned} \tag{4.2}$$

$$\begin{aligned}w^{(1)}_{1,i,k} &= \delta_{0,k}\bar{\alpha}v_i\Pi^{(2)}_{1,1,k} + \delta_{0,k}\alpha\bar{\beta}\sum_{l=1}^{\infty} b_l v_i\Pi^{(2)}_{1,1,k} + \bar{\alpha}w^{(1)}_{1,i+1,k} + \alpha\bar{\beta}\sum_{l=1}^{\infty} b_l w^{(1)}_{n,i+1,k}\\ &+ (1-\delta_{0,k})\alpha\beta\sum_{l=1}^{k} b_l w^{(1)}_{1,i+1,k-l}; \qquad i \geq 1, k \geq 0,\end{aligned} \tag{4.3}$$

$$\begin{aligned}w^{(1)}_{n,i,k} &= \delta_{0,k}\bar{\alpha}v_i w^{(1)}_{n-1,1,k} + \delta_{0,k}\alpha\bar{\beta}\sum_{l=1}^{\infty} b_l v_i w^{(1)}_{n-1,1,k} + \bar{\alpha}w^{(1)}_{n,i+1,k} + \alpha\bar{\beta}\sum_{l=1}^{\infty} b_l w^{(1)}_{n,i+1,k}\\ &+ (1-\delta_{0,k})\alpha\beta\sum_{l=1}^{k} b_l w^{(1)}_{n,i+1,k-l}; \qquad n \geq 2,\ i \geq 1, k \geq 0,\end{aligned} \tag{4.4}$$

where $\bar{\alpha} = 1-\alpha, \bar{\beta} = 1-\beta$, and $\delta_{i,j}$ denotes the Kronecker delta and the condition is

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\Pi^{(2)}_{0,i,k} + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\Pi^{(2)}_{1,i,k} + \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} w^{(1)}_{n,i,k} = 1. \tag{4.5}$$

4.1. Steady-state orbit size probabilities

The probability generating functions are introduced as following to determine the performance characteristics of the underlying model

$$\begin{aligned}\Phi^{(2)}_{0,i}(u) &= \sum_{k=1}^{\infty}\Pi^{(2)}_{0,i,k}u^k; \qquad i \geq 1, \Phi^{(2)}_{1,i}(u) = \sum_{k=0}^{\infty}\Pi^{(2)}_{1,i,k}u^k; \quad i \geq 1,\\ w^{(1)}_{1,i}(u) &= \sum_{k=0}^{\infty} w^{(1)}_{1,i,k}u^k; \qquad i \geq 1, w^{(1)}_{n,i}(u) = \sum_{k=0}^{\infty} w^{(1)}_{n,i,k}u^k; \quad i \geq 1,\\ \Phi^{(2)}_{h}(x,u) &= \sum_{i=1}^{\infty}\Phi^{(2)}_{h,i}(u)x^i, \quad \text{where } h = 0,1;\\ w^{(1)}_{h}(x,u) &= \sum_{i=1}^{\infty} w^{(1)}_{h,i}(u)x^i, \quad \text{where } h = 1,n; \text{ and}\\ \Phi^{(2)}_{2}(x,u) &= \sum_{n=1}^{\infty} w^{(1)}_{n}(x,u).\end{aligned}$$

The performance measures such as orbit size and waiting time in the orbit are obtained from the probability generating function of the number of customers in the orbit. The following theorem provides the results for probability generating function of orbit size of the underlying queueing model with multiple vacations.

Theorem 4.1. *The steady-state probability distribution of the Markov chain*

$$\Gamma_m^{(2)} = \left(S_m^{(2)}, R_{k,m}^{(2)}, N_o^{(2)}\right)$$

at an arbitrary slot has the following probability generating functions:

$$\Phi_0^{(2)}(x,u) = \frac{R(x)-R(\bar{\alpha})}{x-\bar{\alpha}} \frac{\bar{\alpha}x[u(\nu(u)-(\tau)V(\nu(u)))-\alpha\beta B(u)S(\nu(u))(1-V(\tau))]}{(1-V(\tau))\Omega(u)}\Pi_{1,1,0}^{(2)}, \tag{4.6}$$

$$\Phi_1^{(2)}(x,u) = \frac{S(x)-S(\nu(u))}{x-\nu(u)} \frac{\begin{array}{c} x[\alpha\beta\nu(u)B(u)V(\tau)+\bar{\alpha}(1-\beta B(u))(\nu(u) \\ -(\tau)V(\nu(u)))R(\bar{\alpha})+\beta B(u)(\bar{\alpha}\nu(u)-(\tau)V(\nu(u)))]\end{array}}{(1-V(\tau))\Omega(u)}\Pi_{1,1,0}^{(2)}, \tag{4.7}$$

$$\Phi_2^{(2)}(x,u) = \frac{V(x)-V(\nu(u))}{x-\nu(u)} \frac{x\tau}{(1-V(\tau))}\Pi_{1,1,0}^{(2)}, \tag{4.8}$$

where the unknown constant

$$\Pi_{1,1,0}^{(2)} = \frac{(1-\bar{\alpha}R(\bar{\alpha}))(1-V(\tau))}{V(\tau)T_1-T_2}. \tag{4.9}$$

Proof of the Theorem 4.1. Multiplying the equations (4.1)–(4.4) by u^k, summing over k, $k=0,1,\dots$ and the following set of equations are obtained

$$\begin{aligned}\Phi_{0,i}^{(2)}(u) &= \bar{\alpha}\Phi_{0,i+1}^{(2)}(u) + \bar{\alpha}r_i\left[\Phi_{1,1}^{(2)}(u) - \Pi_{1,1,0}^{(2)}\right] \\ &\quad + \bar{\alpha}r_i\left[\sum_{n=1}^{\infty} w_{n,1}^{(1)}(u) - \sum_{n=1}^{\infty} w_{n,1,0}^{(1)}\right]; \qquad i\ge 1,\end{aligned} \tag{4.10}$$

$$\begin{aligned}\Phi_{1,i}^{(2)}(u) &= \alpha\beta\frac{B(u)}{u}s_i\Phi_0^{(2)}(1,u) + \bar{\alpha}s_i\frac{\Phi_{0,1}^{(2)}(u)}{u} + \alpha\beta\frac{B(u)}{u}s_i\Pi_{1,1,0}^{(2)} + \nu(u)\Phi_{1,i+1}^{(2)}(u) \\ &\quad + \left(\frac{\alpha\beta B(u)+\bar{\alpha}r_0}{u}\right)s_i\left[\Phi_{1,1}^{(2)}(u) + \sum_{n=1}^{\infty} w_{n,1}^{(1)}(u)\right] \\ &\quad - \left(\frac{\alpha\beta B(u)+\bar{\alpha}r_0}{u}\right)s_i\left(\Pi_{1,1,0}^{(2)} + \sum_{n=1}^{\infty} w_{n,1,0}^{(1)}\right); \qquad i\ge 1,\end{aligned} \tag{4.11}$$

$$w_{1,i}^{(1)}(u) = \tau v_i\left(\Pi_{1,1,0}^{(2)}\right) + \nu(u)w_{1,i+1}^{(1)}(u); \qquad i\ge 1, \tag{4.12}$$

$$w_{n,i}^{(1)}(u) = \tau v_i\left(w_{n,1,0}^{(1)}\right) + \nu(u)w_{n,i+1}^{(1)}(u); \qquad i\ge 1,\ n\ge 2. \tag{4.13}$$

To find $\Pi_{1,1,0}^{(2)}$ and $w_{n,1,0}^{(1)}$, multiplying the equations (4.12) by x^i, summing over i from $i=1,2,\dots$

$$\frac{(x-\nu(u))w_1^{(1)}(x,u)}{x} = \tau V(x)\Pi_{1,1,0}^{(2)} - \nu(u)w_{1,1}^{(1)}(u). \tag{4.14}$$

Substituting $x=\nu(u)$ in (4.14),

$$w_{1,1}^{(1)}(u) = \frac{(\tau)V(\nu(u))}{\nu(u)}\Pi_{1,1,0}^{(2)}. \tag{4.15}$$

Substituting (4.15) in (4.14),

$$w_1^{(1)}(x,u) = \frac{(V(x) - V(\nu(u)))}{x - \nu(u)} x\tau \Pi_{1,1,0}^{(2)}. \tag{4.16}$$

Differentiating (4.16) w.r.t x and taking $x = u = 0$, we get $w_{1,1,0}^{(1)} = V(\tau)\Pi_{1,1,0}^{(2)}$. □

Similarly multiplying the equation (4.13) by x^i, summing over i from $i = 1, 2, \ldots$, the reduced PGF is obtained as follows

$$w_n^{(1)}(x,u) = \frac{\big(V(x) - V(\nu(u))\big)}{x - \nu(u)} x\tau w_{n-1,1,0}^{(1)};\ n \geq 2. \tag{4.17}$$

Differentiating (4.17) w.r.t x and taking $x = u = 0$, the conditions obtained $w_{n,1,0}^{(1)} = V(\tau)w_{n-1,1,0}^{(1)} = V(\tau)^2 w_{n-2,1,0}^{(1)} = \ldots = V(\tau)^{n-1}w_{1,1,0}^{(1)} = V(\tau)^n \Pi_{1,1,0}^{(2)}$. Equation (4.17) becomes

$$w_n^{(1)}(x,u) = \frac{\big(V(x) - V(\nu(u))\big)}{x - \nu(u)} xV(\tau)^{n-1}\Pi_{1,1,0}^{(2)};\quad n \geq 2. \tag{4.18}$$

The probability generating function of the orbit size when the server is in vacation is derived by summing the equation (4.18) over n from $n = 2, 3, \ldots$ and adding with (4.16)

$$\Phi_2^{(2)}(x,u) = \frac{V(x) - V(\nu(u))}{x - \nu(u)} \frac{x\tau}{(1 - V(\tau))}\Pi_{1,1,0}^{(2)}. \tag{4.19}$$

To complete the theorem, equation (4.11) is multiplied by x^i, summing over i from 1 to ∞ and applying the results of $w_{n,1,0}^{(1)}$

$$\begin{aligned}\left(\frac{x - \bar{\alpha}}{x}\right)\Phi_0^{(2)}(x,u) &= \bar{\alpha}(R(x) - r_0)\left[\Phi_{1,1}^{(1)}(u) + \sum_{n=1}^{\infty} w_{n,1}^{(1)}(u)\right] - \bar{\alpha}\Phi_{0,1}^{(2)}(u)\\ &\quad - \bar{\alpha}(R(x) - r_0)\frac{1}{1 - V(\tau)}\Pi_{1,1,0}^{(2)}.\end{aligned} \tag{4.20}$$

$\Phi_0^{(1)}(1,u)$ can be obtained from (4.20) by substituting $x = 1$. Multiplying the equation (4.11) by x^i, summing over i from 1 to ∞

$$\begin{aligned}\left(\frac{x - \nu(u)}{x}\right)\Phi_1^{(2)}(x,u) &= \left(\frac{1}{u} - \frac{\beta B(u)}{u}\right)\bar{\alpha}S_(x)\Phi_{0,1}^{(2)}(u)\\ &\quad + \left[\left(\frac{\beta(1 - \bar{\alpha}r_0)B(u)}{u} + \frac{\bar{\alpha}r_0}{u}\right)S(x) - \nu(u)\right]\Phi_{1,1}^{(2)}(u)\\ &\quad + \left[\left(\frac{\beta(1 - \bar{\alpha}r_0)B(u)}{u} + \frac{\bar{\alpha}r_0}{u}\right)S(x)\right]\sum_{n=1}^{\infty} w_{n,1}^{(1)}(u)\\ &\quad - \left[\left(\frac{\beta(1 - \bar{\alpha}r_0)B(u)}{u} + \frac{\bar{\alpha}r_0}{u}\right)\frac{1}{1 - V(\tau)} - \left(\frac{\alpha\beta B(u)}{u}\right)\right]S(x)\Pi_{1,1,0}^{(2)}.\end{aligned} \tag{4.21}$$

Substitute the result $w_{n,1}^{(1)}(u)$ in (4.20) and (4.21). Solving the system for $\Phi_{0,1}^{(2)}(u)$ and $\Phi_{1,1}^{(2)}(u)$ by taking $x = \bar{\alpha}, x = \nu(u)$ in (4.20) and (4.21) respectively.

$$\Phi_{0,1}^{(2)}(u) = \frac{R(\bar{\alpha} - r_0)\alpha[u(\nu(u) - (\tau)V(\nu(u))) - \alpha\beta B(u)S(\nu(u))(1 - V(\tau))]}{(1 - V(\tau))\Omega(u)}\Pi_{1,1,0}^{(2)}, \tag{4.22}$$

$$\Phi_{1,1}^{(2)}(u) = \frac{\begin{array}{c} x[\alpha\beta\nu(u)B(u)V(\tau) + \bar{\alpha}(1-\beta B(u))(\nu(u) \\ -(\tau)V(\nu(u)))R(\bar{\alpha}) + \beta B(u)(\bar{\alpha}\nu(u) - (\tau)V(\nu(u)))] \end{array}}{(1-V(\tau))\Omega(u)} \Pi_{1,1,0}^{(2)}. \tag{4.23}$$

Substituting these results in (4.20) and (4.21), the PGFs of the number of customers in the orbit are concluded as

$$\Phi_0^{(2)}(x,u) = \frac{R(x)-R(\bar{\alpha})}{x-\bar{\alpha}} \frac{\bar{\alpha}x[u(\nu(u)-(\tau)V(\nu(u))) - \alpha\beta B(u)S(\nu(u))(1-V(\tau))]}{(1-V(\tau))\Omega(u)} \Pi_{1,1,0}^{(2)}, \tag{4.24}$$

$$\Phi_1^{(2)}(x,u) = \frac{S(x)-S(\nu(u))}{x-\nu(u)} \frac{\begin{array}{c} x[\alpha\beta\nu(u)B(u)V(\tau) + \bar{\alpha}(1-\beta B(u))(\nu(u) \\ -(\tau)V(\nu(u)))R(\bar{\alpha}) + \beta B(u)(\bar{\alpha}\nu(u) - (\tau)V(\nu(u)))] \end{array}}{(1-V(\tau))\Omega(u)} \Pi_{1,1,0}^{(2)}. \tag{4.25}$$

The proof has completed after the determination of the unknown constant $\Pi_{0,0}^{(1)}$ from the steady-state condition (4.5). The equation (4.5) is given in terms of probability generating functions as

$$\Phi_0^{(2)}(1,1) + \Phi_1^{(2)}(1,1) + \Phi_2^{(2)}(1,1) = 1. \tag{4.26}$$

$\Phi_0^{(2)}(1,1), \Phi_1^{(2)}(1,1)$, and $\Phi_2^{(2)}(1,1)$ are obtained from (4.24), (4.25), and (4.19) taking $x = u = 1$

$$\Pi_{1,1,0}^{(2)} = \frac{(1-\bar{\alpha}R(\bar{\alpha}))(1-V(\tau))}{(V(\tau)T_1 - T_2}. \tag{4.27}$$

4.2. Stability condition

The value of the constant $\Pi_{1,1,0}^{(2)}$ in (4.27) must be greater than zero to satisfy equation (4.5). $\Pi_{1,1,0}^{(2)} = \frac{(1-\bar{\alpha}R(\bar{\alpha}))(1-V(\tau))}{(V(\tau)T_1-T_2} > 0$ implies the result that $(1-\bar{\alpha}R(\bar{\alpha}))(1-V(\tau)) > 0$. It can be reduced as $\bar{\alpha}R(\bar{\alpha}) < 1$, which indicates that the system is stable for any set of parameters. This is called the ergodicity, of the Markov chain $\Gamma_m^{(2)}$.

4.3. Marginal generating functions

Taking limit $x \to 1$, from (4.6) to (4.8) in Theorem 4.1, under the stability condition, the marginal generating function of the orbit size when the server in different state is obtained.

(1) The marginal generating function of the orbit size when the server is in idle is given as

$$\Phi_0^{(2)}(1,u) = \frac{1-R(\bar{\alpha})}{1-\bar{\alpha}} \frac{\bar{\alpha}x[u(\nu(u)-(\tau)V(\nu(u))) - \alpha\beta B(u)S(\nu(u))(1-V(\tau))]}{(1-V(\tau))\Omega(u)} \Pi_{1,1,0}^{(2)}. \tag{4.28}$$

(2) The marginal generating function of the orbit size when the server is in busy state with regular service is obtained as follows

$$\Phi_1^{(2)}(1,u) = \frac{1-S(\nu(u))}{1-\nu(u)} \frac{\begin{array}{c} [\alpha\beta\nu(u)B(u)V(\tau) + \bar{\alpha}(1-\beta B(u))(\nu(u) \\ -(\tau)V(\nu(u)))R(\bar{\alpha}) + \beta B(u)(\bar{\alpha}\nu(u) - (\tau)V(\nu(u)))] \end{array}}{(1-V(\tau))\Omega(u)} \Pi_{1,1,0}^{(2)}. \tag{4.29}$$

(3) When the server is under vacation, the marginal generating function of the orbit size is given below

$$\Phi_2^{(2)}(1,u) = \frac{1-V(\nu(u))}{1-\nu(u)} \frac{\tau}{(1-V(\tau))} \Pi_{1,1,0}^{(2)}. \tag{4.30}$$

(4) The probability generating function of the orbit size is derived as

$$\Psi^{(2)}(u) = \Phi_0^{(2)}(1,u) + \Phi_1(1,u) + \Phi_2^{(2)}(1,u), \tag{4.31}$$

$$= \frac{N_1(u)}{D_2(u)}\Pi_{1,1,0}^{(2)}, \tag{4.32}$$

where $D_2(u) = \alpha(1-\nu(u))(1-V(\tau))\Omega(u)$.

(5) The probability generating function of the system size is obtained as follows:

$$\Phi^{(2)}(u) = \Phi_0^{(2)}(1,u) + u\Phi_1^{(2)}(1,u) + \Phi_2^{(2)}(1,u). \tag{4.33}$$

4.4. Performance characteristics

Using the steady solution derived in Theorem 4.1, performance measures such as system state probabilities, mean orbit size, mean system size, and waiting time in the orbit have been obtained as follows

(1) Probability that the server is in idle state is given by

$$\Phi_0^{(2)}(1,1) = \frac{\bar{\alpha}\beta(1-R(\bar{\alpha}))V(\tau)}{\Omega(1)(1-V(\tau))}\Pi_{1,1,0}^{(2)}. \tag{4.34}$$

(2) Probability that the server is in busy state is given by

$$\Phi_1^{(2)}(1,1) = \frac{\alpha^2\beta^2 B_1 S_1\big[(1-\bar{\alpha}R(\bar{\alpha}))\bar{\beta} - V(\tau)\big]}{\Omega(1)(1-V(\tau))}\Pi_{1,1,0}^{(2)}. \tag{4.35}$$

(3) Probability that the server is under vacation is given by

$$\Phi_2^{(2)}(1,1) = \frac{\alpha\beta(\tau)B_1V_1}{(1-V(\tau))}\Pi_{1,1,0}^{(2)}. \tag{4.36}$$

(4) The arrival rate of customers to the orbit is given by

$$P_{\text{orbit}} = \alpha\beta\Phi_1^{(2)}(1,1). \tag{4.37}$$

(5) The mean number of customers in the orbit group is given by $E(N) = \Psi'(1)$.

(6) The mean waiting time of a customer in the orbit is given by Little's formula as

$$W_{\text{orbit}} = \frac{E(N)}{P_{\text{orbit}}}. \tag{4.38}$$

4.5. Particular cases

Suppose that $\beta = 1$, and $V(\nu(u)) = 1$, there is no arrival of impatient customers and no vacation, the proposed model becomes $Geo^{[X]}/G/1$ with general retrial times. The following PGFs are obtained from the equations (4.6), and (4.7) which is similar to the results in Aboul Hassen *et al.* [2].

$$\Phi_0^{(2)}(x,u) = \frac{[R(x)-R(\bar{\alpha})]\alpha x[u - B(u)S(\nu(u))]\Pi_{0,0}^{(2)}}{[x-\bar{\alpha}][(\bar{\alpha}R(\bar{\alpha}))(1-B(u))S(\nu(u)) - (u\nu(u) - B(u)S(\nu(u)))]}, \tag{4.39}$$

$$\Phi_1^{(2)}(x,u) = \frac{[S(x)-S(\nu(u))]\alpha x R(\bar{\alpha})(1-B(u))\Big(\nu(u)\Pi_{0,0}^{(2)}\Big)}{[x-\nu(u)][(\bar{\alpha}R(\bar{\alpha}))(1-B(u))S(\nu(u)) - (u\nu(u) - B(u)S(\nu(u)))]}, \tag{4.40}$$

where the value of $\Pi_{0,0}^{(2)}$ is

$$\Pi_{0,0}^{(2)} = \frac{1 - \alpha B_1 S_1 - B_1\bar{\alpha}(1-R(\bar{\alpha}))}{R(\bar{\alpha})}. \tag{4.41}$$

5. J vacations model (JVM)

Performance analysis of $Geo^X/G/1$ retrial queue with impatient customers and J vacations is analyzed in this section. Batch arrivals occur according to geometric process with rate α. An arriving batch of L customers finds the server is not occupied, one of the customers from the batch gets immediate service and the rest of the $L-1$ customers join the orbit with probability β. On the other hand, if the server is not available at the time of batch arrival, the decision is made by the batch whether to join the orbit with probability β or to abandon from the system with the probability $1-\beta=\bar{\beta}$. The customers that leave the system never return later. It is assumed that customers in the orbit are persistent. At the server's service completion period, if the number of customers in the orbit is zero, then the server avails of the first vacation. If there is no arrival of the customer at the period of the server's first vacation completion, the server takes another vacation and continues with the maximum J-vacations. The server resumes to idle state either when arrival there to the system or after the completion of Jth vacation. The schematic diagram of the J-vacations policy is shown in Figure 2. It is considered that the inter-arrival time, service time, retrial time, and vacation time follow general probability distribution and it is assumed that these times are independent of one another.

At the time m^+, the state of the server is denoted by,

$$S_m^{(3)} = \begin{cases} 0 & \text{if the server is idle;} \\ 1 & \text{if the server is busy;} \\ 2 & \text{if the server is on vacation.} \end{cases}$$

Let $N_o^{(3)}$ denote the number of customers in the orbit, $N_v^{(3)}$, the number of vacations, $R_{0,m}^{(3)}$, the remaining time for retrial, $R_{1,m}^{(3)}$, the remaining service time of the customer, and $R_{2,m}^{(3)}$, the remaining vacation time of the server. The system is modeled by the stochastic process $\Gamma_m^{(3)} = \left(S_m^{(3)}, R_{k,m}^{(3)}, N_o^{(3)}\right)$.

$\Gamma_m^{(3)}$ is a Markov chain and state space is

$$\zeta_{JVM}^{(3)} = \{(0,0) \cup (0,i,k) \cup (1,i,k) \cup (2,i,k); i \geq 1, k \geq 0\}.$$

To find the stationary distribution, the system steady-state probabilities are defined as follows

$$\begin{aligned}
\Pi_{0,0}^{(3)} &= \lim_{m\to\infty} P\left\{S_m^{(3)}=0, N_o^{(3)}=0\right\}, \\
\Pi_{0,i,k}^{(3)} &= \lim_{m\to\infty} P\left\{S_m^{(3)}=0, R_{0,m}^{(3)}=i, N_o^{(3)}=k\right\}; && i\geq 1,\ k\geq 1, \\
\Pi_{1,i,k}^{(3)} &= \lim_{m\to\infty} P\left\{S_m^{(3)}=1, R_{1,m}^{(3)}=i, N_o^{(3)}=k\right\}; && i\geq 1,\ k\geq 0, \\
w_{j,i,k}^{(2)} &= \lim_{m\to\infty} P\left\{S_m^{(3)}=2, N_v^{(3)}=J, R_{2,m}^{(3)}=i, N_o^{(3)}=k\right\}; && i\geq 1,\ k\geq 0,
\end{aligned}$$

The balance equations of the underlying system are obtained as follows:

$$\Pi_{0,0}^{(3)} = \bar{\alpha}\Pi_{0,0}^{(3)} + \bar{\alpha} w_{j,1,0}^{(2)}, \tag{5.1}$$

$$\Pi_{0,i,k}^{(3)} = \bar{\alpha}\Pi_{0,i+1,k}^{(3)} + \bar{\alpha} r_i \Pi_{1,1,k}^{(3)} + \bar{\alpha} r_i \sum_{n=1}^{j} w_{n,1,k}^{(2)}; \qquad i\geq 1, k\geq 1, \tag{5.2}$$

$$\begin{aligned}
\Pi_{1,i,k}^{(3)} &= \delta_{0,k}\alpha\beta b_{k+1} s_i \Pi_{0,0}^{(3)} + (1-\delta_{0,k})\alpha\beta s_i \sum_{l=1}^{k} b_l \sum_{j=1}^{\infty} \Pi_{0,j,k-l+1}^{(3)} + \bar{\alpha} s_i \Pi_{0,1,k+1}^{(3)} \\
&\quad + \bar{\alpha}\Pi_{1,i+1,k}^{(3)} + \alpha\bar{\beta}\sum_{l=1}^{\infty} b_l \Pi_{1,i+1,k}^{(3)} + (1-\delta_{0,k})\alpha\beta \sum_{l=1}^{k} b_l \Pi_{1,i+1,k-l}^{(3)}
\end{aligned}$$

$$+ \bar{\alpha} r_0 s_i \Pi^{(3)}_{1,1,k+1} + \alpha\beta \sum_{l=1}^{k} b_l s_i \Pi^{(3)}_{1,1,k-l+1}$$

$$+ \bar{\alpha} r_0 s_i \sum_{n=1}^{j} w^{(2)}_{n,1,k+1} + \alpha\beta s_i \sum_{l=1}^{k} b_l \sum_{n=1}^{j} w^{(2)}_{n,1,k-l+1}; \qquad i \geq 1, k \geq 0, \tag{5.3}$$

$$w^{(2)}_{1,i,k} = \delta_{0,k} \bar{\alpha} v_i \Pi^{(3)}_{1,1,k} + \delta_{0,k} \alpha\bar{\beta} \sum_{l=1}^{\infty} b_l v_i \Pi^{(3)}_{1,1,k} + \bar{\alpha} w_{1,i+1,k} + \alpha\bar{\beta} \sum_{l=1}^{\infty} b_l w^{(2)}_{n,i+1,k}$$

$$+ (1-\delta_{0,k})\alpha\beta \sum_{l=1}^{k} b_l w^{(2)}_{1,i+1,k-l}; \qquad i \geq 1, k \geq 0, \tag{5.4}$$

$$w^{(2)}_{n,i,k} = \delta_{0,k} \bar{\alpha} v_i w^{(2)}_{n-1,1,k} + \delta_{0,k} \alpha\bar{\beta} \sum_{l=1}^{\infty} b_l v_i w^{(2)}_{n-1,1,k} + \bar{\alpha} w^{(2)}_{n,i+1,k}$$

$$+ \alpha\bar{\beta} \sum_{l=1}^{j} b_l w^{(2)}_{n,i+1,k} + (1-\delta_{0,k})\alpha\beta \sum_{l=1}^{k} b_l w^{(2)}_{n,i+1,k-l}; \qquad 2 \leq n \leq j,\ i \geq 1, k \geq 0, \tag{5.5}$$

where $\bar{\alpha} = 1-\alpha$ and $\delta_{i,j}$ denotes the Kronecker delta and the condition is

$$\Pi^{(3)}_{0,0} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \Pi^{(3)}_{0,i,k} + \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \Pi^{(3)}_{1,i,k} + \sum_{n=1}^{j}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} w^{(2)}_{n,i,k} = 1. \tag{5.6}$$

5.1. Steady-state orbit size probabilities

The probability generating functions are introduced as follows to obtain the performance measures of the underlying model

$$\begin{aligned}
\Phi^{(3)}_{0,i}(u) &= \sum_{k=1}^{\infty} \Pi^{(3)}_{0,i,k} u^k; \qquad i \geq 1, \Phi^{(3)}_{1,i}(u) = \sum_{k=0}^{\infty} \Pi^{(3)}_{1,i,k} u^k; \quad i \geq 1,\\
w^{(2)}_{1,i}(u) &= \sum_{k=0}^{\infty} w^{(2)}_{1,i,k} u^k; \qquad i \geq 1, w^{(2)}_{n,i}(u) = \sum_{k=0}^{\infty} w^{(2)}_{n,i,k} u^k; \quad i \geq 1,\\
\Phi^{(3)}_{h}(x,u) &= \sum_{i=1}^{\infty} \Phi^{(2)}_{h,i}(u) x^i, \quad \text{where } h = 0,1\\
w^{(2)}_{h}(x,u) &= \sum_{i=1}^{\infty} w^{(2)}_{h,i}(u) x^i, \quad \text{where } h = 1, n \text{ and}\\
\Phi^{(3)}_{2}(x,u) &= \sum_{n=1}^{j} w^{(2)}_{n}(x,u).
\end{aligned}$$

Steady-state orbit size probability generating functions have also been derived to obtain the performance characteristics of the system. The orbit size probability generating functions for the underlying J vacations model are derived in the following theorem.

Theorem 5.1. *The steady-state probability distribution of the Markov chain* $\Gamma^{(3)}_m = \left(S^{(3)}_m, R^{(3)}_{k,m}, N^{(3)}_o\right)$ *at an arbitrary slot has the following probability generating functions:*

$$\Phi^{(3)}_0(x,u) = \frac{R(x)-R(\bar{\alpha})}{x-\bar{\alpha}} \frac{\bar{\alpha}x\big[u\nu(u)V(\tau)^{J+1} + \big(\bar{\alpha}\beta B(u)(1-V(\tau))S(\nu(u)) - (\tau)uV(\nu(u))\big)V(\tau)^J - u(\nu(u)-(\tau)V(\nu(u))\big]}{(1-V(\tau))V(\tau)^J\Omega(u)} \Pi^{(3)}_{0,0}, \tag{5.7}$$

$$\Phi_1^{(3)}(x,u) = \frac{S(x)-S(\nu(u))}{x-\nu(u)} \frac{\alpha x\big[\nu(u)(\eta(u)-\bar{\alpha}\beta B(u))V(\tau)^{J+1} + (\bar{\alpha}\beta\nu(u)B(u) - \tau\eta(u))V(\tau)^J + (\eta(u)(\tau-\nu(u)))\big]}{\bar{\alpha}(1-V(\tau))V(\tau)^J\Omega(u)}\Pi_{0,0}^{(3)}, \tag{5.8}$$

$$\Phi_2^{(3)}(x,u) = \frac{V(x)-V(\nu(u))}{x-\nu(u)} \frac{\alpha x(\tau)\big(1-V(\tau)^J\big)}{\bar{\alpha}(1-V(\tau))V(\tau)^J}\Pi_{0,0}^{(3)}, \tag{5.9}$$

where

$$\Pi_{0,0}^{(3)} = \frac{\bar{\alpha}\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))(V(\tau))^J(1-V(\tau))}{(1+(V(\tau))^{J+1})T_1 - T_2 + T_3(V(\tau))^{J+1} + T_4(V(\tau))^J}. \tag{5.10}$$

Proof of the Theorem 5.1. From the Kolmogorov equation (5.1), the condition $w_{j,1,0}^{(2)} = \frac{\alpha}{\bar{\alpha}}\Pi_{0,0}^{(3)}$ is obtained. multiplying the equations (5.2)–(5.5) by u^k, summing up k, and the set of equations in terms of auxiliary generating functions has determined as

$$\begin{aligned}
\Phi_{0,i}^{(3)}(u) &= \bar{\alpha}\Phi_{0,i+1}^{(3)}(u) + \bar{\alpha}r_i\Big[\Phi_{1,1}^{(3)}(u) - \Pi_{1,1,0}^{(3)}\Big] \\
&\quad + \bar{\alpha}r_i\left[\sum_{n=1}^{j} w_{n,1}^{(2)}(u) - \sum_{n=1}^{j} w_{n,1,0}^{(2)}\right]; \qquad i \ge 1,
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\Phi_{1,i}^{(2)}(u) &= \alpha\beta\frac{B(u)}{u}s_i\Phi_0^{(3)}(1,u) + \bar{\alpha}s_i\frac{\Phi_{0,1}^{(3)}(u)}{u} + \nu(u)\Phi_{1,i+1}^{(3)}(u) \\
&\quad + \alpha\beta\frac{B(u)}{u}s_i\Pi_{0,0}^{(3)} + \left(\frac{\alpha\beta B(u)+\bar{\alpha}r_0}{u}\right)s_i\left[\Phi_{1,1}^{(3)}(u) + \sum_{n=1}^{j} w_{n,1}^{(2)}(u)\right] \\
&\quad - \left(\frac{\alpha\beta B(u)+\bar{\alpha}r_0}{u}\right)s_i\left(\Pi_{1,1,0}^{(3)} + \sum_{n=1}^{j} w_{n,1,0}^{(2)}\right); \qquad i \ge 1,
\end{aligned} \tag{5.12}$$

$$w_{1,i}^{(2)}(u) = \tau v_i\Pi_{1,1,0}^{(3)}) + \nu(u)w_{1,i+1}^{(1)}(u); \quad i \ge 1, \tag{5.13}$$

$$w_{n,i}^{(2)}(u) = \tau v_i w_{n,1,0}^{(2)}) + \nu(u)w_{n,i+1}^{(2)}(u); \qquad i \ge 1,\ 2 \le n \le j, \tag{5.14}$$

To find $\Pi_{1,1,0}^{(3)}$ and $w_{n,1,0}^{(2)}$, multiplying the equation (5.13) by x^i, summing over i from $i = 1, 2, \ldots$

$$\frac{(x-\nu(u))w_1^{(2)}(x,u)}{x} = \tau V(x)\Pi_{1,1,0}^{(3)} - \nu(u)w_{1,1}^{(2)}(u). \tag{5.15}$$

Substituting $x = \nu(u)$ in (5.15)

$$w_{1,1}^{(2)}(u) = \frac{(\tau)V(\nu(u))}{\nu(u)}\Pi_{1,1,0}^{(3)}. \tag{5.16}$$

Substituting (5.16) in (5.15)

$$w_1^{(2)}(x,u) = \frac{\big(V(x)-V(\nu(u))\big)}{x-\nu(u)}x\tau\Pi_{1,1,0}^{(3)}. \tag{5.17}$$

Differentiating (5.17) w.r.t x and taking $x = u = 0$, we get $w_{1,1,0}^{(2)} = V(\tau)\Pi_{1,1,0}^{(3)}$. □

Similarly multiplying the equation (5.14) by x^i, summing over i from $i = 1, 2, \ldots$, the reduced PGF is obtained as follows

$$\frac{(x-\nu(u))w_n^{(2)}(x,u)}{x} = \tau V(x)w_{n-1,1,0}^{(2)} - \nu(u)w_{n,1}^{(2)}(u) \quad 2 \le n \le j. \tag{5.18}$$

Substituting $x=\nu(u)$ in (5.18)

$$w_{n,1}^{(2)}(u)=\frac{(\tau)V(\nu(u))}{\nu(u)}w_{n-1,1,0}^{(2)}. \tag{5.19}$$

Substituting (5.19) in (5.18)

$$w_{n}^{(2)}(x,u)=\frac{\big(V(x)-V(\nu(u))\big)}{x-\nu(u)}x\tau w_{n-1,1,0}^{(2)}. \tag{5.20}$$

Differentiating (5.20) w.r.t x and taking $x=u=0$, the conditions obtained $w_{n,1,0}^{(2)}=V(\tau)w_{n-1,1,0}^{(2)}=V(\tau)^2w_{n-2,1,0}^{(2)}=\ldots=V(\tau)^{n-1}w_{1,1,0}^{(2)}=V(\tau)^n\Pi_{1,1,0}^{(3)}$ and $w_{j,1,0}^{(2)}=V(\tau)^j\Pi_{1,1,0}^{(3)}$. Equation (5.20) becomes

$$w_{n}^{(2)}(x,u)=\frac{\big(V(x)-V(\nu(u))\big)}{x-\nu(u)}xV(\tau)^{n-1}\Pi_{1,1,0}^{(3)};\quad 2\leq n\leq j. \tag{5.21}$$

Since $w_{j,1,0}^{(2)}=\frac{\alpha}{\bar{\alpha}}\Pi_{0,0}^{(3)}$ and $w_{j,1,0}^{(2)}=V(\tau)^j\Pi_{1,1,0}^{(3)}$ implies that $\Pi_{1,1,0}^{(3)}=\frac{\alpha}{\bar{\alpha}V(\tau)^j}\Pi_{0,0}^{(3)}$. The probability generating function of the orbit size when the server is in vacation is derived as follows by summing up the equation (5.21) over n from $n=2,3,\ldots j$ and adding with (5.17)

$$\Phi_{2}^{(3)}(x,u)=\frac{V(x)-V(\nu(u))}{x-\nu(u)}\frac{\alpha x(\tau)\big(1-V(\tau)^J\big)}{\bar{\alpha}(1-V(\tau))V(\tau)^J}\Pi_{0,0}^{(3)}. \tag{5.22}$$

Further, to complete the proof, equation (5.11) has been multiplied by x^i, summing over i from 1 to ∞ and applying the results of $w_{n,1,0}^{(2)}$

$$\begin{aligned}\left(\frac{x-\bar{\alpha}}{x}\right)\Phi_{0}^{(3)}(x,u)=\bar{\alpha}(R(x)-r_0)&\left[\Phi_{1,1}^{(3)}(u)+\sum_{n=1}^{j}w_{n,1}^{(2)}(u)\right]-\bar{\alpha}\Phi_{0,1}^{(3)}(u)\\&-\alpha(R(x)-r_0)\frac{1-V(\tau)^J}{(1-V(\tau))V(\tau)^J}\Pi_{0,0}^{(3)}.\end{aligned} \tag{5.23}$$

Multiplying the equation (5.12) by x^i, summing over i from 1 to ∞.$\Phi_0^{(3)}(1,u)$ can be obtained from (5.23) by substituting $x=1$.

$$\begin{aligned}\left(\frac{x-\nu(u)}{x}\right)\Phi_{1}^{(3)}(x,u)=&\left(\frac{1}{u}-\frac{\beta B(u)}{u}\right)\bar{\alpha}S_(x)\Phi_{0,1}^{(3)}(u)\\&+\left[\left(\frac{\beta(1-\bar{\alpha}r_0)B(u)}{u}+\frac{\bar{\alpha}r_0}{u}\right)S(x)-\nu(u)\right]\Phi_{1,1}^{(3)}(u)\\&+\left[\left(\frac{\beta(1-\bar{\alpha}r_0)B(u)}{u}+\frac{\bar{\alpha}r_0}{u}\right)S(x)\right]\sum_{n=1}^{j}w_{n,1}^{(2)}(u)\\&-\left[\left(\frac{\beta(1-\bar{\alpha}r_0)B(u)}{u}+\frac{\bar{\alpha}r_0}{u}\right)\frac{1-V(\tau)^J}{(1-V(\tau))V(\tau)^J}-\left(\frac{\alpha\beta B(u)}{u}\right)\right]S(x)\Pi_{0,0}^{(3)}.\end{aligned} \tag{5.24}$$

Substitute the result $w_{n,1}^{(2)}(u)$ in (5.23) and (5.24). Solving the system for $\Phi_{0,1}^{(3)}(u)$ and $\Phi_{1,1}^{(3)}(u)$ by taking $x=\bar{\alpha}$, $x=\nu(u)$ in (5.23) and (5.24) respectively.

$$\Phi_{0,1}^{(3)}(u)=\frac{\begin{aligned}&\bar{\alpha}\big[u\nu(u)V(\tau)^{J+1}+(\bar{\alpha}\beta B(u)(1-V(\tau))S(\nu(u))\\&-(\tau)uV(\nu(u)))V(\tau)^J-u(\nu(u)-(\tau)V(\nu(u))\big]\end{aligned}}{(1-V(\tau))V(\tau)^J\Omega(u)}\Pi_{0,0}^{(3)}, \tag{5.25}$$

$$\Phi_{1,1}^{(3)}(u) = \frac{\begin{array}{r}\alpha\big[\nu(u)(\eta(u)-\bar{\alpha}\beta B(u))V(\tau)^{J+1}+(\bar{\alpha}\beta\nu(u)B(u)\\ -\tau\eta(u))V(\tau)^{J}+(\eta(u)(\tau-\nu(u)))\big]\end{array}}{\bar{\alpha}(1-V(\tau))V(\tau)^{J}\Omega(u)}\Pi_{0,0}^{(3)}. \tag{5.26}$$

Substituting these results in (5.23) and (5.24), the probability generating functions are concluded as follows:

$$\Phi_{0}^{(3)}(x,u) = \frac{R(x)-R(\bar{\alpha})}{x-\bar{\alpha}}\frac{\begin{array}{r}\bar{\alpha}x\big[u\nu(u)V(\tau)^{J+1}+(\bar{\alpha}\beta B(u)(1-V(\tau))S(\nu(u))\\ -(\tau)uV(\nu(u)))V(\tau)^{J}-u(\nu(u)-(\tau)V(\nu(u))\big]\end{array}}{(1-V(\tau))V(\tau)^{J}\Omega(u)}\Pi_{0,0}^{(3)}, \tag{5.27}$$

$$\Phi_{1}^{(3)}(x,u) = \frac{S(x)-S(\nu(u))}{x-\nu(u)}\frac{\begin{array}{r}\alpha x\big[\nu(u)(\eta(u)-\bar{\alpha}\beta B(u))V(\tau)^{J+1}+(\bar{\alpha}\beta\nu(u)B(u)\\ -\tau\eta(u))V(\tau)^{J}+(\eta(u)(\tau-\nu(u)))\big]\end{array}}{\bar{\alpha}(1-V(\tau))V(\tau)^{J}\Omega(u)}\Pi_{0,0}^{(3)}, \tag{5.28}$$

The unknown constant $\Pi_{0,0}^{(3)}$ is determined from the steady state condition (5.7) The proof is completed after obtaining the unknown constant $\Pi_{0,0}^{(3)}$ from the steady-state condition (5.7). The equation (5.7) is expressed in terms of probability generating functions as follows:

$$\Pi_{0,0}^{(3)}+\Phi_{0}^{(3)}(1,1)+\Phi_{1}^{(3)}(1,1)+\Phi_{2}^{(3)}(1,1)=1. \tag{5.29}$$

$\Phi_{0}^{(3)}(1,1), \Phi_{1}^{(3)}(1,1)$, and $\Phi_{2}^{(3)}(1,1)$ are obtained from (5.27), (5.28), and (5.22) taking $x=u=1$

$$\Pi_{0,0}^{(3)} = \frac{\bar{\alpha}\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))(V(\tau))^{J}(1-V(\tau))}{(1+(V(\tau))^{J+1})T_1-T_2+T_3(V(\tau))^{J+1}+T_4(V(\tau))^{J}}. \tag{5.30}$$

5.2. Stability condition

Equation (4.5) becomes valid if the value of the constant $\Pi_{0,0}^{(3)}$ in (5.30) is greater than zero. It is obtained $\bar{\alpha}\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))(V(\tau))^{J}(1-V(\tau))>0$ from $\Pi_{0,0}^{(3)}=\frac{\bar{\alpha}\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))(V(\tau))^{J}(1-V(\tau))}{(1+(V(\tau))^{J+1})T_1-T_2+T_3(V(\tau))^{J+1}+T_4(V(\tau))^{J}}>0$. Since α,β and $(V(\tau))$ are probabilities, $(1-\bar{\alpha}R(\bar{\alpha}))>0$ which gives the result $\bar{\alpha}R(\bar{\alpha})<1$. This is called the stability condition of the Markov chain $\Gamma_m^{(3)}$.

5.3. Marginal generating functions

Applying limit $x\to 1$ from (5.7) to (5.9) in Theorem 5.1, the marginal generating functions of the orbit size at different states of the server are obtained under the stability condition as follows:

(1) The marginal generating function of the orbit size when the server is in idle state is given as

$$\Phi_{0}^{(3)}(1,u) = 1+\frac{1-R(\bar{\alpha})}{1-\bar{\alpha}}\frac{\begin{array}{r}\bar{\alpha}\big[u\nu(u)V(\tau)^{J}+(\bar{\alpha}\beta B(u)(1-V(\tau))S(\nu(u))\\ -(\tau)uV(\nu(u)))(1-V(\tau))^{J}-u(\nu(u)-(\tau)V(\nu(u))\big]\end{array}}{(1-V(\tau))V(\tau)^{J}\Omega(u)}\Pi_{0,0}^{(3)}. \tag{5.31}$$

(2) The marginal generating function of the orbit size when the server is busy with regular service is obtained as

$$\Phi_{1}^{(3)}(1,u) = \frac{1-S(\nu(u))}{1-\nu(u)}\frac{\begin{array}{r}\alpha x\big[\nu(u)(\eta(u)-\bar{\alpha}\beta B(u))V(\tau)^{J+1}+(\bar{\alpha}\beta\nu(u)B(u)\\ -\tau\eta(u))V(\tau)^{J}+(\eta(u)(\tau-\nu(u)))\big]\end{array}}{\bar{\alpha}(1-V(\tau))V(\tau)^{J}\Omega(u)}\Pi_{0,0}^{(3)}. \tag{5.32}$$

(3) When the server is under vacation, the marginal generating function of the orbit size is given as

$$\Phi_{2}^{(3)}(1,u) = \frac{1-V(\nu(u))}{1-\nu(u)}\frac{\alpha(1-\alpha\beta)\big(1-V(\tau)^{J}\big)}{\bar{\alpha}(1-V(\tau))V(\tau)^{J}}\Pi_{0,0}^{(3)}. \tag{5.33}$$

(4) The probability generating function of the mean orbit size is derived as

$$\Psi^{(3)}(u) = \Phi_0^{(3)}(1,u) + \Phi_1^{(3)}(1,u) + \Phi_2^{(3)}(1,u), \tag{5.34}$$

$$= \frac{N_1(u)+\xi_2(u)}{D_3(u)}\Pi_{0,0}^{(3)}, \tag{5.35}$$

where $D_3(u) = \bar{\alpha}(1-\nu(u))\nu(u)(1-V(\tau))V(\tau)^J\Omega(u)$.

(5) The probability generating function of the mean system size is obtained as

$$\Phi^{(3)}(u) = \Pi_{0,0}^{(3)} + \Phi_0^{(3)}(1,u) + u\Phi_1^{(3)}(1,u) + \Phi_2^{(3)}(1,u). \tag{5.36}$$

5.4. Performance characteristics

Using the steady solution derived in Theorem 5.1, performance measures such as system state probabilities, mean orbit size, mean system size and waiting time in the orbit are obtained as follows

(1) Probability that the server is in idle state is given as

$$\Pi_{0,0}^{(3)} + \Phi_0^{(3)}(1,1) = \frac{\alpha\big[(\beta - R(\bar{\alpha}))V(\tau)^{J+1} + \bar{\beta}R(\bar{\alpha})V(\tau)^J - \beta(1-R(1-\alpha))\big]}{\Omega(1)(1-V(\tau))V(\tau)^J}\Pi_{0,0}^{(3)}. \tag{5.37}$$

(2) Probability that the server is busy in service is determined as

$$\Phi_1^{(3)}(1,1) = \frac{\alpha^2\beta B_1 S_1 \begin{array}{l}\big[\big(\bar{\alpha}\beta V(\tau) + \bar{\beta}(\bar{\alpha}(\tau))R(\bar{\alpha}) + \alpha\beta\big)V(\tau)^J \\ \qquad + (\bar{\alpha}\bar{\beta}R(\bar{\alpha}) + \beta)V(\tau)^{J+1} - \alpha\beta\big]\end{array}}{\bar{\alpha}\bar{\beta}\Omega(1)V(\tau)^J(1-V(\tau))}\Pi_{0,0}^{(3)}. \tag{5.38}$$

(3) Probability that the server is under vacation is found as

$$\Phi_2^{(3)}(1,1) = \frac{\alpha^2\beta(\tau)B_1V_1\big(1-V(\tau)^J\big)}{\bar{\alpha}V(\tau)^JV(\tau)}\Pi_{0,0}^{(3)}. \tag{5.39}$$

(4) The rate of arrival to the orbit is given by

$$P_{\text{orbit}} = \alpha\beta\Phi_1^{(3)}(1,1). \tag{5.40}$$

(5) The mean number of customers in the orbit group is given by $E(N) = \Psi'(1)$.

(6) Mean waiting time of a customer in the orbit is given by Little's formula

$$W_{\text{orbit}} = \frac{E(N)}{P_{\text{orbit}}}. \tag{5.41}$$

5.5. Particular cases

Taking $\beta = 1$, and $B(u) = u$, there is a single arrival without impatient customers, the presented model is reduced to $Geo/G/1$ with J-vacations. From equations (5.7) to (5.9), the following PGFs are obtained, which matches to the results in Yue and Zhang [31].

$$\Phi_0^{(3)}(x,u) = \frac{R(x)-R(\bar{\alpha})}{x-\bar{\alpha}} \frac{\begin{array}{c}\alpha x u\Big[\Big(1-V(\bar{\alpha})^{J+1}\Big)\nu(u) - \bar{\alpha}V(\nu(u))\Big(1-V(\bar{\alpha})^J\Big) \\ -\Big(1-V(\bar{\alpha})^{J+1}\Big) - \bar{\alpha}\Big(1-V(\bar{\alpha})^J\Big)S(\nu(u))\Big]\end{array}}{(1-V(\bar{\alpha}))V(\bar{\alpha})^J(\bar{\alpha}R(\bar{\alpha})(1-u)S(\nu(u)) + u(S(\nu(u)) - \nu(u)))}\Pi_{0,0}^{(3)},$$

$$\Phi_1^{(3)}(x,u) = \frac{S(x)-S(\nu(u))}{x-\nu(u)} \frac{\begin{array}{c}\alpha x\big[u(\nu(u)-V(\nu(u)))\big(1-V(\bar{\alpha})^J\big)\\ +R(\bar{\alpha}(1-u)((1-V(\bar{\alpha})^{J+1})\nu(u)-\bar{\alpha}V(\nu(u))(1-V(\bar{\alpha})^J)))\big]\end{array}}{(1-V(\tau))V(\tau)^J(\bar{\alpha}R(\bar{\alpha})(1-u)S(\nu(u))+u(S(\nu(u))-\nu(u))} \Pi_{0,0}^{(3)},$$

$$\Phi_2^{(3)}(x,u) = \frac{V(x)-V(\nu(u))}{x-\nu(u)} \frac{\alpha x\Big(1-V(\bar{\alpha})^J\Big)}{(1-V(\bar{\alpha}))V(\bar{\alpha})^J} \Pi_{0,0}^{(3)},$$

where

$$\Pi_{0,0}^{(3)} = \frac{((1-V(\bar{\alpha}))V(\bar{\alpha})^J[\bar{\alpha}R(\bar{\alpha})+\alpha-\alpha S_1]}{R(\bar{\alpha})\Big(\Big(1-V(\bar{\alpha})^{J+1}\Big)-\bar{\alpha}\Big(1-V(\bar{\alpha})^J\Big)\Big)-\alpha(1-V_1)\Big(1-V(\bar{\alpha})^J\Big)}.$$

6. Results and discussions

The three models described in the previous sections differ with one another in their vacation pattern. In steady-state, the three vacation policies such as single vacation model (SVM), multiple vacation model (MVM), and J-Vacation model (JVM) are compared based on their performance characteristics. The comparison results are presented in this section.

Case (i) Mean orbit size. In queueing scenario, the efficiency of the model seems to be evident from the measure mean orbit size. From (3.33), (4.32), and (5.35), the marginal generating function of mean orbit size with respect to each model is obtained as

$$\Psi^{(1)}(u) = \frac{N_1(u)-\xi_1(u)}{D_1(u)} \Pi_{0,0}^{(1)}, \tag{6.1}$$

$$\Psi^{(2)}(u) = \frac{N_1(u)}{D_2(u)} \Pi_{1,1,0}^{(2)}, \tag{6.2}$$

$$\Psi^{(3)}(u) = \frac{N_1(u)+\xi_2(u)}{D_3(u)} \Pi_{0,0}^{(3)}. \tag{6.3}$$

The mean orbit size under each model can be determined by $\Psi^{(1)'}(1), \Psi^{(2)'}(1)$, and $\Psi^{(3)'}(1)$. In order to compare these results, the ratio is obtained as

$$\frac{\Psi^{(1)'}(1)}{\Psi^{(2)'}(1)} = \frac{2\alpha E_1 - E_2}{E_3} \frac{1-V(\tau)}{V(\tau)} \frac{\Pi_{0,0}^{(1)}}{\Pi_{1,1,0}^{(2)}}, \tag{6.4}$$

$$\frac{\Psi^{(1)'}(1)}{\Psi^{(3)'}(1)} = \frac{\alpha\beta B_1(E_4V(\tau)+E_5)}{E_4V(\tau)^{J+1}+E_5+E_6V(\tau)^J} \frac{\bar{\alpha}}{\alpha} \frac{V(\tau)^J(1-V(\tau))}{V(\tau)} \frac{\Pi_{0,0}^{(1)}}{\Pi_{0,0}^{(3)}}, \tag{6.5}$$

where

$$\begin{aligned}
E_1 &= \beta((1-\bar{\alpha}R(\bar{\alpha}))+\alpha(S_1-1))V(\tau)-\bar{\beta}(V_1+\alpha\beta(S_1-V_1))(1-\bar{\alpha}R(\bar{\alpha})),\\
E_2 &= \alpha^4\beta^2(B_1)^2\big[\bar{\alpha}\bar{\beta}(S_1-1)R(\bar{\alpha})V(\tau)+\beta(1-\bar{\alpha}R(\bar{\alpha})+\alpha(S_1-1))\big]^2,\\
E_3 &= \frac{1}{2}\bar{\alpha}(\alpha^2\beta\bar{\beta}(B_1)^2S_2+\alpha\bar{\beta}(S_1-1)B_2-2B_1)+\alpha\beta\big[\alpha(B_1)^2+\alpha B_2-\alpha\bar{\alpha}\beta(B_1)^2S_1R(\bar{\alpha})\\
&\quad +\alpha^2\beta(B_1)^2S_2+\alpha(\beta(B_1)^2S_1+(S_1-1)((B_1)^2+B_2))\big],\\
E_4 &= (1-\bar{\beta}S_1)\bar{\alpha}R(\bar{\alpha})+\beta(1+\alpha(S_1-1))\\
E_5 &= \big[\alpha\beta^2(S_1-V_1)+\beta(1+V_1-\alpha(1-V-1))-V_1\big](1-\bar{\alpha}R(\bar{\alpha})),\\
E_6 &= \alpha\bar{\beta}[V_1+\bar{\alpha}R(\bar{\alpha})+(\alpha\beta(S_1-V_1))(1-\bar{\alpha}R(\bar{\alpha}))].
\end{aligned}$$

From the equations (3.9) and (4.27), it is obtained that $\frac{\Pi^{(1)}_{0,0}}{\Pi^{(2)}_{1,1,0}} = \frac{\bar{\beta}V(\tau)}{1-V(\tau)} \frac{V(\tau)T_1-T_2}{(1+V(\tau))T_1-T_2+T_3V(\tau)}$.
Similarly, it also found that $\frac{\Pi^{(1)}_{0,0}}{\Pi^{(3)}_{0,0}} = \frac{V(\tau)}{V(\tau)^J(1-V(\tau))} \frac{(1+(V(\tau))^{J+1})T_1-T_2+T_3(V(\tau))^{J+1}+T_4(V(\tau))^J}{(1+(V(\tau)))T_1-T_2+T_3(V(\tau))}$. By observation, the relations $E_3 > E_1$ and $E_3 > E_2$ hold true.
These results and the equation (6.4) imply that $\frac{\Psi^{(1)'}(1)}{\Psi^{(2)'}(1)} < 1$. The mean orbit size is less under SVM by comparing with the mean orbit size under MVM.
The relation $\frac{\Psi^{(1)'}(1)}{\Psi^{(3)'}(1)} < 1$ is true from equation (6.5), which shows that the mean orbit size is smaller in SVM when compared with JVM.

Case (ii) Vacation state. In the MVM model, the entire idle time of the server is utilized, where as in SVM and JVM, the server utilizes the partial idle time. This can be presented quantitatively by comparing the measures that the fraction of time the server is under vacation. From (3.36), (4.36), and (5.39), the probability that the server under vacation (SV) with respect to each model is simplified as follows:

$$P^1(\text{SV}) = \frac{\alpha\bar{\beta}\tau B_1(1-\bar{\alpha}R(\bar{\alpha}))V_1}{(1+V(\tau))T_1-T_2+T_3V(\tau)}, \tag{6.6}$$

$$P^2(\text{SV}) = \frac{\alpha\bar{\beta}\tau B_1(1-\bar{\alpha}R(\bar{\alpha}))V_1}{V(\tau)T_1-T_2}, \tag{6.7}$$

$$P^3(\text{SV}) = \frac{\alpha^2\beta\bar{\beta}\tau B_1(1-\bar{\alpha}R(\bar{\alpha}))V_1(1-(V(\tau))^J)}{(1+(V(\tau))^{J+1})T_1-T_2+T_3(V(\tau))^{J+1}+(V(\tau))^JT_4}. \tag{6.8}$$

Equations (6.6) and (6.7) imply the following relation $\frac{P^1(\text{SV})}{P^2(\text{SV})} = \frac{V(\tau)T_1-T_2}{(1+V(\tau))T_1-T_2+T_3V(\tau)} < 1$ and also equations (6.6) and (6.8) establish the relation $\frac{P^1(\text{SV})}{P^3(\text{SV})} = \frac{\alpha\bar{\beta}\tau B_1(1-\bar{\alpha}R(\bar{\alpha}))V_1}{\alpha^2\beta\bar{\beta}\tau B_1(1-\bar{\alpha}R(\bar{\alpha}))V_1(1-(V(\tau))^J)} \frac{(1+(V(\tau))^{J+1})T_1-T_2+T_3(V(\tau))^{J+1}+(V(\tau))^JT_4}{(1+V(\tau))T_1-T_2+T_3V(\tau)} < 1$. It can be observed that the probability that the server on vacation is lesser in SVM when compared with the probability of server on vacation with MVM and the probability of server on vacation with JVM.

Case (iii) Busy state. From (3.35), (4.35), and (5.38), the probability that the server is busy by providing service (SB) with respect to each model is obtained as follows:

$$P^1(\text{SB}) = \frac{(\alpha^2\beta B_1S_1\big(V(\tau)-\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))\big)) + (\alpha\beta\beta_1B_1S_1)\big(\alpha\beta+\bar{\alpha}\bar{\beta}R(\bar{\alpha})V(\tau)\big)}{(1+V(\tau))T_1-T_2+T_3V(\tau)}, \tag{6.9}$$

$$P^2(\text{SB}) = \frac{\alpha^2\beta B_1S_1\big(V(\tau)-\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))\big)}{V(\tau)T_1-T_2}, \tag{6.10}$$

$$P^3(\text{SB}) = \frac{\big(\alpha^2\beta B_1S_1\big((V(\tau))^{J+1}-\bar{\beta}(1-\bar{\alpha}R(\bar{\alpha}))\big)\big) + (\alpha\beta\beta_1B_1S_1)\big(\alpha\beta+\bar{\alpha}\bar{\beta}R(\bar{\alpha})(V(\tau))^J\big) \\ (\alpha\beta\beta_1B_1S_1)\big(\alpha\beta+\bar{\alpha}\bar{\beta}R(\bar{\alpha})(V(\tau))^{J+1}\big)}{(1+(V(\tau))^{J+1})T_1-T_2+(V(\tau))^{J+1}T_3+(V(\tau))^JT_4}. \tag{6.11}$$

The equations (6.9)–(6.11) gives the relations $\frac{P^1(\text{SB})}{P^2(\text{SB})} < 1$ and $\frac{P^1(\text{SB})}{P^3(\text{SB})} < 1$.

Case (iv) Idle state. From (3.34), (4.34), and (5.37), the probability that the server in idle (SI) with respect to each model is determined as follows

$$P^1(\text{SI}) = \frac{\bar{\alpha}(1-R(\bar{\alpha}))V(\tau)+\bar{\alpha}(\bar{\beta}V(\tau)+\beta(1-R(\bar{\alpha})))}{\beta[(1+V(\tau))T_1-T_2+T_3V(\tau)]}, \tag{6.12}$$

$$P^2(\text{SI}) = \frac{\bar{\alpha}(1-R(\bar{\alpha}))V(\tau)}{V(\tau)T_1-T_2}, \tag{6.13}$$

TABLE 1. Arrival rate *vs.* probability that the server is in idle state.

α	0.30	0.35	0.40	0.45	0.50	0.55	0.60
P^1(SI)	0.4101	0.3309	0.2672	0.2166	0.1766	0.1451	0.1203
P^2(SI)	0.3197	0.2506	0.1961	0.1533	0.1268	0.0979	0.0721
P^3(SI)	0.3522	0.2732	0.2116	0.1638	0.1196	0.0931	0.0753

TABLE 2. Arrival rate *vs.* probability that the server is in busy state.

α	0.20	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.3
P^1(SB)	0.5611	0.5839	0.6060	0.6272	0.6476	0.6671	0.6857	0.7035	0.7205	0.7366	0.7520
P^2(SB)	0.5836	0.6134	0.6424	0.6703	0.6973	0.7232	0.7481	0.7721	0.7950	0.8169	0.8379
P^3(SB)	0.5581	0.5842	0.6093	0.6332	0.6560	0.6777	.6983	0.7178	0.7362	0.7536	0.7700

$$P^3(\mathrm{SI}) = \frac{\bar{\alpha}(1-R(\bar{\alpha}))(V(\tau))^{J+1} + \bar{\alpha}\big(\bar{\beta}(V(\tau))^{J+1} + \beta(1-R(\bar{\alpha}))\big) - \bar{\beta}R(\bar{\alpha})(V(\tau))^{J+1}}{\beta[(1+(V(\tau))^{J+1})T_1 - T_2 + (V(\tau))^{J+1}T_3 + (V(\tau))^J T_4]}. \tag{6.14}$$

Equations (6.12) and (6.13) imply the result $\frac{P^1(\mathrm{SI})}{P^2(\mathrm{SI})} = \frac{\bar{\alpha}(1-R(\bar{\alpha}))V(\tau)+\bar{\alpha}\left(\bar{\beta}V(\tau)+\beta(1-R(\bar{\alpha}))\right)}{\bar{\alpha}(1-R(\bar{\alpha}))V(\tau)} \frac{V(\tau)T_1-T_2}{\beta[(1+V(\tau))T_1-T_2+T_3V(\tau)]} > 1$ and also equations (6.12) and (6.14) indicate that the following result holds true $\frac{P^1(\mathrm{SV})}{P^3(\mathrm{SV})} = \frac{\alpha\bar{\beta}\tau B_1(1-\bar{\alpha}R(\bar{\alpha}))V_1}{\alpha^2\beta\bar{\beta}\tau B_1(1-\bar{\alpha}R(\bar{\alpha}))V_1(1-(V(\tau))^J)} \frac{(1+(V(\tau))^{J+1})T_1-T_2+T_3(V(\tau))^{J+1}+(V(\tau))^J T_4}{\beta[(1+(V(\tau))^{J+1})T_1-T_2+(V(\tau))^{J+1}T_3+(V(\tau))^J T_4]} > 1.$

From $P^1(\mathrm{SI}) > P^2(\mathrm{SI})$ and $P^1(\mathrm{SI}) > P^3(\mathrm{SI})$, it can be observed that the probability that the server waiting for the customer is greater in SVM when compared with the probability that the server is waiting for the customer with the MVM and JVM.

7. NUMERICAL RESULTS AND DISCUSSIONS

This section explores three queueing models numerically to show their effectiveness. The following assumptions and notations are used for numerical analysis,

Arrival rate	α
customer impatience's probability	β
Retrial rate	r
Service time follows a geometric distribution with parameter	s
Vacation time follows a geometric distribution with parameter	v

The values of all the parameters have taken to fulfill the stability condition.

Taking $\beta = 0.9, r = 0.9, B_1 = 5, S_1 = 3, V_1 = 2$, Table 1 displays the steady-state probabilities with respect to different arrival rates. It shows the results the probability that the server is in an idle state under SVM, MVM, and JVM. It is observed that the probability that the server is waiting for the customer to provide service is greater in the single vacation model. From Table 2, it appears that the probability that the server is in a busy is higher in the single vacation model than those in J vacations model and multiple vacations model. The probability that the server is under vacation is lower in a single vacation model but is greater in the MVM, as can be viewed in Table 3. In Tables 4 and 5, the three vacation models SVM, MVM, and JVM have been compared based on the characteristic mean orbit size. Considering $\beta = 0.9, r = 0.9, B_1 = 5, S_1 = 3, S_2 = 15, V_1 = 2, V_2 = 6$, Table 4 displays the effect of arrival rate on the mean orbit size and it evident orbit size increases with increasing values of α. $E(N)$ has been observed concerning each vacation model and it is apparent from Table 4 that the

TABLE 3. Arrival rate *vs.* probability that the server is under vacation state.

α	0.30	0.35	0.40	0.45	0.50	0.55	0.60
P^1(SV)	0.0716	0.0684	0.0639	0.0587	0.0534	0.0480	0.0428
P^2(SV)	0.2480	0.2603	0.2692	0.2767	0.2846	0.2945	0.3087
P^3(SV)	0.1222	0.1119	0.1003	0.0886	0.0774	0.0671	0.0577

TABLE 4. Effect of arrival rate on mean orbit size.

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
P^1(SV)	29.30	56.05	86.22	124.9	172.7	230.2	297.2	374.1
P^2(SV)	33.49	56.50	86.46	125.5	174.1	232.7	302.0	385.6
P^3(SV)	33.27	56.69	86.56	125.3	173.3	231.0	298.3	375.4

TABLE 5. Effect of retrial rate on mean orbit size.

r	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
P^1(SV)	142.0	134.9	127.8	120.8	113.7	106.7	99.81	92.97	86.22	76.33
P^2(SV)	141.4	134.4	127.3	120.3	113.4	106.5	99.68	92.99	86.46	80.17
P^3(SV)	142.7	135.6	128.5	121.4	114.3	107.3	100.3	93.39	86.56	79.86

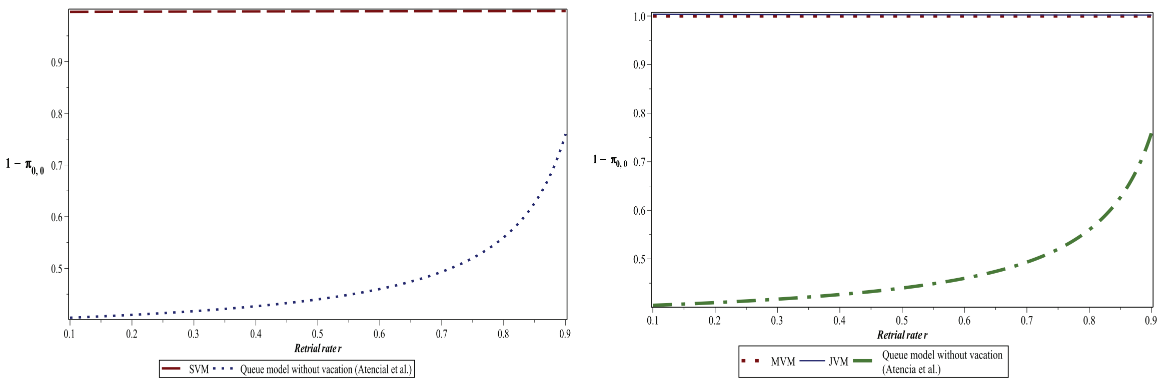

FIGURE 3. Effect of retrail on (a) Probability that the server busy. (b) Probability that the server busy.

single vacation model possesses a lesser mean orbit size when compared with other vacation models, MVM and JVM. Table 5 has plotted mean orbit size against the retrial rate. $E(N)$ values tend to decrease with the increasing values of the retrial rate. Table 5 shows a similar effect as seen in Table 4 that the mean orbit size is smaller in the single vacation model. It can be observed that single vacation model is a superior model to the multiple vacations model and J-vacations from the numerical observation on mean orbit size and other performance characteristics.

This model focusses on the importance of vacation queueing models. The value $\Pi_{0,0}$ represents the probability that the server is free and that no customer in the orbit. $1 - \Pi_{0,0}$ is the probability measure that the server is occupied. Taking the values of the parameter as in Atencia *et al.* [8], $\alpha = 0.2, B_1 = 1, S_1 = 2$, and assuming $\beta = 0.9, S_2 = 15, V_1 = 2, V_2 = 6$, Figure 3a displays the values $1 - \Pi_{0,0}$ with respect to the proposed SVM

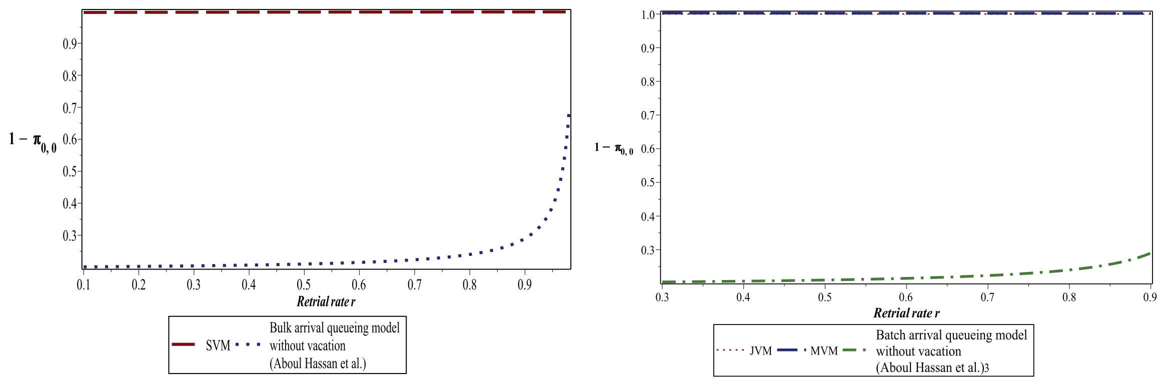

FIGURE 4. Effect of retrail on (a) Probability that the server busy. (b): Probability that the server busy.

and queueing model without vacation [8]. It can be observed that as the retrial rate increases, the utilization factor also increases. The utilization factor, the probability that the server is busy, is more in SVM than those observed by Atencia *et al.* [8]. Similarly, Figure 3b shows the values $1-\Pi_{0,0}$ with respect to the proposed MVM, JVM, and queueing model without vacation [8]. It can be concluded that vacation models are more appropriate at a place where the server should be utilized properly. Using the parameter values $\alpha = 0.1, B_1 = 1, S_1 = 2$ as in Aboul-Hassan [2] and assuming $\beta = 0.9, S_2 = 15, V_1 = 2, V_2 = 6$, the utilization factors are presented in Figure 4a based on proposed SVM and batch arrival model given by Aboul-Hassan [2]. It can be noticed that $1-\Pi_{0,0}$ increases if the retrial rate increases. Also Figure 4b shows the comparison between the proposed MVM, JVM, and batch arrival model given by Aboul-Hassan [2]. The utilization of the server can be made more effective by fitting the model using Vacations.

8. Conclusion and future work

This paper majorly would help designers to make decisions regarding allocate server under various conditions of vacations.The study considered a discrete-time bulk arrival $Geo^X/G/1$ with three types of vacations, single, multiple, and J vacations. The probability generating functions and marginal generating functions have been derived for each model. In a steady-state, performance characteristics of the three models have been obtained. A theoretical comparative analysis among three models based on the measures was also presented. From the theoretical observation on mean orbit size and the other performance characteristics, it was observed that a single vacation is superior to the multiple vacations model and J vacations model in discrete-time bulk arrival $Geo^X/G/1$ queueing context. The result concluded in the paper will be more helpful to the designer to choose the appropriate model. The consistency of the theory developed has also been also verified through numerical illustration.

Future studies can consider the reward for work can be included during the vacation. Further, optimal cost of the model and waiting time distribution can be incorporated in the study. The comparisons can also be revised conditionally. Also, the general arrival process also can be included to enhance the model. The work can also be carried out using simulation with general distribution.

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References

[1] A.K. Aboul-Hassan, S. Raiba and F. Taboly, A discrete-time Geo/G/1 queue with general retrial times and balking customers. *J. Korean Stat. Soc.* **37** (2008) 335–348.

[2] A.K. Aboul-Hassan, S. Raiba and F. Taboly, Performance evaluation of a discrete-time $Geo^X/G/1$ retrial queue with general retrial times. *Comput. Math. Appl.* **58** (2009) 548–557.

[3] A.S. Alfa, Applied Discrete-time Queues. Springer (2015).

[4] A.S. Alfa, Queueing Theory for Telecommunications – Discrete Time Modelling of a Single Node System. Springer (2010).

[5] G. Anupam, G. Choudhaury and S. Dharmaraja, Performance analysis of DRX mechanism using batch arrival vacation queueing system with N-policy in LTE-A networks. *Ann. Telecommun.* **75** (2020) 353–367.

[6] J.R. Artalejoa, A.N. Dudinb and V.I. Klimenokb, Stationary analysis of a retrial queue with preemptive repeated attempts. *Oper. Res. Lett.* **28** (2001) 173–180.

[7] J.R. Artalejo, I. Atencia and P. Moreno, A discrete-time $Geo^{[X]}/G/1$ retrial queue with control of admission. *Appl. Math. Modell.* **29** (2005) 1100–1120.

[8] I. Atencia and P. Moreno, A discrete-time Geo/G/1 retrial queue with general retrial times. *Queueng Syst.* **48** (2004) 5–21.

[9] F.M. Chang and J.C. Ke, On a batch retrial model with J vacations. *J. Comput. Appl. Math.* **232** (2009) 402–414.

[10] G. Choudhurya, L. Tadj and K. Dekaa, A batch arrival retrial queueing system with two phases of service and service interruption. *Comput. Math. Appl.* **59** (2010) 437–450.

[11] I. Dimitriou, On the power series approximations of a structured batch arrival two-class retrial system with weighted fair orbit queues. *Perform. Eval.* **132** (2019) 38–56.

[12] B.T. Doshi, Queueing systems with vacations: a survey. *Queueing Syst.* **1** (1986) 29-66.

[13] G.I. Falin and J.G.C. Templeton, Retrial Queues. Chapman and Hall, London (1997).

[14] S. Gao and J. Wang, Discrete-time $Geo^{[X]}/G/1$ retrial queue with general retrial times, working vacations and vacation interruption. *Qual. Technol. Quant. Manag.* **10** (2013) 495–512.

[15] A. Gomez-Corral, A bibliographical guide to the analysis of retrial queues through matrix analytic techniques. *Ann. Oper. Res.* **141** (2006) 163–191.

[16] M. Haridass and R. Arumuganathan, Analysis of a single server batch arrival retrial queueing system with modified vacations and N-policy. *RAIRO-Oper. Res.* **49** (2014) 279–296.

[17] J.J. Hunter, Mathematical Techniques of Applied Probability: Discrete-time Models: Techniques and Applications. Vol. 2. Academic Press, New York (1983).

[18] V. Jailaxmi and R. Arumuganathan, Analysis of a retrial queue with multiple vacations and state dependent arrivals. *RAIRO-Oper. Res.* **49** (2015) 619–634.

[19] T. Jiang, Tail asymptotics for a batch service polling system with retrials and non-persistent customers. *J. Math. Anal. Appl.* **459** (2008) 893–905.

[20] Z. Liu and Y. Song, Geo/Geo/1 retrial queue with non-persistent customers and working vacations. *J. Appl. Math. Comput.* **42** (2013) 103–115.

[21] G. Malik and S. Upadhyaya, Single vacation policy for discrete-time retrial queue with two types of customers. In: Strategic System Assurance and Business Analytics. Springer, Singapore (2020) 335–349.

[22] R. Nobel, Retrial queueing models in discrete time: a short survey of some late arrival models. *Ann. Oper. Res.* **247** (2016) 37–63.

[23] E. Rosenberg and U. Yechiali, The $M^X/G/1$ queue with single and multiple vacations under the LIFO service regime. *Oper. Res. Lett.* **14** (1993) 171–179.

[24] L.D. Servi and S.G. Finn, M/M/1 queues with working vacations. *Perform. Eval.* **50** (2002) 41–52.

[25] R. Sudhesh, A. Azhagappan and S. Dharmaraja, Transient Analysis of M/M/1 queue with working vacation, heterogeneous service and customers' impatience. *RAIRO-Oper. Res.* **51** (2017) 591–606.

[26] H. Tagaki and D. Wu, M/G/1 queue with multiple working vacations. *Perform. Eval.* **63** (2006) 654–681.

[27] H. Takagi, Queueing analysis: a foundation of performance evaluation. In: Vacations and Priority Systems, Part 1. Vol. I. Amsterdam, North Holland (1991).

[28] M. Takahashi, H. Osawa and T. Fujisawa, $Geo^{[X]}/G/1$ retrial queue with non-preemptive priority. *Asia Pac. J. Oper. Res.* **16** (1999) 215–234.

[29] T.Y. Wang, J.C. Ke and F.M. Chang, Discrete-time Geo/G/1 queue with randomized vacations and at most J vacations. *Appl. Math. Modell.* **35** (2011) 2297–2308.

[30] T. Yang and H. Li, Steady-State queue size distribution of discrete-time $PH/Geo/1$ retrial queues. *Math. Comput. Model.* **30** (1999) 51–63.

[31] D. Yue and F. Zhang, A discrete-time Geo/G/1 retrial queue with J-vacation policy and general retrial times. *J. Syst. Sci. Complex.* **26** (2013) 556–571.

[32] F. Zhang and Z. Zhu, A Discrete-Time Geo/G/1 retrial queue with two different types of vacations. *Math. Probl. Eng.* **13** (2015) 1–12.

[33] F. Zhang, D. Yue and Z. Zhu, A discrete time Geo/G/1 retrial queue with single vacation and starting failure. *J. Inf. Comput. Sci.* **8** (2011) 2751–2758.