RAIRO-Oper. Res. 56 (2022) 1397–1410
https://doi.org/10.1051/ro/2022064

RAIRO Operations Research
www.rairo-ro.org

NONLINEAR PROGRAMMING PROBLEM FOR STRONGLY E-INVEX SETS AND STRONGLY E-PREINVEX FUNCTIONS

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Abstract. The concepts of strongly E-invex sets, strongly E-invex, strongly E-preinvex, and strongly pseudo E-preinvex functions are introduced in this paper. We have included several non-trivial examples to support our definitions. The family of strongly E-invex sets has been shown to form a vector space over R, and other interesting properties have been addressed. The epigraph of strongly E-preinvex function has been derived, as well as the relationship between the strongly E-preinvex function and the strongly pseudo E-preinvex function has been established. To show an important relationship between strongly E-invex and strongly E-preinvex functions, a new Condition A has been introduced. A nonlinear programming problem for strongly E-preinvex functions is explored as an application. Under a few conditions, it has been proved that the local minimum point is the global minimum for a nonlinear programming problem.

Mathematics Subject Classification. 26B25, 26D07, 90C25.

Received March 24, 2021. Accepted May 4, 2022.

1. Introduction

The extension of convexity has been developed more in recent years and has been used in optimization theory, economics, engineering sciences, and Riemannian manifolds for solving practical problems [5,7–9,18,23]. Because of its vast applications, Youness [24] defined E-convexity, which has various applications in different branches of mathematical sciences; see [1, 19]. Later, Youness [26] extended this concept to strongly E-convexity and discussed their applications. Motivated from Youness's works on strongly E-convexity, Majeed [16] introduced strongly E-convex hull, strongly E-convex cone, strongly E-convex cone hull and discussed their interesting properties. Further, Hussain and Iqbal [6] extended the strongly E-convex functions to quasi strongly E-convex functions and considered a nonlinear programming problem for these newly defined functions. Kilickman and Saleh [12,13] extended this concept from Euclidean space to Riemannian manifolds with the help of geodesics and the technique of convexity, which is called geodesic strongly E-convexity and geodesic semilocal E-preinvexity and established several properties with an application to the nonlinear fractional multi-objective programming problem.

Thus, the generalized geodesic strongly E-convexity got new directions on Riemannian manifolds. Israel and Mond [3] defined the concept of preinvex functions, which was named by Jeyakumar [11]. Later, Weir and

Keywords. E-invex sets, strongly E-invex sets, strongly E-preinvex functions, nonlinear programming problem (NLPP).

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Mond [22] defined preinvex functions for a multi-objective optimization problem. Another important extension of E-convex set and E-convex function are E-invex set and E-preinvex function, were given by Fulga and Preda [4]. Many properties and results of nonlinear optimization theory have been developed for E-invex sets and E-preinvex functions. Later, Majeed and Enad [17] extended the concept of E-preinvexity to semi E^h-b preinvexity and discussed some important properties and results of these functions and used them in the application of an optimization problem.

Motivated and inspired by the above research works, see in [2, 15, 19–21], we introduced the class of strongly E-invex sets, strongly E-invex, strongly E-preinvex, and strongly pseudo E-preinvex functions. This paper is divided as follows: Section 2 contains preliminaries. The related properties and important characterization of strongly E-preinvex functions have been discussed in Section 3. In nonlinear programming, if the objective function is strongly E-preinvex and the constraint set is strongly E-invex set, then the nonlinear programming problem is called strongly E-preinvex programming problem. The description of the set of the optimal solutions of the strongly E-preinvex programming problem is studied in Section 4.

2. Preliminaries

Let R^n denotes the Euclidean space of dimension n, let $E : R^n \to R^n$ be a map from R^n into R^n. Youness [26] introduced the concept of strongly E-convexity which generalized E-convexity as follows:

Definition 2.1 ([26])**.** A non empty set $M \subseteq R^n$ is called a strongly E-convex set with respect to (w.r.t.) a map $E : R^n \to R^n$, if for any $u, v \in M$, $\alpha \in [0, 1]$ & $\mu \in [0, 1]$, we have

$$\mu(\alpha u + E(u)) + (1 - \mu)(\alpha v + E(v)) \in M.$$

Definition 2.2 ([26])**.** Let $M \subseteq R^n$ be a strongly E-convex set. A real valued function $f : M \subseteq R^n \to R$ is called a strongly E-convex w.r.t. a map $E : R^n \to R^n$ on M, if for every $u, v \in M$, $\alpha \in [0, 1]$ & $\mu \in [0, 1]$, we have

$$f(\mu(\alpha u + E(u)) + (1 - \mu)(\alpha v + E(v))) \leq \mu f(E(u)) + (1 - \mu) f(E(v)).$$

If the above inequality is strict for every $u, v \in M$, $\alpha u + E(u) \neq \alpha v + E(v)$, $\alpha \in [0, 1]$ & $\mu \in (0, 1)$. Then, the function f is called a strictly strongly E-convex.

For $\alpha = 0$, the strongly E-convexity change to E-convexity.

Fulga *et al.* [4] extended the concept of E-covexity defined by Youness [25], to E-preinvexity, given as follows:

Definition 2.3 ([4])**.** A non empty subset $M \subseteq R^n$ is called E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$, if for every $u, v \in M$ and $\mu \in [0, 1]$,

$$E(v) + \mu\Psi(E(u), E(v)) \in M.$$

Definition 2.4 ([4])**.** Let M be an E-invex set. A real valued function $f : M \to R$ is called E-preinvex w.r.t. Ψ on M, if $\forall\ u, v \in M$ and $\mu \in [0, 1]$, we have

$$f(E(v) + \mu\Psi(E(u), E(v))) \leq \mu f(E(u)) + (1 - \mu) f(E(v)).$$

3. Main results

In this section, we introduce strongly E-invex set, strongly E-preinvex function and construct some suitable examples to show the existence of these definitions. Several interesting properties and results are discussed.

Definition 3.1. A set $M \subseteq R^n$ is called strongly E-invex w.r.t. $\Psi : R^n \times R^n \to R^n$, if for every $u, v \in M,\ \alpha \in [0,1]$ & $\mu \in [0,1]$, we have

$$\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) \in M.$$

If $\Psi(\alpha u + E(u), \alpha v + E(v)) = (\alpha u + E(u) - \alpha v - E(v))$, then the set M reduces to strongly E-convex which is Definition 2.1.

If $\alpha = 0$, then M reduces to E-invex set which is Definition 2.3. If $\alpha = 0$ and $\Psi(E(u), E(v)) = (E(u) - E(v))$, then the set M reduces to E-convex, as defined by Youness [25].

Example 3.2. Let $M = \{(u,v) \in R^2 : u + v \leq 1\} \cup \{(u,v) \in R^2 : u + v \geq -1\}$ and $E : R^2 \to R^2$ be defined by $E(u,v) = (-u, -v)$. It can be shown easily that M is a strongly E-invex set w.r.t. $\Psi : R^2 \times R^2 \to R^2$, where

$$\Psi(u,v) = \begin{cases} u - v, & \text{if } \ u, v \geq 0, \\ -v, & \text{if } \ u, v \leq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Every strongly E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$ is an invex set, when $\alpha = 0$ and $E : R^n \to R^n$ is an identity map. An E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$ need not be strongly E-invex set.

Example 3.3. Let $M = [-2, -1] \cup [1, 2]$. Fulga *et al.* [4] has shown that M is an invex as well as E-invex set w.r.t. $\Psi : R \times R \to R$ defined by

$$\Psi(u,v) = \begin{cases} u - v, & \text{if } \ u \geq 0, v \geq 0, \ \text{ or } \ u \leq 0, v \leq 0, \\ -2 - v, & \text{if } \ u > 0, v \leq 0, \ \text{ or } \ u \geq 0, v < 0, \\ 1 - v, & \text{if } \ u < 0, v \geq 0, \ \text{ or } \ u \leq 0, v > 0, \end{cases}$$

and $E : R \to R$ is defined by

$$E(u) = \begin{cases} u^2, & \text{if } \ |u| \leq \sqrt{2}, \\ -1, & \text{if } \ |u| > \sqrt{2}. \end{cases}$$

However, the set M is not strongly E-invex at $u = 2, v = 3/2$ and $\alpha = \mu = 1/2$.

Example 3.4. Let $M = [-2, -1) \cup [1, 2]$, Ψ and E be same as in Example 3.6. Then, M is not E-invex as well as not strongly E-invex.

Lemma 3.5. *Let $M \subseteq R^n$ be strongly E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$, then $E(M) \subseteq M$.*

Proof. We know that $E(v) \in E(M)$ for any $v \in M$. By the Definition 3.17, $\forall\ u, v \in M$, $\alpha \in [0,1]$ & $\mu \in [0,1]$, we get

$$\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) \in M.$$

For $\alpha = 0,\ \mu = 0,\ E(v) \in M$. Hence, $E(M) \subseteq M$. □

Lemma 3.6. *Let $\{M_i\}_{i \in I}$ be the family of strongly E-invex sets w.r.t. $\Psi : R^n \times R^n \to R^n$, $M_i \subseteq R^n$, for every $i \in I$, then $M = \bigcap_{i \in I} M_i$ is strongly E-invex set w.r.t. Ψ.*

Proof. The proof is obvious. □

Remark 3.7. The union of strongly E-invex sets w.r.t. Ψ need not be strongly E-invex set.

If M is strongly E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$ and $E(M) \subseteq M$. Then, $E(M)$ need not be a strongly E-invex set as shown in the following example.

Example 3.8. Let $M = \{(u,v) \in R^2 : u^2 + v^2 \leq 1\}$ be a set and $E(M) = \{(u,v) \in R^2 : u^2 + v^2 = 1\} \subseteq M$. Then, M is a strongly E-invex set w.r.t. $\Psi : R^2 \times R^2 \to R^2$ defined by

$$\Psi(u,v) = \begin{cases} u - v, & \text{if } \ u,v \geq 0, \ \text{ or } \ u,v \leq 0 \\ -v, & \text{otherwise,} \end{cases}$$

where $E : R^2 \to R^2$ is a map defined by $E(u,v) = (-u,0)$. But $E(M)$ is not strongly E-invex set at $\alpha = 1/2, \ \ \mu = 1$ and $u = v = 1$.

Let $M_1, M_2 \subseteq R^n$ be two non empty subsets of R^n. We know that the sum of M_1 and M_2 is given as follows:

$$M_1 + M_2 = \{u + v : u \in M_1, \ v \in M_2\}.$$

Lemma 3.9. *For a linear map $E : R^n \to R^n$, if the sets $M_1, M_2 \subseteq R^n$ are strongly E-invex w.r.t. $\Psi_1 : R^n \times R^n \to R^n$ and $\Psi_2 : R^n \times R^n \to R^n$, respectively. Then their sum is also strongly E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$, where $\Psi = \Psi_1 + \Psi_2$.*

Proof. Given that M_1 and M_2 are strongly E-invex sets w.r.t. Ψ_1, Ψ_2 respectively and E is a linear map. For every $u + v, p + q \in M_1 + M_2$, where $u, p \in M_1, \ v, q \in M_2$, $\forall \ \alpha \in [0,1]$ & $\mu \in [0,1]$, we have

$$\begin{aligned} &\alpha p + E(p) + \mu\Psi_1(\alpha u + E(u), \alpha p + E(p)) + \alpha q + E(q) + \mu\Psi_2(\alpha v + E(v), \alpha q + E(q)) \\ &\quad = \alpha(p+q) + E(p+q) + \mu\{\Psi_1(\alpha u + E(u), \alpha p + E(p)) + \Psi_2(\alpha v + E(v), \alpha q + E(q))\} \\ &\quad = \alpha(p+q) + E(p+q) + \mu\{\Psi(\alpha(u+v) + E(u+v), \alpha(p+q) + E(p+q))\} \in M_1 + M_2, \end{aligned}$$

where $\Psi : R^n \times R^n \to R^n$ is defined as

$$\Psi(\alpha(u+v) + E(u+v), \alpha(p+q) + E(p+q)) = \Psi_1(\alpha u + E(u), \alpha p + E(p)) + \Psi_2(\alpha v + E(v), \alpha q + E(q)).$$

Therefore, the sum $M_1 + M_2$ of M_1, M_2 is strongly E-invex set w.r.t. $\Psi = \Psi_1 + \Psi_2$. □

Lemma 3.10. *For a linear map $E : R^n \to R^n$, let $M \subseteq R^n$ be a strongly E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$, where $\Psi(\alpha x + E(x), \alpha y + E(y))$ is homogeneous. Then, the set $\beta M = \{\beta u : u \in M, \ \beta \in R\}$ is strongly E-invex w.r.t. Ψ.*

Proof. Let $u', v' \in \beta M$, such that $u' = \beta u, v' = \beta v$. Since E is a linear map and $\Psi(\alpha u + E(u), \alpha v + E(v))$ is homogeneous, we get

$$\alpha v' + E(v') + \mu\Psi(\alpha u' + E(u'), \alpha v' + E(v')) = \beta(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))), \ \forall \ \alpha \in [0,1] \ \& \ \mu \in [0,1].$$

Since M is a strongly E-invex set, we have

$$\beta(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) \in \beta M.$$

$\Longrightarrow$ βM is a strongly E-invex set w.r.t. Ψ. □

Theorem 3.11. *Let $\mathcal{M}$ be the family of all strongly E-invex sets w.r.t. vector functions $\Psi : R^n \times R^n \to R^n$ that are homogeneous. Then, the family $\mathcal{M}$ forms a vector space over R w.r.t. the operations defined in Lemmas 3.17 and 3.10.*

Proof. The family $\mathcal{M}$ is closed w.r.t. the addition defined in Lemma 3.17 and w.r.t. the scalar multiplication defined in Lemma 3.10 respectively. Hence, the family $\mathcal{M}$ forms a vector space over R. □

In the next theorem, we discuss a representation of a strongly E-invex set.

Theorem 3.12. *If the set M is a strongly E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$. Then, M has representation*

$$M = \bigcup_{\alpha v+E(v)\in M} \Big\{M_{\alpha v+E(v)} : \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\},$$

where

$$M_{\alpha v+E(v)} = \bigcup_{\alpha u+E(u)\in M} \Big\{\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) : \forall\ u \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\},$$

or

$$M = \bigcup_{\alpha u+E(u)\in M}\ \bigcup_{\alpha v+E(v)\in M} \Big\{\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) : \forall\ u, v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\}.$$

Proof. From the Definition 3.17, we have

$$\{\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) : \alpha \in [0,1],\ \mu \in [0,1]\} \subseteq M,$$

$$\bigcup_{\alpha u+E(u)\in M} \{\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) : \forall\ u \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\} \subseteq M.$$

Let $M_{\alpha v+E(v)} = \bigcup_{\alpha u+E(u)\in M} \Big\{\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) : \forall\ u \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\} \subseteq M$,

$$\bigcup_{\alpha v+E(v)\in M} \Big\{M_{\alpha v+E(v)} :\ \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\} \subseteq M. \tag{3.1}$$

The equality sign hold in (3.1). Assume on contrary that

$$\bigcup_{\alpha v+E(v)\in M} \Big\{M_{\alpha v+E(v)} :\ \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\} \subset M.$$

Then, $\exists$ a point $w \in M - \bigcup_{\alpha v+E(v)\in M} \Big\{M_{\alpha v+E(v)} : \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\}$ such that

$$w \notin \bigcup_{\alpha v+E(v)\in M} \Big\{M_{\alpha v+E(v)} :\ \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\}.$$

For $w \in M$, we have

$$\alpha w + E(w) + \mu\Psi(\alpha u + E(u), \alpha w + E(w)) \in M_{\alpha w+E(w)}.$$

Since $M_{\alpha w+E(w)} \subseteq \bigcup_{\alpha v+E(v)\in M} \Big\{M_{\alpha v+E(v)} :\ \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\}$, which implies $\alpha w + E(w) + \mu\Psi(\alpha u + E(u), \alpha w + E(w)) \in \bigcup_{\alpha v+E(v)\in M} \Big\{M_{\alpha v+E(v)} :\ \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\}$.

Then, $w \in \bigcup_{\alpha v+E(v)\in M} \Big\{M_{\alpha v+E(v)} :\ \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\Big\}$, which is a contradiction. Therefore,

$$M = \bigcup_{\alpha v+E(v)\in M} \{M_{\alpha v+E(v)} :\ \forall\ v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\},$$

$$M = \bigcup_{\alpha v+E(v)\in M}\ \bigcup_{\alpha u+E(u)\in M} \{\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) : \forall\ u, v \in M,\ \alpha \in [0,1],\ \mu \in [0,1]\}.$$

□

Definition 3.13. Let $M \subseteq R^n$ be a non empty strongly E-invex set. A function $f : M \subseteq R^n \to R$ is called strongly E-preinvex w.r.t. Ψ on M, if $\forall\ u, v \in M,\ \alpha \in [0,1]\ \&\ \mu \in [0,1]$, we have

$$f(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) \le \mu f(E(u)) + (1-\mu)f(E(v)).$$

If the above inequality is strict for all $u, v \in M,\ \alpha u + E(u) \neq \alpha v + E(v),\ \forall\ \alpha \in [0,1]\ \&\ \mu \in (0,1)$. Then, the function f is called strictly strongly E-preinvex.

If $\Psi(\alpha u + E(u), \alpha v + E(v)) = (\alpha u + E(u) - \alpha v - E(v))$, then the function f changes to strongly E-convex, defined in Definition 2.2.

Remark 3.14. For $\alpha = 0$, strongly E-preinvex function reduces to E-preinvex function, defined in Definition 2.4.

Example 3.15. Let $f : R \to R$ be defined by

$$f(u) = \begin{cases} 1-u, & \text{if} \quad 0 \le u \le 1, \\ 1+u, & \text{if} \quad -1 \le u \le 0, \end{cases}$$

and $E : R \to R$ be a map defined by $E(u) = 0$. The function f is strongly E-preinvex w.r.t. $\Psi : R \times R \to R$, where

$$\Psi(u, v) = \begin{cases} u - v, & \text{if} \quad u \ge 0, v \ge 0, \\ -v, & \text{if} \quad u \le 0, v \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.16. Note that a strongly E-preinvex function w.r.t. $\Psi : R^n \times R^n \to R^n$ need not be a strongly E-convex function.

Theorem 3.17. *If $f : M \subseteq R^n \to R$ is strongly E-preinvex function on a strongly E-invex set M, then $f(\alpha v + E(v)) \le f(E(v))$, $\forall\ v \in M\ \&\ \alpha \in [0,1]$.*

Proof. Since f is strongly E-preinvex function on a strongly E-invex set M, $\forall\ u, v \in M,\ \alpha \in [0,1]\ \&\ \mu \in [0,1]$, we have

$$\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) \in M,$$

and

$$f(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) \le \mu f(E(u)) + (1-\mu)f(E(v)).$$

If $\mu = 0$, then we get $f(\alpha v + E(v)) \le f(E(v))\ \forall\ v \in M\ \&\ \alpha \in [0,1]$. □

Theorem 3.18. *Let $M \subseteq R^n$ be a strongly E-invex set. If $f_i : M \subseteq R^n \to R$, $1 \le i \le m$, are non negative strongly E-preinvex functions on M, then linear combination of strongly E-preinvex functions is also strongly E-preinvex function, i.e., for $a_i \ge 0,\ 1 \le i \le m$, then the function*

$$h = \sum_{i=1}^{m} a_i f_i$$

is a strongly E-preinvex on M.

Proof. Since $f_i,\ 1 \le i \le m$, are strongly E-preinvex functions on strongly E-invex set M, then $\forall\ u, v \in M$, $\alpha \in [0,1]\ \&\ \mu \in [0,1]$, we have

$$\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) \in M,$$

and

$$\begin{aligned} h(\alpha v+E(v)+\mu\Psi(\alpha u+E(u),\alpha v+E(v))) &= \sum_{i=1}^{m} a_i f_i(\alpha v+E(v)+\mu\Psi(\alpha u+E(u),\alpha v+E(v))) \\ &\leq \mu\sum_{i=1}^{m} a_i f_i(E(u)) + (1-\mu)\sum_{i=1}^{m} a_i f_i(E(v)), \\ &= \mu h(E(u)) + (1-\mu)h(E(v)). \end{aligned}$$

Thus, the function h is strongly E-preinvex on M. □

The prepseudoinvex function was defined by Jeyakumar [11]. Now, we introduce strongly pseudo E-preinvex function on strongly E-invex set, which is given as follows:

Definition 3.19. Let $M \subseteq R^n$ be a strongly E-invex set. A function $f : M \subseteq R^n \to R$ is called strongly pseudo E-preinvex w.r.t. Ψ on M, if $\exists$ strictly positive function $h : R^n \times R^n \to R$ such that

$$\begin{aligned} f(E(u)) < f(E(v)) \implies\ & f(\alpha v+E(v)+\mu\Psi(\alpha u+E(u),\alpha v+E(v))) \leq f(E(v)) + \mu(\mu-1)h(E(u),E(v)), \\ & \forall\ u,v \in M, \alpha\in[0,1]\ \&\ \mu\in[0,1]. \end{aligned}$$

For $\alpha = 0$ it reduces to E-prepseudoinvex function.

In the next theorem, we discuss a relationship between strongly E-preinvex function and strongly pseudo E-preinvex function.

Theorem 3.20. *Let $M \subseteq R^n$ be a strongly E-invex set and $f : M \subseteq R^n \to R$ be a strongly E-preinvex function w.r.t. Ψ on M. Then, f is strongly pseudo E-preinvex function on M.*

Proof. Let $f(E(u)) < f(E(v))$. Since f is a strongly E-preinvex function on M, for every $u, v \in M$, $\alpha \in [0,1]$ & $\mu \in [0,1]$, we have

$$\begin{aligned} f(\alpha v+E(v)+\mu\Psi(\alpha u+E(u),\alpha v+E(v))) &\leq \mu f(E(u)) + (1-\mu)f(E(v)) \\ &= f(E(v)) + \mu(f(E(u)) - f(E(v))) \\ &\leq f(E(v)) + \mu(1-\mu)(f(E(u)) - f(E(v))) \\ &= f(E(v)) + \mu(\mu-1)(f(E(v)) - f(E(u))) \\ &= f(E(v)) + \mu(\mu-1)h(E(u),(E(v)), \end{aligned}$$

where $h(E(u),E(v)) = f(E(v)) - f(E(u)) > 0$. Therefore, the function f is strongly pseudo E-preinvex on M. □

Theorem 3.21. *Let $M \subseteq R^n$ be a strongly E-invex set and $\{f_i\}_{i\in I}$ be a collection of functions defined on M such that $\forall\ u \in M$, $\sup_{i\in I}\ f_i(u)$ exists in R. Let $f : M \to R$ be a function defined by $f(u) = \sup_{i\in I} f_i(u), \forall\ u \in M$. If $f_i : M \to R$, for every $i \in I$, are strongly E-preinvex functions on M, then f is strongly E-preinvex on M.*

Proof. Suppose that $f_i : M \to R$, for every $i \in I$, are strongly E-preinvex functions on M, then $\forall\ u, v \in M$, $\alpha \in [0,1]$ & $\mu \in [0,1]$, we have

$$\begin{aligned} f_i(\alpha v+E(v)+\mu\Psi(\alpha u+E(u),\alpha v+E(v))) &\leq \mu f_i(E(u)) + (1-\mu)f_i(E(v)) \\ \sup_{i\in I} f_i(\alpha v+E(v)+\mu\Psi(\alpha u+E(u),\alpha v+E(v))) &\leq \mu\sup_{i\in I} f_i(E(u)) + (1-\mu)\sup_{i\in I} f_i(E(v)) \\ &= \mu f(E(u)) + (1-\mu)f(E(v)), \\ f(\alpha v+E(v)+\mu\Psi(\alpha u+E(u),\alpha v+E(v))) &\leq \mu f(E(u)) + (1-\mu)f(E(v)). \end{aligned}$$

□

Definition 3.22. Let $N \subseteq R^n \times R$, $E : R^n \to R^n$, and $E_0 : R \to R$. A set N is called strongly G-invex set w.r.t. Ψ, if $\forall\ (u,\gamma),(v,\delta) \in N$, $\alpha \in [0,1]$ & $\mu \in [0,1]$

$$(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)), \quad \mu E_0(\gamma) + (1-\mu)E_0(\delta)) \in N.$$

The set $M \subseteq R^n$ is strongly E-invex if and only if $M \times R$ is strongly G-invex set w.r.t. Ψ.

The $epi(f)$ is defined by

$$epi(f) = \{(u,\gamma) \in M \times R : f(u) \leq \gamma\}.$$

Next, we discuss an important characterization of strongly E-preinvex function in terms of its $epi(f)$.

Theorem 3.23. *Let $M \subseteq R^n$ be a strongly E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$. Let $f : M \subseteq R^n \to R$ be a function and $E_0 : R \to R$ be a map such that $E_0(f(u)+t) = f(E(u)) + t$, where $t \geq 0$. The function f is strongly E-preinvex on M if and only if $epi(f)$ is strongly G-invex on $M \times R$.*

Proof. Let $(u,\gamma),(v,\delta) \in epi(f)$. Since M is strongly E-invex set, we have

$$\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) \in M,$$

for all $u, v \in M$, $\alpha \in [0,1]$ & $\mu \in [0,1]$. Let $E_0(\gamma)$ and $E_0(\delta)$ be such that

$$f(E(u)) \leq E_0(\gamma), \ f(E(v)) \leq E_0(\delta).$$

Then, $(E(u), E_0(\gamma)), (E(v), E_0(\delta)) \in epi(f)$. Since the function f is strongly E-preinvex on M, we get

$$\begin{aligned} f(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) &\leq \mu f(E(u)) + (1-\mu) f(E(v)) \\ &\leq \mu E_0(\gamma) + (1-\mu)E_0(\delta). \end{aligned}$$

Thus,

$$(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)), \mu E_0(\gamma) + (1-\mu)E_0(\delta)) \in epi(f).$$

Therefore, the set $epi(f)$ is strongly G-invex on $M \times R$.

Conversely, let $epi(f)$ be a strongly G-invex set on $M \times R$. Let $u, v \in M$, $\alpha \in [0,1]$ & $\mu \in [0,1]$. Then, $(u, f(u)), (v, f(v)) \in epi(f)$. Since $epi(f)$ is strongly G-invex on $M \times R$, we have

$$(\alpha v + E(v) + \alpha\Psi(\alpha u + E(u), \alpha v + E(v)), \ \alpha E_0(f(u)) + (1-\alpha)E_0(f(v))) \in epi(f),$$

which implies that

$$\begin{aligned} f(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) &\leq \mu E_0(f(u)) + (1-\mu)E_0(f(v)), \\ &\leq \mu f(E(u)) + (1-\mu) f(E(v)). \end{aligned}$$

Thus, the function f is strongly E-preinvex on M. □

Theorem 3.24. *Let $\{M_{i\in I}\}$ be a collection of strongly G-invex sets in $R^n \times R$ w.r.t. $\Psi : R^n \times R^n \to R^n$, then the intersection $\bigcap_{i \in I} M_i$ of M_i is a strongly G-invex set.*

Proof. Let $(u,\gamma),(v,\delta) \in \bigcap_{i\in I} M_i$, $\forall\ i \in I$, then $(u,\gamma),(v,\delta) \in M_i$, $\forall\ i \in I$.

Since each M_i are strongly G-invex sets, we have

$$(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)), \mu E_0(\gamma) + (1-\mu)E_0(\delta)) \in M_i, \quad \forall\ \alpha \in [0,1] \ \&\ \mu \in [0,1],$$

which implies

$$\Big(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)), \mu E_0(\gamma) + (1-\mu)E_0(\delta)\Big) \in \bigcap_{i\in I} M_i, \quad \forall\ \alpha \in [0,1] \ \&\ \mu \in [0,1].$$

□

Theorem 3.25. *Let $E : R^n \to R^n$, $E_0 : R \to R$ be maps such that $E_0(f(u)+t) = f(E(u)) + t$, where $t \geq 0$. Suppose that the family $\{f_i\}_{i\in I}$ of strongly E-preinvex functions are bounded above on a strongly E-invex set $M \subseteq R^n$. Then, the real valued function $f(u) = \sup_{i\in I} f_i(u)$ is a strongly E-preinvex on M.*

Proof. Since each f_i, $i \in I$, are strongly E-preinvex functions on strongly E-invex set M, then by Theorem 3.23, an $epi(f_i)$ of f_i,

$$epi(f_i) = \big\{(u,\gamma) \in M \times R : f_i(u) \leq \gamma\big\}$$

is strongly G-invex on $M \times R$.

Thus,

$$\begin{aligned}\bigcap_{i\in I} epi(f_i) &= \bigcap_{i\in I} \Big\{(u,\gamma) \in M \times R : f_i(u) \leq \gamma\Big\}\\ &= \big\{(u,\gamma) \in M \times R : f(u) \leq \gamma\big\},\end{aligned}$$

where $f(u) = \sup_{i\in I} f_i(u)$ is also strongly G-invex. By Theorem 3.23, the function f is strongly E-preinvex on M. $\square$

Theorem 3.26. *Let $M \subseteq R^n$ be strongly E-invex set and $f : R^n \to R$ be a strongly E-preinvex w.r.t. Ψ on M. Let $\phi : R \to R$ be a positively homogeneous non-decreasing function, then the composition $\phi \circ f$ is strongly E-preinvex on M.*

Proof. Since f is strongly E-preinvex function on strongly E-invex M. Then, for every $u, v \in M,\ \alpha \in [0,1]$ & $\mu \in [0,1]$ we have

$$f(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) \leq \mu f(E(u)) + (1-\mu) f(E(v)).$$

Since ϕ is positively homogeneous non-decreasing function, we obtain

$$\begin{aligned}\phi \circ f\big(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)\big) &\leq \phi \circ \big(\mu f(E(u)) + (1-\mu) f(E(v)\big)\\ &= \mu\phi(f(E(u))) + (1-\mu)\phi(f(E(v)))\\ &= \mu\phi \circ f(E(u))\big) + (1-\mu)\phi \circ f(E(v))\big).\end{aligned}$$

$\square$

Theorem 3.27. *Let $g_i : R^n \to R,\ 1 \leq i \leq k$ be strongly E-preinvex functions on R^n w.r.t. Ψ. If $E(M) \subseteq M$. Then, the set*

$$M = \big\{u \in R^n : g_i(u) \leq 0, \quad 1 \leq i \leq k\big\}$$

is strongly E-invex.

Proof. Since $g_i(u),\ 1 \leq i \leq k$, are strongly E-preinvex functions, for every $u, v \in M \subseteq R^n,\ \alpha \in [0,1]$ & $\mu \in [0,1]$, we have

$$\begin{aligned}g_i(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) &\leq \mu(g_i(E(u)) + (1-\mu)(g_i(E(v))\\ &\leq 0,\end{aligned}$$

where we used the assumption $E(S) \subseteq S$ to get the right hand side of the above inequality. Hence,

$$\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) \in M.$$

Therefore, the set M is strongly E-invex. $\square$

Now, we define the strongly E-invex function on strongly E-invex set as follows:

Definition 3.28. Let $M \subseteq R^n$ be a strongly E-invex set and $f : M \to R$ be differentiable function on M. A function f is called strongly E-invex w.r.t. Ψ on M, if $\forall\, u, v \in M$ and $\alpha \in [0, 1]$, we have

$$\nabla f(E(v))\ \Psi(\alpha u + E(u), \alpha v + E(v))^T \leq f(E(u)) - f(E(v)).$$

If $\alpha = 0$, then the function f is E-invex, defined by Jaiswal *et al.* [10].

Motivated by Mohan *et al.* [18] and Kumari *et al.* [14], we introduce the **Condition A** as follows:

Condition A. For a onto map $E : M \to M$, let $M \subseteq R^n$ be a strongly E-invex set w.r.t. $\Psi : R^n \times R^n \to R^n$, suppose for every $u, v \in M,\ \alpha \in [0, 1]\ \&\ \mu \in [0, 1],\ \exists\ \bar{v} \in M$ such that

$$E(\bar{v}) = \alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) \in M.$$

Then, the map Ψ satisfies condition A if:

$$\begin{aligned} \mathbf{A1}: &\quad \Psi(\alpha v + E(v), \alpha\bar{v} + E(\bar{v})) = -\mu(\alpha\bar{v} + \Psi(\alpha u + E(u), \alpha v + E(v))), \\ \mathbf{A2}: &\quad \Psi(\alpha u + E(u), \alpha\bar{v} + E(\bar{v})) = (1-\mu)(\alpha\bar{v} + \Psi(\alpha u + E(u), \alpha v + E(v))). \end{aligned}$$

Together **A1** and **A2** is called **Condition A**.

For $\alpha = 0,\quad E(u) = u$, Condition A reduces to the Condition C defined by Mohan and Neogy [18].

In the next theorem, we show an important relation of differentiable strongly E-invex function with Condition A, which is strongly E-preinvex function.

Theorem 3.29. *Let $M \subseteq R^n$ be an open strongly E-invex set w.r.t. Ψ and an onto map $E : M \to M$. If the function $f : M \to R$ is differentiable strongly E-invex w.r.t. Ψ on M and Ψ satisfies Condition A. Then, the function f is strongly E-preinvex w.r.t. Ψ on M.*

Proof. Let $u, v \in M$. Since M is a strongly E-invex set and E be an onto map, $\exists\ \bar{v} \in M$ such that

$$E(\bar{v}) = \alpha v + E(v) + \Psi(\alpha u + E(u), \alpha v + E(v)) \in M,$$

for every $u, v \in M,\ \alpha \in [0, 1]\ \&\ \mu \in [0, 1]$.

By Definition of strongly E-invexity of f, for $u, \bar{v} \in M$, we get

$$\nabla f(E(\bar{v}))\Psi(\alpha u + E(u), \alpha\bar{v} + E(\bar{v}))^T \leq f(E(u)) - f(E(\bar{v})). \tag{3.2}$$

Similarly, strongly E-invexity of f, for $v, \bar{v} \in M$, we get

$$\nabla f(E(\bar{v}))\Psi(\alpha v + E(v), \alpha\bar{v} + E(\bar{v}))^T \leq f(E(v)) - f(E(\bar{v})). \tag{3.3}$$

On multiplying (3.2) by μ, (3.3) by $(1 - \mu)$ and adding them, we get

$$\nabla f(E(\bar{v}))\big(\mu\Psi(\alpha u + E(u), \alpha\bar{v} + E(\bar{v}))^T + (1-\mu)\Psi(\alpha v + E(v), \alpha\bar{v} + E(\bar{v}))^T\big) \leq \mu f(E(u)) + (1-\mu)f(E(v)) - f(E(\bar{v})). \tag{3.4}$$

Since Ψ satisfies Condition A, we have

$$\mu\Psi(\alpha u + E(u), \alpha\bar{v} + E(\bar{v}))^T + (1-\mu)\Psi(\alpha v + E(v), \alpha\bar{v} + E(\bar{v}))^T = 0.$$

Hence, (3.4) reduces to

$$f(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) \leq \mu f(E(u)) + (1-\mu)f(E(v)).$$

Therefore, the function f is strongly E-preinvex w.r.t. Ψ on M. □

Remark 3.30. Unlike differentiable E-preinvex functions, differentiable strongly E-preinvex functions need not be strongly E-invex functions.

4. Strongly E-preinvex programming problem

In this section, we consider the nonlinear programming problem as an application for strongly E-preinvex functions, known as strongly E-preinvex programming problem, which generalizes the results obtained by Fulga *et al.* [4] and Youness [25].

$$(P) \begin{cases} \text{Min } f_0(u), \\ f_j(u) \leq 0, \quad 1 \leq j \leq m, \\ u \in R^n, \end{cases} \tag{4.1}$$

where $f_0 : R^n \to R$ and $f_j : R^n \to R,\ 1 \leq j \leq m$, are strongly E-preinvex functions on R^n.

Let X represents non empty set of feasible solutions *i.e.*,

$$X = \{u \in R^n : f_j(u) \leq 0,\ 1 \leq j \leq m\}. \tag{4.2}$$

Lemma 4.1. *For a linear map $E_0 : R \to R$, the set X is a strongly E-invex set.*

Proof. Since $f_j : R^n \to R, \quad 1 \leq j \leq m$, are strongly E-preinvex functions on R^n, by Theorem 3.23, the $epi(f_j)$ are strongly G-invex sets. Let $(u, 0), (v, 0) \in epi(f_j)$ and $u, v \in X$, we have

$$(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)),\ \mu E_0(0) + (1-\mu)E_0(0)) \in epi(f_j), \quad \forall\ \alpha \in [0,1]\ \&\ \mu \in [0,1],$$
$$(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)),\ 0) \in epi(f_j), \quad \forall\ \alpha \in [0,1]\ \&\ \mu \in [0,1],$$

or

$$f_j(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) \leq 0.$$

Hence, the set X is strongly E-invex. □

Theorem 4.2. *Let $f_j : R^n \to R,\ 0 \leq j \leq m$, be strongly E-preinvex functions on R^n and X be the non-empty set of feasible solutions defined by (4.2). Then, X_{opt} is a strongly E-invex set, where X_{opt} represents the set of optimal solutions of the problem* (4.1).

Proof. Let $u, v, E(u), E(v) \in X_{\text{opt}} \subset X$, then we have $f_0(u) = f_0(v) = f_0(E(u)) = f_0(E(v)) = \min_{w \in X} f_0(w)$, denoted by f_0.

Since $u, v, E(u), E(v) \in X$, for every $\alpha \in [0,1]\ \&\ \mu \in [0,1]$, we get

$$f_0(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) \leq \mu f_0(E(u)) + (1-\mu) f_0(E(v)) = f_0.$$

Therefore, for any $\alpha \in [0,1]\ \&\ \mu \in [0,1]$, we obtained

$$f_0(\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v))) = f_0,$$

i.e.,

$$\alpha v + E(v) + \mu\Psi(\alpha u + E(u), \alpha v + E(v)) \in X_{\text{opt}}.$$

Hence, the set X_{opt} is strongly E-invex. □

Theorem 4.3. *Let $f_0 : R^n \to R$ be a strictly strongly E-preinvex function on R^n, $f_j : R^n \to R, \quad 1 \leq j \leq m$, be strongly E-preinvex functions on R^n and the set X be a non empty. If u^* is a local minimum point of* (4.1), *then u^* is a strict global minimum point of* (4.1).

Proof. Since u^* is a local minimum point of (4.1), then we have $u^* \in R^n$, $f_j(u^*) \leq 0$, $1 \leq j \leq m$ and $\exists\ \epsilon > 0$ such that

$$f_0(u^*) \leq f_0(v), \quad \forall\ v \in S_\epsilon(u^*) \cap X - \{u^*\},$$

where $S_\epsilon(u^*) = \{v \in R^n : \|v - u^*\| < \epsilon\}$ is an open sphere centred at u^* with radius ϵ. Suppose that there exists $w^* \in X$, $w^* \neq u^*$, such that $f_0(w^*) < f_0(u^*)$. Since $w^*, u^* \in X$, then by Lemma 3.5, $\exists$ $w, u \in X$ such that $w^* = E(w)$, $u^* = E(u)$. For any fixed $\alpha \in [0, 1]$ & $\mu \in [0, 1]$, due to Lemma 4.1, we have

$$\alpha u + u^* + \mu\Psi(\alpha w + w^*, \alpha u + u^*) \in X.$$

For $\alpha w + w^* \neq \alpha u + u^*$, we have

$$f_0(\alpha u + u^* + \mu\Psi(\alpha w + w^*, \alpha u + u^*)) < \mu f_0(w^*) + (1 - \mu) f_0(u^*) = f_0^*. \tag{4.3}$$

The following four cases will arise:

Case 4.1: $\alpha = 0$ and $\Psi(w^*, u^*) = 0$, then (4.3) gives a contradiction.

Case 4.2: $\alpha = 0$ and $\Psi(w^*, u^*) \neq 0$, we consider $\bar{\mu} = \min\left\{1, \dfrac{\epsilon}{\|\Psi(w^*, u^*)\|}\right\}$ and, for any $\mu \in [0, \bar{\mu})$, we have

$$\begin{aligned}\|u^* + \mu\Psi(w^*, u^*) - u^*\| &= \mu\|\Psi(w^*, u^*)\| \\ &< \bar{\mu}\|\Psi(w^*, u^*)\| \\ &\leq \epsilon.\end{aligned}$$

In this case, we obtained that for any $\mu \in (0, \bar{\mu})$,

$$u^* + \mu\Psi(w^*, u^*) \in S_\epsilon(x^*) \cap X - \{u^*\},$$

and, from (4.3) ,

$$f_0(u^* + \mu\Psi(w^*, u^*)) < f_0^*,$$

contradicting our assumption that u^* is a strict global minimum point of (4.1).

Case 4.3: $\alpha \neq 0$ and $\Psi(\alpha w + w^*, \alpha u + u^*) = 0$, we choose $\bar{\alpha} = \min\left\{1, \dfrac{\epsilon}{\|u\|}\right\}$ and, for any $\mu \in [0, \bar{\alpha})$, we have

$$\begin{aligned}\|\alpha u + u^* - u^*\| &= \mu\|u\| \\ &< \bar{\mu}\|u\| \\ &\leq \epsilon.\end{aligned}$$

In this case, we obtained that for any $\alpha \in (0, \bar{\alpha})$,

$$\alpha u + u^* \in S_\epsilon(u^*) \cap X - \{u^*\},$$

and from (4.3)

$$f_0(\alpha u + u^*) < f_0^*,$$

again contradicting our assumption that u^* is a strict global minimum point of (4.1).

Case 4.4: $\alpha \neq 0$ and $\Psi(\alpha w + w^*, \alpha u + u^*) \neq 0$, we choose $\bar{\alpha} = \min\left\{1, \dfrac{\epsilon_1}{\|u\|}\right\}$ and for any $\alpha \in [0, \bar{\alpha})$, we have

$$\begin{aligned}\|\alpha u + u^* + \Psi(\alpha w + w^*, \alpha u + u^*) - u^*\| &\leq \alpha\|u\| + \mu\|\Psi(\alpha w + w^*, \alpha u + u^*)\| \\ &< \bar{\alpha}\|u\| + \mu\|\Psi(\bar{\alpha} w + w^*, \bar{\alpha} u + u^*)\|\end{aligned}$$

$$\leq \epsilon_1 + \mu \left\| \Psi\left(\frac{\epsilon_1}{\|u\|}w + w^*, \frac{\epsilon_1}{\|u\|}u + u^*\right) \right\|,$$

again we choose $\bar{\mu} = \min\left\{1, \dfrac{\epsilon_2}{\left\| \Psi\left(\frac{\epsilon_1}{\|u\|}w + w^*, \frac{\epsilon_1}{\|u\|}u + u^*\right) \right\|}\right\}$, and for any $\mu \in [0, \bar{\mu})$, we have

$$\leq \epsilon_1 + \bar{\mu} \left\| \Psi\left(\frac{\epsilon_1}{\|u\|}w + w^*, \frac{\epsilon_1}{\|u\|}u + u^*\right) \right\|$$
$$\leq \epsilon_1 + \epsilon_2 = \epsilon.$$

In this case, we obtained that for any $\alpha \in (0, \bar{\alpha})$ and $\mu \in (0, \bar{\mu})$,

$$\alpha u + u^* + \mu \Psi(\alpha w + w^*, \alpha u + u^*) \in S_\epsilon(u^*) \cap X - \{u^*\}$$

and from (4.3)

$$f_0(\alpha u + u^* + \mu \Psi(\alpha w + w^*, \alpha u + u^*)) < f_0^*,$$

again contradicting our assumption that u^* is a strict global minimum point of (4.1).

□

5. Conclusion

The strong concepts of E-invexity and E-preinvexity have been introduced and several interesting properties have been discussed. Sufficient examples have been presented in support of our definitions. It has been shown that differentiable strongly E-invex function with Condition A is strongly E-preinvex function, while the converse is not true. An application to nonlinear programming problems for strongly E-preinvex functions has also been presented. Our results generalize the previously known results proved by different authors and can be explored over Riemannian manifolds in the future.

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