RAIRO-Oper. Res. 56 (2022) 1857–1875
https://doi.org/10.1051/ro/2022050

**RAIRO Operations Research**
www.rairo-ro.org

# Z-EQUILIBRIUM IN RANDOM BI-MATRIX GAMES: DEFINITION AND COMPUTATION

Farida Achemine[1] and Moussa Larbani[2,3,*]

**Abstract.** This paper deals with bi-matrix games with random payoffs. Using probability tools, we propose a solution based on the concept of Z-equilibrium. Then, we give sufficient conditions of its existence. Further, the problem of computation of this solution is transformed into the determination of Pareto optimal solutions of a deterministic bi-criteria minimization problem. Finally, we provide illustrative numerical examples.

**Mathematics Subject Classification.** 91A05.

Received March 15, 2021. Accepted April 1, 2022.

## 1. Introduction

Bi-matrix games played an important role in the development of game theory. Many real-world conflict situations are analyzed by bi-matrix games such as the prisoner dilemma game. A bi-matrix game is characterized by two matrices $A$ and $B$ representing the payoffs of the players, players I and player II, respectively. A nice feature of matrix games is that Nash equilibrium [19] always exists in pure or mixed strategies and it can be computed *via* the resolution of a quadratic optimization problem.

Nash equilibrium is the most prominent concept of solution in game theory. However, it may not exist in pure strategies and may not be pareto optimal, that is, it may be dominated. In this paper, we consider the concept of Z-equilibrium introduced by Zhukovskii [29]. In contrast to Nash equilibrium, it always exists in pure strategies in finite games and it is always Pareto optimal. This equilibrium is said to be "active" in the sense that, any deviation of a player from her/his Z-equilibrium strategy, the other player has a specific punishing strategy that decreases or maintains her/his payoff. Whereas, in Nash equilibrium, the reaction of a player to any deviation of the other player is the same. Moreover, it is interesting to note that the set of Z-equilibria is a subset of the $\alpha$-core of Aumann [3] in two-player games. Moreover if, we consider a stronger version of the $\alpha$-core where a coalition can block a solution if it can guarantee greater or equal payoff for all its members with a strictly greater payoff for at least one member, then the Z-equilibrium generalizes the $\alpha$-core in $n$-person games with $n > 2$.

*Keywords.* Bi-matrix game, chance constrained game, Z-equilibrium, normal random variable, Cauchy random variable.

[1] Laboratory of Pure and Applied Mathematics, Mouloud Mammeri University of Tizi-Ouzou, Tizi-Ouzou, Algeria.

[2] School of Mathematics and Statistics, Carleton Unversity, Ottawa, Canada.

[3] Ecole Nationale Supérieure de Statistique et d'économie appliquée, Koléa, Tipaza, Algeria.

[*]Corresponding author: larbani61@hotmail.com

A complete study of Z-equilibrium in continuous and deterministic games is given in Zhukovskii [29] (see a formal definition in Sect. 2) and Zhukovskii and Tchickry [30]. Ferhat and Radjef [11] have generalized Z-equilibrium to multiple criteria games in mixed strategies. In Bouchama *et al.* [5] an equivalence between the solution of a constraint satisfaction problem and the Z-equilibrium of its associated game is established.

However, in many real games situations, it is very difficult to determine an exact value of the payoffs. Therefore, some approaches to Z-equilibrium in the case of lack of precision and certainty on the payoffs are considered in Larbani and Lebbah [16], Larbani and Achemine [15], Achemine *et al.* [1], Nessah *et al.* [20] and Achemine *et al.* [2].

The first work on Z-equilibrium in games involving uncertainty is due to Larbani and Lebbah [16]. They considered games with uncertain payoffs of the form $u_i(x, y)$, where $y$ is an unknown parameter that varies in some set $Y \subset \mathbb{R}^m$. They introduced a concept called ZS-equilibrium. Further, Larbani and Achemine [15] introduced and investigated the notion of ZP-equilibrium. This concept is generalized to fuzzy games in Achemine *et al.* [1]. Nessah *et al.* [20] considered games with uncertain parameters, where the players can form coalitions, the introduced concept is called coalitional ZP-equilibrium. Recently, Achemine *et al.* [2] investigated Z-equilibrium in the class of bi-matrix games with uncertain payoffs in the sense of Liu [17]. Liu uncertainty theory is different from probability theory; it is based on credibility measure that measures fuzzy events that are subjective in nature.

In many practical situations, the players' payoffs are better modeled using random variables. Wholesale electricity markets are good examples of this area [8, 9, 18, 27]. One way to handle such games is using expected payoff criterion [9, 10, 13, 21, 27, 28]. The expected payoff criterion is not suitable when the random payoff has a large variance. In this case, it is more interesting to consider payoffs that can be obtained with a certain confidence level. Such situations are modeled using chance-constraints games. The first works on chance-constrained games concern zero-sum chance-constrained games. These games were developed by Blau [4], Cassidy *et al.* [6], Charnes *et al.* [7], and Song [26]. Nash equilibrium in bi-matrix games and in $n$-person finite games have been investigated using chance-programming in Singh *et al.* [25] and Singh and Lisser [24].

Z-equilibrium has not been investigated in games with random payoffs. The contribution of this paper is to initiate the study of Z-equilibrium in games with random payoffs. As a first step, we consider a bi-matrix game in which the payoffs are random variables. The main difficulty in the study of these games is the comparison between payoff values associated with different strategies of the players. Using probability tools, for such games, we introduce a concept of equilibrium based on Z-equilibrium and we establish sufficient conditions for its existence. Furthermore, we show that the computation of this equilibrium can be transformed into the computation of a Pareto optimal solution of a bi-criteria optimization problem. We use chance-constrained programming to formulate Z-equilibrium. However, our approach differs from the existing ones. Following the satisficing principle of Simon [23], we first ask the players to provide satisfaction levels in terms of payoffs, then we formulate new payoffs as probabilities of achieving those levels. In existing works, payoffs are formulated as values that are achieved with given confidence levels.

The rest of the paper is organized as follows. The next section is devoted to the description and the introduction of the proposed solution called RZ-equilibrium (random Z-equilibrium). In Section 3, we present sufficient conditions for the existence of this equilibrium. In Section 4, we show that the computation of the RZ-equilibrium can be formulated as a bi-criteria optimization problem that can be solved using methods of multiple criteria optimization. A numerical example is given in Section 5. Section 6 discusses related work. Section 7 concludes the paper.

## 2. Problem description and its solution

### 2.1. Z-equilbrium in a deterministic game

To help the unfamiliar reader understand the Z-equilibrium [29], we recall its definition for a deterministic strategic two-person game.

Consider the two-person strategic game

$$G =< N, X_1 \times X_2, (U_1(x_1, x_2), U_2(x_1, x_2)) >,$$

where $N = \{1, 2\}$ is the set of players; $X_i \subset \mathbb{R}^{n_i}, n_i \in \mathbb{N}^*, X_i$ the set of strategies of the $i$-th player, $i = 1, 2$. $U_i : X_1 \times X_2 \longrightarrow \mathbb{R}$, is the payoff function of the $i$-th player. The aim of each player is to maximize her/his payoff function.

**Notation.** In the following, we use the notation: for all $(\overline{r}_1, \overline{r}_2)$, $(r_1, r_2) \in \mathbb{R}^2$,

$$(\overline{r}_1, \overline{r}_2) \precsim (r_1, r_2) \Longleftrightarrow (\forall k \in \{1, 2\}, \quad \overline{r}_k \leq r_k \quad \text{and} \quad \exists l \in \{1, 2\}, \quad \overline{r}_l < r_l).$$

**Definition 2.1.** $x^* \in X_1 \times X_2$ is said to be a Z-equilibrium for the game $G$ if and only if the following two conditions hold.

(1) $\begin{cases} \forall x_1 \in X_1, \quad \exists x_2 \in X_2, \quad U_1(x_1, x_2) \leq U_1(x^*); \\ \forall x_2 \in X_2, \quad \exists x_1 \in X_1, \quad U_2(x_1, x_2) \leq U_2(x^*). \end{cases}$

(2) It does not exist a strategy profile $(x_1, x_2) \in X_1 \times X_2$, such that

$$(U_1(x^*), U_2(x^*)) \precsim (U_1(x_1, x_2), U_2(x_1, x_2)).$$

**Remark 2.2.**
- The condition 1 of Definition 2.1 means that for any deviation $x_i$ of the $i$-th player ($i \in \{1, 2\}$) from her/his equilibrium strategy, the other player can punish her/him by choosing a specific strategy $x_{N-i}$ that prevents him/her from being better off. This condition guarantees the stability of Z-equilibrium.
- Condition 2 of Definition 2.1 means that $x^*$ is Pareto optimal for the players, that is, $x^*$ is not dominated in payoff space. It is interesting to note that the set of Z-equilibria is a subset of the $\alpha$-core of Aumann [3] in two-player games. Further, if we consider a stronger version of the $\alpha$-core where a coalition can block a solution if it can guarantee a greater or equal payoff for all its members with a strictly greater payoff for at least one of them, then the Z-equilibrium generalizes the $\alpha$-core in $n$-person games with $n \geq 2$.
- A Nash equilibrium that is Pareto optimal is a Z-equilibrium. Indeed, each player can use her/his Nash equilibrium strategy to punish the other player for deviating from the equilibrium.

**Example 2.3.** Consider the prisoner dilemma game.

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cc} \quad b_1 & \quad\; b_2 \end{array} \\ \begin{pmatrix} (10, 10) & (1, 15) \\ (15, 1) & (4, 4) \end{pmatrix} \end{array}.$$

Note that the profile $(a_2, b_2)$ is a Nash equilibrium. It is easy to see that $(a_1, b_1)$ is a Z-equilibrium. That is, Z-equilibrium captures the cooperative profile in the prisoner dilemma game. Note that experimental evidence has shown that in 50% of the cases, players choose the cooperative profile $(a_1, b_1)$ (Z-equilibrium) rather than Nash equuilibrium in a two person prisoner dilemma game Sally [22]. When the game is repeated, following Z-equilibrium, player I (resp. player II) can punish player II (resp. player I) by selecting $a_2$ (resp. $b_2$) to stabilise the game at $(a_1, b_1)$.

**Example 2.4.** Consider the following bi-matrix game.

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} (3, 2) & (2, 7) & (5, 4) \\ (1, 2) & (4, 1) & (6, 1) \end{pmatrix} \end{array}$$

where $X_1 = \{a_1, a_2\}$ and $X_2 = \{b_1, b_2, b_3\}$ are the set of pure strategies for row player and column player, respectively.

This game has no Nash equilibrium. The strategy profile $(a_1, b_3)$ is a Z-equilibrium.

- Condition 1 of Definition 2.1 guarantees the stability of Z-equilibrium. Indeed, for each deviation $x_1 \in X_1$ (resp. $x_2 \in X_2$) of the row player (resp. column player) from her/his Z-equilibrium strategy, the other player has a counter strategy $t_2 \in X_2$ (resp. $t_1 \in X_1$) to punishes her/him. Indeed,
  for the deviation of the row player to $a_2 \in X_1$, the column player has the strategy $b_1 \in X_2$, such that $U_1(a_2, b_1) = 1 \leq U_1(a_1, b_3) = 5$;
  for the deviation of the column player to $b_2 \in X_2$, the row player has the strategy $a_2 \in X_2$, such that $U_2(a_2, b_2) = 1 \leq U_2(a_1, b_3) = 4$;
  for the deviation of the column player to $b_1 \in X_2$, the row player has the strategy $a_1 \in X_1$, such that $U_2(a_1, b_1) = 2 \leq U_2(a_1, b_3) = 4$.
- Clearly, condition 2 of Definition 2.1 is satisfied by the pair of pure strategies $(a_1, b_3)$. Indeed, $(a_1, b_3)$ is Pareto optimal for the players, that is, this pair is not dominated in payoff space.

As the game has no Nash equilibrium, the players can adopt Z-equilibrium as a solution for its desirable properties, especially if the game is repeated.

The following theorem guarantees the existence of Z-equilibrium [29].

**Theorem 2.5.** *Assume that*

(i) *the sets of strategies $X_1$ and $X_2$ are non empty and compact;*
(ii) *the functions $U_1$ and $U_2$ are continuous on $X_1 \times X_2$.*

*Then, $G$ has at least one Z-equilibrium.*

## 2.2. The bi-matrix game with random payoffs

In the classical bi-matrix games, the payoffs of the two players are real numbers. They are precisely known. However, real-life decisions problems often involve randomness. Neglecting randomness in modeling when it exists, may lead to poor quality decisions. Therefore, in this section, we focus on bi-matrix games, where the payoffs are random variables.

We consider a bi-matrix game in which the players have exactly defined their pure strategies but are uncertain about the induced payoffs. In the following, we assume that this uncertainty is modeled by random variables.

A random bi-matrix game is given by

$$\left\langle \{\mathrm{I}, \mathrm{II}\}, X \times Y, \left(\widetilde{A}, \widetilde{B}\right) \right\rangle, \tag{2.1}$$

where $X = \{1, 2, \dots, m\}$ and $Y = \{1, 2, \dots, n\}$ are the sets of pure strategies for player I and player II, respectively.

Let $(\Omega, F, \mathbb{P})$ be a probability space, $\widetilde{a}_{ij} : \Omega \longrightarrow \mathbb{R}$ and $\widetilde{b}_{ij} : \Omega \longrightarrow \mathbb{R}$, $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ are random variables on $(\Omega, F, P)$.

The random payoff matrices to the row player I and column player II are $\widetilde{A} = [\widetilde{a}_{ij}]$ and $\widetilde{B} = [\widetilde{b}_{ij}]$, respectively.

We denote the sets of mixed strategies of players I and II, which represent weights assigned to their pure strategies, by

$$P = \left\{ p^T = (p_1, \dots, p_m),\ \sum_{i=1}^{m} p_i = 1,\ p_i \in [0, 1] \right\} \quad \text{and}$$
$$Q = \left\{ q^T = (q_1, \dots, q_n),\ \sum_{j=1}^{n} q_j = 1,\ q_j \in [0, 1] \right\},$$

respectively, where $T$ represents the transpose operator. They can also be interpreted as probabilities that players choose their particular pure strategies. Then, a mixed strategy game with random payoffs is given as follows

$$\left\langle \{\mathrm{I}, \mathrm{II}\}, P \times Q, \left(\widetilde{A}, \widetilde{B}\right) \right\rangle.$$

The payoffs induced when players I and II choose the mixed strategies $p \in P$ and $q \in Q$ are $p^T\tilde{A}q$ and $p^T\tilde{B}q$, respectively.

In this game, it is assumed that the players are rational and each of them knows the set of strategies of the other player. It is also assumed that each player knows the distribution of every random entry in $\widetilde{A}$ and $\widetilde{B}$. The aim of each player is to maximize her/his payoff.

As a solution for the game (2.1), we propose a concept based on the notion of Z-equilibrium [29], which takes into account the random aspect of the game. For this purpose, we formulate the payoff of each player using a chance constraint. Following Simon [23] satisficing principle, for predetermined satisfaction levels $\delta_1$ (resp. $\delta_2$) $\in \mathbb{R}$, we assume that player I aims to maximize the probability of the random event $\left\{\omega : p^T\tilde{A}q \geq \delta_1\right\}$ and player II aims to maximize the probability of the random event $\left\{\omega : p^T\tilde{B}q \geq \delta_2\right\}$.

**Definition 2.6.** For predetermined satisfaction levels $\delta_1$, $\delta_2$, a pair $(p^*, q^*)$ is called a RZ-equilibrium (random Z-equilibrium) of the game (2.1) at $(\delta_1, \delta_2)$ satisfaction levels, if it satisfies

(1)

$$\begin{cases} \forall p \in P, \quad \exists q \in Q, \quad \mathbb{P}\{\omega : p^{*T}\tilde{A}q^* \geq \delta_1\} \geq \mathbb{P}\{\omega : p^T\tilde{A}q \geq \delta_1\}; \\ \forall q \in Q, \quad \exists p \in P, \quad \mathbb{P}\{\omega : p^{*T}\tilde{B}q^* \geq \delta_2\} \geq \mathbb{P}\{\omega : p^T\tilde{B}q \geq \delta_2\}. \end{cases}$$

(2) There is no strategy profile $(p, q) \in P \times Q$, such that

$$\left(\mathbb{P}\left\{\omega : p^{*T}\tilde{A}q^* \geq \delta_1\right\}, \quad \mathbb{P}\left\{\omega : p^{*T}\tilde{B}q^* \geq \delta_2\right\}\right) \precsim \left(\mathbb{P}\left\{\omega : p^T\tilde{A}q \geq \delta_1\right\}, \quad \mathbb{P}\left\{\omega : p^T\tilde{B}q \geq \delta_2\right\}\right).$$

Definition 2.6 means that $(p^*, q^*)$ is a Z-equilibrium of the following chance-constrained game

$$\left\langle \{\mathrm{I}, \mathrm{II}\}, P \times Q, (u_1^{\delta_1}, u_2^{\delta_2}) \right\rangle,$$

where $u_1^{\delta_1}(p, q) = \mathbb{P}\left\{\omega : p^T\tilde{A}q \geq \delta_1\right\}$ and $u_2^{\delta_2}(p, q) = \mathbb{P}\left\{\omega : p^T\tilde{B}q \geq \delta_2\right\}$.

**Remark 2.7.**
- The first condition of Definition 2.6 means that for any deviation $p$ (resp. $q$) of the player I (player II) from her/his equilibrium strategy $p^*$ ($q^*$ resp.), the other player can punish her/him by choosing some strategy $q$ (resp. $p$), so that at the resulting profile $(p, q)$, her/his payoff is less (or equal) than at the profile $(p^*, q^*)$. This condition guarantees the stability of the Z-equilibrium $(p^*, q^*)$.
- If the maxmin values

$$\lambda_1 = \max_p \min_q \mathbb{P}\left\{\omega : p^T\tilde{A}q \geq \delta_1\right\} \quad \text{and} \quad \lambda_2 = \max_q \min_p \mathbb{P}\left\{\omega : p^T\tilde{B}q \geq \delta_2\right\}$$

exist, then the first condition of Definition 2.6 is equivalent to

$$\begin{cases} \lambda_1 \leq \mathbb{P}\left\{\omega : p^{*T}\tilde{A}q^* \geq \delta_1\right\}; \\ \lambda_2 \leq \mathbb{P}\left\{\omega : p^{*T}\tilde{B}q^* \geq \delta_2\right\}; \end{cases}$$

which means that the strategy profile $(p^*, q^*)$ satisfies the principle of individual rationality.
- The second condition of Definition 2.6 means that $(p^*, q^*)$ is Pareto optimal for the players in the game $\left\langle \{\mathrm{I}, \mathrm{II}\}, P \times Q, \left(u_1^{\delta_1}, u_2^{\delta_2}\right) \right\rangle$, with $u_1^{\delta_1}(p, q) = \mathbb{P}\left\{\omega : p^T\tilde{A}q \geq \delta_1\right\}$ and $u_2^{\delta_2}(p, q) = \mathbb{P}\left\{\omega : p^T\tilde{B}q \geq \delta_2\right\}$.

Note that in existing literature of finite random games Singh *et al.* [25] and Singh and Lisser [24], in equilibrium definition, the maximum achieved value with a given confidence level is considered as payoff, while we use probabilities of achieving a given satisfaction level as payoff. We discuss this aspect in Section 6 in more detail.

## 3. Existence of the RZ-equilibrium

In this section, the problem of the existence of RZ-equilibrium is investigated. Using probability tools, sufficient conditions for the existence of RZ-equilibrium are established in two important cases: (i) the entries of the payoff matrices are normally distributed random variables; (ii) the entries of the payoffs matrices are Cauchy distributed random variables.

### 3.1. Payoffs following normal distribution

In the following theorem, we present sufficient conditions of RZ-equilibrium existence in the game (2.1) when payoffs follow a normal distribution.

**Theorem 3.1.** *Assume that the random variables* $\widetilde{a}_{ij}$ *and* $\widetilde{b}_{ij}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ *are independent and satisfy* $\widetilde{a}_{ij} \sim N\left(\mu_{a_{ij}}, \sigma^2_{a_{ij}}\right)$ *and* $\widetilde{b}_{ij} \sim N\left(\mu_{b_{ij}}, \sigma^2_{b_{rj}}\right)$, *that is, they are normally distributed on* $(\Omega, F, P)$, *with* $\sigma_{a_{ij}} > 0$ *and* $\sigma_{b_{ij}} > 0$, *for* $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

*Then, the game (2.1) has at least one RZ-equilibrium at* $(\delta_1, \delta_2)$ *levels, for all* $(\delta_1, \delta_2) \in \mathbb{R} \times \mathbb{R}$.

*Proof.* Under the conditions of Theorem 3.1, using the properties of the Gaussian random variables, $p^T \widetilde{A} q = \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij}$ is a Gaussian random variable.

Then for given levels $(\delta_1, \delta_2) \in \mathbb{R} \times \mathbb{R}$, we have the following chain of equalities

$$
\begin{aligned}
u_1^{\delta_1}(p,q) &= \mathbb{P}\left\{\omega : p^T \tilde{A} q \geq \delta_1\right\} = \mathbb{P}\left\{\omega : \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij} \geq \delta_1\right\} \\
&= \mathbb{P}\left\{\omega : \frac{\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij} - \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \mu_{a_{ij}}}{\sqrt{\operatorname{Var}\left(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij}\right)}} \geq \frac{\delta_1 - \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \mu_{a_{ij}}}{\sqrt{\operatorname{Var}\left(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij}\right)}}\right\} \\
&= 1 - \mathbb{P}\left\{\omega : \frac{\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij} - \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \mu_{a_{ij}}}{\sqrt{\operatorname{Var}\left(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij}\right)}} \leq \frac{\delta_1 - \sum_{1=}^{i=m}\sum_{j=1}^{j=n} p_i q_j \mu_{a_{ij}}}{\sqrt{\operatorname{Var}\left(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij}\right)}}\right\} \\
&= 1 - \mathbb{P}\left\{\omega : \frac{\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij} - M_1(p,q)}{\sqrt{V_1(p,q)}} \leq \frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right\},
\end{aligned}
$$

where

$$
M_1(p,q) = \mathbb{E}\left(\sum_{1=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij}\right) = \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \mu_{a_{ij}}
$$

and

$$V_1(p,q) = \mathrm{Var}\left(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij}\right) = \sum_{i=1}^{i=m}\sum_{j=1}^{j=n}(p_i q_j)^2 \sigma_{a_{ij}}^2.$$

As $\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij}$ is $\mathcal{N}(M_1(p,q), V_1(p,q))$, then $\frac{\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{a}_{ij} - M_1(p,q)}{\sqrt{V_1(p,q)}}$ is a zero mean unit variance Gaussian variable, that is $\mathcal{N}(0,1)$, then

$$u_1^{\delta_1}(p,q) = 1 - \Phi\left(\frac{\delta_1 - M_1}{\sqrt{V_1(p,q)}}\right),$$

where $\Phi$ is the standardized normal distribution function. Let $g_1^{\delta_1}(p,q)) = \frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}$.

The function $(p,q) \longmapsto \Phi(g_1^{\delta_1}(p,q))$ is continuous on $P \times Q$.

Similarly, we prove that

$$u_2^{\delta_2}(p,q) = \mathbb{P}\left\{w : p^T \widetilde{B} q \geq \delta_2\right\} = 1 - \Phi\left(\frac{\delta_2 - M_2(p,q)}{\sqrt{V_2(p,q)}}\right).$$

Let $g_2^{\delta_2}(p,q)) = \frac{\delta_2 - M_2(p,q)}{V_2(p,q)}$. The function $(p,q) \longmapsto \Phi\left(g_2^{\delta_2}(p,q)\right)$ is also continuous on $P \times Q$, where

$$M_2(p,q) = \mathbb{E}\left(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{b}_{ij}\right) = \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \mu_{b_{ij}}$$

and

$$V_2(p,q) = \mathrm{Var}\left(\sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \widetilde{b}_{ij}\right) = \sum_{i=1}^{i=m}\sum_{j=1}^{j=n}(p_i q_j)^2 \sigma_{b_{ij}}^2.$$

The existence of RZ-equilibrium of the game (2.1) at $(\delta_1, \delta_2)$ levels is equivalent to the existence of a Z-equilibrium in the game

$$\left\langle \{\mathrm{I}, \mathrm{II}\}, P \times Q, \left(1 - \Phi \circ g_1^{\delta_1}, 1 - \Phi \circ g_2^{\delta_2}\right)\right\rangle.$$

Since all the conditions of Zhukovskii theorem [29] (see Thm. 2.5 in Sect. 2) are satisfied in this game, we conclude that at least one RZ-equilibrium at levels $(\delta_1, \delta_2)$ exists. □

## 3.2. Payoffs following Cauchy distribution

In the following theorem, we present sufficient conditions of RZ-equilibrium existence in the game (2.1) when payoffs follow Cauchy distribution.

**Theorem 3.2.** *Assume that $\widetilde{a}_{ij}$ and $\widetilde{b}_{ij}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ are independent Cauchy random variables on $(\Omega, F, P)$, where the location and scale parameters of $\widetilde{a}_{ij}, i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}$ are $\mu_{a_{ij}} \in \mathbb{R}$ and $\sigma_{a_{ij}} > 0$ respectively and the location and scale parameters of $\widetilde{b}_{ij}, i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, n\}$ are $\mu_{b_{ij}} \in \mathbb{R}$ and $\sigma_{b_{ij}} > 0$ respectively. Then, the game (2.1) has at least one RZ-equilibrium at $(\delta_1, \delta_2)$ levels, for all $(\delta_1, \delta_2) \in \mathbb{R} \times \mathbb{R}$.*

*Proof.* Let $(\delta_1, \delta_2) \in \mathbb{R} \times \mathbb{R}$. It is well known that a linear combination of independent Cauchy random variables is a Cauchy random variable [14], then, for a given strategy profile $(p,q)$,

– the payoff $p^T\widetilde{A}q$ of the row player follows a Cauchy distribution with location parameter $\mu_a(p,q)=\sum_{i=1}^{i=m}\sum_{j=1}^{j=n}p_iq_j\mu_{a_{ij}}$ and scale parameter $\sigma_a(p,q)=\sum_{i=1}^{i=m}\sum_{j=1}^{j=n}p_iq_j\sigma_{a_{ij}}$;
– the payoff $p^t\widetilde{B}q$ of the column player follows a Cauchy distribution with location parameter $\mu_b(p,q)=\sum_{i=1}^{i=m}\sum_{j=1}^{j=n}p_iq_j\mu_{b_{ij}}$ and scale parameter $\sigma_b(p,q)=\sum_{1=i}^{i=m}\sum_{j=1}^{j=n}p_iq_j\sigma_{b_{ij}}$.

Therefore,

$$\begin{aligned} u_1^{\delta_1}(p,q) &= \mathbb{P}\left\{\omega : p^T\widetilde{A}q \geq \delta_1\right\} = \mathbb{P}\left\{\omega : \sum_{i=1}^{i=m}\sum_{j=1}^{j=n}p_iq_j\widetilde{a}_{ij}(\omega)\geq\delta_1\right\} \\ &= \mathbb{P}\left\{\omega : \frac{\sum_{i=1}^{i=m}\sum_{j=1}^{j=n}p_iq_j\widetilde{a}_{ij}(\omega)-\mu_a(p,q)}{\sigma_a(p,q)} \geq \frac{\delta_1-\mu_a(p,q)}{\sigma_a(p,q)}\right\} \\ &= 1-\mathbb{P}\left\{\omega : \frac{\sum_{1=i}^{i=m}\sum_{j=1}^{j=n}p_iq_j\widetilde{a}_{ij}(\omega)-\mu_a(p,q)}{\sigma_a(p,q)} \leq \frac{\delta_1-\mu_a(p,q)}{\sigma_a(p,q)}\right\}, \end{aligned}$$

where $\frac{\sum_{i=1}^{i=m}\sum_{j=1}^{j=n}p_iq_j\widetilde{a}_{ij}(\omega)-\mu_a(p,q)}{\sigma_a(p,q)}$ follows a standard Cauchy distribution.

Let $T(.)$ be the distribution function of a standard Cauchy random variable and $f_1^{\delta_1}(p,q)=\frac{\delta_1-\mu_a(p,q)}{\sigma_a(p,q)}$. Then,

$$u_1^{\delta_1}(p,q)=\mathbb{P}\left\{\omega : p^T\widetilde{A}q\geq\delta_1\right\}=1-T\left(\frac{\delta_1-\mu_a(p,q)}{\sigma_a(p,q)}\right)=1-T\circ f_1^{\delta_1}(p,q).$$

Similarly, we show that

$$u_2^{\delta_2}(p,q)=\mathbb{P}\left\{\omega : p^T\widetilde{B}q\geq\delta_2\right\}=1-T\left(\frac{\delta_2-\mu_b(p,q)}{\sigma_b(p,q)}\right)=1-T\circ f_2^{\delta_2}(p,q),\quad \text{with}\quad f_2^{\delta_2}(p,q)=\frac{\delta_2-\mu_b(p,q)}{\sigma_b(p,q)}.$$

Since $T(.)$ is continuous, we deduce that these two functions are continuous on $P\times Q$. Proceeding as in the proof of Theorem 3.1, we deduce the existence of at least one RZ-equilibrium of game (2.1) at $(\delta_1,\delta_2)$ levels. □

**Remark 3.3.** If the Cauchy random variables $\widetilde{a}_{ij}, i\in\{1,\cdots,m\}, j\in\{1,\cdots,n\}$ are independent and identically distributed (i.i.d.) and the Cauchy random variables $\widetilde{b}_{ij}, i\in\{1,\cdots,m\}, j\in\{1,\cdots,n\}$ are i.i.d., the functions $\mathbb{P}\left\{\omega : p^T\widetilde{A}q\geq\delta_1\right\}=1-T\left(\frac{\delta_1-\mu_a(p,q)}{\sigma_a(p,q)}\right)$ and $\mathbb{P}\left\{\omega : p^T\widetilde{B}q\geq\delta_1\right\}=1-T\left(\frac{\delta_1-\mu_b(p,q)}{\sigma_b(p,q)}\right)$ are constant. Consequently, every strategy profile $(p,q)\in P\times Q$ is a RZ-equilibrium of the game (2.1) at $(\delta_1,\delta_2)$ levels.

## 4. Computation of RZ-equilibria

In this section, we show that the problem of determination of a RZ-equilibrium can be transformed into a problem of computation of a Pareto optimal solution of a bi-criteria minimization problem. Then, from this problem, we derive an algorithm for the computation of RZ-equilibria.

**Remark 4.1.** – If the conditions of Theorem 3.1 (or Thm. 3.2) are satisfied, the maxmin values

$$\lambda_1 = \max_{p\in P}\min_{q\in Q}\mathbb{P}\left\{\omega : p^T\tilde{A}q \geq \delta_1\right\} \quad \text{and} \quad \lambda_2 = \max_{q\in Q}\min_{p\in P}\mathbb{P}\left\{\omega : p^T\tilde{B}q \geq \delta_2\right\}$$

exist and condition 1 of Definition 2.6 is equivalent to

$$\begin{cases} \lambda_1 \leq \mathbb{P}\left\{\omega : p^{*T}\tilde{A}q^* \geq \delta_1\right\}; \\ \lambda_2 \leq \mathbb{P}\left\{\omega : p^{*T}\tilde{B}q^* \geq \delta_2\right\}. \end{cases}$$

– In RZ-equilibrium, player I (resp. player II) guarantees a gain greater than or equal to what she/he can obtain using her/his maxmin (secure) strategy which is given by $p^* \in Arg\max_{p\in P}\min_{q\in Q}\mathbb{P}\left\{\omega : p^T\tilde{A}q \geq \delta_1\right\}$ $\left(\text{resp. } q^* \in \text{Arg}\max_{q\in Q}\min_{p\in P}\mathbb{P}\left\{\omega : p^T\tilde{B}q \geq \delta_2\right\}\right)$.

In order to deal with the problem of computation of a RZ-equilibrium, we recall that a bi-criteria optimisation (minimization/maximization) is an optimization problem that involves two objective functions. The general formulation of a bi-criteria optimization problem is

$$\begin{cases} \text{opt}(L_1(z), L_2(z)); \\ \text{s.t.} \quad z \in E, \end{cases}$$

where $E$ is the feasible set of decision vectors, $E \subset \mathbb{R}^n$, $n \in \mathbb{N}^*$;

$$L = (L_1, L_2) : E \longrightarrow \mathbb{R}^n, \quad L(z) = (L_1(z), \quad L_2(z)), \quad z \in E.$$

In the rest of this paper, we use the following notation of the bi-criteria optimization problem.

$$\langle E, (L_1, L_2)\rangle.$$

The concept of solution in bi-criteria optimization is based on Pareto optimality.

### 4.1. Payoffs following normal distribution

**Theorem 4.2.** *Under the assumptions of Theorem 3.1, $(p^*, q^*)$ is a RZ-equilibrium of the game (2.1), if and only if it is a Pareto optimal solution of the bi-criteria minimization problem*

$$\left\langle H, \left(g_1^{\delta_1}, g_2^{\delta_2}\right)\right\rangle; \tag{4.1}$$

*where*

$$H = \left\{(p,q) \in P \times Q : \gamma_1 \geq g_1^{\delta_1}(p,q) \quad \text{and} \quad \gamma_2 \geq g_2^{\delta_2}(p,q)\right\}$$

*and*

$$\gamma_1 = \min_{p\in P}\max_{q\in Q} g_1^{\delta_1}(p,q) \quad \text{and} \quad \gamma_2 = \min_{q\in Q}\max_{p\in P} g_2^{\delta_2}(p,q).$$

*Proof.* Indeed, from Remark 2.2 and Definition 2.6, we have the following equivalence $(p^*, q^*)$ is a RZ-equilibrium of the game (2.1) at $(\delta_1, \delta_2)$ levels $\iff$ $(p^*, q^*)$ is Pareto optimal solution for the maximization problem $\langle M, (F_1, F_2)\rangle$, with $M = \left\{(p,q) \in P \times Q : u_1^{\delta_1}(p,q) \geq \lambda_1 \text{ and } u_2^{\delta_2}(p,q) \geq \lambda_2\right\}$, $u_1^{\delta_1}(p,q) = 1 - \Phi\left(\frac{\delta_1 - M_1(p,q)}{V_1(p,q)}\right) = 1 - \Phi \circ g_1^{\delta_1}(p,q)$ and $u_2^{\delta_2}(p,q) = 1 - \Phi\left(\frac{\delta_2 - M_2(p,q)}{V_2(p,q)}\right) = 1 - \Phi \circ g_2^{\delta_2}(p,q)$.

First, we prove that $H = M$, considering that $\Phi$ is continuous and strictly increasing on $\mathbb{R}$, and $M_1$ and $V_1$ are continuous on the compact set $P \times Q$ and $V_1 > 0$, we obtain the equalities

$$\begin{aligned}\lambda_1 &= \max_{p\in P}\min_{q\in Q}\left(1 - \Phi\left(\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right)\right)\\ &= 1 - \min_{p\in P}\max_{q\in Q}\left(\Phi\left(\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right)\right)\\ &= 1 - \Phi\left(\min_{p\in P}\max_{q\in Q}\left(\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right)\right).\end{aligned}$$

Then,

$$U_1^{\delta_1}(p,q) = 1 - \Phi\left(\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right) \geq \lambda_1$$
$$\iff$$
$$1 - \Phi\left(\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right) \geq 1 - \Phi\left(\min_{p\in P}\max_{q\in Q}\left(\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right)\right)$$
$$\iff$$
$$\Phi\left(\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right) \leq \Phi\left(\min_{p\in P}\max_{q\in Q}\left(\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\right)\right)$$
$$\iff$$
$$g_1^{\delta_1}(p,q) = \frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}} \leq \gamma_1 = \min_{p\in P}\max_{q\in Q}\frac{\delta_1 - M_1(p,q)}{\sqrt{V_1(p,q)}}\cdot$$

In the same way, we have

$$u_2^{\delta_2}(p,q) \geq \lambda_2 \iff g_2^{\delta_2}(p,q) = \frac{\delta_2 - M_2(p,q)}{\sqrt{V_2(p,q)}} \leq \gamma_2 = \min_{q\in Q}\max_{p\in P}\frac{\delta_2 - M_2(p,q)}{\sqrt{V_2(p,q)}}\cdot$$

Then $M = \left\{(p,q) \in P \times Q : \gamma_1 \geq g_1^{\delta_1}(p,q) \text{ and } \gamma_2 \geq g_2^{\delta_2}(p,q)\right\} = H$. Next, we prove that a Pareto optimal solution of the bi-criteria maximization problem $\left\langle M, \left(u_1^{\delta_1}, u_2^{\delta_2}\right)\right\rangle$ is a Pareto optimal solution for the bi-criteria minimization problem $\left\langle H, \left(g_1^{\delta_1}, g_2^{\delta_2}\right)\right\rangle$. For all $(p,q) \in M$, the vector inequality

$$\left(u_1^{\delta_1}(p^*,q^*), u_2^{\delta_2}(p^*,q^*)\right) \precsim \left(u_1^{\delta_1}(p,q), u_2^{\delta_2}(p,q)\right), \quad \text{is impossible}$$

is equivalent to for all $(p,q) \in M$, the vector inequality

$$\left(\Phi \circ g_1^{\delta_1}(p,q), \Phi \circ g_2^{\delta_2}(p,q)\right) \precsim \left(\Phi \circ g_1^{\delta_1}(p^*,q^*), \Phi \circ g_2^{\delta_2}(p^*,q^*)\right), \quad \text{is impossible.}$$

Then, using the properties of the standardized normal distribution $\Phi$, it is also equivalent to for all $(p,q) \in M$, the system of inequalities

$$\left(g_1^{\delta_1}(p,q), g_2^{\delta_2}(p,q)\right) \precsim \left(g_1^{\delta_1}(p^*,q^*), g_2^{\delta_2}(p^*,q^*)\right), \quad \text{is impossible,}$$

which concludes the proof. □

*4.1.1. Algorithm*

To find Pareto optimal solutions of problem (4.1), we use the scalarization approach by choosing a pair of weights $(\beta_1, \beta_2) \in [0,1] \times [0,1]$ for $g_1^{\delta_1}$ and $g_2^{\delta_2}$, such that $\beta_1 + \beta_2 = 1$. Thus, to compute RZ-equilibria, we just need to solve a deterministic optimization problem. We present the computation of RZ-equilibria in the form of an algorithm as follows.

**Algorithm 1**

(1) Initialization: Let $\tilde{A}$, $\tilde{B}$, $\beta_1 \geq 0$, $\beta_2 \geq 0$ ($\beta_1 + \beta_2 = 1$), $\delta_1$ and $\delta_2$ be given.
(2) Compute $g_1^{\delta_1}(p,q)$ and $g_2^{\delta_2}(p,q)$, $\gamma_1$ and $\gamma_2$.
(3) Solve the problem

$$(P1)\begin{cases} \min(\beta_1 g_1^{\delta_1}(p,q) + \beta_2 g_2^{\delta_2}(p,q)) \\ \text{subject to} \\ \gamma_1 \geq g_1^{\delta_1}(p,q) \\ \gamma_2 \geq g_2^{\delta_2}(p,q) \\ (p,q) \in P \times Q. \end{cases}$$

Any solution of $(P1)$ is a RZ-equilibrium of the game (2.1).

## 4.2. Payoffs following Cauchy distribution

**Theorem 4.3.** *Under the assumptions of Theorem 3.2, $(p^*, q^*)$ is a RZ-equilibrium of the game (2.1), if and only if it is a Pareto optimal solution of the bi-criteria minimization problem*

$$\left\langle S, \left(f_1^{\delta_1}, f_2^{\delta_2}\right)\right\rangle, \tag{4.2}$$

*where*

$$S = \left\{(p,q) \in P \times Q : \theta_1 \geq f_1^{\delta_1}(p,q) \quad \text{and} \quad \theta_2 \geq f_2^{\delta_2}(p,q)\right\}$$

*and*

$$\theta_1 = \min_{p\in P}\max_{q\in Q} f_1^{\delta_1}(p,q), \quad \theta_2 = \min_{q\in Q}\max_{p\in P} f_2^{\delta_2}(p,q).$$

$$f_1^{\delta_1}(p,q) = \frac{\delta_1 - \mu_a(p,q)}{\sigma_a(p,q)} \quad \text{and} \quad f_2^{\delta_2}(p,q) = \frac{\delta_2 - \mu_b(p,q)}{\sigma_b(p,q)}.$$

*Proof.* Using the Cauchy distribution properties, the proof of this result is analogous to that of Theorem 4.2. □

*4.2.1. Algorithm*

Here we give an algorithm for finding RZ-equilibria by finding Pareto optimal solutions of the problem (4.2), using the scalarization approach by choosing a pair of weights $(\beta_1, \beta_2) \in [0,1] \times [0,1]$ for $f_1^{\delta_1}$ and $f_2^{\delta_2}$, respectively, such that $\beta_1 + \beta_2 = 1$.

**Algorithm 2**

(1) Initialization: Let $\widetilde{A}$, $\widetilde{B}$, $\beta_1 \geq 0$, $\beta_2 \geq 0$ $(\beta_1 + \beta_2 = 1)$, $\delta_1$ and $\delta_2$ be given.
(2) Compute $f_1^{\delta_1}(p,q)$, $f_2^{\delta_2}(p,q)$, $\theta_1$ and $\theta_2$.
(3) Solve the problem

$$(P2)\begin{cases} \min(\beta_1 f_1^{\delta_1}(p,q) + \beta_2 f_2^{\delta_2}(p,q)) \\ \text{subject to} \\ \theta_1 \geq f_1^{\delta_1}(p,q) \\ \theta_2 \geq f_2^{\delta_2}(p,q) \\ (p,q) \in P \times Q. \end{cases}$$

Any solution of the problem (P2) problem is a RZ-equilibrium of the game (2.1).

## 5. Numerical examples

**Example 5.1.** In order to show the applicability of the proposed approach, let us assume that, in the game (2.1), the payoffs matrices of player I and player II are, respectively

$$\tilde{A} = \begin{pmatrix} \mathcal{N}(1,1) & \mathcal{N}(0,2) \\ \mathcal{N}(3,\sqrt{2}) & \mathcal{N}(4,1) \end{pmatrix},$$

and

$$\tilde{B} = \begin{pmatrix} \mathcal{N}(1,2) & \mathcal{N}(3,1) \\ \mathcal{N}(2,1) & \mathcal{N}(4,2) \end{pmatrix}.$$

Assume that the conditions of Theorem 4.2 are satisfied for the given satisfaction levels $\delta_1$ and $\delta_2$.

Let $p = (p_1, p_2)^T$ and $q = (q_1, q_2)^T$, with $p_2 = 1 - p_1$ and $q_2 = 1 - q_1$. Then,

$$g_1^{\delta_1}(p,q) = \frac{\delta_1 - 2p_1q_1 + q_1 + 4p_1 - 4}{\sqrt{8p_1^2q_1^2 + 5p_1^2 + 3q_1^2 - 10p_1^2q_1 - 6p_1q_1^2 + 4p_1q_1 - 2q_1 - 2p_1 + 1}},$$
$$g_2^{\delta_2}(p,q) = \frac{\delta_2 + p_1 + 2q_1 - 4}{\sqrt{10p_1^2q_1^2 + 5p_1^2 + 5q_1^2 - 10p_1^2q_1 - 10p_1q_1^2 - 8p_1 - 8q_1 + 16p_1q_1 + 4}}$$

and

$$\gamma_1 = \min_{p_1\in[0,1]} \max_{q_1\in[0,1]} g_1^{\delta_1}(p,q) \quad \text{and} \quad \gamma_2 = \min_{q_1\in[0,1]} \max_{p\in[0,1]} g_2^{\delta_2}(p,q).$$

Assume that the players set their satisfaction levels at $\delta_1 = 10$ and $\delta_2 = 10$.

A Pareto optimal solution of Problem (4.1), with $\beta_1 = 0.5$ and $\beta_2 = 0.5$, is given by solving the problem $(P1)$, which is equivalent to

$$(P1)\begin{cases} \min\left(\frac{\beta_1(\delta_1 - 2p_1q_1 + q_1 + 4p_1 - 4)}{\sqrt{8p_1^2q_1^2+5p_1^2+3q_1^2-10p_1^2q_1-6p_1q_1^2+4p_1q_1-2q_1-2p_1+1}}\right. \\ \left.+\frac{\beta_2(\delta_2+p_1+2q_1-4)}{\sqrt{10p_1^2q_1^2+5p_1^2+5q_1^2-10p_1^2q_1-10p_1q_1^2-8p_1-8q_1+16p_1q_1+4}}\right) \\ \text{subject to} \\ \gamma_1 \geq \frac{\delta_1-2p_1q_1+q_1+4p_1-4}{\sqrt{8p_1^2q_1^2+5p_1^2+3q_1^2-10p_1^2q_1-6p_1q_1^2+4p_1q_1-2q_1-2p_1+1}}; \\ \gamma_2 \geq \frac{\delta_2+p_1+2q_1-4}{\sqrt{10p_1^2q_1^2+5p_1^2+5q_1^2-10p_1^2q_1-10p_1q_1^2-8p_1-8q_1+16p_1q_1+4}}; \\ (p_1,q_1) \in [0,1]^2. \end{cases}$$

First, we use the mesh of the set $[0,1]\times[0,1]$ for computing $\gamma_1$ and $\gamma_2$. We get $\gamma_1 = 7.776$ and $\gamma_2 = 7.6155$.

To obtain the global optimal solutions of $(P1)$, we use BARON solver [12], which is a computational system for solving non-convex optimization problems for global optimality. The solution is the pure strategy profile $(p^*, q^*) = ((0,1),(0,1))$, which is a RZ-equilibrium at $(10,10)$ satisfaction levels and probabilities $\left(\mathbb{P}\left(p^{*T}\tilde{A}q^* \geq 10\right), \mathbb{P}\left(p^{*T}\tilde{B}q^* \geq 10\right)\right) = (0, 0.001)$.

These probabilities are not acceptable, which means that the players are too optimistic. To increase them, the players need to decrease the satisfaction levels $\delta_1$ and $\delta_2$.

Consider the levels $(\delta_1, \delta_2) = (3,2)$ with $\beta_1 = \beta_2 = \frac{1}{2}$. The RZ-equilibrium is $(p^*, q^*) = ((0,1),(0,1))$ and the probabilities are $\mathbb{P}\left(p^{*T}\tilde{A}q^* \geq 3\right) = 0.8413$ and $\mathbb{P}\left(p^{*T}\tilde{B}q^* \geq 2\right) = 0.8413$.

**Example 5.2.** In order to show the applicability of the proposed approach in the case where the payoffs follow Cauchy distribution, let us assume that in the game (2.1) the payoffs matrices of player I and player II are, respectively

$$\widetilde{A} = \begin{pmatrix} \mathcal{C}(1,2) & \mathcal{C}(2,3) \\ \mathcal{C}(3,1) & \mathcal{C}(4,2) \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} \mathcal{C}(4,4) & \mathcal{C}(3,1) \\ \mathcal{C}(3,2) & \mathcal{C}(5,3) \end{pmatrix}.$$

Assume that the conditions of Theorem 4.2 are satisfied for the given satisfaction levels $\delta_1$ and $\delta_2$.

We have

$$f_1^{\delta_1}(p,q) = \frac{\delta_1 + 2p_1 + q_1 - 4}{p_1 - q_1 + 2}, \quad f_2^{\delta_2}(p,q) = \frac{\delta_2 - 3p_1q_1 + 2p_1 + 2q_1 - 5}{4p_1q_1 - q_1 - 2p_1 + 3},$$
$$\theta_1 = \min_{p_1 \in [0,1]} \max_{q_1 \in [0,1]} f_1^{\delta_1}(p,q) \quad \text{and} \quad \theta_2 = \min_{q_1 \in [0,1]} \max_{p_1 \in [0,1]} f_2^{\delta_2}(p,q).$$

Assume that the players set their satisfaction levels at $\delta_1 = 10$ and $\delta_2 = 10$. A Pareto optimal solution of Problem (4.2), with $\beta_1 = 0.5$ and $\beta_2 = 0.5$, is given by solving the following problem

$$(P2) \begin{cases} \min\left(\frac{\beta_1(\delta_1+2p_1+q_1-4)}{p_1-q_1+2} + \frac{\beta_2(\delta_2-3p_1q_1+2p_1+2q_1-5)}{4p_1q_1-q_1-2p_1+3}\right) \\ \text{subject to} \\ \theta_1 \geq \frac{\delta_1+2p_1+q_1-4}{p_1-q_1+2} \\ \theta_2 \geq \frac{\delta_2+3p_1q_1+2p_1+2q_1-5}{4p_1q_1-q_1-2p_1+3}. \\ (p_1,q_1) \in [0,1]^2. \end{cases}$$

We use the mesh of the set $[0,1] \times [0,1]$ for computing $\theta_1$ and $\theta_2$. Thus, $\theta_1 = 4,5$ and $\theta_2 = 2,47151564$.

To obtain the global optimal solutions of $(P2)$, we use BARON solver [12]. The solution is the pure strategy profile $(p^*, q^*) = ((0,1),(0,1))$, which is an RZ-equiibrium at $(10,10)$ satisfaction levels and

$$\left(\mathbb{P}\left(p^{*T}\tilde{A}q^* \geq 10\right), \left(\mathbb{P}(p^{*T}\tilde{B}q^* \geq 10)\right)\right) = (0.1024, 0.1720).$$

## 6. Related work

Z-equilibrium in games with uncertain payoffs in the form $u_i(x,y)$, where $x$ is a strategy profile and $y$ is a parameter with unknown behavior has been investigated in Larbani and Lebbah [16]. In these games, the payoffs are not completely uncertain, they are represented by a parameter only. In the present work, the payoffs do not involve any parameter. In this sense, it is more general. Recently, Achemine *et al.* [2] have investigated Z-equilibrium in bi-matrix games with uncertain payoffs in the sense of Liu [17]. The present work fundamentally differs from this work as it deals with games with random (uncertainty of probability type) payoffs. The difference between probability theory and Liu uncertainty theory is that the latter is based on a credibility measure that is introduced to measure the credibility of a fuzzy event, while the former deals with random phenomena. Fuzzy set theory mainly deals with subjective uncertainty *i.e.*, imprecision in human judgment and evaluation of events and phenomena. Therefore, the scopes of application of the two papers differ considerably as they model two different types of uncertainty. Thus, compared to Achemine *et al.* [2], the present paper is a new theoretical and application contribution.

Further, the literature on games with random payoffs is mainly concentrated on the investigation of Nash equilibrium, its existence and computation. We mention and compare our work to two prominent works on this area of research, Singh and Lisser [24] that deals with bi-matrix games with random payoffs and Singh *et al.* [25] that investigates $n$-person finite games.

The main differences between the present paper and the mentioned two papers are that (i) the two papers deal with existence, and computation of Nash equilibrium, while ours deals with Z-equilibrium, and (ii) in the two papers the payoffs are defined by fixing a probability (confidence) level, then finding the maximum payoff that can be obtained for this level; formally, they use the formulas

$$u_i^{\alpha_i}(p,q) = \sup\left\{v_i : \mathbb{P}\left(p^T\tilde{A}q \geq v_i\right) \geq \alpha_i\right\} \tag{6.1}$$

$$u_i^{\alpha_i}(\tau) = \sup\left\{v_i : \mathbb{P}(\omega | r_i^{\omega}(\tau,\omega) \geq v_i) \geq \alpha_i\right\} \tag{6.2}$$

respectively, where $\alpha_i$ is a given confidence level. In our paper, we use the formula

$$u_i^{\delta_i}(p,q) = \mathbb{P}\left(p^T\tilde{A}q \geq \delta_i\right) \tag{6.3}$$

where $\delta_i$ is a given satisfaction level in terms of payoff.

As (6.1) and (6.2) are based on the same idea, we will compare (6.1) to (6.3) because they deal with the same type of games, bi-matrix games. In (6.1), the used payoff to define Nash equilibrium is defined as the supremum of the set of $v_i$ values such that $\mathbb{P}\left(p^T\tilde{A}q \geq v_i\right) \geq \alpha_i$. Then a RZ-equilibrium in the sense of Singh and Lisser payoffs can be defined as follows.

**Definition 6.1.** For predetermined confidence levels $\alpha_1$, $\alpha_2$, a pair $(p^*, q^*)$ is called a RZ-equilibrium (random Z-equilibrium) of the game (2.1) at $(\alpha_1, \alpha_2)$ levels, if it satisfies

(1) $\begin{cases} \forall p \in P, \quad \exists q \in Q, \quad u_1^{\alpha_1}(p,q) \leq u_1^{\alpha_1}(p^*,q^*) \\ \forall q \in Q, \quad \exists p \in P, \quad u_2^{\alpha_2}(p,q) \leq u_2^{\alpha_q}(p^*,q^*). \end{cases}$

(2) There is no strategy profile $(p,q) \in P \times Q$, such that

$$\left(u_1^{\alpha_1}(p^*,q^*), \quad u_2^{\alpha_2}(p^*,q^*)\right) \precsim \left(u_1^{\alpha_1}(p,q), \quad u_2^{\alpha_2}(p,q)\right).$$

That is, using Singh and Lisser's definition of payoffs, the players must first provide confidence levels (values of the probability parameters $\alpha_i$, $i = 1, 2$), then if a pair $(p^*, q^*)$ of mixed strategies is a Z-equilibrium, the payoffs $u_i^{\alpha_i}(p^*, q^*)$, $i = 1, 2$ satisfy the confidence levels $\alpha_i, i = 1, 2$, that is $\mathbb{P}\left(p^{*T}Aq^* \geq u_1^{\alpha_1}(p^*,q^*)\right) \geq \alpha_1$, and $\mathbb{P}\left(p^{*T}Bq^* \geq u_2^{\alpha_2}(p^*,q^*)\right) \geq \alpha_2$.

In our work, we proceed in the other way around, we first ask the players to provide their satisfaction levels, $\delta_i, i = 1, 2$, in terms of payoffs and then define the RZ-equilibrium in terms of probabilities of events where payoffs satisfy those levels. Then, if a pair $(p^*, q^*)$ of mixed strategies is a RZ-equilibrium, the payoffs are expressed in terms of probabilities $u_i^{\delta_i}(p^*,q^*) = \mathbb{P}\left(p^{*T}\tilde{A}q^* \geq \delta_i\right)$, and $u_2^{\delta_2}(p^*,q^*) = \mathbb{P}\left(p^{*T}\tilde{B}q^* \geq \delta_2\right)$.

Our approach that is based on considering probabilities as payoffs in defining RZ-equilibrium makes theoretical and practical sense. In fact, asking the players to provide a satisfaction level, we use the Simon's "satisficing principle" [23] which means that in non-trivial real-life decision problems, decision-makers look for satisficing alternatives rather than maximizing ones. As uncertain payoff games are highly complex because they involve strategic uncertainty and payoff uncertainty, this principle is highly relevant and appropriate. Further, it also makes sense to express the attainment of the given satisfaction level in terms of probability as the payoffs are of probability uncertainty type. Consequently, as we are in a game context, it makes sense to define Z-equilibrium in terms of the probability of attainment of satisfaction levels.

Now let us investigate the two definitions of RZ-equilibrium, our Definition 2.6 and Sing and Lisser inspired Definition 6.1. For the sake of discussion, assume that the entries of the payoff matrices in the game (2.1) satisfy the conditions of Theorem 3.1. The relation between the two RZ-equilibrium definitions is that with each given pair of confidence levels $(\alpha_1, \alpha_2)$, and RZ-equilibrium $(p^*, q^*)$ in the sense of Definition 6.1, we can associate a Z-equilibrium $(\overline{p}^*, \overline{q}^*)$ in the sense of our Definition 2.6 as follows. Let

$$u_i^{\alpha_i}(p^*, q^*) = \delta_i, \quad i = 1, 2,$$

then Theorem 3.1 guarantees the existence of RZ-equilibrium in the sense of Definition 2.6 with satisfaction levels $\delta_1 = u_1^{\alpha_1}(p^*, q^*)$, $\delta_2 = u_2^{\alpha_2}(p^*, q^*)$. That is, with each RZ-equilibrium in the sense of Definition 6.1, we associate a RZ-equilibrium in the sense of Definition 2.6 that guarantees (in probability sense) the payoffs of the former RZ-equilibrium, that is $\mathbb{P}\left\{\omega : p^{*T}\tilde{A}q^* \geq \delta_1\right\}$ and $\mathbb{P}\left\{\omega : p^{*T}\tilde{B}q^* \geq \delta_2\right\}$. And based on RZ-equilibrium Pareto optimality condition, each equilibrium is not Pareto-dominated by the other in the sense of the definition that is used to find it, that is, on the one hand the inequality

$$\left(u_1^{\alpha_1}(p^*, q^*), \quad u_2^{\alpha_2}(p^*, q^*)\right) \precsim \left(u_1^{\alpha_1}(\overline{p}^*, \overline{q}^*), \quad u_2^{\alpha_2}(\overline{p}^*, \overline{q}^*)\right).$$

is impossible. On the other hand, the inequality

$$\left(\mathbb{P}\left\{\omega : \overline{p}^{*T}\tilde{A}\overline{q}^* \geq \delta_1\right\}, \quad \mathbb{P}\left\{\omega : \overline{p}^{*T}\tilde{B}\overline{q}^* \geq \delta_2\right\}\right) \precsim \left(\mathbb{P}\left\{\omega : p^{*T}\tilde{A}q^* \geq \delta_1\right\}, \quad \mathbb{P}\left\{\omega : p^{*T}\tilde{B}q^* \geq \delta_2\right\}\right).$$

is impossible.

Therefore, from this point of view, no equilibrium has an advantage over the other in terms of payoffs. One may argue that the probabilities $\mathbb{P}\left\{\omega : \overline{p}^{*T}\tilde{A}\overline{q}^* \geq \delta_1\right\}$ and $\mathbb{P}\left\{\omega : \overline{p}^{*T}\tilde{B}\overline{q}^* \geq \delta_2\right\}$ at the Z-equilibrium in the sense of Definition 2.6 can be small and may not be attractive to the players even if the satisfaction levels, $\delta_i, i = 1, 2$ are large. As stated above, because of the Pareto optimality of RZ-equilibrium, even if the probabilities are small at the equilibrium, they are not Pareto dominated by the probabilities of any other strategy profile $(p, q)$. Another argument is that, in fact, these probabilities depend on the satisfaction levels $\delta_i, i = 1, 2$ and the probability distributions of the payoffs. For example, in the case the entries of the payoff matrices of the game (2.1) are normally distributed, for the first player, we have $\mathbb{P}\left(\overline{p}^{*T}\tilde{A}\overline{q}^* \geq \delta_1\right) = 1 - \Phi\left(\frac{\delta_1 - \mathbb{E}\left(\overline{p}^{*T}\tilde{A}\overline{q}^*\right)}{\sqrt{V_1(\overline{p}, \overline{q})}}\right)$, (see the proof of Thm. 3.1), then the larger the number $\frac{\delta_1 - \mathbb{E}\left(\overline{p}^{*T}A\overline{q}^*\right)}{\sqrt{V_1(\overline{p}, \overline{q})}}$, the smaller the probabilities. In other words, to get larger probabilities, the players should choose smaller satisfaction levels than the averages $\mathbb{E}\left(\overline{p}^{*T}\tilde{A}\overline{q}^*\right)$ and $\mathbb{E}\left(\overline{p}^{*T}\tilde{B}\overline{q}^*\right)$.

Now, we explain how to get RZ-equilibrium with desired probabilities.

For player I, we have

$$\begin{aligned}\mathbb{E}\left(p^T\tilde{A}q\right) &= \mathbb{E}\left(\sum_{1=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \tilde{a}_{ij}\right)\\ &= \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \mu_{a_{ij}} \leq \sum_{i=1}^{i=m}\sum_{j=1}^{j=n} p_i q_j \max_{i,j}\mu_{a_{ij}} = \max_{i,j}\mu_{a_{ij}} = \mu_{a,\max}.\end{aligned}$$

Similarly, for player II, we have $\mathbb{E}\left(p^T\tilde{B}q\right) \leq \max_{i,j}\mu_{b_{ij}} = \mu_{b,\max}$.

We have also

$$\mathbb{P}\left\{\omega : p^T\tilde{A}q \geq \delta_1\right\} = 1 - \Phi\left(\frac{\delta_1 - M_1(p, q)}{\sqrt{V_1(p, q)}}\right) \leq 1 - \Phi\left(\frac{\delta_1 - \mu_{a,\max}}{\sqrt{V_1(p, q)}}\right)$$

and

$$\mathbb{P}\left\{\omega : p^T\tilde{B}q \geq \delta_1\right\} = 1 - \Phi\left(\frac{\delta_2 - M_2(p,q)}{\sqrt{V_2(p,q)}}\right) \leq 1 - \Phi\left(\frac{\delta_2 - \mu_{b,\max}}{\sqrt{V_2(p,q)}}\right).$$

Therefore, $\mu_{a,\max}$ and $\mu_{b,\max}$ can be used as reference points for computing RZ-equilibria with desired probabilities for given satisfaction levels. For example, for $\delta_1 = \mu_{a,\max}$, $\delta_2 = \mu_{b,\max}$, $\mathbb{P}\left(p^T\tilde{A}q \geq \delta_1\right) \leq 0.5$ and $\mathbb{P}\left(p^T\tilde{B}q \geq \delta_1\right) \leq 0.5$.

Clearly, the smaller than $\mu_{a,\max}$, $\mu_{b,\max}$ the satisfaction levels $\delta_1$, $\delta_2$, the larger the probabilities $\mathbb{P}\left(p^T\tilde{A}q \geq \delta_1\right)$ and $\mathbb{P}\left(p^T\tilde{B}q \geq \delta_1\right)$, respectively.

Let us illustrate this statement by the game of Example 5.1.

**Example 6.2.** Consider the game of Example 5.1 and compute RZ-equilibrium for decreasing values of the satisfaction levels $\delta_i, i = 1, 2$ with the same weights $\beta_i = \frac{1}{2},\ i = 1, 2$. We have $\mu_{a,\max} = 4$ and $\mu_{b,\max} = 4$.

(1) For $\delta_1 = 10, \delta_2 = 10$, we have already computed the RZ-equilibrium in Example 5.1. It is $(\overline{p}^*, \overline{q}^*) = ((0,1),(0,1))$. Then

$$\mathbb{P}\left(\overline{p}^{*T}\tilde{A}\overline{q}^* \geq 10\right) = 1 - \Phi\left(\frac{10 - M_1((0,1),(0,1))}{\sqrt{V_1((0,1),(0,1))}}\right) = 1 - \Phi\left(\frac{10-4}{1}\right) = 1 - \Phi(6) = 0.$$

$$\mathbb{P}\left(\overline{p}^{*T}\tilde{B}\overline{q}^* \geq 10\right) = 1 - \Phi\left(\frac{10 - M_2((0,1),(0,1))}{\sqrt{V_2((0,1),(0,1))}}\right) = 1 - \Phi\left(\frac{10-4}{2}\right) = 1 - \Phi(3) = 0.001.$$

Then

$$\left(\mathbb{P}\left(\overline{p}^{*T}\tilde{A}\overline{q}^* \geq 10\right), \mathbb{P}\left(\overline{p}^{*T}\tilde{B}\overline{q}^* \geq 10\right)\right) = (0, 0.001).$$

(2) For $\delta_1 = 5$ and $\delta_2 = 4$, the RZ-equilibrium is $(\overline{p}^*, \overline{q}^*) = ((0,1),(0,1))$. Then,

$$\left(\mathbb{P}\left(\overline{p}^{*T}\tilde{A}\overline{q}^* \geq 5\right), \mathbb{P}\left(\overline{p}^{*T}\tilde{B}\overline{q}^* \geq 4\right)\right) = (0.1587, 0.5).$$

(3) $\delta_1 = 1$, $\delta_2 = 2$, the RZ-equilibrium is $(\overline{p}^*, \overline{q}^*) = ((0.0257, 0.9743), (0.2161, 0.7839))$, then

$$\mathbb{P}\left(\overline{p}^{*T}\tilde{A}\overline{q}^* \geq 1\right) = 1 - \Phi(-3) = \Phi(3) = 0.9986 \simeq 1.$$

$$\mathbb{P}\left(\overline{p}^{*T}\tilde{B}\overline{q}^* \geq 2\right) = 1 - \Phi(-1) = \Phi(1) = 0.8413 \quad \text{and}$$

$$\left(\mathbb{P}\left(\overline{p}^{*T}\tilde{A}\overline{q}^* \geq 1\right), \mathbb{P}\left(\overline{p}^{*T}\tilde{B}\overline{q}^* \geq 2\right)\right) = (1, 0.8413).$$

Clearly, the probabilities decrease as $\delta_i, i = 1, 2$ increase. Thus, starting from $\delta_1 = \mu_{a\max}$, $\delta_2 = \mu_{b\max}$, players can get RZ-equilibrium with desired probabilities by increasing or decreasing $\delta_i, i = 1, 2$.

Finally, an advantage of the payoffs in Definition 2.6 is that they are simpler as they are expressed by probabilities only, while the payoffs in Definition 6.1 are expressed by probabilities and the "sup" operation. Thus, the existence conditions and computation of RZ-equilibrium would be more difficult *via* the latter definition than *via* the former.

Next, we provide an example with RZ-equilibrium and Nash equilibrium.

**Example 6.3.** Consider the random bi-matrix game

$$\tilde{A} = \begin{pmatrix} \mathcal{N}(1,2) & \mathcal{N}(1,2) \\ \mathcal{N}(1,2) & \mathcal{N}(1,2) \end{pmatrix},$$

and

$$\tilde{B} = \begin{pmatrix} \mathcal{N}(2,1) & \mathcal{N}(2,1) \\ \mathcal{N}(2,1) & \mathcal{N}(2,1) \end{pmatrix}.$$

According to Theorem 3.2 in Singh and Lisser [24], for all $\alpha_1, \alpha_2 \in [0.5, 1]$, $(p, q) = \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ is a Nash equilibrium. Using Lisser *et al.* [25], the payoffs of the players at this equilibrium are computed as $(1.6915, 2.1728)$.

To compute RZ-equilibrium, we use the problem $(P1)$, which is equivalent to

$$(P3) \begin{cases} \min \Bigg( \frac{\beta_1(\delta_1 - 1)}{\sqrt{4(4p_1^2q_1^2 + 2p_1^2 + 2q_1^2 - 4p_1^2q_1 - 4p_1q_1^2 + 4p_1q_1 - 2q_1 - 2p_1 + 1)}} \\ + \frac{\beta_2(\delta_2 - 2)}{\sqrt{(4p_1^2q_1^2 + 2p_1^2 + 2q_1^2 - 4p_1^2q_1 - 4p_1q_1^2 + 4p_1q_1 - 2q_1 - 2p_1 + 1)}} \Bigg) \\ \text{subject to} \\ \gamma_1 \geq \frac{(\delta_1 - 1)}{\sqrt{4(4p_1^2q_1^2 + 2p_1^2 + 2q_1^2 - 4p_1^2q_1 - 4p_1q_1^2 + 4p_1q_1 - 2q_1 - 2p_1 + 1)}}; \\ \gamma_2 \geq \frac{(\delta_2 - 2)}{\sqrt{(4p_1^2q_1^2 + 2p_1^2 + 2q_1^2 - 4p_1^2q_1 - 4p_1q_1^2 + 4p_1q_1 - 2q_1 - 2p_1 + 1)}}; \\ (p_1, q_1) \in [0, 1]^2. \end{cases}$$

Comparing Nash equilibrium $(p^*, q^*)$ with RZ-equilibrium, we obtain.

(1) For $\beta_1 = \beta_2 = \frac{1}{2}$ and $\delta_1 = 0.5$, $\delta_2 = 0.5$, $(p*, q*) = ((0.5, 0.5), (0.5, 0.5))$ is a RZ-equilibrium with probabilities $\left(\mathbb{P}\left(p^{*T}\tilde{A}q^* \geq 0.5\right), \mathbb{P}\left(p^{*T}\tilde{B}q^* \geq 0.5\right)\right) = (0.6915, 0.9986)$.

(2) When $\beta_1 = \beta_2 = \frac{1}{2}$ and $\delta_1 = 1.6915$, $\delta_2 = 2.1728$, the RZ-equilibrium that guarantees the payoffs $(1.6915, 2.1728)$ of the Nash equilibrium $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$ is $(\overline{p}*, \overline{q}*) = ((0, 1), (0, 1))$ with probabilities

$$\left(\mathbb{P}\left(\overline{p}^{*T}\tilde{A}\overline{q}^* \geq 1.6915\right), \mathbb{P}\left(\overline{p}^{*T}\tilde{B}\overline{q}^* \geq 2.1728\right)\right) = (0.3669, 0.4325).$$

Thus, in (1) with low satisfaction levels, $\delta_1 = \delta_2 = 0.5$, Nash equilibrium is also a RZ-equilibrium with high probabilities. However, in (2), Nash equilibrium payoffs are achieved with low probabilities. These results can be explained by the facts that (i) in general, Nash equilibrium is not Pareto optimal, while RZ-equilibrium is, and (ii) a Nash equilibrium that is Pareto optimal is a Z-equilibrium.

## 7. Conclusion

In this paper, we have introduced the concept of RZ-equilibrium for a bi-matrix game with random payoffs which is based on the notion of Z-equilibrium. Sufficient existence conditions are established in the cases where the payoffs follow normal or Cauchy distributions. In both cases, we show that the chance-constrained game can be formulated as an equivalent bi-criteria minimization problem, from which we derived algorithms for the computation of RZ-equilibrium. In the future, we intend to extend this work to the case of multi-criteria bi-matrix games with random payoffs. Finding conditions under which a game has at least one pure strategy RZ-equilibrium is an interesting challenging research problem.

*Acknowledgements.* The authors would like to thank the two referees for their comments that helped to substantially improve this article.

## References

[1] F. Achemine, A. Merakeb and A. Hamaz, A new equilibrium for an $n$-person game with fuzzy parameters. *Am. J. Math. Manage. Sci.* **32** (2013) 118–132.

[2] F. Achemine, A. Merakeb, M. Larbani and P. Marthon, Z-equilibria in Bi-matrix games with uncertain payoffs. *RAIRO: OR* **54** (2020) 393–412.

[3] R.J. Aumann, The core of a cooperative game without side payements. *Trans. Am. Math. Soc.* **96** (1961) 539–552.

[4] R.A. Blau, Random-payoff two person zero-sum games. *Oper. Res.* **22** (1974) 1243–1251.

[5] K. Bouchama, M.S. Radjef and L. Sais, Z-equilibrium for a CSP game. In: International Symposium on Artificial Intelligence and Mathematics Fort Lauderdale, FL (2016).

[6] R.G. Cassidy, C.A. Field and M.J.L. Kirby, Solution of a satisficing model for random payoff games. *Manage. Sci.* **19** (1972) 266–271.

[7] A. Charnes, M.J.L. Kirby and W.M. Raike, Zero-zero chanceconstrained games. *Theory Probab. App.* **13** (1968) 628–646.

[8] P. Couchman, B. Kouvaritakis, M. Cannon and F. Prashad, Gaming strategy for electric power with random demand. *IEEE Trans. Power Syst.* **20** (2005) 1283–1292.

[9] D. De Wolf and Y. Smeers, A stochastic version of a Stackelberg-Nash-Cournot equilibrium model. *Manage. Sci.* **43** (1997) 190–197.

[10] V. DeMiguel and H. Xu, A stochastic multiple leader Stackelberg model: Analysis, computation, and application. *Oper. Res.* **57** (2009) 1220–1235.

[11] A. Ferhat and M.S. Radjef, Z-equilibrium for a mixed strategic multicriteria game. EURO 25 – Vilnius (2012) 8–11.

[12] V. Ghildyal and N.V. Sahinidis, Solving global optimization problems with baron. In: From Local to Global Optimization. Vol. 53 of *Nonconvex Optimization and Its Applications*, edited by A. Migdalas, P.M. Pardalos and P. Värbrand. Springer, Boston, MA (2001).

[13] B. Jadamba and F. Raciti, Variational inequality approach to stochastic Nash equilibrium problems with an application to Cournot oligopoly. *J. Optim. Theory App.* **165** (2015) 1050–1070.

[14] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, 2nd edition. Vol. 1, John Wiley and Sons Inc. (1994).

[15] M. Larbani and F. Achemine, Solving a two person cooperative game under uncertainty. In: Multiple Criteria Decision Making in the New Millennium, edited by M. Koksalan and S. Zionts. Springer Verlag (2001) 314–324.

[16] M. Larbani and H. Lebbah, A concept of equilibrium for a game under uncertainty. *Eur. J. Oper. Res.* **1** (1999) 145–156.

[17] B. Liu, Uncertainty Theory, 2nd edition. Springer-Verlag, Berlin (2007).

[18] M. Mazadi, W.D. Rosehart, H. Zareipour, O.P. Malik and M. Oloomi, Impact of wind integration on electricity markets: A chance-constrained Nash-Cournot model. *Int. Trans. Electr. Energy Syst.* **23** (2013) 83–96.

[19] J.F. Nash, Non-cooperative games. *Ann. Math.* **54** (1951) 286–295.

[20] R. Nessah, M. Larbani and T. Tazdait, Coalitional ZP-equilibrium in games and its existence. *Int. Game Theory Rev.* **17** (2015) 1–18.

[21] U. Ravat and U.V. Shanbhag, On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. *SIAM J. Optim.* **21** (2011) 1168–1199.

[22] D. Sally, Conversation and cooperation in social Dilemma. A meta analysis from 1958 to 1992. *Ration. Soc.* **7** (1995) 58–92.

[23] H. Simon, Administrative Behavior: A Study of Decision-Making Processes in Administrative Organization. Macmillan, New York (1947).

[24] V.V. Singh and A. Lisser, A characterization of Nash equilibrium for the games with random payoffs. *J. Optim. Theory App.* **178** (2018) 998–1013. Springer.

[25] V. Singh, O. Jouini and A. Lisser, Existence of Nash equilibrium for chance-constrained games. *Oper. Res. Lett.* **44** (2016) 640–644.

[26] T. Song, Random payoff matrix games. In: Systems and Management Science by Extremal Methods. Springer Science + Business Media, LLC (1992) 291–308.

[27] J. Valenzuela and M. Mazumdar, Cournot prices considering generator availability and demand uncertainty. *IEEE Trans. Power Syst.* **22** (2007) 116–125.

[28] H. Xu and D. Zhang, Stochastic Nash equilibrium problems: Sample average approximation and applications. *Comput. Optim. App.* **55** (2013) 597–645.

[29] V.I. Zhukovskii, Some problems of non-antagonistic differential games. In: Matematiceskie metody versus issledovanii operacij [Mathematical Methods in Operations Research], edited by P. Kenderov. Bulgarian Academy of Sciences, Sofia (1985) 103–195.

[30] V.I. Zhukovskii and A.A. Tchikry, Linear-Quadratic Differential Games. Naoukova Doumka, Kiev (1994).