COST OPTIMIZATION INVENTORY MODEL FOR DETERIORATING ITEMS WITH TRAPEZOIDAL DEMAND RATE UNDER COMPLETELY BACKlogged SHORTAGES IN CRISP AND FUZZY ENVIRONMENT

Boina Anil Kumar and Susanta Kumar Paikray*

Abstract. Recently, various deterministic inventory models were developed for deteriorating items with the uniform demand pattern (either increasing or decreasing) throughout the cycle. However, such types of models are not suitable for many real business problems. In particular, the demand patterns of various items are not steady throughout the cycle. In many inventory models, ordinarily, the demand rises first, then it becomes static and finally decreases, and such types of demands can be portrayed by considering trapezoidal functions. Moreover, the costs associated with the inventory become imprecise due to several socio-economical factors. As a result, the optimal solution obtained by the classical inventory model may not fit the actual scenario. Keeping this in view, we develop here an inventory model for deteriorating items having the trapezoidal type of demand function in both crisp and fuzzy environments by considering three possible cases of shortages which are completely backlogged. Furthermore, in view of the comparative study of both scenarios, different data sets of constraints are examined for optimal results. Also, it is observed that the optimal results of the fuzzy model are more appropriate to real-world inventory problems. Finally, in order to strengthen the present investigation, the managerial insight of fluctuation in parameters is presented analytically via sensitivity analysis.

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1. Introduction

The main objective of traders is to earn profit by running the business smoothly without any interruption. Simultaneously, they wish to increase their goodwill and brand value to attract more customers by implementing proper managerial strategies. In general, to have a smooth business affair, the proper inventory management of items is highly desirable. The inventory management depends on several constraints like demand, deterioration, shortages, inflation, and different inventory costs. Also, it differs from business to business. Many inventory models have been developed in this direction by several researchers. Among all, Ford Whitman Harris was the first to propose an EOQ model as mentioned in [10].

The deterioration and decay of items usually prevent them from their original use. The deteriorating nature is varied with respect to the type of items, storage facility, environmental conditions, etc. For instance, fruits,

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dry fruits, vegetables, different types of agricultural products, milk products, and food products have various deteriorations. Also, cold drinks, health drinks, medicines, photographic films, radioactive substances, electronic goods, fashionable products, clothes, plastic products, jewelry have different life spans. Moreover, the deterioration of items also affects the inventory cost. So, the inventory management of deteriorating items is a challengeable task for inventory managers to minimize the total cost. Ghare and Schrader [11] have obtained the optimal results for retailer’s inventory having constant deteriorating items in the early days of 1960.

Again, the demand of the items has also an important role for proper maintaining of the inventory. Different items have various demand patterns, and the demand patterns of items varies with consumers as well as business cycles and so on. The demand for items may increases, decreases, or constant depending on the need of the customers. Also, in some scenarios, it depends on time, selling price, advertisement cost, trade credit financing, and many other factors. Moreover, the demand for the same item may vary from cycle to cycle and market to market. On the other hand, transportation and communication between countries are developing day by day with technological advancement. So, many substitute products from other countries are arriving into the market, and it affects the demand of the items. So, many researchers used to develop inventory models for different demanding conditions along with other constraints.

The shortages of the items in an inventory cycle are quite a general phenomenon and are occurring when the supplier provides limited items for a business cycle or the storage capacity of the retailer’s stock point is smaller than the actual requirement. Furthermore, shortages may arise in the inventory due to the fuzziness in demand and deterioration. In such cases, all the customers or some of them may wait till the arrival of the next stock or may not wait. Then, different backlogging rates are to be considered for shortages in the inventory depending upon the business environment. Thus, shortages and backlogging also affect the inventory cost.

In classical inventory models, researchers considered the demand, deterioration, and other constraints as deterministic in nature. These constraints are formulated by the experts based on existing data and other parameters. But, these data may inaccurate or inadequate to formulate the constraints of the model due to the impreciseness of available data and rapid changes in market conditions. For instance, the costs associated with inventory are fuzzy in nature due to uncertainty in global economic conditions. Thus, the optimal results obtained in classical inventory models may be inappropriate for their applications. That is, the ordering quantity, total cost, and cycle time are different from the actual requirements. As a result, the retailers may lose or get a reduction in their profit by applying the results of classical inventory models. Hence, it becomes a critical task for inventory managers to obtain optimal results for inventory models having imprecise costs and other parameters. In such scenarios, the fuzzy set theory is the best tool to deal with the impreciseness of costs and other parameters in inventory models. Initially, Lee and Yao [26] used the fuzzy concepts to deal with the impreciseness in demand and production quantity in their inventory model. Later on, the concepts of fuzzy set theory have been used by many researchers in developing inventory models involving the impreciseness of parameters to find more accurate optimal results.

In real business sectors, most of the inventories follow a trapezoidal demand pattern. That is, it has a growing demand at the beginning of a cycle, followed by a constant demand rate, and then declines towards end of the cycle. For instance, fashionable items, seasonal goods, new products, and many more have such types of demand. Moreover, the shortage of items and their fulfillment is quite natural in any business cycle. Thus, in relevance to these real scenarios, we first consider an inventory model for constant deteriorating items with trapezoidal type demand rate under fully backlogged shortages with deterministic costs and parameters. Furthermore, as the classical approach is not sufficient to deal with real-world inventory problems as discussed in the earlier paragraph, we devise the proposed inventory problem in a fuzzy environment by incorporating imprecise costs to obtain the optimal decisions. We have considered three different possible cases of shortages arising under trapezoidal demand in both crisp and fuzzy environments. Also, we present various numerical examples to discuss the effect of imprecise costs on the optimum results.

In view of our proposed investigation, the review of existing literature is presented in Section 2. Section 3, provides the assumptions and notations used in the model. Section 4 describes the formulation of the model in different scenarios. In Section 5, we explain the solution procedure for the crisp model. The corresponding
fuzzy model with its solution procedure is discussed in Section 6. The effect of imprecise costs on the optimal results is discussed through numerical illustrations in Section 7. Section 8 provides managerial insights into the sensitivity behavior of different parameters. Finally, Section 9 describes the brief conclusion with managerial suggestions and implications.

2. Literature review

The objective of inventory management is to fulfill the demand of the customers at any time under various constraints and minimize the total inventory cost. All the items in nature do not have the same kind of demand. So, many researchers developed inventory models for various demand items with different constraints under several settings intending for minimal cost and other perspectives.

In recent years, the constant demand for imperfect quality having constant deterioration of items were considered by Khanna et al. [20], Tiwari et al. [49], and Jaggi et al. [15] in their inventory models. But, Yu [52] considered constant demand for constant deteriorating items to obtain the optimal results for his inventory problem. The stock dependent demand inventory models corresponding to time dependent deteriorating items and constant deteriorating items were studied by Indrajitsingha et al. [13], Shaikh et al. [45] respectively. Further, the inventory problems with price dependent demand were discussed by Barik et al. [3] and Routray et al. [38] for deteriorating items under different scenarios. Whereas, both the stock and price dependent demand were assumed by Mishra et al. [33] for constant deteriorating items in their proposed inventory problem. Also, the optimal cost estimated for imperfect quality and constant deteriorating items when demand depends upon both time and stock in a business cycle were presented by Khurana [22]. However, the stock, price and time-dependent demand having constant deteriorating inventory problems were examined by Chen et al. [8]. Furthermore, Banerjee and Agarwal [5], Singh and Kumar [47], Jaggi et al. [16], and Indrajitsingha et al. [14] found optimal results for constant deteriorating items having selling price dependent demand. Nevertheless, the selling price dependent demand assumed by Sahoo et al. [40] for linear deteriorating items, and Khanna et al. [21] for imperfect quality and constant deteriorating items in their proposed problems. Moreover, both advertising cost and price dependent demand were discussed by Chanda and Kumar [7] for fashionable products, Shaikh et al. [43, 44] for Weibull and constant deteriorating items, and Kumar et al. [23] for defective products. Rajan and Uthayakumar [36], and Kaliraman et al. [18] developed inventory models for constant deteriorating items under exponentially growing demand. Also, the exponentially declining demand for an inventory of imperfect quality having constant deterioration was earlier taken into account by Jaggi et al. [17]. For different inventory models with different demand and deteriorations under various inventory constraints, the attention of interested readers is drawn towards the works of Barik et al. [1, 2], Routray et al. [37], and Mishra et al. [28–31].

The demand considered by different authors cited above is same throughout the cycle. But, in reality, it may not be the case for all the items in every cycle. Subsequently, some researchers considered ramp-type demand (that means, the demand increases up to a certain point of time and thereafter, it becomes stable) for their inventory problems. The ramp-type demand was considered by Sharma et al. [46] for constant deteriorating items and Chakraborty et al. [6] for Weibull deteriorating items. Furthermore, the trapezoidal-type demand (that is, in the cycle, the first phase has increased demand, the second phase has constant demand and the third phase has decreased demand) was considered by Mishra et al. [27], Wu et al. [50, 51] and Singh et al. [48] for their inventory models. Whereas, quadratic trapezoidal demand was taken by Debata et al. [9], and Price and Time dependent trapezoidal demand was assumed by Kaushik and Sharma [19].

The shortages in an inventory system affect the inventory cost as well as the business, which may lead to the loss in both the business and brand value of the product. In some cases, the shortages are filled partially or fully at the beginning of the subsequent cycle. In the literature discussed above, the researchers developed inventory models having shortages with different backlogging rates. Also, some inventory models are discussed without shortages. The details are outlined in the Table 1 below.

Most of the inventory parameters were taken as deterministic based on available past data. However, this data may inaccurate or inappropriate in the real sense. Moreover, due to various factors the associated costs under
Table 1. Summary of literature review.

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Year</th>
<th>Demand type</th>
<th>Deterioration type</th>
<th>Backlogging type</th>
<th>Fuzzy model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debata et al. [9]</td>
<td>2015</td>
<td>Quadratic trapezoidal</td>
<td>Constant</td>
<td>Partial</td>
<td>No</td>
</tr>
<tr>
<td>Khurana [22]</td>
<td>2015</td>
<td>Time and stock dependent</td>
<td>Time dependent</td>
<td>Completely backlogged</td>
<td>No</td>
</tr>
<tr>
<td>Mishra et al. [27]</td>
<td>2015</td>
<td>Trapezoidal-type</td>
<td>Constant</td>
<td>Lost sale</td>
<td>No</td>
</tr>
<tr>
<td>Rajan and Uthayakumar [36]</td>
<td>2015</td>
<td>Exponential increasing</td>
<td>Constant</td>
<td>No shortages</td>
<td>No</td>
</tr>
<tr>
<td>Shabani et al. [42]</td>
<td>2015</td>
<td>Fuzzy demand</td>
<td>Fuzzy deterioration</td>
<td>No shortages</td>
<td>Yes</td>
</tr>
<tr>
<td>Khanna et al. [20]</td>
<td>2016</td>
<td>Constant</td>
<td>Constant</td>
<td>Fully</td>
<td>No</td>
</tr>
<tr>
<td>Tiwari et al. [49]</td>
<td>2016</td>
<td>Constant</td>
<td>Constant</td>
<td>Partially backlogged</td>
<td>No</td>
</tr>
<tr>
<td>Wu et al. [50]</td>
<td>2016</td>
<td>Trapezoidal-type</td>
<td>Time-dependent</td>
<td>(i) No shortages</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(ii) Partial</td>
<td></td>
</tr>
<tr>
<td>Benerjee and Agrawal [5]</td>
<td>2017</td>
<td>Price dependent</td>
<td>Constant</td>
<td>Lost sale</td>
<td>No</td>
</tr>
<tr>
<td>Chanda and Kumar [7]</td>
<td>2017</td>
<td>Price and advertisement cost</td>
<td>No</td>
<td>No shortages</td>
<td>Yes</td>
</tr>
<tr>
<td>Jaggi et al. [16]</td>
<td>2017</td>
<td>Selling price</td>
<td>Constant</td>
<td>Completely</td>
<td>No</td>
</tr>
<tr>
<td>Jaggi et al. [15]</td>
<td>2017</td>
<td>Constant</td>
<td>Constant</td>
<td>No shortages</td>
<td>No</td>
</tr>
<tr>
<td>Kaliraman et al. [18]</td>
<td>2017</td>
<td>Exponential</td>
<td>Constant</td>
<td>No shortages</td>
<td>No</td>
</tr>
<tr>
<td>Khanna et al. [21]</td>
<td>2017</td>
<td>Selling price</td>
<td>Constant</td>
<td>Fully</td>
<td>No</td>
</tr>
<tr>
<td>Mishra et al. [33]</td>
<td>2017</td>
<td>Price and stock dependent</td>
<td>Constant</td>
<td>(i) Partial</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(ii) Fully</td>
<td></td>
</tr>
<tr>
<td>Sharma et al. [46]</td>
<td>2017</td>
<td>Ramp type</td>
<td>Constant</td>
<td>Partial</td>
<td>Yes</td>
</tr>
<tr>
<td>Chakraborty et al. [6]</td>
<td>2018</td>
<td>Ramp-type</td>
<td>Weibull</td>
<td>Partial</td>
<td>No</td>
</tr>
<tr>
<td>Indrajitsingha et al. [13]</td>
<td>2018</td>
<td>Stock dependent</td>
<td>Time dependent</td>
<td>No shortages</td>
<td>Yes</td>
</tr>
<tr>
<td>Jaggi et al. [17]</td>
<td>2018</td>
<td>Exponentially declining</td>
<td>Constant</td>
<td>Partially</td>
<td>No</td>
</tr>
<tr>
<td>Shaikh et al. [43]</td>
<td>2018</td>
<td>Advertisement</td>
<td>Constant</td>
<td>Partial</td>
<td>Yes</td>
</tr>
<tr>
<td>Singh and Kumar [47]</td>
<td>2018</td>
<td>Selling price</td>
<td>Constant</td>
<td>Partial</td>
<td>No</td>
</tr>
<tr>
<td>Wu et al. [51]</td>
<td>2018</td>
<td>Trapezoidal</td>
<td>Time dependent</td>
<td>Partial</td>
<td>No</td>
</tr>
<tr>
<td>Chen et al. [8]</td>
<td>2019</td>
<td>Stock, time, price dependent</td>
<td>Constant</td>
<td>No shortages</td>
<td>No</td>
</tr>
<tr>
<td>Indrajitsingha et al. [14]</td>
<td>2019</td>
<td>Selling price</td>
<td>Constant</td>
<td>Partial</td>
<td>Yes</td>
</tr>
<tr>
<td>Sahoo et al. [40]</td>
<td>2019</td>
<td>Selling price</td>
<td>Linear</td>
<td>Partial</td>
<td>No</td>
</tr>
<tr>
<td>Shaikh et al. [45]</td>
<td>2019</td>
<td>Stock dependent</td>
<td>Constant</td>
<td>Partial</td>
<td>No</td>
</tr>
<tr>
<td>Shaikh et al. [44]</td>
<td>2019</td>
<td>Price and advertisement cost</td>
<td>Weibull</td>
<td>Partial</td>
<td>No</td>
</tr>
<tr>
<td>Yu [52]</td>
<td>2019</td>
<td>Constant</td>
<td>Constant</td>
<td>Completely</td>
<td>No</td>
</tr>
<tr>
<td>Kaushik and Sharma [19]</td>
<td>2020</td>
<td>Price and time dependent Trapezoidal</td>
<td>Partial</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>Kumar et al. [24]</td>
<td>2020</td>
<td>Fuzzy</td>
<td>Fuzzy</td>
<td>No shortages</td>
<td>Yes</td>
</tr>
<tr>
<td>Singh et al. [48]</td>
<td>2021</td>
<td>Trapezoidal</td>
<td>Weibull</td>
<td>Completely</td>
<td>No</td>
</tr>
<tr>
<td>This paper</td>
<td></td>
<td>Trapezoidal</td>
<td>Constant</td>
<td>Fully</td>
<td>Yes</td>
</tr>
</tbody>
</table>
different inventory constraints may fuzzy in nature. That is to say, the introduction of substitute products, foreign products, etc. affects the demand. Similarly, the environmental conditions, available facilities, advancement of preservation technology, and other scientific methods have an impact on the deterioration rate. Moreover, the global economical condition affects the different costs involved in the inventory system. Thus, many researchers in the present era have considered the impreciseness of the parameters for the inventory requirements in the fuzzy environment. For some developments in this direction, see the recent works of Chanda and Kumar [7], Sharma et al. [46], Indrajitsingha et al. [12–14], Mishra et al. [32], Shaikh et al. [43], Kumar et al. [23–25], Nayak et al. [35], Routray et al. [39] and Shabani et al. [42]. Apart from these, the government and various companies are changing their policies from time to time, which also lead to fuzziness in inventory models. Also, while planning a project, the managers assess the risk factors of different components including inventory management. In this context, we refer the interested readers to the current works of Mitsumori [34], Seyedimany [41], and Baylan [4].

Motivated essentially by the above mentioned works, we develop here an inventory model for deteriorating items having trapezoidal type of demand function in both crisp and fuzzy environments by considering three possible cases of shortages which are completely backlogged. In the fuzzy environment, the imprecise costs are characterized by the trapezoidal fuzzy numbers, and the signed distance method is used for defuzzification of fuzzy total costs. Furthermore, in view of the comparative study of both scenarios, different data sets of constraints are examined for optimal results. Also, it is observed that the optimal results of the fuzzy model are more appropriate to real-world inventory problems. Finally, in order to strengthen the present investigation, the managerial insight of fluctuation in parameters is presented analytically via sensitivity analysis.

3. Assumptions and notations

3.1. Assumptions

(i) The infinite replenishment facility is available, thus the replenishment occurs instantaneously.

(ii) The associated costs are imprecise in nature.

(iii) The items in the inventory follows trapezoidal type demand \( R(t) \),

\[
R(t) = \begin{cases} 
    c_1 + d_1 t, & 0 \leq t \leq \nu_1, \\
    D, & \nu_1 \leq t \leq \nu_2, \\
    c_2 - d_2 t, & \nu_2 \leq t \leq T \leq \frac{c_2}{d_2}.
\end{cases}
\]

Here, the demand increases linearly in the first interval, is stable in the second interval and linearly decreases in the third interval. The demand changes from increasing to stable at the point \( \nu_1 \) and from stable to decreasing at \( \nu_2 \) (see Fig. 1).

(iv) The rate of deterioration of the items in the inventory is constant.

(v) Shortages occur in the inventory and are backlogged completely to the next cycle.

(vi) Imprecise costs are considered in the fuzzy environment and are characterized by trapezoidal fuzzy numbers.

3.2. Notations

(i) The inventory at any time \( t \), \( 0 \leq t \leq T \) is denoted by \( x(t) \).

(ii) The inventory becomes empty at time \( t = t_1 \).

(iii) The inventory cycle terminates at time \( t = T \).

(iv) The rate of deterioration of the items in the inventory is \( \eta \).

(v) The ordering cost per cycle is \( C_0 \).

(vi) The deterioration cost per unit item is \( k_1 \).

(vii) The holding cost per unit item per unit time is \( k_2 \).

(viii) The shortage cost per unit item per unit time is \( k_3 \).

(ix) The initial inventory of the cycle is \( \Pi \), that is, \( \Pi = x(0) \).
The optimal ordering quantity per cycle is $W$.

The total average cost per unit time is $T_c$.

The total average cost per unit time in the fuzzy environment is $ST_c$.

Costs in the fuzzy environment are $k_1$, $k_2$, and $k_3$ corresponding to the costs $k_1$, $k_2$, and $k_3$ in the crisp environment respectively.

4. FORMULATION OF MATHEMATICAL MODEL

Let the ordering quantity $W$ be received at time $t = 0$. After this, the back ordered quantity $B$ of the previous cycle is delivered to the customer and the inventory cycle starts with remaining $\Pi$ items. From beginning onwards the inventory depletes due to trapezoidal demand $R(t)$ and constant deterioration $\eta$ till it reaches zero at $t = t_1$. Then, the shortages occur in the inventory during the cycle from $t = t_1$ to $t = T$ and are back ordered to the next replenishment. The behavior of the inventory level $x(t)$ at any time during the cycle $[0, T]$ is explained by the following differential equations

\[
\frac{dx(t)}{dt} + \eta x(t) = -R(t) \quad (0 < t < t_1) \tag{4.1}
\]

and

\[
\frac{dx(t)}{dt} = -R(t) \quad (t_1 < t < T) \tag{4.2}
\]

with the condition $x(t_1) = 0$.

Depending upon the values of $t_1$, $\nu_1$, and $\nu_2$, there arise three cases:

**Case 1 ($0 < t_1 \leq \nu_1$)**

Due to trapezoidal demand and constant deterioration, the inventory level diminishes gradually and reaches zero at $t = t_1$ for $0 < t_1 \leq \nu_1$ (see Fig. 2). Then, the corresponding differential equations are

\[
\frac{dx(t)}{dt} + \eta x(t) = -(c_1 + d_1 t) \quad (0 < t < t_1) \tag{4.3}
\]

\[
\frac{dx(t)}{dt} = -(c_1 + d_1 t) \quad (t_1 < t < \nu_1) \tag{4.4}
\]
Figure 2. Inventory level $x(t)$ in Case 1 $(0 < t_1 \leq \nu_1)$.

\[
\frac{dx(t)}{dt} = -D \quad (\nu_1 < t < \nu_2)
\]

\[
\frac{dx(t)}{dt} = -(c_2 - d_2t) \quad (\nu_2 < t < T)
\]

with the boundary condition $x(t_1) = 0$.

The solutions of the above differential equations are

\[
x(t) = \left( \frac{c_1 + d_1 t_1}{\eta} - \frac{d_1}{\eta^2} \right) e^{\eta (t_1 - t)} - \left( \frac{c_1 + d_1 t}{\eta} - \frac{d_1}{\eta^2} \right) \quad (0 \leq t \leq t_1)
\]

\[
x(t) = c_1 t_1 - D t + \frac{d_1}{2} (t_1^2 + \nu_1^2) \quad (t_1 \leq t \leq \nu_1)
\]

\[
x(t) = c_1 t_1 - c_2 t + \frac{d_2}{2} (t^2 + \nu_2^2) + \frac{d_1}{2} (t_1^2 + \nu_1^2) \quad (\nu_1 \leq t \leq \nu_2)
\]

\[
x(t) = c_1 t_1 - c_2 t + \frac{d_2}{2} (t^2 + \nu_2^2) + \frac{d_1}{2} (t_1^2 + \nu_1^2) \quad (\nu_2 \leq t \leq T)
\]

Initial inventory

The inventory starts with $\Pi$ items, where

\[
\Pi = x(0) = \left( \frac{c_1}{\eta} - \frac{d_1}{\eta^2} \right) (e^{\eta t_1} - 1) + \frac{d_1}{\eta} t_1 e^{\eta t_1}.
\]

Deteriorating cost

The total deterioration cost during the cycle due to deterioration of items is $\eta_T$, where

\[
\eta_T = k_1 \left[ \Pi - \int_0^{t_1} R(t) \, dt \right] = k_1 \left[ \Pi - \int_0^{t_1} (c_1 + d_1 t) \, dt \right]
\]

\[
= k_1 \left[ \left( \frac{c_1}{\eta} - \frac{d_1}{\eta^2} \right) (e^{\eta t_1} - 1) + \frac{d_1}{\eta} t_1 e^{\eta t_1} - c_1 t_1 - \frac{d_1 t_1^2}{2} \right].
\]

Inventory carrying cost

The total carrying cost of the items during the interval $[0, t_1]$ is

\[
C_T = k_2 \left[ \int_0^{t_1} x(t) \, dt \right] = k_2 \left[ \left( \frac{c_1 + d_1 t_1}{\eta^2} - \frac{d_1}{\eta^3} \right) (e^{\eta t_1} - 1) - \left( \frac{(c_1 \eta - d_1) t_1}{\eta^2} + \frac{d_1 t_1^2}{2 \eta} \right) \right].
\]
Shortage cost

The opportunity cost due to shortage of items during the interval \([t_1, T]\) is

\[
S_T = k_3 \left[ - \int_{t_1}^{T} x(t) \, dt \right] = k_3 \left[ - \int_{t_1}^{\nu_1} x(t) \, dt - \int_{\nu_1}^{\nu_2} x(t) \, dt - \int_{\nu_2}^{T} x(t) \, dt \right]
\]

\[
= k_3 \left[ \frac{c_1}{2} (t_1 - \nu_1)(t_1 + \nu_1 - 2T) + \frac{d_1}{6} (2t_1^3 - 2\nu_1^3 + 3T\nu_1^2 - 3Tt_1^2) + \frac{c_2}{2} (\nu_2 - T)^2 \right.
\]

\[
+ \frac{d_2}{6} (3T\nu_2^2 - T^3 - 2\nu_2^3) + \frac{D}{2} (\nu_1 - \nu_2)(\nu_1 + \nu_2 - 2T) \right]. \tag{4.14}
\]

Total cost

The total average cost of the inventory is

\[
T_{c_1} = \frac{1}{T} \left\{ C_0 + \eta T + C_T + S_T \right\}
\]

\[
= \frac{1}{T} \left\{ C_0 + k_1 \left[ \left( \frac{c_1}{\eta} - \frac{d_1}{\eta^2} \right) (e^{\eta t_1} - 1) + \frac{d_1}{\eta} t_1 e^{\eta t_1} - c_1 t_1 - \frac{d_1 t_1^2}{2} \right] + k_2 \left[ \left( \frac{c_1 + d_1 t_1}{\eta^2} - \frac{d_1}{\eta^3} \right) (e^{\eta t_1} - 1) \right.
\]

\[
- \left( \frac{(c_1 - d_1)t_1}{\eta^2} + \frac{d_1 t_1^2}{2\eta} \right) \right] + k_3 \left[ \frac{c_1}{2} (t_1 - \nu_1)(t_1 + \nu_1 - 2T) + \frac{d_1}{6} (2t_1^3 - 2\nu_1^3 + 3T\nu_1^2 - 3Tt_1^2) \right.
\]

\[
+ \frac{c_2}{2} (\nu_2 - T)^2 + \frac{d_2}{6} (3T\nu_2^2 - T^3 - 2\nu_2^3) + \frac{D}{2} (\nu_1 - \nu_2)(\nu_1 + \nu_2 - 2T) \right]. \tag{4.15}
\]

Back ordered quantity

The amount of backlogging, which is to be fulfilled in subsequent cycle is

\[
B = \int_{t_1}^{T} R(t) \, dt = \int_{t_1}^{\nu_1} R(t) \, dt + \int_{\nu_1}^{\nu_2} R(t) \, dt + \int_{\nu_2}^{T} R(t) \, dt
\]

\[
= c_1(\nu_1 - t_1) + \frac{d_1}{2} (\nu_1^2 - t_1^2) + D(\nu_2 - \nu_1) + c_2(T - \nu_2) - \frac{d_2}{2} (T^2 - \nu_2^2). \tag{4.16}
\]

Economic ordering quantity

The total Economic Ordering Quantity for the inventory management is

\[
W_1 = \Pi + B + \left( \frac{c_1}{\eta} - \frac{d_1}{\eta^2} \right) (e^{\eta t_1} - 1) + \frac{d_1}{\eta} t_1 e^{\eta t_1}
\]

\[
+ c_1(\nu_1 - t_1) + \frac{d_1}{2} (\nu_1^2 - t_1^2) + D(\nu_2 - \nu_1) + c_2(T - \nu_2) - \frac{d_2}{2} (T^2 - \nu_2^2). \tag{4.17}
\]

Case 2 \((\nu_1 < t_1 \leq \nu_2)\)

Due to trapezoidal demand and constant deterioration, the inventory level diminishes gradually and reaches zero at \(t = t_1\) for \(\nu_1 < t_1 \leq \nu_2\) (see Fig. 3). Then, the corresponding differential equations are

\[
\frac{dx(t)}{dt} + \eta x(t) = -(c_1 + d_1 t) \quad (0 < t < \nu_1) \tag{4.18}
\]

\[
\frac{dx(t)}{dt} + \eta x(t) = -D \quad (\nu_1 < t < t_1) \tag{4.19}
\]

\[
\frac{dx(t)}{dt} = -D \quad (t_1 < t \leq \nu_2) \tag{4.20}
\]
Figure 3. Inventory level \(x(t)\) in Case 2 \((\nu_1 < t_1 \leq \nu_2)\).

\[
\frac{dx(t)}{dt} = -(c_2 - d_2 t) \quad (\nu_2 < t < T)
\]

with the boundary condition \(x(t_1) = 0\).

The solutions of the above differential equations are

\[
x(t) = \left(\frac{D e^{\eta t_1}}{\eta} - \frac{d_1 e^{\eta t_1}}{\eta^2}\right) e^{-\eta t} + \frac{d_1}{\eta^2} - \frac{c_1 + d_1 t}{\eta} \quad (0 \leq t \leq \nu_1) \tag{4.22}
\]

\[
x(t) = D \left(e^{\eta(t_1-t)} - 1\right) \quad (\nu_1 \leq t \leq t_1) \tag{4.23}
\]

\[
x(t) = D(t_1 - t) \quad (t_1 \leq t \leq \nu_2) \tag{4.24}
\]

\[
x(t) = D t_1 - c_2 t + \frac{d_2}{2} \left(t^2 + \nu_2^2\right) \quad (\nu_2 \leq t \leq T). \tag{4.25}
\]

Initial inventory

The inventory starts with \(\Pi\) items, where

\[
\Pi = x(0) = \frac{D e^{\eta t_1}}{\eta} - \frac{d_1 e^{\eta t_1}}{\eta^2} + \frac{d_1}{\eta^2} - \frac{c_1}{\eta}. \tag{4.26}
\]

Deteriorating cost

The total deterioration cost during the cycle due to deterioration of items is \(\eta_T\), where

\[
\eta_T = k_1 \left[\Pi - \int_0^{t_1} R(t) \, dt\right] = k_1 \left[\Pi - \int_0^{\nu_1} (c_1 + d_1 t) \, dt - \int_{\nu_1}^{t_1} D \, dt\right]
\]

\[
= k_1 \left[\frac{D e^{\eta t_1}}{\eta} - \frac{d_1 e^{\eta t_1}}{\eta^2} + \frac{d_1}{\eta^2} - \frac{c_1}{\eta} - D t_1 + \frac{d_1 \nu_2^2}{2}\right]. \tag{4.27}
\]

Inventory carrying cost

The total carrying cost of the items during the interval \([0, t_1]\) is

\[
C_T = k_2 \left[\int_0^{t_1} x(t) \, dt\right] = k_2 \left[\int_0^{\nu_1} x(t) \, dt + \int_{\nu_1}^{t_1} x(t) \, dt\right]
\]
\[ = k_2 \left[ \frac{D e^{\eta t}}{\eta^2} - \frac{d_1 e^{\nu_1 t}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1 + d_1}{\eta^2} + \frac{d_1}{2\eta} \right]. \] (4.28)

**Shortage cost**

The opportunity cost due to shortage of items during the interval \([t_1, T]\) is

\[ S_T = k_3 \left[ - \int_{t_1}^{T} x(t) \, dt \right] = k_3 \left[ - \int_{t_1}^{\nu_2} x(t) \, dt - \int_{\nu_2}^{T} x(t) \, dt \right] \]
\[ = k_3 \left[ \frac{D}{2} (\nu_2 - t_1)^2 + \frac{c_2}{2} (T - \nu_2)^2 + \frac{d_2}{6} (3T^2 - T^3) - 2\nu_2^3 \right] + D(\nu_2 - t_1)(T - \nu_2). \] (4.29)

**Total cost**

The total average cost of the inventory is

\[ T_{c2} = \frac{1}{T} \{ C_0 + \eta T + C_T + S_T \} \]
\[ = \frac{1}{T} \left\{ C_0 + k_1 \left[ \frac{D e^{\eta t}}{\eta^2} - \frac{d_1 e^{\nu_1 t}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1 + d_1}{\eta^2} + \frac{d_1}{2\eta} \right] + k_2 \left[ \frac{D e^{\nu_1 t}}{\eta^3} - \frac{d_1 e^{\nu_1 t}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1 + D t_1}{\eta^2} \right] + \frac{d_1}{2\eta} \right\} \]
\[ \left\{ k_3 \left[ \frac{D}{2} (\nu_2 - t_1)^2 + \frac{c_2}{2} (T - \nu_2)^2 + \frac{d_2}{6} (3T^2 - T^3) - 2\nu_2^3 \right] + D(\nu_2 - t_1)(T - \nu_2) \right\}. \] (4.30)

**Back ordered quantity**

The amount of backlogging, which is to be fulfilled in subsequent cycle is

\[ B = \int_{t_1}^{T} R(t) \, dt = \int_{t_1}^{\nu_2} R(t) \, dt + \int_{\nu_2}^{T} R(t) \, dt = D(\nu_2 - t_1) + c_2(T - \nu_2) = \frac{d_2}{2} (T^2 - \nu_2^2). \] (4.31)

**Economic Ordering Quantity**

The total Economic Ordering Quantity for the inventory management is

\[ W_2 = \Pi + B = \frac{D e^{\eta t}}{\eta} - \frac{d_1 e^{\nu_1 t}}{\eta^2} + \frac{d_1}{\eta^2} - \frac{c_1}{\eta} + D(\nu_2 - t_1) + c_2(T - \nu_2) - \frac{d_2}{2} (T^2 - \nu_2^2). \] (4.32)

**Case 3 (\(\nu_2 < t_1 \leq T\))**

Due to trapezoidal demand and constant deterioration, the inventory level diminishes gradually and reaches zero at \(t = t_1\) for \(\nu_2 < t_1 \leq T\) (see Fig. 4). Then, the corresponding differential equations are

\[ \frac{dx(t)}{dt} + \eta x(t) = -(c_1 + d_1 t) \quad (0 < t < \nu_1) \] (4.33)
\[ \frac{dx(t)}{dt} + \eta x(t) = -D \quad (\nu_1 < t < \nu_2) \] (4.34)
\[ \frac{dx(t)}{dt} + \eta x(t) = -(c_2 - d_2 t) \quad (\nu_2 < t \leq t_1) \] (4.35)
\[ \frac{dx(t)}{dt} = -(c_2 - d_2 t) \quad (t_1 < t < T) \] (4.36)

with the boundary condition \(x(t_1) = 0\).

The solutions of the above differential equations are

\[ x(t) = \left[ \left( \frac{c_2 - d_2 t_1}{\eta} \right) e^{\eta t} - \frac{d_1}{\eta^2} e^{\nu_1 t} - \frac{d_2}{\eta^2} e^{\nu_2 t} \right] e^{-\eta t} + \frac{d_1}{\eta^2} - \frac{c_1 + d_1 t}{\eta} \quad (0 \leq t \leq \nu_1) \] (4.37)
The total carrying cost of the items during the interval \([0, t_1]\) is given by:

\[
C_T = k_2 \left[ \int_0^{t_1} x(t) \, dt \right] = k_2 \left[ \int_0^{t_1} x(t) \, dt + \int_{\nu_1}^{\nu_2} x(t) \, dt + \int_{\nu_2}^{t_1} x(t) \, dt \right] = k_2 \left[ \left( \frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2} \right) e^{\eta_1} - \frac{d_1}{\eta^2} e^{\eta_1} - \frac{d_2}{\eta^2} e^{\nu_2} + \frac{1}{\eta} - c_1 \eta \right] + \frac{d_2 \nu_2^2}{2} - c_2 t_1 + \frac{d_2}{2} \left( t_1^2 + \nu_2^2 \right). \tag{4.43}
\]

**Shortage cost**

The opportunity cost due to shortage of items during the interval \([t_1, T]\) is given by:

\[
S_T = k_3 \left[ - \int_{t_1}^T x(t) \, dt \right] = k_3 \left[ \frac{c_2 (T - t_1)^2}{2} + \frac{d_2 t_2^2}{2} (T - t_1) + \frac{d_2}{6} (t_1^3 - T^3) \right]. \tag{4.44}
\]
Total cost

The total average cost of the inventory is

\[ T_{c3} = \frac{1}{T} \{ C0 + \eta T + C_T + S_T \} \]

\[ = \frac{1}{T} \left\{ C0 + k_1 \left[ \left( \frac{c_2 - d_2 t_1}{\eta^2} + \frac{d_2}{\eta^2} \right) e^{\eta t_1} - \frac{d_1}{\eta^2} e^{\eta v_1} - \frac{d_2}{\eta^2} e^{\eta v_2} + \frac{d_1 - c_1 \eta}{\eta^2} + \frac{d_1 \nu_1^2}{2} - c_2 t_1 + \frac{d_3}{2} (t_1^2 + \nu_2^2) \right] \right. \]

\[ + k_2 \left[ \left( \frac{c_2 - d_2 t_1}{\eta^2} + \frac{d_2}{\eta^2} \right) e^{\eta t_1} - \frac{d_1}{\eta^2} e^{\eta v_1} - \frac{d_2}{\eta^2} e^{\eta v_2} + \frac{d_1 - c_1 \eta}{\eta^2} + \frac{d_1 \nu_1^2}{2\eta} - c_2 t_1 + \frac{d_2}{2\eta} (t_1^2 + \nu_2^2) \right] \]

\[ + k_3 \left[ \frac{c_2}{2} (T - t_1)^2 + \frac{d_2 t_1^2}{2} (T - t_1) + \frac{d_2}{6} (t_1^3 - T^3) \right] \right\}. \quad (4.45) \]

Back ordered quantity

The amount of backlogging, which is to be fulfilled in subsequent cycle is

\[ B = \int_{t_1}^{T} R(t) \, dt = c_2 (T - t_1) - \frac{d_2}{2} (T^2 - t_1^2). \quad (4.46) \]

Economic Ordering Quantity

The total Economic Ordering Quantity for the inventory management is

\[ W_3 = \Pi + B = \left( \frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2} \right) e^{\eta t_1} - \frac{d_1}{\eta^2} e^{\eta v_1} - \frac{d_2}{\eta^2} e^{\eta v_2} + \frac{d_1 - c_1 \eta}{\eta^2} + c_2 (T - t_1) - \frac{d_2}{2} (T^2 - t_1^2). \quad (4.47) \]

5. SOLUTION OF THE MODEL

The following procedure is explained to get the minimum average cost of the model.

(i) Find \( t_1^{(1)} \) such that, \( \frac{dT_1}{dt_1} t_1^{(1)} = 0 \) and \( \frac{d^2T_1}{dt_1^2} t_1^{(1)} > 0 \).

(ii) Find \( t_1^{(2)} \) such that, \( \frac{dT_2}{dt_1} t_1^{(2)} = 0 \) and \( \frac{d^2T_2}{dt_1^2} t_1^{(2)} > 0 \).

(iii) Find \( t_1^{(3)} \) such that, \( \frac{dT_3}{dt_1} t_1^{(3)} = 0 \) and \( \frac{d^2T_3}{dt_1^2} t_1^{(3)} > 0 \).

(iv) If \( 0 \leq t_1^{(1)} \leq \nu_1 \), then calculate \( W_1 \left( t_1^{(1)} \right) \) and \( T_1 \left( t_1^{(1)} \right) \).

(v) If \( \nu_1 \leq t_1^{(2)} \leq \nu_2 \), then calculate \( W_2 \left( t_1^{(2)} \right) \) and \( T_2 \left( t_1^{(2)} \right) \).

(vi) If \( \nu_2 \leq t_1^{(3)} \leq T \), then calculate \( W_3 \left( t_1^{(3)} \right) \) and \( T_3 \left( t_1^{(3)} \right) \).

(vi) Let \( t_1^{(j)} = \arg \min \{ T_1 \left( t_1^{(1)} \right), T_2 \left( t_1^{(2)} \right), T_3 \left( t_1^{(3)} \right) \} \). Then, set \( t_1 = t_1^{(j)} \), \( W = W_j \left( t_1^{(j)} \right) \) and \( T_c = \min \{ T_1 \left( t_1^{(1)} \right), T_2 \left( t_1^{(2)} \right), T_3 \left( t_1^{(3)} \right) \} \).

Thus, \( t_1 \) is the optimal positive inventory time, \( W \) is the optimal economic ordering quantity and \( T_c \) is the optimal total inventory cost.

The Flow Chart of the crisp model is presented in the Figure 5.

6. FUZZY MODEL

Due to the globalization of business, many multinational companies (MNC) having different types of new products are opening their retail outlets in the existing and new marketplaces. As a result, space scarcity increases and it affect the holding cost. Also, due to the rapid growth in technological advancement and unpredictable climate conditions, the deterioration cost is fuzzy in nature. Similarly, different inventory costs are
Figure 5. Flow chart of crisp model.
imprecise in nature due to many social factors. Hence, we considered deteriorating cost \((k_1)\), holding cost \((k_2)\) and shortage cost \((k_3)\) as fuzzy parameters \(\bar{k}_1, \bar{k}_2\) and \(\bar{k}_3\) respectively. Then by proceeding in the similar fashions of the crisp model, we get the following fuzzy total costs:

Case 1 \((0 < t_1 \leq \nu_1)\)

\[
\overline{T_{c1}} = \frac{1}{T} \{C_0 + \eta T + C_T + S_T\} \\
= \frac{1}{T} \left\{ C_0 + \bar{k}_1 \left[ \left( \frac{c_1}{\eta} - \frac{d_1}{\eta^2} \right) (e^{\eta t_1} - 1) + \frac{d_1}{\eta} t_1 e^{\eta t_1} - c_1 t_1 - \frac{d_1 t_1^2}{2} \right] + \bar{k}_2 \left[ \left( \frac{c_1 + d_1 t_1}{\eta^2} - \frac{d_1}{\eta^3} \right) (e^{\eta t_1} - 1) \right. \right. \\
- \left. \left. \left( \frac{(c_1\eta - d_1) t_1}{\eta^2} + \frac{d_1 t_1^2}{2\eta} \right) \right] + \bar{k}_3 \left[ \frac{c_1}{2} (t_1 - \nu_1)(t_1 + \nu_1 - 2T) + \frac{d_1}{6} (2t_1^3 - 2\nu_1^3 + 3T \nu_1^2 - 3T t_1^2) \right. \right. \\
+ \left. \left. \frac{c_2}{2} (\nu_2 - T)^2 + \frac{d_2}{6} (3T \nu_2 - T^3 - 2\nu_2^3) + \frac{D}{2} (\nu_1 - \nu_2)(\nu_1 + \nu_2 - 2T) \right] \right\}. \tag{6.1}
\]

Case 2 \((\nu_1 < t_1 \leq \nu_2)\)

\[
\overline{T_{c2}} = \frac{1}{T} \{C_0 + \eta T + C_T + S_T\} \\
= \frac{1}{T} \left\{ C_0 + \bar{k}_1 \left[ \frac{D e^{\eta t_1}}{\eta^2} - \frac{d_1 e^{\eta t_1}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1}{\eta^2} + \frac{D t_1}{2\eta} \right] + \bar{k}_2 \left[ \frac{D e^{\eta t_1}}{\eta^2} - \frac{d_1 e^{\eta t_1}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1}{\eta^2} - \frac{D t_1}{\eta} \right. \right. \\
+ \left. \left. \frac{d_1 \nu_1^2}{2\eta} \right] + \bar{k}_3 \left[ \frac{D}{2} (\nu_2 - t_1)^2 + \frac{c_2}{2} (T - \nu_2)^2 + \frac{d_2}{6} (3T \nu_2 - T^3 - 2\nu_2^3) + D (\nu_2 - t_1)(T - \nu_2) \right] \right\}. \tag{6.2}
\]

Case 3 \((\nu_2 < t_1 \leq T)\)

\[
\overline{T_{c3}} = \frac{1}{T} \{C_0 + \eta T + C_T + S_T\} \\
= \frac{1}{T} \left\{ C_0 + \bar{k}_1 \left[ \left( \frac{c_2}{\eta^2} - \frac{d_2 t_1}{\eta} \right) + \frac{d_2}{\eta^3} \right] e^{\eta t_1} - \frac{d_1}{\eta^2} e^{\eta t_1} - \frac{d_2}{\eta^2} e^{\eta t_2} + \frac{d_1}{\eta^3} + \frac{d_1 \nu_1^2}{2} - c_2 t_1 + \frac{d_2}{2} (t_1^2 + \nu_2^2) \right. \\
+ \left. \bar{k}_2 \left[ \left( \frac{c_2}{\eta^2} - \frac{d_2 t_1}{\eta} \right) + \frac{d_2}{\eta^3} \right] e^{\eta t_1} - \frac{d_1}{\eta^2} e^{\eta t_1} - \frac{d_2}{\eta^2} e^{\eta t_2} + \frac{d_1}{\eta^3} + \frac{d_1 \nu_1^2}{2} - c_2 t_1 + \frac{d_2}{2} (t_1^2 + \nu_2^2) \right. \\
+ \left. \bar{k}_3 \left[ \frac{c_2}{2} (T - t_1)^2 + \frac{d_2 t_1^2}{2} (T - t_1) + \frac{d_2}{6} (t_1^3 - T^3) \right] \right\}. \tag{6.3}
\]

6.1. Defuzzification

The fuzzy parameters are characterized by trapezoidal fuzzy numbers as follows: \(\bar{k}_1 = (k_{11}, k_{12}, k_{13}, k_{14})\), \(\bar{k}_2 = (k_{21}, k_{22}, k_{23}, k_{24})\), and \(\bar{k}_3 = (k_{31}, k_{32}, k_{33}, k_{34})\). Using these trapezoidal fuzzy numbers and Signed Distance Method, the defuzzified total costs are:

\[
ST_{c1} = \frac{1}{4} [T_{c11} + T_{c12} + T_{c13} + T_{c14}] \quad (0 < t_1 \leq \nu_1) \tag{6.4}
\]

\[
ST_{c2} = \frac{1}{4} [T_{c21} + T_{c22} + T_{c23} + T_{c24}] \quad (\nu_1 < t_1 \leq \nu_2) \tag{6.5}
\]

\[
ST_{c3} = \frac{1}{4} [T_{c31} + T_{c32} + T_{c33} + T_{c34}] \quad (\nu_2 < t_1 \leq T). \tag{6.6}
\]
Here, $T_{cij}$ are obtained by replacing $\overline{k_1}$, $\overline{k_2}$, and $\overline{k_3}$ by $k_{1j}$, $k_{2j}$, and $k_{3j}$ in $T_{ci}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

Moreover, the solution of the fuzzy model can be obtained in similar lines to the crisp model, so we skip the details involved.

The Flow Chart of the fuzzy model is presented in the Figure 6.

7. Numerical illustration

Based on our proposed solution procedure and by using Mathematica 11.1.1 software, the numerical illustrations of both models are presented below.

Example 7.1 (Case 1 ($0 < t_1 \leq \nu_1$)).

(a) Crisp model

The values of the parameters involved in demand function are $c_1 = 750$, $d_1 = 50$, $c_2 = 1500$, $d_2 = 150$, and $D = 900$, the time at which demand pattern changes in the cycle are $\nu_1 = 3$, and $\nu_2 = 4$, the total cycle time is $T = 5$, the ordering cost is $C_0 = 2000$, the rate of deterioration of items in the inventory is $\eta = 0.32$, and the deterioration cost, holding cost, and shortage cost per unit item are $k_1 = 6$, $k_2 = 4$, and $k_3 = 8$ respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

$$
t_1^{(1)} = 2.37219 \quad W_1 = 5180.49 \quad T_{c1} = 8921.58 \\
t_1^{(2)} = 2.37219 \quad W_2 = 5167.81 \quad T_{c2} = 8912.79 \\
t_1^{(3)} = 2.37219 \quad W_3 = 4857.57 \quad T_{c3} = 8427.93,
$$

and the optimal solution: $t_1 = 2.37219 \quad W = 5180.49 \quad T_c = 8921.58$.

That is, on ordering $W = 5180.49$ items initially, the minimized total cost of inventory is $T_c = 8921.58$ and the inventory becomes empty at $t_1 = 2.37219$.

(b) Fuzzy model

The values of the parameters involved in demand function are $c_1 = 750$, $d_1 = 50$, $c_2 = 1500$, $d_2 = 150$, and $D = 900$, the time at which demand pattern changes in the cycle are $\nu_1 = 3$, and $\nu_2 = 4$, the total cycle time is $T = 5$, the ordering cost is $C_0 = 2000$, the rate of deterioration of items in the inventory is $\eta = 0.32$, and the fuzzy deterioration, holding and shortage costs are $\overline{k_1} = (k_{11}, k_{12}, k_{13}, k_{14}) = (5, 6, 7, 8)$, $\overline{k_2} = (k_{21}, k_{22}, k_{23}, k_{24}) = (3, 4, 5, 6)$, and $\overline{k_3} = (k_{31}, k_{32}, k_{33}, k_{34}) = (7, 8, 9, 10)$ respectively.

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

$$
t_1^{(1)} = 2.42686 \quad W_1 = 5235.44 \quad ST_{c1} = 8206.36 \\
t_1^{(2)} = 2.42686 \quad W_2 = 5224.65 \quad ST_{c2} = 8200.25 \\
t_1^{(3)} = 2.37219 \quad W_3 = 4857.57 \quad ST_{c3} = 7807.0,
$$

and the optimal solution: $t_1 = 2.42686 \quad W = 5235.44 \quad ST_c = 8206.36$.

That is, on ordering $W = 5235.44$ items initially, the minimized total cost of inventory is $ST_c = 8206.36$ and the inventory becomes empty at $t_1 = 2.42686$.

Example 7.2 (Case 1 ($0 < t_1 \leq \nu_1$)).
Figure 6. Flow chart of fuzzy model.
(a) Crisp model

The values of the parameters involved in demand function are \( c_1 = 450, d_1 = 5, c_2 = 522.5, d_2 = 15 \), and \( D = 462.5 \), the time at which demand pattern changes in the cycle are \( \nu_1 = 2.5 \) and \( \nu_2 = 4 \), the total cycle time is \( T = 5 \), the ordering cost is \( C_0 = 1800 \), the rate of deterioration of items in the inventory is \( \eta = 0.25 \), and the deterioration cost, holding cost, and shortage cost per unit item are \( k_1 = 13 \), \( k_2 = 7 \), and \( k_3 = 10 \) respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

\[
\begin{align*}
    t_1^{(1)} &= 2.12524 & W_1 &= 2600.03 & T_{c1} &= 6722.63 \\
    t_1^{(2)} &= 2.12524 & W_2 &= 2599.76 & T_{c2} &= 6722.38 \\
    t_1^{(3)} &= 2.12524 & W_3 &= 2573.37 & T_{c3} &= 6624.57,
\end{align*}
\]

and the optimal solution: \( t_1 = 2.12524 \quad W = 2600.03 \quad T_c = 6722.63 \).

That is, on ordering \( W = 2600.03 \) items initially, the minimized total cost of inventory is \( T_c = 6722.63 \) and the inventory becomes empty at \( t_1 = 2.12524 \).

(b) Fuzzy model

The values of the parameters involved in demand function are \( c_1 = 450, d_1 = 5, c_2 = 522.5, d_2 = 15 \), and \( D = 462.5 \), the time at which demand pattern changes in the cycle are \( \nu_1 = 2.5 \) and \( \nu_2 = 4 \), the total cycle time is \( T = 5 \), the ordering cost is \( C_0 = 1800 \), the rate of deterioration of items in the inventory is \( \eta = 0.25 \), and the fuzzy deterioration, holding and shortage costs are \( \overline{k_1} = (k_{11}, k_{12}, k_{13}, k_{14}) = (12, 13, 14, 15) \), \( \overline{k_2} = (k_{21}, k_{22}, k_{23}, k_{24}) = (6, 7, 8, 9) \), and \( \overline{k_3} = (k_{31}, k_{32}, k_{33}, k_{34}) = (9, 10, 11, 12) \) respectively.

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

\[
\begin{align*}
    t_1^{(1)} &= 2.11435 & W_1 &= 2596.52 & ST_{c1} &= 7068.56 \\
    t_1^{(2)} &= 2.11435 & W_2 &= 2596.24 & ST_{c2} &= 7068.28 \\
    t_1^{(3)} &= 2.12524 & W_3 &= 2573.37 & ST_{c3} &= 6963.36,
\end{align*}
\]

and the optimal solution: \( t_1 = 2.11435 \quad W = 2596.52 \quad ST_c = 7068.56 \).

That is, on ordering \( W = 2596.52 \) items initially, the minimized total cost of inventory is \( ST_c = 7068.56 \) and the inventory becomes empty at \( t_1 = 2.11435 \).

Example 7.3 (Case 2 \((\nu_1 < t_1 \leq \nu_2)\)).

(a) Crisp model

The values of the parameters involved in demand function are \( c_1 = 350, d_1 = 25, c_2 = 537.5, d_2 = 50 \), and \( D = 387.5 \), the time at which demand pattern changes in the cycle are \( \nu_1 = 1.5 \) and \( \nu_2 = 3 \), the total cycle time is \( T = 5 \), the ordering cost is \( C_0 = 1000 \), the rate of deterioration of items in the inventory is \( \eta = 0.2 \), and the deterioration cost, holding cost, and shortage cost per unit item are \( k_1 = 5 \), \( k_2 = 4 \), and \( k_3 = 8 \) respectively.
Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

\[ t_1^{(1)} = 2.7301 \quad W_1 = 2166.98 \quad T_{c1} = 3397.52 \]
\[ t_1^{(2)} = 2.7301 \quad W_2 = 2155.76 \quad T_{c2} = 3422.53 \]
\[ t_1^{(3)} = 2.7301 \quad W_3 = 2154.38 \quad T_{c3} = 3421.98, \]

and the optimal solution: \( t_1 = 2.7301 \quad W = 2155.76 \quad T_c = 3422.53 \).

That is, on ordering \( W = 2155.76 \) items initially, the minimized total cost of inventory is \( T_c = 3422.53 \) and the inventory becomes empty at \( t_1 = 2.7301 \).

(b) Fuzzy model

The values of the parameters involved in demand function are \( c_1 = 350, \, d_1 = 25, \, c_2 = 537.5, \, d_2 = 50, \) and \( D = 387.5 \), the time at which demand pattern changes in the cycle are \( \nu_1 = 1.5, \) and \( \nu_2 = 3, \) the total cycle time is \( T = 5, \) the ordering cost is \( C_0 = 1000, \) the rate of deterioration of items in the inventory is \( \eta = 0.2, \) and the fuzzy deterioration, holding and shortage costs are \( k_1 = (k_{11}, k_{12}, k_{13}, k_{14}) = (4, 5, 6, 7), \) \( k_2 = (k_{21}, k_{22}, k_{23}, k_{24}) = (3, 4, 5, 6), \) and \( k_3 = (k_{31}, k_{32}, k_{33}, k_{34}) = (7, 8, 9, 10) \) respectively.

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

\[ t_1^{(1)} = 2.67245 \quad W_1 = 2149.74 \quad ST_{c1} = 3697.66 \]
\[ t_1^{(2)} = 2.67245 \quad W_2 = 2139.76 \quad ST_{c2} = 3721.2 \]
\[ t_1^{(3)} = 2.7301 \quad W_3 = 2154.38 \quad ST_{c3} = 3722.55, \]

and the optimal solution: \( t_1 = 2.67245 \quad W = 2139.76 \quad ST_c = 3721.2 \).

That is, on ordering \( W = 2139.76 \) items initially, the minimized total cost of inventory is \( ST_c = 3721.2 \) and the inventory becomes empty at \( t_1 = 2.67245 \).

Example 7.4 (Case 3 (\( \nu_2 < t_1 \leq T \))).

(a) Crisp model

The values of the parameters involved in demand function are \( c_1 = 350, \, d_1 = 25, \, c_2 = 505, \, d_2 = 50, \) and \( D = 355, \) the time at which demand pattern changes in the cycle are \( \nu_1 = 0.2, \) and \( \nu_2 = 3, \) the total cycle time is \( T = 7, \) the ordering cost is \( C_0 = 1000, \) the rate of deterioration of items in the inventory is \( \eta = 0.2, \) and the deterioration cost, holding cost, and shortage cost per unit item are \( k_1 = 5, \, k_2 = 4, \) and \( k_3 = 8 \) respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

\[ t_1^{(1)} = 3.64581 \quad W_1 = 2794.7 \quad T_{c1} = 3588.26 \]
\[ t_1^{(2)} = 3.64581 \quad W_2 = 2695.41 \quad T_{c2} = 3997.43 \]
\[ t_1^{(3)} = 3.64581 \quad W_3 = 2685.12 \quad T_{c3} = 4003.21, \]

and the optimal solution: \( t_1 = 3.64581 \quad W = 2685.12 \quad T_c = 4003.21 \).

That is, on ordering \( W = 2685.12 \) items initially, the minimized total cost of inventory is \( T_c = 4003.21 \) and the inventory becomes empty at \( t_1 = 3.64581 \).
(b) Fuzzy model

The values of the parameters involved in demand function are $c_1 = 350$, $d_1 = 25$, $c_2 = 505$, $d_2 = 50$, and $D = 355$, the time at which demand pattern changes in the cycle are $\nu_1 = 0.2$, and $\nu_2 = 3$, the total cycle time is $T = 7$, the ordering cost is $C_0 = 1000$, the rate of deterioration of items in the inventory is $\eta = 0.2$, and the fuzzy deterioration, holding and shortage costs are $k_1 = (k_{11}, k_{12}, k_{13}, k_{14}) = (4, 5, 6, 7)$, $k_2 = (k_{21}, k_{22}, k_{23}, k_{24}) = (3, 4, 5, 6)$, and $k_3 = (k_{31}, k_{32}, k_{33}, k_{34}) = (7, 8, 9, 10)$ respectively.

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

$t_1^{(1)} = 3.56874 \quad W_1 = 2758.83 \quad ST_{c1} = 3945.71$

$t_1^{(2)} = 3.56874 \quad W_2 = 2666.48 \quad ST_{c2} = 4360.79$

$t_1^{(3)} = 3.56874 \quad W_3 = 2658.67 \quad ST_{c3} = 4365.08,$

and the optimal solution: $t_1 = 3.56874 \quad W = 2658.67 \quad ST_{c} = 4365.08$.

That is, on ordering $W = 2658.67$ items initially, the minimized total cost of inventory is $ST_{c} = 4365.08$ and the inventory becomes empty at $t_1 = 3.56874$.

**Example 7.5** (Case 3 ($\nu_2 < t_1 \leq T$)).

(a) Crisp model

The values of the parameters involved in demand function are $c_1 = 600$, $d_1 = 35$, $c_2 = 1155$, $d_2 = 130$, and $D = 635$, the time at which demand pattern changes in the cycle are $\nu_1 = 1$, and $\nu_2 = 4$, the total cycle time is $T = 8$, the ordering cost is $C_0 = 1600$, the rate of deterioration of items in the inventory is $\eta = 0.25$, and the deterioration cost, holding cost, and shortage cost per unit item are $k_1 = 5$, $k_2 = 3$, and $k_3 = 9$ respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

$t_1^{(1)} = 4.32281 \quad W_1 = 6467.17 \quad T_{c1} = 7624.64$

$t_1^{(2)} = 4.32281 \quad W_2 = 6220.7 \quad T_{c2} = 8140.97$

$t_1^{(3)} = 4.32281 \quad W_3 = 6208.04 \quad T_{c3} = 8142.91,$

and the optimal solution: $t_1 = 4.32281 \quad W = 6208.04 \quad T_{c} = 8142.91$.

That is, on ordering $W = 6208.04$ items initially, the minimized total cost of inventory is $T_{c} = 8142.91$ and the inventory becomes empty at $t_1 = 4.32281$.

(b) Fuzzy model

The values of the parameters involved in demand function are $c_1 = 600$, $d_1 = 35$, $c_2 = 1155$, $d_2 = 130$, and $D = 635$, the time at which demand pattern changes in the cycle are $\nu_1 = 1$, and $\nu_2 = 4$, the total cycle time is $T = 8$, the ordering cost is $C_0 = 1600$, the rate of deterioration of items in the inventory is $\eta = 0.25$, and the fuzzy deterioration, holding and shortage costs are $k_1 = (k_{11}, k_{12}, k_{13}, k_{14}) = (4, 5, 6, 7)$, $k_2 = (k_{21}, k_{22}, k_{23}, k_{24}) = (2, 3, 4, 5)$, and $k_3 = (k_{31}, k_{32}, k_{33}, k_{34}) = (8, 9, 10, 11)$ respectively.
Table 2. Effect of change in values of parameters on optimal results.

<table>
<thead>
<tr>
<th>Sensitivity of ( \eta )</th>
<th>Sensitivity of ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>( t_1 )</td>
</tr>
<tr>
<td>0.2</td>
<td>3.56874</td>
</tr>
<tr>
<td>0.3</td>
<td>3.16979</td>
</tr>
<tr>
<td>0.4</td>
<td>2.83815</td>
</tr>
<tr>
<td>0.5</td>
<td>2.56256</td>
</tr>
<tr>
<td>0.6</td>
<td>2.33206</td>
</tr>
</tbody>
</table>

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

\[
\begin{align*}
  t_1^{(1)} &= 4.47896 & W_1 &= 6703.25 & ST_{c_1} &= 6714.4 \\
  t_1^{(2)} &= 4.47896 & W_2 &= 6419.51 & ST_{c_2} &= 7253.55 \\
  t_1^{(3)} &= 4.32281 & W_3 &= 6208.04 & ST_{c_3} &= 7276.67,
\end{align*}
\]

and the optimal solution: \( t_1 = 4.32281 \) \( W = 6208.04 \) \( ST_c = 7276.67 \).

That is, on ordering \( W = 6208.04 \) items initially, the minimized total cost of inventory is \( ST_c = 7276.67 \) and the inventory becomes empty at \( t_1 = 4.32281 \).

Remark. In view of the above mentioned examples, we have the following outcomes.

(i) In Example 7.1, the values of positive inventory time \( t_1 \) and economic ordering quantity \( W \) of fuzzy model are greater than the crisp model. But, the total inventory cost \( ST_c \) in fuzzy model is less than the \( T_c \) of crisp model.

(ii) In Examples 7.2–7.4, the values of positive inventory time \( t_1 \) and economic ordering quantity \( W \) of fuzzy model are less than the crisp model. But, the total inventory cost \( ST_c \) in fuzzy model is greater than the \( T_c \) of crisp model.

(iii) In Example 7.5, the values of positive inventory time \( t_1 \) and economic ordering quantity \( W \) of both crisp and fuzzy models are equal. But, the total inventory cost \( ST_c \) in fuzzy model is less than the \( T_c \) of crisp model.

8. Sensitivity analysis

It is very important in an inventory system for a retailer to know the behavior of the system parameters which impact the optimal strategies. Thus, in order to illustrate the applicability of the model and to locate some significant managerial ramifications in the model, the sensitivity analysis with a variety of different parameters is to be carried out. In view of this, here we consider Example 7.4 to study the effect of different parameters on optimal results of the inventory.
Figure 7. $\eta$ vs. $t_1$.

Figure 8. $\eta$ vs. $W$.

Figure 9. $\eta$ vs. $ST_c$.

Figure 10. $T$ vs. $t_1$.

Figure 11. $T$ vs. $W$.

Figure 12. $T$ vs. $ST_c$. 
Figure 13. $\nu_1$ vs. $t_1$.

Figure 14. $\nu_1$ vs. $W$.

Figure 15. $\nu_1$ vs. $ST_c$.

Figure 16. $\nu_2$ vs. $t_1$.

Figure 17. $\nu_2$ vs. $W$.

Figure 18. $\nu_2$ vs. $ST_c$. 
From Table 2 and Figures 7–18, we draw the following conclusions:

(η) The increase of deterioration rate results, decrease in positive inventory time $t_1$, and increase in economic ordering quantity $W$ and total cost $ST_c$ in fuzzy environment (see Tab. 2 and Figs. 7–9).

(T) The increase of total cycle time $T$ results, an increase in positive inventory time $t_1$, economic ordering quantity $W$ and total cost $ST_c$ in fuzzy environment (see Tab. 2 and Figs. 10–12).

(ν₁) The increase in increasing demand period $ν_1$ results, increase in economic ordering quantity $W$ and total cost $ST_c$ both in fuzzy environment. But, there is no change in positive inventory time $t_1$ (see Tab. 2 and Figs. 13–15).

(ν₂) The increase in constant demand period $ν_2$ results, increase in economic ordering quantity $W$ and total cost $ST_c$ both in fuzzy environment. But, there is no change in positive inventory time $t_1$ (see Tab. 2 and Figs. 16–18).

Managerial insights

The following managerial insights have been found from the sensitivity examination of various parameters (refer Tab. 2).

(i) Increases in the rate of deterioration lead to a significant increase in inventory cost, order quantity, and the number of inventory shortages (refer Figs. 7–9). This result suggests the inventory managers should take extra precautions to prevent the deterioration from intensifying in order to keep the total inventory cost under control.

(ii) Inventory cost, order quantity, and positive inventory time all rise when the planning horizon is extended (refer Figs. 10–12). Hence, retailers must pay more attention to the duration of the planning horizon and acquire and store items accordingly in order to keep their business costs under control.

(iii) The cost of inventory and order quantity rise as the duration of linear growing demand or steady demand rises. But, there will be no change in positive inventory time (refer Figs. 13–18). This means that when the length of increasing demand or stable demand increases, retailers will need to order more inventory and have to spend more money on inventory upkeep to match the demand.

9. Conclusion

Inventory management of decaying products with trapezoidal demand is a crucial part of most of the firms in the present real-world business. On the other hand, the imprecision of various costs has a substantial impact on the inventory’s ideal performance. In this context, we built an inventory model with trapezoidal demand with constant deterioration in both crisp and fuzzy environments by integrating fully backlogged shortages. The crisp model is developed by considering deterministic costs, and the fuzzy model is developed by considering imprecised costs. Further, the impreciseness in costs was taken as trapezoidal fuzzy numbers, and the resultant total inventory costs was defuzzified by using the signed distance method. In addition, the procedure for obtaining optimal strategy was explained for both crisp and fuzzy situations. Thereafter, several inventory constraint sets were examined numerically for the validation of the proposed model in both scenarios. Further, the managerial insights have been drawn using sensitivity analysis to deal with the situations that may arise due to the variation in parameters in a business cycle. Moreover, we found the following managerial suggestions and managerial implications from the numerical experiments.

Managerial suggestions

The optimal strategies of both classical and fuzzy models are different for the same set of constraints. Because, the crisp model doesn’t consider the impreciseness of parameters where as the fuzzy model considers it. Further, it is evident from the numerical illustration that the difference in strategies of crisp and fuzzy models are not identical and are varies with constraint sets. That is, the values for inventory cost, shortage time point, and order quantity in the fuzzy model may less or more or equal to the corresponding crisp model depending upon
the constraint set. Thus, it is concluded that the optimal strategies obtained in the crisp model are suitable for inventory problems where the parameters are known with complete certainty and the optimal strategies obtained in the fuzzy model are suitable for inventory problems where the parameters are known with uncertainty. That means, the optimal strategy obtained in the crisp model is inadequate to implement in most of the real-life inventory problems as they have impreciseness in costs and other parameters. Hence, we suggest the inventory managers to adopt the fuzzy environment by taking account of impreciseness in costs and other parameters in order to get more accuracy in optimal strategy for real-world inventory problems.

Managerial implications

In the present paper, we have used trapezoidal fuzzy numbers to characterize the impreciseness of cost parameters, and the Signed Distance method was employed for defuzzification of fuzzified costs. But, the trapezoidal fuzzy numbers may not be adequate to quantify the impreciseness of parameters in all the scenarios. Also, other defuzzification methods may give better optimal solutions. However, there are a variety of fuzzy numbers to represent impreciseness, for instance, triangular, pentagonal, Hexagonal fuzzy numbers, etc. Also, there are different defuzzification methods available. Hence, researchers can use the work in this paper to obtain a more accurate optimal strategy for real-world inventory issues with varying restrictions by using appropriate fuzzy numbers and defuzzification techniques.

Moreover, it has been identified from the sensitivity analysis that the retailers have to order more quantities and spend more cost on inventory maintenance subject to the increase in parametric values. In view of some similar works (see, [27, 50, 51]) in this direction, our study enhance the scope for more real world inventory problems involving imprecise costs with completely backlogged shortages. Basically, the main contribution of this article is that, the fuzzy approach has been adopted to obtain the optimal decision policy for the proposed model with imprecise costs under three different cases.

Future research directions

The present model can be extended with no shortages, shortages without any backlogging for various inventory items under different demands and deteriorations. Again, one may consider the production inventory model under a similar approach. The model can further be extended by incorporating trade credit financing under various inventory constraints. Another possible extension can be done by implementing preservation techniques for deteriorating inventories.

REFERENCES


