RAIRO-Oper. Res. 56 (2022) 1373–1395
https://doi.org/10.1051/ro/2022062

RAIRO Operations Research
www.rairo-ro.org

ON SECOND-ORDER RADIAL-ASYMPTOTIC PROTO-DIFFERENTIABILITY OF THE BORWEIN PERTURBATION MAPS

Thanh-Hung Pham* and Thanh-Sang Nguyen

Abstract. This paper deals with second-order sensitivity analysis of parameterized vector optimization problems. We prove that the Borwein efficient solution map and the Borwein efficient perturbation map of a parametric vector optimization problem are second-order radial-asymptotic proto-differentiable under some suitable qualification conditions. Some examples are also given for illustrating the obtained results.

Mathematics Subject Classification. 90Q46, 90C26, 90C29, 90C30.

Received April 8, 2021. Accepted May 3, 2022.

1. Introduction

In parametric vector optimization problems, sensitivity analysis is not only theoretically interesting but also practically important. Here, sensitivity analysis is the research of derivatives of the perturbation maps. It provides informations about derivatives of the perturbation maps of the parametric optimization problem. Due to its importance not only for theoretical aspect, but also for practical application, sensitivity analysis has been considered by numerous researchers. There are two main approaches in sensitivity analysis: the dual space approach and the primal space approach. In the dual space approach, we refer to the books [23, 24] and the recent papers [7, 9, 10, 12]. In the primal space approach, the first results for sensitivity analysis *via* contingent derivative have been given by Tanino [30, 31]. The TP-derivative has been introduced in [27], which has proved to be useful in vector optimization and set-valued analysis. In [8], Chuong has established formulae for inner and outer evaluating the TP-derivative of the efficient point multifunction in parametric vector optimization problem. The behavior of perturbation maps in nonsmooth convex problems has been investigated in [17, 28]. Recently, the formulas for computing the generalized Clarke epiderivative of the efficient point multifunction have been given by Chuong [11]. Very recently, Tung and Hung [38] have concerned with sensitivity analysis in parametric vector optimization problems *via* τ^w-contingent derivative. Some results in second-order sensitivity analysis for vector optimization problem *via* second-order contingent derivative have been considered in [21, 32, 39, 40]. In [29], Sun and Li have investigated generalized second-order contingent epiderivatives of frontier and solution maps in parametric vector optimization problem. Recently, higher-order sensitivity analysis in parametric vector optimization problems and parametric set-valued optimization problems has occupied attention of researchers. In

Keywords. Parametric vector optimization problem, second-order radial-asymptotic derivative, Borwein efficient solution map, Borwein efficient perturbation map, sensitivity analysis.

Faculty of Pedagogy and Faculty of Social Sciences & Humanities, Kien Giang University, Chau Thanh, Kien Giang, Vietnam.
*Corresponding author: thanhhungpham.math@gmail.com; pthung@vnkgu.edu.vn

[4,33], some results in higher-order sensitivity analysis have been given by using the higher-order variational sets and asymptotic variational sets. In [13], properties of higher-order contingent-type derivatives of the perturbation and weak perturbation maps of a parameterized optimization problem have been obtained by using the higher-order contingent-type derivatives. In [1], Anh has obtained sensitivity results of set-valued optimization problem in terms of Studniarski derivatives. In [2], the higher-order contingent derivative of a parametrized set-valued optimization problem has been studied. Very recently, the second-order composed contingent derivatives of the perturbation maps/the weak perturbation maps have been given in [3, 25]. In [41], Wang and Zhang have obtained second-order sensitivity results for parametric multi-objective optimization problems under the Benson proper efficiency in terms of second-order composed radial derivative.

Another important topic in the primal space approach is to study the proto-differentiability of perturbation maps, the first result for proto-differentiability of perturbation maps has been presented by Rockafellar in [26]. In [19], Levy and Rockafellar have investigated proto-differentiability generalized equations. In [18], Huy and Lee have obtained sufficient conditions for the proto-differentiability of the generalized perturbation map. The proto-differentiability of the efficient solution map/the efficient frontier map and the sufficient conditions in order to approximate to the proto-derivative of the efficient frontier maps have been given in [14]. In [15], Huy and Lee have been investigated proto-differentiability of generalized perturbation multifunction. Recently [22], authors have established the semi-differentiability of the marginal mapping in the parametric multiobjective optimization problem. However, it is worth noting that there exist only few papers in the literature devoting to the study of the higher-order proto-differentiability for perturbation maps. In [20], the second-order proto-differentiability properties and the second-order semi-differentiability properties have been discussed for generalized perturbation maps. In [34], Tung has studied the second-order proto-differentiability of the efficient solution map and the efficient frontier map under some appropriate qualification conditions. The higher-order proto-differentiability of the perturbation maps/the proper perturbation maps/the weak perturbation maps were investigated in [35]. Recently, the higher-order proto-differentiability/the higher-order asymptotic proto-differentiability of the weak efficient solution maps/the weak perturbation maps were considered in [36]. Very recently, the second-order composed proto-differentiability of the proper perturbation maps/the proper efficient solution maps were discussed in [37]. For the considerations of the second-order proto-differentiability, we observe only references [20, 34, 37]. On the other hand, the higher-order proto-differentiability properties of the Borwein efficient solution maps and the Borwein efficient frontier maps of parametric vector optimization problems have not been yet investigated in [34–37]. Recently, the second-order radial-asymptotic derivative was introduced and used in qualification conditions in [32] to obtain some quantitative results in analyzing the second-order contingent derivative of the proper perturbation map. Moreover, to the best of our knowledge, there is no paper dealing with the second-order radial-asymptotic proto-differentiability of perturbation maps in parametric vector optimization problems. In addition, it is well known that the range of the set of Borwein minimal points is smaller than minimal points, so the discussion of the sensitivity analysis makes a lot of sense under the Borwein efficiency.

Inspired by the above observations, we provide some new results for the second-order radial-asymptotic proto-differentiability of the Borwein efficient solution map and the Borwein efficient frontier map of parameterized vector optimization problem in this paper under some suitable qualification conditions. In addition, the sufficient conditions for approximating the second-order radial-asymptotic proto-derivative of the Borwein efficient frontier map are also given.

The plan of paper is organized as follows. In Section 2, we recall several concepts of the derivatives of multifunctions and their properties which are needed in the sequel. In Section 3, we establish the seond-order radial-asymptotic proto-differentiability of the Borwein efficient solution map and the Borwein efficient frontier map. The sufficient conditions in order to approximate to the second-order radial-asymptotic proto-derivative of the Borwein efficient frontier map are also presented in Section 3.

2. Preliminaries

Throughout this paper, let P, X and Y be Euclidean spaces equipped with the usual norms, where the space Y is partially ordered by nontrivial pointed closed convex cone $K \subseteq Y$ with nonempty interior $\text{int}K$. The norms of all Euclidean spaces are denoted by $\|.\|$. The origins of all Euclidean spaces are denoted by 0. B_X, B_Y stands for the closed unit ball in X, Y. Closure and boundary of $A \subseteq X$ are denoted by $\text{cl}A$ and ∂A. Furthermore, $\text{cone}A = \{ka | k \geq 0, a \in A\}$. $\mathbb{N}, \mathbb{R}$, and $\mathbb{R}_+$ are used for sets of natural numbers, real numbers, and nonnegative real numbers, respectively.

Definition 2.1 (see [16])**.** Let Ω be a nonempty subset of Y.

(i) An element $y \in \Omega$ is said to be a K-minimal point of Ω, if $(\Omega - y) \cap (-K) = \{0\}$. The set of all K-minimal points of Ω is denoted by $\text{Min}_K\Omega$.
(ii) An element $y \in \Omega$ is said to be a Borwein K-minimal point of Ω, if $\text{cl}\,\text{cone}(\Omega - y) \cap (-K) = \{0\}$. The set of all Borwein K-minimal points of Ω is denoted by $\text{BoMin}_K\Omega$.

It is easy to see that $\text{BoMin}_K\Omega \subset \text{Min}_K\Omega$ and the inclusion may be strict as in the following example.

Example 2.2 (see [38])**.** Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and $\Omega = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^2 \leq x_1 \leq 1\big\}$. Then, we can check that

$$\begin{aligned}\text{Min}_K\Omega &= \big\{x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2^2, 0 \leq x_1 \leq 1, -1 \leq x_2 \leq 0\big\},\\ \text{BoMin}_K\Omega &= \big\{x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2^2, 0 < x_1 \leq 1, -1 \leq x_2 < 0\big\}.\end{aligned}$$

Hence,

$$\text{BoMin}_K\Omega \subsetneqq \text{Min}_K\Omega.$$

Definition 2.3 (see [5,6])**.** Let $f : X \to Y$ be a vector-valued map. f is said to be twice Fréchet differential at $\bar{x} \in X$, if there exist two linear continuous operators $\nabla f(\bar{x}) : X \to Y$ and $\nabla^2 f(\bar{x}) : X \times X \to Y$, such that

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2}\nabla^2 f(\bar{x})(x - \bar{x}, x - \bar{x}) + o\big(\|x - \bar{x}\|^2\big),$$

where $o(\|x - \bar{x}\|^2)$ satisfies $\dfrac{o(\|x - \bar{x}\|^2)}{\|x - \bar{x}\|^2} \to 0$ when $x \to \bar{x}$. $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$ are the Fréchet derivative and second-order Fréchet derivative, respectively. f is said twice Fréchet differentiable on X if f is twice Fréchet differentiable at any $x \in X$. If $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$ are continuous at $\bar{x}$ then f is said to be twice continuously Fréchet differentiable at $\bar{x}$.

Let $G : P \rightrightarrows Y$ be a multifunction. The effective domain, graph, and epigraph of G are defined by

$$\begin{aligned}\text{dom}G &:= \{p \in P \mid G(p) \neq \emptyset\},\\ \text{gph}G &:= \{(p, y) \in P \times Y \mid y \in G(p)\},\\ \text{epi}G &:= \{(p, y) \in P \times Y \mid p \in \text{dom}G, y \in G(p) + K\}.\end{aligned}$$

The profile map of G is $G + K$, defined by $(G + K)(p) := G(p) + K$.

Definition 2.4 (see [34])**.** Let $G : P \to Y$ be a vector-valued function. G is said to be monotone if for any $p_1, p_2 \in P$, one has $\langle G(p_2) - G(p_1), p_2 - p_1\rangle \geq 0$. G is said to be strictly monotone if for any $p_1, p_2 \in P$, and $p_1 \neq p_2$, one has $\langle G(p_2) - G(p_1), p_2 - p_1\rangle > 0$.

Definition 2.5 (see [5,34])**.** Let $G : P \rightrightarrows Y$ be a set-valued map, $(\bar{p}, \bar{y}) \in \text{gph}G$ and $(\bar{u}, \bar{v}) \in P \times Y$.

(i) The second-order contingent derivative of G at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})\in P\times Y$ is the set-valued map $D^2G(\bar{p},\bar{y},\bar{u},\bar{v}):P\rightrightarrows Y$ defined by

$$D^2G(\bar{p},\bar{y},\bar{u},\bar{v})(p):=\Big\{y\in Y\mid \exists t_n\to 0^+,\exists (p_n,y_n)\to(p,y),\quad \forall n\in\mathbb{N},\ \text{such that}\ \bar{y}+t_n\bar{v}+\frac{1}{2}t_n^2y_n\in G\Big(\bar{p}+t_n\bar{u}+\frac{1}{2}t_n^2p_n\Big)\Big\},\qquad \forall p\in P.$$

(ii) The second-order contingent adjacent derivative of G at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})\in P\times Y$ is the set-valued map $D^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v}):P\rightrightarrows Y$ defined by

$$D^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p):=\Big\{y\in Y\mid \forall t_n\to 0^+,\exists (p_n,y_n)\to(p,y),\quad \forall n\in\mathbb{N},\ \text{such that}\ \bar{y}+t_n\bar{v}+\frac{1}{2}t_n^2y_n\in G\Big(\bar{p}+t_n\bar{u}+\frac{1}{2}t_n^2p_n\Big)\Big\},\qquad \forall p\in P.$$

(iii) The second-order contingent lower derivative of G at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})\in P\times Y$ is the set-valued map $D^{2(l)}G(\bar{p},\bar{y},\bar{u},\bar{v}):P\rightrightarrows Y$ defined by

$$D^{2(l)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p):=\Big\{y\in Y\mid \forall t_n\to 0^+,\quad \forall p_n\to p,\exists y_n\to y,\quad \forall n\in\mathbb{N},\ \text{such that}\ \bar{y}+t_n\bar{v}+\frac{1}{2}t_n^2y_n\in G\Big(\bar{p}+t_n\bar{u}+\frac{1}{2}t_n^2p_n\Big)\Big\},\qquad \forall p\in P.$$

Definition 2.6 (see [32])**.** Let $G:P\rightrightarrows Y$ be a set-valued map, $(\bar{p},\bar{y})\in \text{gph}G$ and $(\bar{u},\bar{v})\in P\times Y$.

(i) The second-order radial-asymptotic derivative of G at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})\in P\times Y$ is the set-valued map $D_S^2G(\bar{p},\bar{y},\bar{u},\bar{v}):P\rightrightarrows Y$ defined by

$$D_S^2G(\bar{p},\bar{y},\bar{u},\bar{v})(p):=\Big\{y\in Y\mid \exists t_n\to 0^+,\exists r_n>0,\exists (p_n,y_n)\to(p,y),\quad \forall n\in\mathbb{N},\ \text{such that}\ \bar{y}+t_n\bar{v}+\frac{1}{2}t_nr_ny_n\in G\Big(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\Big),t_nr_np_n\to 0\Big\},\quad \forall p\in P.$$

(ii) The second-order radial-asymptotic adjacent derivative of G at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})\in P\times Y$ is the set-valued map $D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v}):P\rightrightarrows Y$ defined by

$$D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p):=\Big\{y\in Y\mid \forall t_n\to 0^+,\quad \forall r_n>0,\exists (p_n,y_n)\to(p,y),\quad \forall n\in\mathbb{N},\ \text{such that}\ \bar{y}+t_n\bar{v}+\frac{1}{2}t_nr_ny_n\in G\Big(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\Big),t_nr_np_n\to 0\Big\},\quad \forall p\in P.$$

(iii) The second-order radial-asymptotic lower derivative of G at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})\in P\times Y$ is the set-valued map $D_S^{2(l)}G(\bar{p},\bar{y},\bar{u},\bar{v}):P\rightrightarrows Y$ defined by

$$D_S^{2(l)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p):=\Big\{y\in Y\mid \forall t_n\to 0^+,\quad \forall r_n>0,\quad \forall p_n\to p,\exists y_n\to y,\quad \forall n\in\mathbb{N},\ \text{such that}\ \bar{y}+t_n\bar{v}+\frac{1}{2}t_nr_ny_n\in G\Big(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\Big),t_nr_np_n\to 0\Big\},\quad \forall p\in P.$$

Remark 2.7 ([34])**.** From definitions we derive,

(i) $D^{2(l)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subseteq D^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subseteq D^2G(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subseteq D^2_SG(\bar{p},\bar{y},\bar{u},\bar{v})(p), \quad \forall p \in P.$

(ii) $D^{2(l)}_SG(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subseteq D^{2(b)}_SG(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subseteq D^2_SG(\bar{p},\bar{y},\bar{u},\bar{v})(p), \quad \forall p \in P.$

However, the reverse inclusions in Remark 2.7(ii) may not hold. The following examples illustrate the cases.

Example 2.8. Let $P = \mathbb{R}, Y = \mathbb{R}, \mathcal{I} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and $G : P \rightrightarrows Y$ be defined by

$$G(p) = \begin{cases} \{0\}, & \text{if } p \leq 0,\\ \{-3p\}, & \text{if } p \in \mathcal{I},\\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, for $(\bar{p},\bar{y}) = (0,0) \in \text{gph}G$, $(\bar{u},\bar{v}) = (1,-3)$ and $p = 1$, we have

$$D^2_SG(\bar{p},\bar{y},\bar{u},\bar{v})(1) = \{-3\}.$$

Taking

$$t_k = \frac{-1 + \sqrt{1 + 3p_k \ln\left(1 + \frac{1}{k}\right)}}{p_k} \to 0^+, \quad \forall p_k \to 1,$$

and

$$r_k = \frac{2\sqrt{1 + 3p_k \ln\left(1 + \frac{1}{k}\right)}}{p_k} > 0, \quad \forall p_k \to 1,$$

then

$$\begin{aligned} \bar{p} + t_k\bar{u} + \frac{1}{2}t_kr_kp_k &= t_k + \frac{1}{2}t_kr_kp_k\\ &= \frac{-1 + \sqrt{1 + 3p_k \ln\left(1 + \frac{1}{k}\right)}}{p_k}\\ &\quad + \frac{-\sqrt{1 + 3p_k \ln\left(1 + \frac{1}{k}\right)} + 1 + 3p_k \ln\left(1 + \frac{1}{k}\right)}{p_k}\\ &= 3\ln\left(1 + \frac{1}{k}\right) \not\subseteq \mathcal{I}, \quad \forall p_k \to 1. \end{aligned}$$

Hence, $G(\bar{p} + t_k\bar{u} + \frac{1}{2}t_kr_kp_k) = \emptyset$. Consequently,

$$D^{2(b)}_SG(\bar{p},\bar{y},\bar{u},\bar{v})(1) = \emptyset.$$

Hence,

$$D^2_SG(\bar{p},\bar{y},\bar{u},\bar{v})(1) \not\subseteq D^{2(b)}_SG(\bar{p},\bar{y},\bar{u},\bar{v})(1).$$

Example 2.9. Let $P = \mathbb{R}, Y = \mathbb{R}^2$ and $G : P \rightrightarrows Y$ be defined by

$$G(p) = \begin{cases} \left\{(y_1,y_2) \in \mathbb{R}^2 : y_1 \leq p^2, y_2 \leq 0\right\}, & \text{if } p < 0,\\ \left\{(y_1,y_2) \in \mathbb{R}^2 : y_1 \leq 0, y_2 \geq p^2\right\}, & \text{if } p \geq 0. \end{cases}$$

Let $(\bar{p},\bar{y})=(0,(0,0))$ and $(\bar{u},\bar{v})=(0,(0,0))$. Then,

$$D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p)=\begin{cases}\{(y_1,y_2)\in\mathbb{R}^2: y_1\le 0, y_2\le 0\}, & \text{if } p<0,\\ \{(y_1,y_2)\in\mathbb{R}^2: y_1\le 0, y_2\in\mathbb{R}\}, & \text{if } p=0,\\ \{(y_1,y_2)\in\mathbb{R}^2: y_1\le 0, y_2\ge 0\}, & \text{if } p>0,\end{cases}$$

and

$$D_S^{2(l)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p)=\begin{cases}\{(y_1,y_2)\in\mathbb{R}^2: y_1\le 0, y_2\le 0\}, & \text{if } p<0,\\ \{(y_1,y_2)\in\mathbb{R}^2: y_1\le 0, y_2=0\}, & \text{if } p=0,\\ \{(y_1,y_2)\in\mathbb{R}^2: y_1\le 0, y_2\ge 0\}, & \text{if } p>0.\end{cases}$$

Therefore,

$$D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(0)\not\subseteq D_S^{2(l)}G(\bar{p},\bar{y},\bar{u},\bar{v})(0).$$

Definition 2.10. Let $G:P\rightrightarrows Y$ be a set-valued map, $(\bar{p},\bar{y})\in \operatorname{gph}G$ and $(\bar{u},\bar{v})\in P\times Y$.

(i) The map G is said to be second-order radial-asymptotic proto-differentiable at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})$ if for any $p\in P$,

$$D_S^2G(\bar{p},\bar{y},\bar{u},\bar{v})(p)=D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p),$$

and its second-order radial-asymptotic proto-derivative is denoted by $G^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}$.

(ii) The map G is said to be second-order radial-asymptotic semi-differentiable at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})$ if for any $p\in P$,

$$D_S^2G(\bar{p},\bar{y},\bar{u},\bar{v})(p)=D_S^{2(l)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p).$$

and its second-order radial-asymptotic semi-derivative is denoted by $\widehat{G}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}$.

Definition 2.11 (see [27])**.** Let $G:P\rightrightarrows Y$ be a set-valued map and $(\bar{p},\bar{y})\in \operatorname{gph}G$.

(i) The TP-derivative of G at $(\bar{p},\bar{y})$ is the set-valued map $D_SG(\bar{p},\bar{y}):P\rightrightarrows Y$ defined by

$$\begin{aligned}D_SG(\bar{p},\bar{y})(p):=\{y\in Y\mid \exists t_n>0, \exists(p_n,y_n)\to(p,y),\quad \forall n\in\mathbb{N}, \text{ such that}\\ \bar{y}+t_ny_n\in G(\bar{p}+t_np_n), t_np_n\to 0\},\quad \forall p\in P.\end{aligned}$$

(ii) The adjacent TP-derivative of G at $(\bar{p},\bar{y})$ is the set-valued map $D_S^bG(\bar{p},\bar{y}):P\rightrightarrows Y$ defined by

$$\begin{aligned}D_S^bG(\bar{p},\bar{y})(p):=\{y\in Y\mid \forall t_n>0, \exists(p_n,y_n)\to(p,y),\quad \forall n\in\mathbb{N}, \text{ such that}\\ \bar{y}+t_ny_n\in G(\bar{p}+t_np_n), t_np_n\to 0\},\quad \forall p\in P.\end{aligned}$$

(iii) The map G is said to be TP-proto-differentiable at $(\bar{p},\bar{y})$ if for any $p\in P$,

$$D_SG(\bar{p},\bar{y})(p)=D_S^bG(\bar{p},\bar{y})(p).$$

Remark 2.12. (i) G is TP-proto-differentiable at $(\bar{p},\bar{y})$ if G is second-order radial-asymptotic proto-differentiable $(\bar{p},\bar{y})$ in the direction $(0,0)$.

(ii) If G is second-order radial-asymptotic semi-differentiable at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})$, then G is second-order radial-asymptotic proto-differentiable at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})$.

Definition 2.13 (see [5])**.** Set-valued map $G:P\rightrightarrows Y$ is said to be local Lipschitz at $(\bar{p},\bar{y})\in \operatorname{gph}G$, if there exist a real constant $M>0$ and a neighborhood U of $\bar{p}$ such that

$$G(p_1)\subseteq G(p_2)+M\|p_1-p_2\|B_Y,\quad \forall p_1,p_2\in U.$$

Proposition 2.14. *Let $G : P \rightrightarrows Y$ be the set-valued map, $(\bar{p}, \bar{y}) \in \text{gph}G$ and $(\bar{u}, \bar{v}) \in P \times Y$. Suppose that G is local Lipschitz at $(\bar{p}, \bar{y}) \in \text{gph}G$. If G is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$, then G is second-order radial-asymptotic semi-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$.*

Proof. Obviously, $D_S^{2(l)} G(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subseteq D_S^2 G(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$, for any $p \in P$. Therefore, we only need to prove,

$$D_S^2 G(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subseteq D_S^{2(l)} G(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p),$$

for any $p \in P$. In fact, for any $y \in D_S^2 G(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$, there exist $t_n \to 0^+, r_n > 0, (p_n, y_n) \to (p, y)$, such that

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n \in G\left(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n\right) \text{ and } t_n r_n p_n \to 0, \quad \forall n \in \mathbb{N}. \tag{2.1}$$

Let $\{p'_n\} \subset P$ be any sequence such that $p'_n \to p$. Since G is local Lipschitz at $(\bar{p}, \bar{y})$, there exist a $M > 0$ and a neighborhood U of $\bar{p}$ such that

$$G(p_1) \subseteq G(p_2) + M\|p_1 - p_2\| B_Y, \quad \forall p_1, p_2 \in U. \tag{2.2}$$

Hence, there exists a $n_0 > 0$ such that $\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n, \bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p'_n \in U$, for any $n \geq n_0$. From (2.1) and (2.2), we have, for any $n \geq n_0$, there exists a sequence $\{b_n\}$ with $b_n \in B_Y$ such that $t_n r_n p'_n \to 0$ and

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n (y_n - M\|p'_n - p_n\| b_n) \in G\left(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p'_n\right).$$

Let $y'_n := y_n - M\|p'_n - p_n\| b_n$. From $p'_n \to p$ and $p_n \to p$, we have $y'_n \to y$. Thus, for any $t_n \to 0^+, r_n > 0$ and $p'_n \to p$, there exists $y'_n \to y$ such that $t_n r_n p'_n \to 0$ and

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y'_n \in G\left(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p'_n\right).$$

Therefore, $y \in D_S^{2(l)} G(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Hence,

$$D_S^2 G(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subseteq D_S^{2(l)} G(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p).$$

□

Inspired by Definition 2.3 in [21], we introduce the following definition of second-order radial-asymptotic directionally compact of a set-valued map.

Definition 2.15. Let $G : P \rightrightarrows Y$ be the set-valued map, $(\bar{p}, \bar{y}) \in \text{gph}G$ and $(\bar{u}, \bar{v}) \in P \times Y$. G is said to be second-order radial-asymptotic directionally compact at $(\bar{p}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in the direction $p \in P$ if for all sequences $t_n \to 0^+, r_n > 0$ and $p_n \to p$, every sequences $\{y_n\}$ with $\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n \in G\left(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n\right)$ and $t_n r_n p_n \to 0$, there exists a convergent subsequence of $\{y_n\}$.

Proposition 2.16. *Let $G : P \rightrightarrows Y$ be the set-valued map, $(\bar{p}, \bar{y}) \in \text{gph}G$ and $(\bar{u}, \bar{v}) \in P \times Y$. Suppose that G is second-order radial-asymptotic directionally compact at $(\bar{p}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in any direction $p \in P$. If G is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$, then $G + K$ is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ and*

$$(G + K)^{2(S)}_{\bar{p}, \bar{y}, \bar{u}, \bar{v}}(p) = G^{2(S)}_{\bar{p}, \bar{y}, \bar{u}, \bar{v}}(p) + K, \quad \forall p \in P.$$

Proof. Since $G(p) \subseteq (G+K)(p)$ and

$$D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p) = D_S^2 G(\bar{p},\bar{y},\bar{u},\bar{v})(p), \quad \forall p \in P,$$

we only need to prove

$$D_S^2(G+K)(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subseteq D_S^2 G(\bar{p},\bar{y},\bar{u},\bar{v})(p) + K, \quad \forall p \in P \tag{2.3}$$

and

$$D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p) + K \subseteq D_S^{2(b)}(G+K)(\bar{p},\bar{y},\bar{u},\bar{v})(p), \quad \forall p \in P. \tag{2.4}$$

Firstly, we prove that (2.3) holds. Let $y \in D_S^2(G+K)(\bar{p},\bar{y},\bar{u},\bar{v})(p)$. Then, there exist $t_n \to 0^+, r_n > 0, (p_n, y_n) \to (p,y), k_n \in K$ for all $n \in \mathbb{N}$, such that

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n(y_n - k_n) \in G\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n\right) \text{ and } t_n r_n p_n \to 0.$$

Denote $\bar{y}_n := y_n - k_n$. Since, G is second-order radial-asymptotic directionally compact at $(\bar{p},\bar{y})$, with respect to $(\bar{u},\bar{v})$, we have $\bar{y}_n \to y' \in Y$. Then, one has $y' \in D_S^2 G(\bar{p},\bar{y},\bar{u},\bar{v})(p)$. Together with $y_n \to y$, we have $k_n \to k \in K$ and $y' = y - k$, which implies that $y - k \in D_S^2 G(\bar{p},\bar{y},\bar{u},\bar{v})(p)$. Therefore, $y \in D_S^2 G(\bar{p},\bar{y},\bar{u},\bar{v})(p) + K, \forall p \in P$.

Secondly, we prove that (2.4) holds. Let $y \in D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p) + K$. Then, there exist $\hat{y} \in D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p)$ and $k \in K$ such that $y = \hat{y} + k$. Thus, there exist $t_n \to 0^+, r_n > 0, (p_n, y_n) \to (p, \hat{y})$ for all $n \in \mathbb{N}$, such that

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n \in G\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n\right) \text{ and } t_n r_n p_n \to 0.$$

Setting $y_n' := y_n + k$, one has $y_n' \to \hat{y} + k$ and

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n' = \bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n + \frac{1}{2}t_n r_n k \in G\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n\right) + K$$

and $t_n r_n p_n \to 0$. Therefore, $y = \hat{y} + k \in D_S^{2(b)}(G+K)(\bar{p},\bar{y},\bar{u},\bar{v})(p)$. □

In Proposition 2.16, if G is not second-order radial-asymptotic directionally compact at $(\bar{p},\bar{y})$ with respect to $(\bar{u},\bar{v})$ in any direction $p \in P$, then Proposition 2.16 may not hold. The following example illustrates the case.

Example 2.17. Let $P = \mathbb{R}^2, Y = \mathbb{R}, K = \mathbb{R}_+$ and $G : P \rightrightarrows Y$ be difined by

$$G(p) = \begin{cases} \{p_1^2 + p_1, -1\}, & \text{if } p_1 = p_2 \geq 0, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where $p = (p_1, p_2) \in \mathbb{R}^2$. Let $(\bar{p},\bar{y}) = ((0,0),0) \in \text{gph}G$ and $(\bar{u},\bar{v}) = ((1,0),1)$. We have, for all $p = (p_1,p_2) \in P$,

$$D_S^2 G(\bar{p},\bar{y},\bar{u},\bar{v})(p) = D_S^{2(b)}G(\bar{p},\bar{y},\bar{u},\bar{v})(p) = \{p_1\}.$$

Therefore, G is second-order radial-asymptotic proto-differentiable at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})$ and $G^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \{p_1\}$. We have, for all $p = (p_1,p_2) \in P$,

$$D_S^2(G+K)(\bar{p},\bar{y},\bar{u},\bar{v})(p) = D_S^{2(b)}(G+K)(\bar{p},\bar{y},\bar{u},\bar{v})(p) = \mathbb{R}.$$

Hence, $G + K$ is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ and $(G + K)^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \mathbb{R}$. Thus, for all $p = (p_1, p_2) \in P$,

$$(G + K)^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) \neq G^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) + K.$$

The reason is that the condition second-order radial-asymptotic directionally compact of G does not hold. Indeed, for the direction $p = (1, 1)$, for every $t_n \to 0^+, r_n > 0$ for $p_n = (p_{1n}, p_{2n}) \to p = (1, 1)$, the sequence $\{y_n\} \subseteq \mathbb{R}$ with

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n = -1 \in G\Big(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n\Big) \text{ and } t_n r_n p_n \to 0,$$

i.e. $y_n = -\dfrac{2}{t_n r_n} - \dfrac{2}{r_n}$, has no convergent subsequence. Thus, the condition second-order radial-asymptotic directionally compact of G is not satisfied.

3. Second-order radial-asymptotic proto-differentiability of Borwein efficient solution map and Borwein efficient frontier map

In this section, we consider the second-order sensitivity analysis of parameterized vector optimization problems. Firstly, some notations and definitions are recollected. Let $f : P \times X \to Y$ be a vector function and $C : P \rightrightarrows X$ be a multifunction. Let $F : P \rightrightarrows Y$ be a multifunction defined by

$$F(p) := f(p, C(p)) = \{y \in Y \mid \exists x \in C(p), y = f(p, x)\}.$$

We consider the following parametric vector optimization problem

$$(\text{PVO}_p) \qquad \text{BoMin}_K\{y \in Y \mid \exists x \in C(p), y = f(p, x)\} = \text{BoMin}_K F(p),$$

where x is decision variable, p is perturbation parameter, f is objective map, C is constraint map and F is feasible set map in objective space. The Borwein efficient perturbation map/the Borwein efficient frontier map $\mathcal{B} : P \rightrightarrows Y$ of a family of parameterized vector optimization problem is defined by

$$\mathcal{B}(p) := \text{BoMin}_K\{y \in Y \mid \exists x \in C(p), y = f(p, x)\} = \text{BoMin}_K F(p),$$

and the Borwein efficient solution map $\mathcal{H}$ is given by

$$\mathcal{H}(p) := \{x \in X \mid x \in C(p), f(p, x) \in \mathcal{B}(p)\}.$$

Definition 3.1. F is said to be K-dominated by $\mathcal{B}$ near $\bar{p} \in P$ if there exists a neighborhood U of $\bar{p}$, such that

$$F(p) \subseteq \mathcal{B}(p) + K, \quad \forall p \in U.$$

Remark 3.2. Since $\mathcal{B}(p) \subseteq F(p)$ for all $p \in P$, the K-dominatedness of F by $\mathcal{B}$ implies that

(i) $F(p) + K = \mathcal{B}(p) + K, \quad \forall p \in U$;
(ii) For $(\bar{p}, \bar{y}) \in \text{gph}\mathcal{B}$ and $(\bar{u}, \bar{v}) \in P \times Y$,

$$D^2_S(F + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^2_S(\mathcal{B} + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \quad \forall p \in P.$$

Proposition 3.3. *Let* $\bar{p} \in P, \bar{x} \in \mathcal{H}(\bar{p})$ *and* $\bar{y} = f(\bar{p}, \bar{x}), (\bar{u}, \bar{w}, \bar{v}) \in P \times X \times Y$. *Suppose that the following conditions hold:*

(i) f *is twice continuously Fréchet differentiable at* $(\bar{p}, \bar{x})$ *and* $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$*;*
(ii) $\nabla_x f(\bar{p}, \bar{x})(.)$ *is strictly monotone on* X*;*

(iii) *C is second-order radial-asymptotic directionally compact at $(\bar{p},\bar{x})$ with respect to $(\bar{u},\bar{w})$ in any direction $p\in P$.*

Then, the Borwein efficient solution map $\mathcal{H}$ is second-order radial-asymptotic proto-differentiable at $(\bar{p},\bar{x})$ in the direction $(\bar{u},\bar{w})$ whenever the Borwein efficient perturbation map $\mathcal{B}$ is second-order radial-asymptotic proto-differentiable at $(\bar{p},\bar{y})$ in the direction $(\bar{u},\bar{v})$. Furthermore, for all $p\in P$,

$$\mathcal{H}^{2(S)}_{\bar{p},\bar{x},\bar{u},\bar{w}}(p)=\Big\{x\in X\mid x\in D^2_S C(\bar{p},\bar{x},\bar{u},\bar{w})(p),\nabla f(\bar{p},\bar{x})(p,x)\in\mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)\Big\}.$$

Proof. Let $p\in P$. Setting $L:=\Big\{x\in X\mid x\in D^2_S C(\bar{p},\bar{x},\bar{u},\bar{w})(p),\nabla f(\bar{p},\bar{x})(p,x)\in\mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)\Big\}$. From Remark 2.7, we only need to prove that, $\forall p\in P$,

$$D^2_S\mathcal{H}(\bar{p},\bar{x},\bar{u},\bar{w})(p)\subseteq L\subseteq D^{2(b)}_S\mathcal{H}(\bar{p},\bar{x},\bar{u},\bar{w})(p). \tag{3.1}$$

Let $x\in D^2_S\mathcal{H}(\bar{p},\bar{x},\bar{u},\bar{w})(p)$. Then, there exist $t_n\to 0^+, r_n>0$ and $(p_n,x_n)\to(p,x)$ such that $t_nr_np_n\to 0$ and

$$\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_nx_n\in\mathcal{H}\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right),\quad \forall n\in\mathbb{N}.$$

Thus, there exists a sequence $\{\widetilde{x}_n\}\subseteq X$ such that $\widetilde{x}_n\in C\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right)$ and

$$\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_nx_n=\widetilde{x}_n,$$

which in turn implies that

$$f\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n,\widetilde{x}_n\right)\in\mathcal{B}\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right).$$

Setting $\widetilde{x}'_n:=\dfrac{\widetilde{x}_n-\bar{x}-t_n\bar{w}}{\frac{1}{2}t_nr_n}$, we have

$$\begin{aligned}\widetilde{x}_n=\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_n\widetilde{x}'_n\in C\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right),\\ f\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n,\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_n\widetilde{x}'_n\right)\in\mathcal{B}\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right).\end{aligned} \tag{3.2}$$

Combining this with (iii), we can suppose that $\widetilde{x}'_n\to\widetilde{x}'$. Then, we have $\widetilde{x}'\in D^2_SC(\bar{p},\bar{x},\bar{u},\bar{w})(p)$. Setting

$$y_n:=\frac{f\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n,\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_n\widetilde{x}'_n\right)-f(\bar{p},\bar{x})-t_n\bar{v}}{\frac{1}{2}t_nr_n},$$

we deduce from (3.2) that

$$\bar{y}+t_n\bar{v}+\frac{1}{2}t_nr_ny_n\in\mathcal{B}\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right). \tag{3.3}$$

Since f is twice continuously Fréchet differentiable at $(\bar{p},\bar{x})$, we obtain that

$$f\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n,\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_n\widetilde{x}'_n\right)=f(\bar{p},\bar{x})+t_n\nabla f(\bar{p},\bar{x})(\bar{u},\bar{w})+\frac{1}{2}t_nr_n\nabla f(\bar{p},\bar{x})(p_n,\widetilde{x}'_n)$$

$$+\frac{1}{2}t_n^2\nabla^2 f(\bar p,\bar x)\left(\left(\bar u+\frac{1}{2}r_np_n,\bar w+\frac{1}{2}r_n\widetilde{x}'_n\right),\left(\bar u+\frac{1}{2}r_np_n,\bar w+\frac{1}{2}r_n\widetilde{x}'_n\right)\right)$$
$$+o\left(\left\|\left(t_n\bar u+\frac{1}{2}t_nr_np_n,t_n\bar w+\frac{1}{2}t_nr_n\widetilde{x}'_n\right)\right\|^2\right). \tag{3.4}$$

From $\bar y+t_n\bar v+\frac{1}{2}t_nr_ny_n = f\left(\bar p+t_n\bar u+\frac{1}{2}t_nr_np_n,\bar x+t_n\bar w+\frac{1}{2}t_nr_n\widetilde{x}'_n\right), \bar y = f(\bar p,\bar x), \bar v = \nabla f(\bar p,\bar x)(\bar u,\bar w)$ and (3.4), we deduce that

$$y_n=\nabla f(\bar p,\bar x)(p_n,\widetilde{x}'_n)+\frac{t_n}{r_n}\nabla^2 f(\bar p,\bar x)\left(\left(\bar u+\frac{1}{2}r_np_n,\bar w+\frac{1}{2}r_n\widetilde{x}'_n\right),\left(\bar u+\frac{1}{2}r_np_n,\bar w+\frac{1}{2}r_n\widetilde{x}'_n\right)\right)$$
$$+\frac{o\left(\left\|\left(t_n\bar u+\frac{1}{2}t_nr_np_n,t_n\bar w+\frac{1}{2}t_nr_n\widetilde{x}'_n\right)\right\|^2\right)}{\frac{1}{2}t_nr_n}.$$

Let $n\to\infty$, we obtain

$$y_n\to\nabla f(\bar p,\bar x)(p,\widetilde{x}').$$

This and (3.3) give us that $\nabla f(\bar p,\bar x)(p,\widetilde{x}')\in\mathcal{B}^{2(S)}_{\bar p,\bar y,\bar u,\bar v}(p)$. Hence, $x\in L$, *i.e.* the first inclusion in (3.1) is fulfilled.

Now, we prove the second inclusion in (3.1). Let $p\in P$ and $x\in L$. Then, $x\in D^2_S C(\bar p,\bar x,\bar u,\bar w)(p)$ and $y:=\nabla f(\bar p,\bar x)(p,x)\in\mathcal{B}^{2(S)}_{\bar p,\bar y,\bar u,\bar v}(p)$. Since $\mathcal{B}$ is second-order radial-asymptotic proto-differentiable at $(\bar p,\bar y)$ in the direction $(\bar u,\bar v)$. Hence, for all $t_n\to 0^+, r_n>0$, there exists $(p_n,y_n)\to(p,y)$ such that $t_nr_np_n\to 0$ and

$$\bar y+t_n\bar v+\frac{1}{2}t_nr_ny_n\in\mathcal{B}\left(\bar p+t_n\bar u+\frac{1}{2}t_nr_np_n\right)\subseteq F\left(\bar p+t_n\bar u+\frac{1}{2}t_nr_np_n\right).$$

This leads the existence of sequence $\{x_n\}\subseteq X$ such that $x_n\in C\left(\bar p+t_n\bar u+\frac{1}{2}t_nr_np_n\right)$ and

$$\bar y+t_n\bar v+\frac{1}{2}t_nr_ny_n=f\left(\bar p+t_n\bar u+\frac{1}{2}t_nr_np_n,x_n\right).$$

Setting $\widetilde{x}_n:=\dfrac{x_n-\bar x-t_n\bar w}{\frac{1}{2}t_nr_n}$, we get

$$\bar y+t_n\bar v+\frac{1}{2}t_nr_ny_n=f\left(\bar p+t_n\bar u+\frac{1}{2}t_nr_np_n,\bar x+t_n\bar w+\frac{1}{2}t_nr_n\widetilde{x}_n\right) \tag{3.5}$$

and

$$x_n=\bar x+t_n\bar w+\frac{1}{2}t_nr_n\widetilde{x}_n\in C\left(\bar p+t_n\bar u+\frac{1}{2}t_nr_np_n\right).$$

Together with (iii), we have $\widetilde{x}_n\to\widetilde{x}$. Thus,

$$\widetilde{x}\in D^{2(b)}_S C(\bar p,\bar x,\bar u,\bar w)(p)\subseteq D^2_S C(\bar p,\bar x,\bar u,\bar w)(p).$$

Moreover, since f is twice continuously Fréchet differentiable at $(\bar p,\bar x)$, we obtain that

$$f\left(\bar p+t_n\bar u+\frac{1}{2}t_nr_np_n,\bar x+t_n\bar w+\frac{1}{2}t_nr_n\widetilde{x}_n\right)=f(\bar p,\bar x)+t_n\nabla f(\bar p,\bar x)(\bar u,\bar w)+\frac{1}{2}t_nr_n\nabla f(\bar p,\bar x)(p_n,\widetilde{x}_n)$$

$$+\frac{1}{2}t_n^2\nabla^2 f(\bar{p},\bar{x})\left(\left(\bar{u}+\frac{1}{2}r_np_n,\bar{w}+\frac{1}{2}r_n\widetilde{x}_n\right),\left(\bar{u}+\frac{1}{2}r_np_n,\bar{w}+\frac{1}{2}r_n\widetilde{x}_n\right)\right)$$
$$+\,o\left(\left\|\left(t_n\bar{u}+\frac{1}{2}t_nr_np_n,t_n\bar{w}+\frac{1}{2}t_nr_n\widetilde{x}_n\right)\right\|^2\right). \tag{3.6}$$

From (3.5), (3.6) and $\bar{y}=f(\bar{p},\bar{x}),\bar{v}=\nabla f(\bar{p},\bar{x})(\bar{u},\bar{w})$, we deduce that

$$y_n=\nabla f(\bar{p},\bar{x})(p_n,\widetilde{x}_n)+\frac{t_n}{r_n}\nabla^2 f(\bar{p},\bar{x})\left(\left(\bar{u}+\frac{1}{2}r_np_n,\bar{w}+\frac{1}{2}r_n\widetilde{x}_n\right),\left(\bar{u}+\frac{1}{2}r_np_n,\bar{w}+\frac{1}{2}r_n\widetilde{x}_n\right)\right)$$
$$+\frac{o\left(\left\|\left(t_n\bar{u}+\frac{1}{2}t_nr_np_n,t_n\bar{w}+\frac{1}{2}t_nr_n\widetilde{x}_n\right)\right\|^2\right)}{\frac{1}{2}t_nr_n}.$$

Let $n\to\infty$, we obtain

$$y_n\to y:=\nabla f(\bar{p},\bar{x})(p,\widetilde{x}).$$

Combining this with $y:=\nabla f(\bar{p},\bar{x})(p,x)$, we have

$$\nabla_x f(\bar{p},\bar{x})(x)=\nabla_x f(\bar{p},\bar{x})(\widetilde{x}).$$

From $\nabla_x f(\bar{p},\bar{x})(.)$ is strictly monotone on X, we get $x=\widetilde{x}$. Therefore, for any $t_n\to 0^+,r_n>0$, there exists $(p_n,\widetilde{x}_n)\to(p,x)$ such that $t_nr_np_n\to 0$,

$$\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_n\widetilde{x}_n\in C\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right)$$

and

$$\bar{y}+t_n\bar{v}+\frac{1}{2}t_nr_ny_n=f\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n,\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_n\widetilde{x}_n\right)\in\mathcal{B}\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right).$$

Therefore, $x\in D_S^{2(b)}\mathcal{H}(\bar{p},\bar{x},\bar{u},\bar{w})(p)$. The proof is complete. □

Now, we present an example to illustrate Proposition 3.3.

Example 3.4. Let $P=\mathbb{R}_+,X=Y=\mathbb{R}^2,K=\mathbb{R}^2_+$ and $f:P\times X\to Y,C:P\rightrightarrows X$ be defined as follows:

$$f(p,x)=(x_1,x_2),\quad \forall p\in P,\quad \forall x=(x_1,x_2)\in X,$$
$$C(p)=\left\{(x_1,x_2)\in\mathbb{R}^2\mid x_1=p^2+p,x_2\geq 3p\right\}.$$

Taking $\bar{p}=0,\bar{x}=(0,0)$ and $(\bar{u},\bar{w})=(1,(1,3))$, we have

$$D_S^2C(\bar{p},\bar{x},\bar{u},\bar{w})(p)=\left\{(x_1,x_2)\in\mathbb{R}^2\mid x_1=p,x_2\geq 3p\right\}.$$

Obviousbly, C is second-order radial-asymptotic directionally compact at $(\bar{p},\bar{x})$ with respect to $(\bar{u},\bar{w})$ in any direction $p\in P$. Since $\bar{y}=f(\bar{p},\bar{x})=0$,

$$\nabla f(p,x)=(\nabla_p f(p,x),\nabla_x f(p,x))=\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}\right),\quad \nabla^2 f(p,x)=0,$$
$$\nabla f(\bar{p},\bar{x})=\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}\right),\qquad \nabla^2 f(\bar{p},\bar{x})=0$$

and $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w}) = (1, 3)$. Hence, condition (i) in Proposition 3.3 holds. Since

$$\nabla_x f(\bar{p}, \bar{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have $\nabla_x f(\bar{p}, \bar{x})(.)$ is strictly monotone on X. By direct calculation, we have, for any $p \in P$,

$$F(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = p^2 + p, y_2 \geq 3p\},$$

and

$$\mathcal{B}(p) = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y = (p^2 + p, 3p)\}.$$

Therefore, for any $p \in P$, one has

$$D_S^2 \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D_S^{2(b)} \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y = (p, 3p)\}.$$

Hence, $\mathcal{B}$ is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ and

$$\mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y = (p, 3p)\}.$$

By direct calculation, one has

$$\mathcal{H}(p) = \{x \in X \mid x \in C(p), f(p, x) \in \mathcal{B}(p)\} = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x = (p^2 + p, 3p)\}.$$

This means that, for all $p \in P$,

$$D_S^2 \mathcal{H}(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) = D_S^{2(b)} \mathcal{H}(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x = (p, 3p)\}.$$

Therefore, $\mathcal{H}$ is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{x})$ in the direction $(\bar{u}, \bar{w})$ and

$$\begin{aligned}\mathcal{H}^{2(S)}_{\bar{p},\bar{x},\bar{u},\bar{w}}(p) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x = (p, 3p)\}, \nabla f(\bar{p}, \bar{x})(p, x) = x,\\ \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x \in D_S^2 C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), \nabla f(\bar{p}, \bar{x})(p, x) \in \mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)\right\}\\ = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x = (p, 3p)\} = \mathcal{H}^{2(S)}_{\bar{p},\bar{x},\bar{u},\bar{w}}(p).\end{aligned}$$

Thus, Proposition 3.3 is satisfied.

Proposition 3.5. *Let $\bar{p} \in P, \bar{x} \in C(\bar{p})$ and $\bar{y} = f(\bar{p}, \bar{x}), (\bar{u}, \bar{w}, \bar{v}) \in P \times X \times Y$. Suppose that the following conditions hold:*

(i) *C is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{x})$ in the direction $(\bar{u}, \bar{w})$;*
(ii) *f is twice continuously Fréchet differentiable at $(\bar{p}, \bar{x})$ and $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$;*
(iii) *C is second-order radial-asymptotic directionally compact at $(\bar{p}, \bar{x})$ with respect to $(\bar{u}, \bar{w})$ in any direction $p \in P$.*

Then, F is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$. Furthermore, for all $p \in P$,

$$F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \{y \in Y \mid x \in D_S^2 C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x)\}.$$

Proof. Let $p \in P$. Setting $L := \{y \in Y \mid x \in D_S^2 C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x)\}$. From Remark 2.7, we only need to prove that, $\forall p \in P$,

$$D_S^2 F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subseteq L \subseteq D_S^{2(b)} F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p). \tag{3.7}$$

Let $y \in D_S^2F(\bar{p},\bar{y},\bar{u},\bar{v})(p)$. Then, there exist $t_n \to 0^+, r_n > 0$ and $(p_n, y_n) \to (p, y)$ such that $t_n r_n p_n \to 0$ and

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n \in F\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n\right).$$

Then, there exists a sequence $\{x_n\} \subseteq X$ such that $x_n \in C\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n\right)$ and

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n = f\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n, x_n\right).$$

Setting $x'_n := \dfrac{x_n - \bar{x} - t_n\bar{w}}{\frac{1}{2}t_n r_n}$, we have

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n = f\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n, \bar{x} + t_n\bar{w} + \frac{1}{2}t_n r_n x'_n\right) \tag{3.8}$$

and

$$x_n = \bar{x} + t_n\bar{w} + \frac{1}{2}t_n r_n x'_n \in C\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n\right).$$

Since C is second-order radial-asymptotic directionally compact at $(\bar{p},\bar{x})$ with respect to $(\bar{u},\bar{w})$. Without loss of generality, we suppose that $x'_n \to x'$. Then, $x' \in D_S^2C(\bar{p},\bar{x},\bar{u},\bar{w})(p)$. Moreover, since f is twice continuously Fréchet differentiable at $(\bar{p},\bar{x})$, one has

$$\begin{aligned} f\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n, \bar{x} + t_n\bar{w} + \frac{1}{2}t_n r_n x'_n\right) &= f(\bar{p},\bar{x}) + t_n\nabla f(\bar{p},\bar{x})(\bar{u},\bar{w}) + \frac{1}{2}t_n r_n \nabla f(\bar{p},\bar{x})(p_n, x'_n) \\ &+ \frac{1}{2}t_n^2\nabla^2 f(\bar{p},\bar{x})\left(\left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x'_n\right), \left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x'_n\right)\right) \\ &+ o\left(\left\|\left(t_n\bar{u} + \frac{1}{2}t_n r_n p_n, t_n\bar{w} + \frac{1}{2}t_n r_n x'_n\right)\right\|^2\right). \end{aligned} \tag{3.9}$$

From (3.8), (3.9) and $\bar{y} = f(\bar{p},\bar{x}), \bar{v} = \nabla f(\bar{p},\bar{x})(\bar{u},\bar{w})$, one has

$$\begin{aligned} y_n = \nabla f(\bar{p},\bar{x})(p_n, x'_n) &+ \frac{t_n}{r_n}\nabla^2 f(\bar{p},\bar{x})\left(\left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x'_n\right), \left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x'_n\right)\right) \\ &+ \frac{o\left(\left\|\left(t_n\bar{u} + \frac{1}{2}t_n r_n p_n, t_n\bar{w} + \frac{1}{2}t_n r_n x'_n\right)\right\|^2\right)}{\frac{1}{2}t_n r_n}. \end{aligned}$$

Let $n \to \infty$, we obtain

$$y_n \to y := \nabla f(\bar{p},\bar{x})(p, x').$$

Therefore, $D_S^2F(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subseteq L$. It follows that the first inclusion in (3.7) holds.

Now, we will prove that $L \subseteq D_S^{2(b)}F(\bar{p},\bar{y},\bar{u},\bar{v})(p)$ holds. Let $p \in P$ and $y \in L$. Then, there exists $x \in D_S^2C(\bar{p},\bar{x},\bar{u},\bar{w})(p)$ such that $y = \nabla f(\bar{p},\bar{x})(p, x)$. Since C is second-order radial-asymptotic proto-differentiable at $(\bar{p},\bar{x})$ in the direction $(\bar{u},\bar{w})$. Thus, for all $t_n \to 0^+, r_n > 0$, there exists $(p_n, x_n) \to (p, x)$ such that $t_n r_n p_n \to 0$ and

$$\bar{x} + t_n\bar{w} + \frac{1}{2}t_n r_n x_n \in C\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n\right),$$

which implies that,

$$f\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n,\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_nx_n\right)\in F\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right). \tag{3.10}$$

Setting

$$y_n:=\frac{f\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n,\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_nx_n\right)-f(\bar{p},\bar{x})-t_n\bar{v}}{\frac{1}{2}t_nr_n}. \tag{3.11}$$

Then,

$$\bar{y}+t_n\bar{v}+\frac{1}{2}t_nr_ny_n\in F\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right).$$

Since f is twice continuously Fréchet differentiable at $(\bar{p},\bar{x})$, we obtain that

$$\begin{aligned}f\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n,\bar{x}+t_n\bar{w}+\frac{1}{2}t_nr_nx_n\right)&=f(\bar{p},\bar{x})+t_n\nabla f(\bar{p},\bar{x})(\bar{u},\bar{w})+\frac{1}{2}t_nr_n\nabla f(\bar{p},\bar{x})(p_n,x_n)\\&+\frac{1}{2}t_n^2\nabla^2 f(\bar{p},\bar{x})\left(\left(\bar{u}+\frac{1}{2}r_np_n,\bar{w}+\frac{1}{2}r_nx_n\right),\left(\bar{u}+\frac{1}{2}r_np_n,\bar{w}+\frac{1}{2}r_nx_n\right)\right)\\&+o\left(\left\|\left(t_n\bar{u}+\frac{1}{2}t_nr_np_n,t_n\bar{w}+\frac{1}{2}t_nr_nx_n\right)\right\|^2\right).\end{aligned} \tag{3.12}$$

From (3.11), (3.12) and $\bar{y}=f(\bar{p},\bar{x}),\bar{v}=\nabla f(\bar{p},\bar{x})(\bar{u},\bar{w})$, we deduce that

$$\begin{aligned}y_n=\nabla f(\bar{p},\bar{x})(p_n,x_n)&+\frac{t_n}{r_n}\nabla^2 f(\bar{p},\bar{x})\left(\left(\bar{u}+\frac{1}{2}r_np_n,\bar{w}+\frac{1}{2}r_nx_n\right),\left(\bar{u}+\frac{1}{2}r_np_n,\bar{w}+\frac{1}{2}r_nx_n\right)\right)\\&+\frac{o\left(\left\|\left(t_n\bar{u}+\frac{1}{2}t_nr_np_n,t_n\bar{w}+\frac{1}{2}t_nr_nx_n\right)\right\|^2\right)}{\frac{1}{2}t_nr_n}.\end{aligned}$$

Let $n\to\infty$, we obtain

$$y_n\to y:=\nabla f(\bar{p},\bar{x})(p,x).$$

Thus, for all $t_n\to 0^+,r_n>0$, there exists $(p_n,y_n)\to(p,y)$ such that $t_nr_np_n\to 0$ and

$$\bar{y}+t_n\bar{v}+\frac{1}{2}t_nr_ny_n\in F\left(\bar{p}+t_n\bar{u}+\frac{1}{2}t_nr_np_n\right).$$

Consequently,

$$y\in D_S^{2(b)}F(\bar{p},\bar{y},\bar{u},\bar{v})(p).$$

It follows that the second inclusion in (3.7) holds. The proof is complete. □

Theorem 3.6. *Let $\bar{p}\in P,\bar{x}\in\mathcal{H}(\bar{p})$ and $\bar{y}=f(\bar{p},\bar{x}),(\bar{u},\bar{w},\bar{v})\in P\times X\times Y$. Suppose that the following conditions hold:*

- (i) *C is second-order radial-asymptotic proto-differentiable at $(\bar{p},\bar{x})$ in the direction $(\bar{u},\bar{w})$;*
- (ii) *C is second-order radial-asymptotic directionally compact at $(\bar{p},\bar{x})$ with respect to $(\bar{u},\bar{w})$ in any direction $p\in P$;*
- (iii) *f is twice continuously Fréchet differentiable at $(\bar{p},\bar{x})$ and $\bar{v}=\nabla f(\bar{p},\bar{x})(\bar{u},\bar{w})$;*

(iv) *F is local Lipschitz at $\bar{p}$;*
(v) *F is K-dominated by $\mathcal{B}$ near $\bar{p}$;*
(vi) *For all $p \in P, w - z \in K \cup (-K), \quad \forall w, z \in F(p), w \neq z$.*

Then, the Borwein efficient perturbation map $\mathcal{B}$ is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$. Moreover, for all $p \in P$,

$$\begin{aligned}\mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) &= \mathrm{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)\\ &= \mathrm{BoMin}_K \big\{ y \in Y \mid x \in D^2_S C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x) \big\}.\end{aligned}$$

Proof. First of all, we prove that $\mathcal{B}(p)$ is a single point set for all $p \in P$. Let $w \in \mathcal{B}(p)$, then $w \in F(p)$. Suppose to contrary that $\mathcal{B}(p)$ is not a single point set. Therefore, for any $z \in \mathcal{B}(p) \subset F(p)$ with $w \neq z$, it implies from assumption (vi) that $w - z \in K \cup (-K)$. Hence, $w - z \in K$ or $w - z \in -K$. Obviously, $w - z \neq 0$. If $w - z \in K$, then we have $z - w \in -K$. Thus,

$$\mathrm{cl}\,\mathrm{cone}(F(p) - w) \cap ((-K) \backslash \{0\}) \neq \emptyset,$$

which contradicts the fact that $w \in \mathcal{B}(p)$. If $w - z \in -K$, then we have

$$\mathrm{cl}\,\mathrm{cone}(F(p) - z) \cap ((-K) \backslash \{0\}) \neq \emptyset,$$

which contradicts the fact that $z \in \mathcal{B}(p)$. Thus, $\mathcal{B}(p)$ is a single point set for all $p \in P$.

Let an arbitrary $p \in P$. By (i), (ii) and (iii) and Proposition 3.5, we follow that F is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ and

$$F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \big\{ y \in Y \mid x \in D^2_S C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x) \big\}.$$

Setting $L := \mathrm{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$. From Remark 2.7, we only need to prove that

$$D^2_S \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subseteq L \subseteq D^{2(b)}_S \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p). \tag{3.13}$$

First, we prove that $D^2_S \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subseteq L, \forall p \in P$. Let $y \in D^2_S \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Then, there exist $t_n \to 0^+, r_n > 0$ and $(p_n, y_n) \to (p, y)$ such that $t_n r_n p_n \to 0$ and

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n \in \mathcal{B}\bigg(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n\bigg) \subseteq F\bigg(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n\bigg).$$

This implies that $y \in D^2_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Arguing by contradiction, suppose that

$$y \notin \mathrm{BoMin}_K D^2_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p).$$

Thus, there exist $h_m > 0$ and $\widehat{y}_m \in D^2_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$ such that

$$\lim_{m \to \infty} h_m(\widehat{y}_m - y) \in -K \backslash \{0\}. \tag{3.14}$$

Since F is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ and $\widehat{y}_m \in D^2_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), D^2_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^{2(b)}_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$, one yields, for all $t_n \to 0^+, r_n > 0$ there exists $(\widehat{p}_{m_n}, \widehat{y}_{m_n}) \to (p, \widehat{y}_m)$ such that $t_n r_n \widehat{p}_{m_n} \to 0$ and

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n \widehat{y}_{m_n} \in F\bigg(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n \widehat{p}_{m_n}\bigg), \quad \forall n \in \mathbb{N}. \tag{3.15}$$

Since F is K-dominated by $\mathcal{B}$ near $\bar{p} \in P$, there exists the neighborhood U_1 of $\bar{p}$ such that

$$F(p) \subseteq \mathcal{B}(p) + K, \quad \forall p \in U_1. \tag{3.16}$$

Because F is local Lipschitz at $\bar{p}$, one implies that exist the neighborhood U_2 of $\bar{p}$ and $M > 0$ such that

$$F(p_1) \subseteq F(p_2) + M\|p_1 - p_2\|B_Y, \quad \forall p_1, p_2 \in U_2. \tag{3.17}$$

Naturally, there exists $n_0 > 0$ with $n_0 \in \mathbb{N}$, such that

$$\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_n\widehat{p}_{m_n}, \bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n \in U_1 \cap U_2, \quad \forall n > n_0, \quad \forall m \in \mathbb{N}. \tag{3.18}$$

Thus, it follows from (3.15), (3.18), (3.17) and (3.16), there exists $b_n \in B_Y$ in order that, for every n large enough,

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_n(\widehat{y}_{m_n} - M\|\widehat{p}_{m_n} - p_n\|b_n) \in F\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n\right) \subseteq \mathcal{B}\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n\right) + K, \quad \forall m \in \mathbb{N}. \tag{3.19}$$

Because $\mathcal{B}(p)$ is a single point set for all $p \in P$. So, by (3.19), one has

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_n(\widehat{y}_{m_n} - M\|\widehat{p}_{m_n} - p_n\|b_n) - \left(\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_ny_n\right) = \frac{1}{2}t_nr_n(\widehat{y}_{m_n} - M\|\widehat{p}_{m_n} - p_n\|b_n - y_n) \in K.$$

Thus,

$$\widehat{y}_{m_n} - M\|\widehat{p}_{m_n} - p_n\|b_n - y_n \to \widehat{y}_m - y, \quad \forall m \in \mathbb{N}.$$

Since K is a pointed closed convex cone in Euclidean space Y, we deduce $\widehat{y}_m - y \in K, \quad \forall m \in \mathbb{N}$. Therefore, we derive from $h_m > 0$ and K is a pointed closed convex cone that

$$\lim_{m\to\infty} h_m(\widehat{y}_m - y) \in K,$$

which contradicts (3.14). Thus, $y \in \text{BoMin}_K D_S^2F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = L$, which completes the first inclusion in (3.13).

Now, we prove $L \subseteq D_S^{2(b)}\mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \forall p \in P$. Let $y \in L = \text{BoMin}F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$. Then, $y \in F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$. By the second-order radial-asymptotic proto-differentiability of F at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$, for any $t_n \to 0^+, r_n > 0$, there exists $(p_n, y_n) \to (p, y)$ such that $t_nr_np_n \to 0$ and

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_ny_n \in F\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n\right), \quad \forall n \in \mathbb{N}.$$

Hence,

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_ny_n \in F\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n\right) + K, \quad \forall n \in \mathbb{N}.$$

On the other hand, from the K-dominatedness by $\mathcal{B}$ near $\bar{p}$ of F and $\mathcal{B}(p) \subseteq F(p), \forall p \in P$, there exists the neighborhood U of $\bar{p}$ such that

$$F(p) + K = \mathcal{B}(p) + K, \quad \forall p \in U.$$

Thus,

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_ny_n \in \mathcal{B}\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n\right) + K, \quad \forall n \in \mathbb{N}.$$

Therefore, one follows the existence of $k_n \in K$ such that

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_n\widetilde{y}_n \in \mathcal{B}\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n\right) \subseteq F\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n\right), \widetilde{y}_n := y_n - k_n, \quad \forall n \in \mathbb{N}. \tag{3.20}$$

Thus, there exists $x_n \in X$ such that

$$\bar{x} + t_n\bar{w} + \frac{1}{2}t_nr_nx_n \in C\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n\right),$$

and

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_n\widetilde{y}_n = f\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n, \bar{x} + t_n\bar{w} + \frac{1}{2}t_nr_nx_n\right). \tag{3.21}$$

As the second-order radial-asymptotic directionally compactness of C at $(\bar{p}, \bar{x})$ with respect to $(\bar{u}, \bar{w})$, we derive that the sequence $\{x_n\} \subseteq X$ has a convergent subsequence, also denoted by x_n and $x_n \to x$, which in turn implies that $x \in D_S^{2(b)}C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) \subseteq D_S^2C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p)$. Moreover, since f is twice continuously Fréchet differentiable at $(\bar{p}, \bar{x})$, one has

$$\begin{aligned} f\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n, \bar{x} + t_n\bar{w} + \frac{1}{2}t_nr_nx_n\right) &= f(\bar{p}, \bar{x}) + t_n\nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w}) + \frac{1}{2}t_nr_n\nabla f(\bar{p}, \bar{x})(p_n, x_n) \\ &+ \frac{1}{2}t_n^2\nabla^2 f(\bar{p}, \bar{x})\left(\left(\bar{u} + \frac{1}{2}r_np_n, \bar{w} + \frac{1}{2}r_nx_n\right), \left(\bar{u} + \frac{1}{2}r_np_n, \bar{w} + \frac{1}{2}r_nx_n\right)\right) \\ &+ o\left(\left\|\left(t_n\bar{u} + \frac{1}{2}t_nr_np_n, t_n\bar{w} + \frac{1}{2}t_nr_nx_n\right)\right\|^2\right). \end{aligned} \tag{3.22}$$

From (3.21), (3.22) and $\bar{y} = f(\bar{p}, \bar{x}), \bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$, one has

$$\begin{aligned} \widetilde{y}_n = \nabla f(\bar{p}, \bar{x})(p_n, x_n) &+ \frac{t_n}{r_n}\nabla^2 f(\bar{p}, \bar{x})\left(\left(\bar{u} + \frac{1}{2}r_np_n, \bar{w} + \frac{1}{2}r_nx_n\right), \left(\bar{u} + \frac{1}{2}r_np_n, \bar{w} + \frac{1}{2}r_nx_n\right)\right) \\ &+ \frac{o\left(\left\|\left(t_n\bar{u} + \frac{1}{2}t_nr_np_n, t_n\bar{w} + \frac{1}{2}t_nr_nx_n\right)\right\|^2\right)}{\frac{1}{2}t_nr_n}. \end{aligned}$$

Let $n \to \infty$, we obtain

$$\widetilde{y}_n \to \widetilde{y} := \nabla f(\bar{p}, \bar{x})(p, x).$$

On the other hand, from (3.20), one gets

$$\widetilde{y} \in D_S^{2(b)}\mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subseteq D_S^{2(b)}F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p).$$

By the closedness of K and $k_n = y_n - \widetilde{y}_n \to y - \widetilde{y}$, one has $y - \widetilde{y} \in K$. Since K is a pointed closed convex cone, there exists $h > 0$ such that

$$h(y - \widetilde{y}) \in K.$$

We prove that $y = \widetilde{y}$. Indeed, suppose to the contrary that $y \neq \widetilde{y}$. Then, one gets

$$h(\widetilde{y} - y) \in -K\backslash\{0\}.$$

Since $D_S^{2(b)}F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$, it follows that there exists $\widetilde{y} \in F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$ such that

$$h(\widetilde{y} - y) \in -K\backslash\{0\}.$$

Hence, $y \notin \mathrm{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$, a contradiction. So, $y = \widetilde{y} \in D_S^{2(b)}\mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. The proof is complete. $\square$

Remark 3.7. Assumption (vi) in Theorem 3.6 is difference from assumption (vi) in Theorem 3.2 of [34], assumption (vi) in Proposition 3.1, Proposition 3.2 of [35] and assumption (vi) in Proposition 3.3 of [37]. Thus, Theorem 3.6 improves the results in [34, 35, 37].

The following example shows that the condition (vi) in Theorem 3.6 is essential.

Example 3.8. Let $P = \mathbb{R}_+, X = Y = \mathbb{R}^2, K = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq 3y_1 \geq 0\}$ and $f : P \times X \to Y, C : P \rightrightarrows X$ be defined as follows:

$$f(p,x) = (x_1, x_2), \quad \forall p \in P, \quad \forall x = (x_1, x_2) \in X,$$
$$C(p) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq p^2, x_2 \geq 3p^2 + 3p\} \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 3x_1 \geq 0\}.$$

Taking $\bar{p} = 0, \bar{x} = (0,0)$ and $(\bar{u}, \bar{w}) = (1,(1,3))$, we have

$$D^2_S C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) = D^{2(b)}_S C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 3p\} \cup \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 3x_1\},$$

leading that the condition (i) in Theorem 3.6 is satisfied. Since $\bar{y} = f(\bar{p}, \bar{x}) = 0$,

$$\nabla f(p,x) = (\nabla_p f(p,x), \nabla_x f(p,x)) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \quad \nabla^2 f(p,x) = 0,$$
$$\nabla f(\bar{p}, \bar{x}) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \qquad \nabla^2 f(\bar{p}, \bar{x}) = 0,$$

and $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w}) = (1,3)$. Hence, condition (iii) in Theorem 3.6 holds. It is easy to prove that the assumptions (ii), (iv) and (v) in Theorem 3.6 are satisfied. By direct calculation, we have, for any $p \in P$,

$$F(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq p^2, y_2 \geq 3p^2 + 3p\} \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq 3y_1 \geq 0\},$$

and

$$\mathcal{B}(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > p^2 + p, y_2 = 3p^2 + 3p\} \cup \{(0,0)\}.$$

Take $(1,3) \in F(p)$ and $(0,3) \in F(p)$, we have

$$(0,3) - (1,3) = (-1,0) \notin K \cup (-K).$$

Thus, the assumption (vi) in Theorem 3.6 is not satisfied. For any $p \in P$, one has

$$D^2_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^{2(b)}_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq 3p\} \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq 3y_1\},$$
$$D^2_S \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^{2(b)}_S \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq p, y_2 = 3p\}.$$

Hence, F and $\mathcal{B}$ are second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ and

$$F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq 3p\} \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq 3y_1\},$$
$$\mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq p, y_2 = 3p\}.$$

Hence,

$$\text{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > p, y_2 = 3p\}.$$

This means that, for all $p \in P$,

$$\mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) \neq \text{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p).$$

Thus, Theorem 3.6 does not hold. Because the assumption (vi) in Theorem 3.6 is not satisfied.

Remark 3.9. For all $p \in P$, $\mathcal{B}(p)$ is a single point set if and only if $w - z \in K \cup (-K), \forall w, z \in F(p), w \neq z$.

Indeed, for all $p \in P$, if $w - z \in K \cup (-K), \forall w, z \in F(p), w \neq z$, then we have $\mathcal{B}(p)$ is a single point set.

Suppose that $\forall p \in P, \mathcal{B}(p)$ is a single point set, then we need prove that $w-z \in K\cup(-K), \forall w, z \in F(p), w \neq z$. Let $z \in \mathcal{B}(p)$, then $z \in F(p)$. Suppose to contrary that $w - z \notin K \cup (-K), \forall w, z \in F(p), w \neq z$. If $w - z \notin K$, then we have $z - w \notin -K$. Thus,

$$\operatorname{cl}\operatorname{cone}(F(p) - w) \cap (-K) = \{0\},$$

it follows $w \in \mathcal{B}(p)$, a contradiction, since $\mathcal{B}(p)$ is a single point set. If $w - z \notin -K$, then we have $z - w \notin K$. Thus,

$$\operatorname{cl}\operatorname{cone}(F(p) - w) \cap K = \{0\}.$$

Since K is pointed cone, we get

$$K \cap (-K) = \{0\} \text{ (see [16] page 15).}$$

Hence, we have

$$\operatorname{cl}\operatorname{cone}(F(p) - w) \cap (-K) = \{0\},$$

it follows $w \in \mathcal{B}(p)$, a contradiction, since $\mathcal{B}(p)$ is a single point set. Thus, $w - z \in K \cup (-K), \quad \forall w, z \in F(p), w \neq z$ for all $p \in P$.

Now, we present an example to explain the given result in Theorem 3.6.

Example 3.10. Let $P = \mathbb{R}_+, X = Y = \mathbb{R}^2, K = \mathbb{R}^2_+$ and $f : P \times X \to Y, C : P \rightrightarrows X$ be defined as follows:

$$f(p, x) = (x_1, x_2), \quad \forall p \in P, \quad \forall x = (x_1, x_2) \in X,$$
$$C(p) = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = p^2 + 2p, x_2 \geq 3p\big\}.$$

Taking $\bar{p} = 0, \bar{x} = (0, 0)$ and $(\bar{u}, \bar{w}) = (1, (2, 3))$, we have

$$D^2_S C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) = D^{2(b)}_S C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) = \big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 2p, x_2 \geq 3p\big\},$$

leading that the condition (i) in Theorem 3.6 is satisfied. Since $\bar{y} = f(\bar{p}, \bar{x}) = 0$,

$$\nabla f(p, x) = (\nabla_p f(p, x), \nabla_x f(p, x)) = \left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right), \quad \nabla^2 f(p, x) = 0,$$
$$\nabla f(\bar{p}, \bar{x}) = \left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right), \qquad \nabla^2 f(\bar{p}, \bar{x}) = 0$$

and $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w}) = (2, 3)$. Hence, condition (iii) in Theorem 3.6 holds. It is easy to prove that the assumption (ii), (iv), (v) and (vi) in Theorem 3.6 are satisfied. By direct calculation, we have, for any $p \in P$,

$$F(p) = \big\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = p^2 + 2p, y_2 \geq 3p\big\},$$

and

$$\mathcal{B}(p) = \big\{y = (y_1, y_2) \in \mathbb{R}^2 \mid y = (p^2 + 2p, 3p)\big\}.$$

Therefore, for any $p \in P$, one has

$$D^2_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^{2(b)}_S F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \big\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 2p, y_2 \geq 3p\big\},$$
$$D^2_S \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^{2(b)}_S \mathcal{B}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \big\{y = (y_1, y_2) \in \mathbb{R}^2 \mid y = (2p, 3p)\big\}.$$

Hence, F and $\mathcal{B}$ are second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$ and

$$F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \big\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 2p, y_2 \geq 3p\big\},$$

$$\mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \big\{y = (y_1, y_2) \in \mathbb{R}^2 \mid y = (2p, 3p)\big\}.$$

Hence,

$$\text{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \big\{y = (y_1, y_2) \in \mathbb{R}^2 \mid y = (2p, 3p)\big\}.$$

This means that, for all $p \in P$,

$$\mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \text{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \big\{y = (y_1, y_2) \in \mathbb{R}^2 \mid y = (2p, 3p)\big\}.$$

So, Theorem 3.6 holds here.

Theorem 3.11. *Let $\bar{p} \in P, \bar{x} \in \mathcal{H}(\bar{p})$ and $\bar{y} = f(\bar{p}, \bar{x}), (\bar{u}, \bar{w}, \bar{v}) \in P \times X \times Y$. Suppose that the following conditions hold:*

(i) *C is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{x})$ in the direction $(\bar{u}, \bar{w})$;*
(ii) *f is twice continuously Fréchet differentiable at $(\bar{p}, \bar{x})$ and $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$;*
(iii) *C is second-order radial-asymptotic directionally compact at $(\bar{p}, \bar{x})$ with respect to $(\bar{u}, \bar{w})$ in any direction $p \in P$;*
(iv) *$\mathcal{B}$ is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$;*
(v) *$\mathcal{B}$ is second-order radial-asymptotic directionally compact at $(\bar{p}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in any direction $p \in P$;*
(vi) *F is K-dominated by $\mathcal{B}$ near $\bar{p}$.*

Then, $\text{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) \subset \mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p), \forall p \in P$.

Proof. Since $\bar{x} \in \mathcal{H}(\bar{p})$, we have $\bar{y} \in \mathcal{B}(\bar{p}) \subseteq F(\bar{p})$. From (i), (ii), (iii) and Proposition 3.5, we follow that F is second-order radial-asymptotic proto-differentiable at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v})$. Let arbitrary $p \in P$ and $y \in \text{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$. Then, $y \in F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$. From (iv), (v) and Proposition 2.16, it follows that

$$(\mathcal{B} + K)^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) + K, \quad \forall p \in P.$$

On the other hand, from the domination around $\bar{p}$ of F and $\mathcal{B}(p) \subseteq F(p), \forall p \in P$, there exists the neighborhood U of $\bar{p}$ such that

$$F(p) + K = \mathcal{B}(p) + K, \quad \forall p \in U.$$

Thus,

$$F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) \subset (F + K)^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = (\mathcal{B} + K)^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) = \mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p) + K, \quad \forall p \in P.$$

This means that, there exist $\widetilde{y} \in \mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$ and $k \in K$ such that $y = \widetilde{y} + k$. By the closedness of K and $k = y - \widetilde{y}$, one has $y - \widetilde{y} \in K$. Since K is a pointed closed convex cone, there exists $h > 0$ such that

$$h(y - \widetilde{y}) \in K.$$

We will prove that $k = 0$. Indeed, if $k \in K\backslash\{0\}$, then

$$h(\widetilde{y} - y) \in -K\backslash\{0\}.$$

Since $\widetilde{y} \in \mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$, for any $t_n \to 0^+, r_n > 0$, there exists $(p_n, \widetilde{y}_n) \to (p, \widetilde{y})$ such that $t_n r_n p_n \to 0$ and

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n \widetilde{y}_n \in \mathcal{B}\left(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n\right) \subseteq F\left(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n\right), \quad \forall n \in \mathbb{N}.$$

It yields that $\widetilde{y} \in F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$ such that

$$h(\widetilde{y} - y) \in -K\backslash\{0\}.$$

Hence, $y \notin \text{BoMin}_K F^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p)$, a contradiction. So $k = 0$, this means that, $y \in \mathcal{B}^{2(S)}_{\bar{p},\bar{y},\bar{u},\bar{v}}(p), p \in P$. The proof is complete. $\square$

Remark 3.12. (i) By some suitable changes, most of the results of Section 3 are still true when the second-order radial-asymptotic proto-differentiability is replaced by the second-order radial-asymptotic semi-differentiability.

(ii) As in Remark 2.7, the results in this paper are different from the results in [34].

(iii) Second-order radial-asymptotic derivative contains both the second-order contingent derivative and the second-order asymptotic derivative. So, second-order radial-asymptotic proto-differentiability properties of the efficient solution maps and the perturbation maps are different from the results in [36] when $m = 2$.

(iv) Second-order radial-asymptotic derivative in this paper is different from second-order composed derivatives in [37]. So, the results in this paper are different from the results in [37].

(v) The proto-differentiability properties of the Borwein efficient solution map and the Borwein efficient frontier map of parametric vector optimization problems have not been yet considered in [34–37]. So, the obtained results improve and extend the results in [34–37].

Acknowledgements. The authors would like to thank the Editors for the help in the processing of the article. The authors are very grateful to the Associate Editor and the Anonymous Referees for many valuable comments and suggestions to improve the original version of the article.

References

[1] N.L.H. Anh, Sensitivity analysis in constrained set-valued optimization via Studniarski derivatives. *Positivity* **21** (2017) 255–272.

[2] N.L.H. Anh, Some results on sensitivity analysis in set-valued optimization. *Positivity* **21** (2017) 1527–1543.

[3] N.L.H. Anh, Second-order composed contingent derivatives of perturbation maps in set-valued optimization. *Comput. Appl. Math.* **38** (2019) 145.

[4] N.L.H. Anh and P.Q. Khanh, Variational sets of perturbation maps and applications to sensitivity analysis for constrained vector optimization. *J. Optim. Theory Appl.* **158** (2013) 363–384.

[5] J.P. Aubin and H. Frankowska, Set-Valued Analysis. Birkhauser, Boston (1990).

[6] J.F. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems. Springer, New York (2000).

[7] T.D. Chuong, Clarke coderivatives of efficient point multifunctions in parametric vector optimization. *Nonlinear Anal.* **74** (2011) 273–285.

[8] T.D. Chuong, Derivatives of the efficient point multifunction in parametric vector optimization problems. *J. Optim. Theory Appl.* **156** (2013) 247–265.

[9] T.D. Chuong, Normal subdifferentials of effcient point multifunctions in parametric vector optimization. *Optim. Lett.* **7** (2013) 1087–1117.

[10] T.D. Chuong and J.-C. Yao, Coderivatives of efficient point multifunctions in parametric vector optimization. *Taiwan. J. Math.* **13** (2009) 1671–1693.

[11] T.D. Chuong and J.-C. Yao, Generalized clarke epiderivatives of parametric vector optimization problems. *J. Optim. Theory Appl.* **147** (2010) 77–94.

[12] T.D. Chuong and J.-C. Yao, Fréchet subdifferentials of efficient point multifunctions in parametric vector optimization. *J. Global Optim.* **57** (2013) 1229–1243.

[13] H.T.H. Diem, P.Q. Khanh and L.T. Tung, On higher-order sensitivity analysis in nonsmooth vector optimization. *J. Optim. Theory Appl.* **162** (2014) 463–488.

[14] N.Q. Huy and G.M. Lee, On sensitivity analysis in vector optimization. *Taiwan. J. Math.* **11** (2007) 945–958.

[15] N.Q. Huy and G.M. Lee, Sensitivity of solutions to a parametric generalized equation. *Set-Valued Anal.* **16** (2008) 805–820.

[16] A. Khan, C. Tammer and C. Zălinescu, Set-Valued Optimization – An Introduction with Applications. Springer, Berlin (2015).

[17] H. Kuk, T. Tanino and M. Tanaka, Sensitivity analysis in parametrized convex vector optimization. *J. Math. Anal. Appl.* **202** (1996) 511–522.

[18] G.M. Lee and N.Q. Huy, On proto-differentiability of generalized perturbation maps. *J. Math. Anal. Appl.* **324** (2006) 1297–1309.

[19] A.B. Levy and R.T. Rockafellar, Sensitivity analysis of solutions to generalized equations. *Trans. Am. Math. Soc.* **345** (1994) 661–671.

[20] S.J. Li and C.M. Liao, Second-order differentiability of generalized perturbation maps. *J. Global Optim.* **52** (2012) 243–252.

[21] S.J. Li, X.K. Sun and J. Zhai, Second-order contingent derivatives of set-valued mappings with application to set-valued optimization. *Appl. Math. Comput.* **218** (2012) 6874–6886.

[22] D.T. Luc, M. Soleimani-damaneh and M. Zamani, Semi-differentiability of the marginal mapping in vector optimization. *SIAM J. Optim.* **28** (2018) 1255–1281.

[23] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. I: Basic Theory. Springer, Berlin (2006).

[24] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation. II: Applications. Springer, Berlin (2006).
[25] Z. Peng and Z. Wan, Second-order composed contingent derivative of the perturbation map in multiobjective optimization. *Asia Pac. J. Oper. Res.* **37** (2020) 2050002.
[26] R.T. Rockafellar, Proto-differentiablility of set-valued mappings and its applications in optimization. *Ann. Inst. Non Linéaire. H. Poincaré Anal.* **6** (1989) 449–482.
[27] D.S. Shi, Contingent derivative of the perturbation map in multiobjective optimization. *J. Optim. Theory Appl.* **70** (1991) 385–396.
[28] D.S. Shi, Sensitivity analysis in convex vector optimization. *J. Optim. Theory Appl.* **77** (1993) 145–159.
[29] X.K. Sun and S.J. Li, Generalized second-order contingent epiderivatives in parametric vector optimization problem. *J. Glob. Optim.* **58** (2014) 351–363.
[30] T. Tanino, Sensitivity analysis in multiobjective optimization. *J. Optim. Theory Appl.* **56** (1988) 479–499.
[31] T. Tanino, Stability and sensitivity analysis in convex vector optimization. *SIAM J. Control Optim.* **26** (1988) 521–536.
[32] L.T. Tung, Second-order radial-asymptotic derivatives and applications in set-valued vector optimization. *Pac. J. Optim.* **13** (2017) 137–153.
[33] L.T. Tung, Variational sets and asymptotic variational sets of proper perturbation map in parametric vector optimization. *Positivity* **21** (2017) 1647–1673.
[34] L.T. Tung, On second-order proto-differentiability of perturbation maps. *Set-Valued Var. Anal.* **26** (2018) 561–579.
[35] L.T. Tung, On higher-order proto-differentiability of perturbation maps. *Positivity* **24** (2020) 441–462.
[36] L.T. Tung, On higher-order proto-differentiability and higher-order asymptotic proto-differentiability of weak perturbation maps in parametric vector optimization. *Positivity* **25** (2021) 579–604.
[37] L.T. Tung, On second-order composed proto-differentiability of proper perturbation maps in parametric vector optimization problems. *Asia Pac. J. Oper. Res.* **38** (2021) 2050040.
[38] L.T. Tung and P.T. Hung, Sensitivity analysis in parametric vector optimization in Banach spaces via τ^w-contingent derivatives. *Turk. J. Math.* **44** (2020) 152–168.
[39] Q.L. Wang and S.J. Li, Second-order contingent derivative of the pertubation map in multiobjective optimization. *Fixed Point Theory Appl.* **2011** (2011) 1–13.
[40] Q.L. Wang and S.J. Li, Sensitivity and stability for the second-order contingent derivative of the proper perturbation map in vector optimization. *Optim. Lett.* **6** (2012) 731–748.
[41] Q.L. Wang and X.Y. Zhang, Second-order composed radial derivatives of the Benson proper perturbation map for parametric multi-objective optimization problems. *Asia Pac. J. Oper. Res.* **37** (2020) 2040011.