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# OPTIMAL POLICIES FOR A DETERMINISTIC CONTINUOUS-TIME INVENTORY MODEL WITH SEVERAL SUPPLIERS: WHEN A SUPPLIER INCURS NO SET-UP COST

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**Abstract.** The subject is a deterministic continuous-time continuous-state inventory control model. Stock is replenished by ordering from one of a number of suppliers incurring a different cost per item and a different set-up cost. Taking the cost of procurement into account, the objective is to minimize the total discounted cost over an infinite planning horizon. The size of the order that is to be placed and the supplier with which it is to be placed are to be decided. Earlier studies of the problem have relied substantially on the assumption that the set-up cost of every supplier is strictly positive. Removing this restriction calls for a significant modification of the adopted approach. This is realized in the present study. It is shown that there is a stable unique optimal policy of a type that encompasses $(s, S)$ and generalized $(s, S)$ policies. Conditions that are necessary and sufficient for it to reduce to each of these types are established. The case of two suppliers is studied in detail, properties of the solution are investigated, numerical examples illustrating various aspects are included, and the connection with antecedent results is assessed.

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## 1. Introduction

An inventory can be defined as a stock of goods held for use in a production process, the provision of a service, or sale. These goods could be raw materials, components, consumable commodities, or finished products. Carrying such an inventory entails the cost of storage, handling, obsolescence, depreciation, deterioration, insurance, taxation, and miscellaneous other transactions. Not carrying an adequate inventory involves the cost of replenishing stock, lost sales, lost production, loss of good will, overtime, extraordinary administration, and other penalties that might be incurred. The goal of efficient inventory management is to balance these costs. Decisions have to be made on when best to replenish stock and the quantity of goods that should be ordered at these junctures. Mathematical modelling can assist in making these decisions [2, 9, 13, 19, 23, 30, 41].

The subject of the present paper is a mathematical model for an inventory comprising a single item. Shortages are simulated by the admission of negative inventory levels. The model may be categorized as deterministic, continuous-time, and continuous-state. In the absence of intervention, the inventory level decreases according

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to an evolution process in which changes in inventory level are governed by a differential equation. The level of stock is continuously monitored, and the cost of holding stock or maintaining a backlog is prescribed as a function of the inventory level. Stock can be replenished by ordering from several available suppliers, each of which offers an unlimited supply. Ordering from each supplier incurs a fixed cost per item and a fixed set-up cost. The objective is to minimize the total cost of procurement, holding the inventory, and permitting shortages over an infinite planning horizon. The challenges of managing an inventory when there are multiple potential suppliers have been reviewed in [10, 22, 33, 36].

When there is a single supplier, the decision problem reduces to the search for an optimal policy of a type that is well-documented, namely an $(s, S)$ policy. This sets two stock levels, $s$ and $S > s$. If the inventory level is greater than $s$, a manager does not intervene. If it is $s$ or less, the manager orders to bring the inventory level up to $S$. An $(s, S)$ policy was introduced by Arrow *et al.* [1]. The further historical development can be traced to [2, 20, 31, 35, 40]. More contemporary exposés are to be found in [8, 28]. The complication with the model with several suppliers is that determining an optimal management policy means not only finding the inventory level from which a replenishment should take place, and the quantity of goods that should be ordered, but also identifying the supplier with which the order should be placed.

Given any two suppliers, the first of which incurs both a greater cost per item and a greater set-up cost than the other, an order would naturally be placed with the second. This can also be said if only one of these costs is greater and the other is the same. By the same token, given any two suppliers incurring the same cost per item and set-up cost, it is economically immaterial with which an order would be placed. The essence of the problem is that given any two suppliers, one incurs a lesser cost per item and a greater set-up cost than the other. This enables the suppliers to be ranked in strictly increasing order of cost per item and strictly decreasing order of set-up cost, or *vice versa*.

The model considered was proposed in [5], where, under the assumption that the set-up cost of every supplier is strictly positive, it was shown that the following alternatives are mutually exclusive.

– There is a unique optimal $(s, S)$ policy involving only one predetermined supplier.
– There is a unique optimal generalized $(s, S)$ policy involving more than one supplier.
– There is no optimal generalized $(s, S)$ policy, let alone an optimal $(s, S)$ policy.

A generalized $(s, S)$ policy involves $N$ suppliers and stock levels

$$s_{(N)} < s_{(N-1)} < \cdots < s_{(1)} < S_{(1)} < S_{(2)} < \cdots < S_{(N)} \tag{1.1}$$

for some natural number $N \geq 2$. If the inventory level is greater than $s_{(1)}$, one does not replenish stock. If it is between $s_{(n+1)}$ and $s_{(n)}$ for $n$ from 1 to $N-1$, one orders from supplier $(n)$ up to the level $S_{(n)}$. If it is less than $s_{(N)}$ one orders from supplier $(N)$ up to the level $S_{(N)}$. This policy may deliberately exclude a selection of the available suppliers. For instance, it could involve just three of five available suppliers, with supplier (1) being number 4 in the original ranking, supplier (2) being number 3, supplier (3) being number 1, and, suppliers 2 and 5 dispensed with. A generalized $(s, S)$ policy was first proposed as a viable optimal inventory-control policy by Porteus [26, 27]. Further discourse can be found in [5, 6, 8, 28].

Building upon the investigation of the model in [5] and with retention of the assumption of strict positivity of every set-up cost, it has since been shown [6] that in the event of a generalized $(s, S)$ policy not being optimal, nonetheless there is a unique optimal policy. This has a feature not previously documented, and has been termed a *hyper-generalized* $(s, S)$ *policy*. Like a generalized $(s, S)$ policy, it entails $N$ suppliers and stock levels (1.1). In addition, it contains stock levels $r_{(n)}$, where

$$s_{(n+1)} \leq r_{(n)} \leq s_{(n)} \quad \text{for } 1 \leq n \leq N-1, \tag{1.2}$$

for which the following applies. If the inventory level is greater than $s_{(1)}$, one does not replenish, as with a generalized $(s, S)$ policy. If it is between $r_{(n)}$ and $s_{(n)}$ for some $n$ indicated, one orders from supplier $(n)$ up to the level $S_{(n)}$. However, if it is between $s_{(n+1)}$ and $r_{(n)}$ for some such $n$, one again does not intervene, just as if the inventory level were greater than $s_{(1)}$. Regardless, if the inventory level is less than $s_{(N)}$, one orders from supplier $(N)$ up to the level $S_{(N)}$, once more as with a generalized $(s, S)$ policy. Such a hyper-generalized $(s, S)$ policy reduces to a generalized $(s, S)$ policy when $r_{(n)} = s_{(n+1)}$ for every $n$ from 1 to $N - 1$.

According to the above definition of a hyper-generalized $(s, S)$ policy with $N$ suppliers, when $r_{(i)} = r_{(i-1)}$ for some $i \in \{2, 3, \dots, N-1\}$, this policy is indistinguishable from that with $N-1$ suppliers in which the inventory levels $r_{(i)}$ and $s_{(i)}$ have been removed and supplier $(n+1)$ has become supplier $(n)$ for $n$ from $i$ to $N-1$. Hence, in addition to (1.1) and (1.2), it can be taken that

$$r_{(N-1)} < r_{(N-2)} < \cdots < r_{(1)}. \tag{1.3}$$

In everyday terms, a hyper-generalized $(s, S)$ policy arises when there are backlogging levels for which it does not pay to replenish in excess of the conventional cut-off level. If the backlog were less, there would be an optimal course of action in placing an order with a supplier incurring a relatively low set-up cost. However, the high cost per item of such a supplier is prohibitive. On the other hand, if the backlog were greater, there would be an optimal course of action in placing an order with a supplier incurring a relatively low cost per item. However, the high set-up cost of a supplier in this category prohibits this too. The best thing to do is to let the backlog accumulate further until such time that the amount that has to be ordered is so great that is it is indeed worthwhile to replenish from a supplier incurring a low cost per item, whereby the high set-up cost can be set off against the size of the order.

To recapitulate, under the assumption that the set-up cost of every supplier is strictly positive:

– When there is no optimal generalized $(s, S)$ policy, there is a unique optimal hyper-generalized $(s, S)$ policy.

The present paper extends the preceding results to the situation that the set-up cost of one of the available suppliers may be negligible. One may think of a number of suppliers at different locations, whereby a supplier offering a lesser cost per item is situated further away from the customer, entailing a greater delivery cost. In this scenario, an order from a supplier located in close proximity to the customer may incur no transportation cost. Alternatively, when a supplier and the customer are subsidiaries of the same company, transportation of goods from one to the other may be merely a bookkeeping transaction or be covered by sundry overheads.

Extension of the current results to the situation that one of the available suppliers may incur a negligible set-up cost is not achievable by simply considering this to be the limit of the case that the supplier has a positive set-up cost. It requires substantive development of the theory expounded in [5,6], and adaptation of the notion of a generalized and a hyper-generalized $(s, S)$ policy. In realizing this, the earlier results will be expanded and consolidated with novel results in a unifying framework.

The problem with a stochastic demand and two suppliers, one of which incurs a negligible set-up cost and the other a significant set-up cost and lesser cost per item, has been investigated heretofore by Fox *et al.* [11]. They concluded that when the sale of unsatisfied demand is lost, an optimal inventory-control policy must be one of three types. The first is an $(s, S)$ policy involving only the supplier with a significant set-up cost. The second is a base policy involving only the supplier with a negligible set-up cost. The third is a mixed-ordering policy involving both suppliers. When excess demand is back-ordered, an optimal inventory-control policy is necessarily of the first or third type. In the present setting, the base policy can be viewed as a degenerate $(s, S)$ policy in which $s = S$, while the mixed-ordering policy can be seen as a degenerate generalized $(s, S)$ policy with $N = 2$ in which $s_{(2)} < s_{(1)} = S_{(1)} < S_{(2)}$, supplier (1) is the supplier with no set-up cost, and supplier (2) is the supplier with a significant set-up cost. We shall show that the admissible alternatives are either a conventional $(s, S)$ policy involving only the supplier with a significant set-up cost, a degenerate generalized $(s, S)$ policy as just described, or a (degenerate) hyper-generalized $(s, S)$ policy with $N = 2$, $s_{(2)} < r_{(1)} \le s_{(1)} = S_{(1)} < S_{(2)}$, and a like configuration of suppliers.

More recently, the problem with a stochastic demand, several suppliers, one of which may incur a negligible set-up cost, periodic review, and a finite planning horizon has been studied by Benjaafar *et al.* [4]. They concluded that for each period, except for a bounded interval of inventory levels, a generalized $(s, S)$ policy is optimal, and, provided an explicit example demonstrating that this result is best possible as far as the exceptional interval of inventory levels is concerned. They further reported extensive numerical experiments testing the ancillary problem with an infinite planning horizon. The latter discrete-time problem has since been studied by Helal *et al.* [15]. For the case of two available suppliers, they have established conditions under which an $(s, S)$ policy involving the supplier with the greater set-up cost is optimal, and, in the event that the demand distribution is exponential, antithetical conditions under which a generalized $(s, S)$ involving both suppliers is optimal.

The counterpart to the problem studied in the present paper in which the objective is to minimize the long-term average cost of procurement, holding the inventory, and permitting shortages in continuous time has been examined for a deterministic demand by Perera *et al.* [24] and for a stochastic demand by a number of authors [3, 14, 16–18, 25, 38, 39]. The prevailing conclusion is that the optimal inventory control policy is given by an $(s, S)$ policy, which may degenerate into a singular policy with $s = S$ when a supplier incurs no set-up cost.

The organization of the remainder of the paper is as follows. Section 2 reviews the formal statement of the inventory control problem. Section 3 elucidates the difficulty in expanding the current theory in which ordering from every supplier incurs a positive set-up cost to the situation in which ordering from a supplier may incur a negligible set-up cost. Sections 4–7 subsequently develop a theory that encompasses both situations in a comprehensive setting. Incidentally, this simplifies and consolidates aspects of the earlier analysis. This part of the paper culminates in the establishment that the inventory control problem with or without a supplier with a negligible set-up cost has a stable unique solution which depends monotonically on the number of suppliers and the costs of each supplier. Section 8 deals with the determination of the conditions under which, with a slight adjustment of the notions, this solution corresponds to an $(s, S)$ policy, a generalized $(s, S)$ policy, or a hyper-generalized $(s, S)$ policy. Section 9 subsequently delves deeper into the occurrence of these policies for the specific case of two suppliers. Computation of the solution and the attendant optimal policy is the subject of Section 10, which includes numerical examples. The connection between the present results and those of Fox *et al.* [11], Benjaafar *et al.* [4] and Helal *et al.* [15] is discussed in Section 11. The paper closes with a conclusion constituting Section 12. A list of notation is included as Appendix A.

## 2. Problem statement

A stock of a single item is considered, whereby the level of stock at time $t$ is $x(t)$. A level $x \geq 0$ corresponds to the number of items held. A level $x < 0$ indicates a shortage of $-x$ items. In the absence of intervention, changes in the stock level are governed by the evolution equation

$$\dot{x}(t) = -G(x(t)), \tag{2.1}$$

where $G$ is a positive continuous function defined on $\mathbb{R}$ accounting for stock-dependent demand and deterioration of items. If $x$ satisfies (2.1) then

$$\tilde{x} = \int_0^x \frac{\mathrm{d}\eta}{G(\eta)}$$

satisfies the equation

$$\dot{x}(t) = -1. \tag{2.2}$$

Conversely, if $\tilde{x}$ is governed by (2.2) and

$$\int_0^x \frac{\mathrm{d}\eta}{G(\eta)} \to \pm\infty \quad \text{as } x \to \pm\infty \tag{2.3}$$

then the reverse transformation leads to (2.1). The condition (2.3) is fulfilled by the commonly used expressions for $G$, which are encapsulated in the generic expression $G(x) = g_0 + g_1 \max\{x, 0\} + g_\beta \max\{x, 0\}^\beta$ for some $g_0 > 0$, $g_1 \geq 0$, $g_\beta \geq 0$ and $0 < \beta < 1$ [12, 34]. Hence, with nominal loss of generality, it is supposed that $x$ evolves according to (2.2).

To replenish stock there are $J$ available suppliers. Placing an order with supplier $j \in \mathcal{J}$, where

$$\mathcal{J} = \{1, 2, \ldots, J\},$$

entails a fixed cost $c_j$ per item and set-up cost $k_j$. The suppliers are ordered so that

$$c_1 < c_2 < \cdots < c_J \tag{2.4}$$

and

$$k_1 > k_2 > \cdots > k_J \geq 0. \tag{2.5}$$

The running cost is given by a continuous nonnegative function $f$ defined on $\mathbb{R}$. This amalgamates the diverse costs of maintaining an inventory when there is stock in hand, and those in the nature of an incurred penalty when there is a shortage. A commonly-used expression is

$$f(x) = \begin{cases} -px & \text{for } x < 0 \\ qx & \text{for } x \geq 0, \end{cases} \tag{2.6}$$

where $p > 0$ and $q > 0$ are constants [3,4,8,11,14,15,19,21,23,28,30,32].

Variability of money-value in time is considered by the exponential discount of costs at a constant rate

$$\alpha > 0. \tag{2.7}$$

This is a well-established method of discounting costs which can be traced to [29] and is a component of the continuous-time inventory models in [7,8,13,15,16,21,28,32,41].

Under the above conditions [5,6], the optimal total future cost $u(x)$ associated with a level of stock $x$ that has evolved according to equation (2.2) will satisfy the equation $\big(Au\big)(x) = f(x)$, where

$$\big(Au\big)(x) = u'(x) + \alpha u(x).$$

On the other hand, placing an order of size $\xi$ with supplier $j$ when the level of stock is $x$ will incur a cost of $k_j + c_j\xi$ and bring the inventory level up to $x + \xi$. Hence, $\xi$ should minimize $k_j + c_j\xi + u(x+\xi)$ [3,4,8,11,15,21,26,28,32]. In other words, one should have $u(x) = \big(M_j u\big)(x)$, where

$$\big(M_j u\big)(x) = k_j + \min\{u(x+\xi) + c_j\xi : \xi \geq 0\}. \tag{2.8}$$

Consequently, defining

$$\big(Mu\big)(x) = \min\big\{\big(M_j u\big)(x) : j \in \mathcal{J}\big\}, \tag{2.9}$$

an optimal impulse control policy is a solution of the quasi-variational inequality (QVI)

$$\begin{cases} Au \leq f \\ u \leq Mu \\ (Au - f)(u - Mu) = 0. \end{cases} \tag{2.10}$$

## 3. The approach

The key to the successful resolution of many a sophisticated mathematical problem is the notion of a solution of the problem. A restrictive notion makes it difficult to find a solution. With regard to applications, this may result in only being able to establish that there is a solution in limited circumstances. On the other hand, a lax notion makes it difficult to exclude a multiplicity of solutions. This can lead to an inconclusive outcome with regard to applications. There is tradeoff in posing a credible notion. Concerning the problem in hand, one would like a notion of a solution of (2.10) that delivers the kind of optimal inventory control policy that one would intuitively expect and can be applied in practice, and, at the same time, is backed by a conclusive theory.

In the earlier paper devoted to (2.10) with strictness in the rightmost inequality in (2.5) [5], the notion of an admissible solution had too narrow a scope to yield a solution under all circumstances. In the later paper [6], widening the scope led to a successful resolution of the problem. The expanded notion is the following.

**Ansatz 3.1.** The solution of (2.10) is a continuous real function $u$ such that $u = Mu$ in $(-\infty, s] \setminus \mathcal{S}$, where $\mathcal{S}$ is the union of a finite number of bounded open subintervals of $(-\infty, s)$, and, $u$ is differentiable, $Au = f$, and $u < Mu$ in $\mathcal{S} \cup (s, \infty)$, for some number $s$.

The precursor to Ansatz 3.1 limited the set $\mathcal{S}$ to the empty set, or, if one so prefers, the number of subintervals comprising $\mathcal{S}$ to zero. Given such a limitation, the ansatz partitions $\mathbb{R}$ into a stopping region $(-\infty, s]$ in which $u = Mu$, and a continuation region $(s, \infty)$ in which $u < Mu$ and $Au = f$. This is a primary feature of an $(s, S)$ policy and a generalized $(s, S)$ policy. Permitting $\mathcal{S}$ to comprise one or more nonempty bounded open subintervals opened the way to the realization of a hyper-generalized $(s, S)$ policy.

When the rightmost inequality in (2.5) is not strict, the theorem below reveals that Ansatz 3.1 is wanting. The proof of this theorem is given in Appendix B.

**Theorem 3.2.** *Suppose that $k_J = 0$. Let $u$ be a real function with the property that $Mu$ is well defined in $\mathbb{R}$. Then $u \leq Mu$ in $\mathbb{R}$ if and only if $u = Mu = M_J u$ everywhere in $\mathbb{R}$.*

Theorem 3.2 implies that when $k_J = 0$, asking that a solution $u$ of (2.10) be such that $u = Mu$ in a subset of $\mathbb{R}$ is superfluous. Conversely, asking that a solution $u$ be such that $u < Mu$ in a subset of $\mathbb{R}$ disqualifies it.

To deal with the failure of Ansatz 3.1 when $k_J = 0$, we propose the following notion covering $k_J > 0$ and $k_J = 0$.

**Ansatz 3.3.** The solution of (2.10) is a continuous real function $u$ with the following properties. The set $\Omega$ of $x \in \mathbb{R}$ for which

$$u(x) = k_\ell + u(x + \xi) + c_\ell \xi \quad \text{for some } \xi > 0 \text{ and } \ell \in \mathcal{J} \tag{3.1}$$

is not empty, $\Omega$ has a finite least upper bound $s$, and $\mathcal{S} = (-\infty, s) \setminus \Omega$ is the union of a finite number of bounded open intervals. Furthermore, $u$ is differentiable and $Au = f$ at the left endpoint of any subinterval of $\mathcal{S} \cup (s, \infty)$.

The theorem below, whose proof is given in Appendix C, confirms that this new ansatz is equivalent to the preceding one when $k_J > 0$.

**Theorem 3.4.** *Suppose that $k_J > 0$. Then $u$ is a solution of (2.10) satisfying Ansatz 3.1 if and only if it is a solution of (2.10) satisfying Ansatz 3.3.*

Armed with Ansatz 3.3, we adopt the strategy previously used to tackle the inventory control problem in [5, 6]. We start, under the mere assumption that $f$ is continuous, by extracting characteristics of a solution of (2.10) that are concealed in the ansatz. This is the subject of the next section. In the subsequent section we instate the hypothesis that led to the successful resolution of the problem when $k_J > 0$, and show that under this hypothesis the QVI has at most one solution with the extracted characteristics (therewith proving the uniqueness of a solution). In the section thereafter, we construct a solution embodying these characteristics (therewith proving existence). A supplementarily section shows that the solution found is stable with respect to perturbations of (2.5), and depends monotonically on $J$ and the components of (2.4) and (2.5).

## 4. Preliminary characterization

Inherently a solution of (2.10) satisfying Ansatz 3.3 embodies a number of features. The most significant of these are summarized in the next theorem. The proof of its forerunner in [6] relies heavily on the assumption that $k_J > 0$, and cannot be easily modified to contend with $k_J = 0$. The search for an alternative has inadvertently uncovered a proof for $k_J \geq 0$ which is simpler than the previous one for $k_J > 0$. This is delivered in Appendix D.

**Theorem 4.1.** *Suppose that (2.4) and (2.5) hold and $f$ is continuous on $\mathbb{R}$. Let $u$ be a solution of (2.10) satisfying Ansatz 3.3. Then $u = y$ in $[s, \infty)$, where $y$ is a solution of the differential equation*

$$y' + \alpha y = f \quad \textit{in} \quad \mathbb{R} \tag{4.1}$$

*satisfying*

$$y'(s) = y'(S) = -c_j, \tag{4.2}$$

$$y(s) = y(S) + k_j + c_j(S - s) \tag{4.3}$$

*and*

$$y(s) = (M_j y)(s) = (My)(s) \tag{4.4}$$

*for some* $j \in \mathcal{J}$ *and* $S \geq s$. *Furthermore,*

$$y \leq u \leq v \text{ in } (-\infty, s), \quad \text{and} \quad u < v \text{ in } \mathcal{S},$$

*where*

$$v(x) = \min\{(M_\ell y)(s) + c_\ell(s - x) : \ell \in \mathcal{J}\}. \tag{4.5}$$

Theorem 4.1 seems to be as far as one can take the characterization of a solution of (2.10) satisfying Ansatz 3.3 without further hypotheses on the function $f$ or the numbers (2.4), (2.5) and (2.7). To progress, we impose the hypothesis under which it was shown in [5] that when each supplier $\ell$ is the sole supplier, the QVI has a unique solution corresponding to an $(s, S)$ policy no matter how large the set-up cost. This reads as follows.

**Hypothesis 4.2.** The function $f_\ell$ defined by

$$f_\ell(x) = f(x) + \alpha c_\ell x \quad \text{for} \quad x \in \mathbb{R},$$

is continuous in $\mathbb{R}$, strictly decreasing in $(-\infty, \gamma_\ell]$, and strictly increasing in $[\gamma_\ell, \infty)$ for some $\gamma_\ell \in \mathbb{R}$, $f_\ell(x) \to \infty$ as $x \to -\infty$, and

$$\int_{\gamma_\ell}^{\infty} e^{\alpha\eta} \, df_\ell(\eta) \geq -\int_{-\infty}^{\gamma_\ell} e^{\alpha\eta} \, df_\ell(\eta).$$

The above inequality is automatically satisfied when $f_\ell(x) \to \infty$ as $x \to \infty$.

Compatibility of (2.4) and (2.7) with the assumption that Hypothesis 4.2 holds for every $\ell \in \mathcal{J}$ requires

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_J. \tag{4.6}$$

**Remark 4.3.** For the archetypical function (2.6), Hypothesis 4.2 holds for every $\ell \in \mathcal{J}$ if and only if

$$p > \alpha c_J \quad \text{and} \quad q > -\alpha c_1, \tag{4.7}$$

under which circumstance $\gamma_\ell = 0$ for every $\ell \in \mathcal{J}$.

The condition (4.7) in one form or another can be found in [8, 11, 15, 21, 26, 28, 32].

## 5. Uniqueness

Given Hypothesis 4.2, we can state the following.

**Lemma 5.1.** *Equation* (4.1) *has a solution satisfying* (4.2) *and* (4.3) *with* $j = \ell$ *for some* $S \geq s$ *if and only if* $S = S_\ell$ *and* $s = s_\ell$ *for a unique ordered pair* $(s_\ell, S_\ell) \in \mathbb{R}^2$. *The numbers* $S_\ell$ *and* $s_\ell$ *are the unique solution of the simultaneous equations*

$$\int_{s_\ell}^{S_\ell} e^{\alpha\eta} \, df_\ell(\eta) = 0 \quad \text{and} \quad f_\ell(s_\ell) = f_\ell(S_\ell) + \alpha k_\ell \tag{5.1}$$

*satisfying*

$$s_\ell \leq \gamma_\ell \leq S_\ell, \tag{5.2}$$

*depend continuously and strictly monotonically on* $k_\ell \geq 0$*,* $S_\ell = s_\ell = \gamma_\ell$ *when* $k_\ell = 0$*, and* $s_\ell \to -\infty$ *when* $k_\ell \to \infty$*. Furthermore, the concurrent solution* $y_\ell$ *of equation* (4.1) *is unique, expressible as*

$$y_\ell(x) = \frac{1}{\alpha}\left\{ f(x) + c_\ell - \mathrm{e}^{-\alpha x} \int_{s_\ell}^{x} \mathrm{e}^{\alpha\eta} \, \mathrm{d} f_\ell(\eta) \right\}, \tag{5.3}$$

*depends continuously and strictly decreasingly on* $s_\ell \leq \gamma_\ell$*, and is such that*

$$y_\ell' < -c_\ell \quad \textit{in} \quad (s_\ell, S_\ell), \quad \textit{and} \quad y_\ell' > -c_\ell \quad \textit{in} \quad (-\infty, s_\ell) \cup (S_\ell, \infty). \tag{5.4}$$

*Proof.* Allowing for the omission of $k_\ell = 0$, expression (5.3) and the dependency of $y_\ell$ on $s_\ell$, the lemma can be found in Section 3 of [5]. Augmentation of the proof accommodates the omissions. With no loss of generality, it can be supposed that $c_\ell = 0$. By extension of Lemma 3.4 of [5], there exists a unique function $\varphi_\ell : (-\infty, \gamma_\ell] \to [\gamma_\ell, \infty)$ such that (4.1) admits a solution $y$ satisfying (4.2) with $j = \ell$ for $s \leq S$ if and only if $s \leq \gamma_\ell$ and $S = \varphi_\ell(s)$. Furthermore, $\varphi_\ell(\gamma_\ell) = \gamma_\ell$, $\varphi_\ell$ is continuous and strictly decreasing, $y$ is unique, $y' < 0$ in $(s, S)$, and $y' > 0$ in $(-\infty, s) \cup (S, \infty)$. Subsequently, condition (4.3) with $j = \ell$ can be formulated as $F_\ell(s) = k_\ell$, where $F_\ell(s) = \{f_\ell(s) - f_\ell(\varphi_\ell(s))\}/\alpha$. By extension of Lemma 3.5 of [5], $F_\ell(\gamma_\ell) = 0$, $F_\ell(s) \to \infty$ as $s \to -\infty$, and $F_\ell$ is continuous and strictly decreasing in $(\infty, \gamma_\ell]$. The upshot is Lemma 5.1 saving (5.3) and the dependency of $y_\ell$ on $s_\ell$. Expression (5.3) can be found by solving (4.1) subject to the initial condition $y'(s_\ell) = -c_\ell$. This gives

$$y_\ell(x) = \mathrm{e}^{-\alpha x} \left\{ \frac{f(s_\ell) + c_\ell}{\alpha} \, \mathrm{e}^{\alpha s_\ell} + \int_{s_\ell}^{x} \mathrm{e}^{\alpha\eta} f(\eta) \, \mathrm{d}\eta \right\}.$$

Expressing the integrand in terms of $f_\ell$ rather than $f$, and applying the formula for integration by parts of Riemann–Stieltjes delivers (5.3). The continuous and monotonic dependence of $y_\ell$ on $s_\ell$ is a consequence. □

**Remark 5.2.** When $f$ assumes the classical form (2.6) and (4.7) holds, the right-hand equation in (5.1) can be solved for $S_\ell$ explicitly, yielding

$$S_\ell = -\{\alpha k_\ell + (p - \alpha c_\ell) s_\ell\}/(q + \alpha c_\ell). \tag{5.5}$$

Consequently, $S_\ell$ can be eliminated from the left-hand equation in (5.1), making $s_\ell$ the unique solution of the transcendental equation

$$\alpha(p - \alpha c_\ell) s_\ell + (q + \alpha c_\ell) \ln\big\{ [p + q - (p - \alpha c_\ell) \mathrm{e}^{\alpha s_\ell}]/(q + \alpha c_\ell) \big\} + \alpha^2 k_\ell = 0 \tag{5.6}$$

in $(-\infty, 0]$. Formula (5.3) gives

$$y_\ell(x) = \begin{cases} \big[p(1 - \alpha x) - (p - \alpha c_\ell) \mathrm{e}^{\alpha(s_\ell - x)}\big]/\alpha^2 & \text{for } x \leq 0 \\ \big[q(\alpha x - 1) + (q + \alpha c_\ell) \mathrm{e}^{\alpha(S_\ell - x)}\big]/\alpha^2 & \text{for } x > 0. \end{cases} \tag{5.7}$$

Supposing that Hypothesis 4.2 holds for every $\ell \in \mathcal{J}$, Lemma 5.1 supplies the existence of a unique solution $y_\ell$ of equation (4.1) satisfying (4.2) and (4.3) with $j = \ell$ for some $S \geq s$ and the uniqueness of the accompanying numbers, $S_\ell$ and $s_\ell$, for every $\ell$. Theorem 4.1 subsequently tells us that (2.10) has a solution $u$ satisfying Ansatz 3.3 only if $s = s_j$, $u = y_j$ in $[s_j, \infty)$, and $u \geq y_j$ in $(-\infty, s_j)$ for some $j \in \mathcal{J}$. The task ahead is to identify $j$.

We begin the quest for $j$ with an observation.

**Lemma 5.3.** *Given any* $j \in \mathcal{J}$ *and* $\ell \in \mathcal{J}$*, either* $y_j < y_\ell$ *and* $y_j' > y_\ell'$ *in* $\mathbb{R}$*,* $y_j > y_\ell$ *and* $y_j' < y_\ell'$ *in* $\mathbb{R}$*, or,* $y_j \equiv y_\ell$ *in* $\mathbb{R}$*.*

*Proof.* The function $y_j - y_\ell$ is a solution of $y' + \alpha y = 0$ in $\mathbb{R}$. Therefore, $\big(y_j - y_\ell\big)(x) = \mathcal{C}\mathrm{e}^{-\alpha x}$ for all $x \in \mathbb{R}$, for some constant $\mathcal{C}$. The trichotomy follows from whether $\mathcal{C}$ is negative, positive or zero. □

**Remark 5.4.** When $f$ takes the prototypical form (2.6) and (4.7) holds, formula (5.7) implies that $y_j < y_\ell$ if and only if $\alpha s_j + \ln(p - \alpha c_j) > \alpha s_\ell + \ln(p - \alpha c_\ell)$.

Lemma 5.3 is relevant to the following.

**Lemma 5.5.** *Suppose that (2.10) has a solution $u$ satisfying Ansatz 3.3 such that $u = y_j$ in $[s_j, \infty)$, and $u \geq y_j$ in $(-\infty, s_j)$ for some $j \in \mathcal{J}$. Let $\ell \in \mathcal{J} \setminus \{j\}$. Then $y_j \leq y_\ell$. Moreover, $y_j = y_\ell$ only if $c_j > c_\ell$ or $k_\ell = 0$.*

*Proof.* When $k_\ell > 0$, it can be shown that $y_j \leq y_\ell$ with equality only if $c_j > c_\ell$, following the proof of Lemma 4.11 of [5]. To deal with the case $k_\ell = 0$, suppose, contrarily, that $y_j > y_\ell$. Then, by Lemma 5.3, $y'_j < y'_\ell$ in $\mathbb{R}$. Since Lemma 5.1 says that $y'_\ell(\gamma_\ell) = -c_\ell$, this gives $y'_j(\gamma_\ell) < -c_\ell$. If $c_\ell \geq c_j$, it follows that $y'_j(\gamma_\ell) < -c_j$. Hence, by (5.4) for $\ell = j$, $\gamma_\ell \geq s_j$. On the other hand, if $c_\ell < c_j$, then $\gamma_\ell \geq \gamma_j$ by (4.6), and $\gamma_j \geq s_j$ by (5.2) with $\ell = j$. So either way, $\gamma_\ell \geq s_j$. This means that $u$ is differentiable at $\gamma_\ell$, and $u'(\gamma_\ell) = y'_j(\gamma_\ell) < -c_\ell$, which contradicts Lemma B.1 in Appendix B. Therefore, by *reductio ad absurdum*, $y_j \leq y_\ell$. ☐

When $k_J > 0$, Lemma 5.5 leads to the conclusion that $j$ must be the greatest minimizer of $y_\ell$ with respect to $\ell \in \mathcal{J}$. This identifies $j$ precisely. When $k_J = 0$ such a definitive conclusion cannot be drawn. Should there be several minimizers, one of which is $J$, the lemma fails to narrow down the selection beyond $J$ and the second greatest minimizer. This presents yet another hurdle in extending the theory from $k_J > 0$ to $k_J \geq 0$. Fortunately, it is the last.

Providentially, we can continue the development merely supposing that $j$ is a minimizer.

**Lemma 5.6.** *Suppose that $j$ is a minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. Then $M_\ell y_j$ is well defined in $\mathbb{R}$ for every $\ell \in \mathcal{J}$. Moreover,*

$$y'_j < -c_\ell \text{ in } (s_j, S_{j,\ell}), \quad \text{and} \quad y'_j > -c_\ell \text{ in } (S_{j,\ell}, \infty) \tag{5.8}$$

*for a sequence of numbers*

$$S_{j,j} = S_j < S_{j,j-1} < S_{j,j-2} < \cdots < S_{j,1}.$$

*Each $S_{j,\ell}$ with $1 \leq \ell \leq j$ is the unique number in $[S_j, \infty)$ for which*

$$\int_{S_j}^{S_{j,\ell}} \mathrm{e}^{\alpha\eta} \, \mathrm{d}f_\ell(\eta) = (c_j - c_\ell)\mathrm{e}^{\alpha S_j}. \tag{5.9}$$

*Proof.* See the proof of Lemmas 4.12 and 4.16 of [5] for the main result, and Corollary 4.20 of [5] for the formula

$$\int_{S_j}^{S_{j,\ell}} \mathrm{e}^{\alpha\eta} \, \mathrm{d}f_j(\eta) = (c_j - c_\ell)\mathrm{e}^{\alpha S_{j,\ell}}. \tag{5.10}$$

Substitution of $f_j(\eta) = f_\ell(\eta) + \alpha(c_j - c_\ell)\eta$ in (5.10) yields (5.9). ☐

**Remark 5.7.** In the specific case that $f$ is given by (2.6) and (4.7) holds, formula (5.9) gives

$$S_{j,\ell} = S_j + \ln\bigl\{(q + \alpha c_j)/(q + \alpha c_\ell)\bigr\}/\alpha. \tag{5.11}$$

In the light of Theorem 4.1, Lemma 5.6 reveals that (4.5) can be more succinctly expressed

$$v(x) = \min\bigl\{v_\ell(x) : 1 \leq \ell \leq j\bigr\}, \tag{5.12}$$

where

$$v_\ell(x) = y_j(S_{j,\ell}) + k_\ell + c_\ell(S_{j,\ell} - x). \tag{5.13}$$

Considering (5.12) and (5.13) as definitions for all $x \in \mathbb{R}$, the function $v$ is piecewise-linear and concave in $\mathbb{R}$. Consequently, $v$ has a right derivative $D_+v$ and a left derivative $D_-v$ everywhere in $\mathbb{R}$. Furthermore, there is a partition

$$\sigma_1 = s_j > \sigma_2 > \sigma_3 > \cdots > \sigma_{\mathsf{N}} \tag{5.14}$$

such that $v$ is affine in each interval

$$I_1 = (\sigma_2, \sigma_1),\ I_2 = (\sigma_3, \sigma_2),\ \ldots,\ I_{\mathsf{N}} = (-\infty, \sigma_{\mathsf{N}}). \tag{5.15}$$

Defining $\mathsf{N}$ as the smallest number for which such a partition exists, this partition is unique. Moreover, since (2.4), (5.12) and (5.13) imply that $v(x)/x \to -c_1$ as $x \to -\infty$,

$$v = v_1 \text{ in } I_{\mathsf{N}}.$$

The function $v$ has additional relevant properties stated in the two lemmas below, in which

$$\mathcal{M} = \{1, 2 \ldots, \mathsf{N}\}. \tag{5.16}$$

**Lemma 5.8.** *Let $x \le S_j$. Then $v(x) \ge y_j(x)$ with equality if and only if $y_j = y_\ell$ and $x = s_\ell$ for some $\ell \in \{1, 2, \ldots, j\}$, in which event $v$ is differentiable at $s_\ell$, $v(s_\ell) = v_\ell(s_\ell)$, and $v'(s_\ell) = v'_\ell(s_\ell) = y'_j(s_\ell) = -c_\ell$.*

*Proof.* The leading statement can be verified by the argument used to prove Lemma 4.24 of [6]. With regard to the subsidiary statement, suppose that $s_\ell < S_j$ and $v(s_\ell) = y_j(s_\ell)$. Then, by the leading statement, $s_\ell$ is a minimum of $v - y_j$ in $(-\infty, S_j)$. Hence, $D_+v - y'_j \ge 0$ and $D_-v - y'_j \le 0$ at $s_\ell$. Inasmuch the concavity of $v$ implies $D_+v \le D_-v$ everywhere, it follows that $v$ is differentiable and $v' = y'_j$ at $s_\ell$. When $s_\ell = S_j$ and $v(s_\ell) = y_j(s_\ell)$, we may likewise deduce that $D_-v - y'_j \le 0$ at $s_\ell$. Hence, $\big(D_-v\big)(s_\ell) \le y'_j(s_\ell) = y'_j(S_j) = -c_j$. However, by (2.4), (5.12) and (5.13), $D_-v \ge D_+v \ge -c_j$ everywhere. Thus, in this case too, $v$ is differentiable and $v' = y'_j$ at $s_\ell$. Subsequently, in both cases, $v'(s_\ell) = y'_j(s_\ell) = y'_\ell(s_\ell) = -c_\ell$. Given the structure of $v$, this leads to $v(s_\ell) = v_\ell(s_\ell)$ and $v'(s_\ell) = v'_\ell(s_\ell)$. □

**Lemma 5.9.** *The combination $f - Av$ is strictly decreasing in $I_m$ for every $m \in \mathcal{M}$. Moreover, $f - Av$ has no upper bound in $I_{\mathsf{N}}$.*

*Proof.* Fix $m \in \mathcal{M}$, and let $\ell \in \{1, 2, \ldots, j\}$ be such that $v = v_\ell$ in $I_m$. Then, by substitution, it can be verified that $f - Av = f_\ell + \mathcal{C}$ in $I_m$ for some constant $\mathcal{C}$. By Hypothesis 4.2, $f_\ell$ is strictly decreasing in $(-\infty, \gamma_\ell]$, and $f_\ell(x) \to \infty$ as $x \to -\infty$. Furthermore, by (4.6), (5.2), and (5.14), $I_m \subset (-\infty, \gamma_\ell]$. Together, these observations confirm the lemma. □

From Lemma 5.9 it follows that $Av \le f$ in $I_m$ for $m \in \mathcal{M}$ if and only if $T_m \ge 0$, where

$$T_m = \big(f - D_-v - \alpha v\big)(\sigma_m). \tag{5.17}$$

By (5.14) and Lemma 5.8,

$$T_1 = \big(f - Ay_j\big)(s_j) = 0. \tag{5.18}$$

Last but not least, the next lemma concerning $v$ is of importance. For clarification, in the statement of this lemma, $Av < f$ is taken in the standard sense in $I_m$ for $m \in \mathcal{M}$. However, at $\sigma_m$ for $m \in \mathcal{M} \setminus \{1\}$, it is to be interpreted as $D_+v + \alpha v < D_-v + \alpha v \le f$. By the above, this is equivalent to $T_m \ge 0$.

**Lemma 5.10.** *There exists a uniquely defined nonnegative integer $L$ and uniquely defined*

$$a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_L < b_L \le a_{L+1} = s_j \tag{5.19}$$

*such that $Av < f$ in $(-\infty, a_1)$, $v$ is differentiable and $Av = f$ at $a_\nu$, $v > Y_\nu$ in $(a_\nu, b_\nu)$, where*

$$Y_\nu \text{ is the unique solution of (4.1) satisfying } Y_\nu(a_\nu) = v(a_\nu), \tag{5.20}$$

*$v = Y_\nu$ at $b_\nu$, and $Av < f$ in $(b_\nu, a_{\nu+1})$ for $\nu = 1, 2, \ldots, L$.*

*Proof.* We refer to [6] for the proof of this lemma, which is a synopsis of Lemmas 4.12–4.14 and Subsection 4.3 prior to Lemma 4.25 of [6]. ☐

In the light of the preceding considerations, $L = 0$ if and only if $T_m \geq 0$ for every $m \in \mathcal{M} \setminus \{1\}$. Furthermore, when $L \geq 1$, every interval $(a_\nu, b_\nu)$ with $1 \leq \nu \leq L$ contains a $\sigma_m$ for which $T_m < 0$. Indeed the least $\sigma_m$ in said interval possesses this property. Complementarily, no interval $[b_\nu, a_{\nu+1}]$ nor $(-\infty, a_1]$ contains such a $\sigma_m$.

The crux of the above is the next lemma.

**Lemma 5.11.** *Continuing on from Lemma 5.10, define*

$$\mathcal{S}_j = (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_L, b_L), \tag{5.21}$$

$$\Omega_j = (-\infty, s_j] \setminus \mathcal{S}_j, \tag{5.22}$$

*and $U_j$ on $\mathbb{R}$ by*

$$U_j = y_j \text{ in } (s_j, \infty), \quad U_j = v \text{ in } \Omega_j, \quad \text{and} \quad U_j = Y_\nu \text{ in } (a_\nu, b_\nu) \text{ for } 1 \leq \nu \leq L.$$

*Suppose that $u$ is a solution of (2.10) satisfying Ansatz 3.3 with $s = s_j$, $u = y_j$ in $[s_j, \infty)$, and $u \geq y_j$ in $(-\infty, s_j)$. Then $u = U_j$ and $\mathcal{S} = \mathcal{S}_j$.*

*Proof.* See Subsection 4.2 of [6]. ☐

Let us recapitulate. We have shown that any solution $u$ of (2.10) satisfying Ansatz 3.3 has the properties stated in Theorem 4.1 for some $j \in \mathcal{J}$. By Lemma 5.5, we know that $j$ must be a minimizer of $y_\ell$ with respect to $\ell \in \mathcal{J}$, and, consequently, by Lemma 5.11, that $u = U_j$. It follows that if $\{y_\ell : \ell \in \mathcal{J}\}$ has a unique minimizer then (2.10) has at most one solution satisfying Ansatz 3.3. If not, rather than attempt to whittle down the number of minimizers by some further means, we adopt a different tactic. We show that whatever the choice of the minimizer, we end up with the same function $u$. The next three lemmas do the trick.

**Lemma 5.12.** *Let $j$ and $\theta < j$ be minimizers of $\{y_\ell : \ell \in \mathcal{J}\}$. Then $s_\theta < s_j \leq S_j < S_\theta$.*

*Proof.* By (5.4) with $\ell = j$, $y_j \leq -c_j$ in $[s_j, S_j]$. Hence, recalling (2.4), $y_\theta = y_j \leq -c_j < -c_\theta$ in $[s_j, S_j]$. By (5.4) with $\ell = \theta$, this necessitates $[s_j, S_j] \subset (s_\theta, S_\theta)$. ☐

**Lemma 5.13.** *Further to Lemma 5.12, let $v$ be the function defined by (5.8), (5.12) and (5.13), $\mathsf{N}$ be the smallest natural number for which there is a partition (5.14) with the property that $v$ is affine in each of the intervals (5.15), and $T_m$ be given by (5.17) for $m \in \mathcal{M}$. Then $s_\theta \in I_m$ and $T_m < 0$ for some $m \in \mathcal{M} \setminus \{1\}$.*

*Proof.* Lemma 5.8 implies that $v$ is differentiable at $s_\theta$. So, $s_\theta \in I_m$ for some $m \in \mathcal{M}$. Furthermore, Lemma 5.8 states that

$$v(s_\theta) = v_\theta(s_\theta) = y_\theta(s_\theta) = y_j(s_\theta) \tag{5.23}$$

and $v'(s_\theta) = y_j'(s_\theta)$. Hence, $\big(f - Av\big)(s_\theta) = \big(f - Ay_j\big)(s_\theta) = 0$. Lemma 5.9 subsequently yields $T_m < 0$. Recalling (5.18), this excludes $m = 1$. ☐

**Lemma 5.14.** *Further to Lemma 5.12, $U_j \equiv U_\theta$ and $U_\theta = y_j$ in $[s_\theta, \infty)$.*

*Proof.* With no loss of generality, we may suppose that $\theta$ and $j$ are consecutive minimizers of $\{y_\ell : \ell \in \mathcal{J}\}$. Retaining the notation $v$ for the function defined by (5.8), (5.12) and (5.13), and the notation $L$ and (5.19) for the numbers given by Lemma 5.10, we add an asterisk superscript to their counterparts when $j$ is replaced by $\theta$. By Lemma 5.12, $s_\theta < s_j$. So $U_j = y_j = y_\theta = U_\theta$ in $[s_j, \infty)$, and $S_{j,\ell} = S_{\theta,\ell}$ for $1 \leq \ell \leq \theta$. This means that we can dispense with notation to distinguish between the functions (5.13) entering the construction of $U_j$ and $U_\theta$. By (2.4), (5.12), (5.13) and (5.23),

$$\big(v_\ell - v_\theta\big)(x) = \big(v_\ell - v_\theta\big)(s_\theta) + (c_\ell - c_\theta)(s_\theta - x) \geq \big(v_\ell - v_\theta\big)(s_\theta) \geq \big(v - v_\theta\big)(s_\theta) = 0 \quad \text{for } \theta \leq \ell \leq j$$

and $x \le s_\theta$. Consequently, $v = \min\{v_\ell : 1 \le \ell \le j\} = \min\{v_\ell : 1 \le \ell \le \theta\} = v^*$ in $(-\infty, s_\theta]$. This leads to $U_j = U_\theta$ in $(-\infty, s_\theta]$. Whereupon, Lemma 5.8 and (5.23) necessitate $s_\theta = a_{L^*+1}$. Therewith, there is no need to distinguish the functions (5.20) given by Lemma 5.10 for $v$ and $v^*$. Furthermore, $L > L^*$, $a_{L^*+1} = a^*_{L^*+1} = s_\theta < s_j$, and, $Y_{L^*+1}(s_\theta) = y_\theta(s_\theta) = y_j(s_\theta)$. The uniqueness of solutions of the initial-value problem for equation (4.1) subsequently implies that $Y_{L^*+1} \equiv y_j$. However, by Lemmas 5.8 and 5.12 and the consecutiveness of $\theta$ and $j$, $y_j < v$ in $(s_\theta, s_j)$. Hence, $b_{L^*+1} = s_j$ and $L = L^* + 1$. Thus, $\mathcal{S}_j = \mathcal{S}_\theta \cup (s_\theta, s_j)$ and $U_j = y_j = y_\theta = U_\theta$ in $(s_\theta, s_j)$. □

The above delivers the desired uniqueness result.

**Theorem 5.15.** *Suppose that* (2.4) *and* (2.5) *hold and $f_\ell$ satisfies Hypothesis* 4.2 *for every $\ell \in \mathcal{J}$. Then* (2.10) *has at most one solution satisfying Ansatz* 3.3.

*Proof.* By the preamble to Lemma 5.12, a solution $u$ of (2.10) satisfying Ansatz 3.3 is necessarily equal to $U_j$ where $j$ is a minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. Pursuant to Lemma 5.14, this prescribes $u$ uniquely. □

## 6. Existence

From the preceding section, we know that if (2.10) has a solution satisfying Ansatz 3.3 then this solution is necessarily the function $U_j$, defined by (5.8), (5.12), (5.13), Lemma 5.10 and Lemma 5.11, for a minimizer $j$ of $y_\ell$ with respect to $\ell \in \mathcal{J}$. In the light of Lemma 5.14, we may drop the subscript from the notation of $U$. To prove that (2.10) admits a solution satisfying Ansatz 3.3 it subsequently suffices to verify that $U$ solves (2.10) and possesses the hallmarks of the ansatz. It is convenient to divide this undertaking into five steps, whereby, without further mention, it is supposed that $j$ is the greatest minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$.

**Lemma 6.1.** *The function $U$ is concave in $(-\infty, s_j]$. Moreover, $U'$ is strictly decreasing in $\mathcal{S}_j$.*

*Proof.* This is part and parcel of the proof of Lemma 5.10. □

**Lemma 6.2.** *Let $\ell \in \mathcal{J}$ and $z > x \ge s_j$. Then*

$$U(x) \le k_\ell + U(z) + c_\ell(z - x) \tag{6.1}$$

*with equality if and only if $\ell = j$, $x = s_j$ and $z = S_j$.*

*Proof.* Recalling that $U = y_j$ in $[s_j, \infty)$, see the proof of Lemma 4.16 of [5]. □

**Lemma 6.3.** *There holds $U \le MU$ in $\mathbb{R}$.*

*Proof.* For every $\ell \in \mathcal{J}$, Lemma 6.2 implies that $U \le M_\ell U$ in $[s_j, \infty)$. To show the same in $(-\infty, s_j)$, pick $x < s_j$. According to Lemma 6.1, $U$ is concave in $(-\infty, s_j]$. This means that $\eta \mapsto U(\eta) + c_\ell \eta$ is likewise concave. Therefore,

$$\begin{aligned}\big(M_\ell U\big)(x) &= k_\ell - c_\ell x + \min\{U(\eta) + c_\ell \eta : \eta \ge x\} \\ &= k_\ell - c_\ell x + \min\{U(\eta) + c_\ell \eta : \eta \ge s_j \text{ or } \eta = x\} \\ &= \min\{\big(M_\ell y_j\big)(s_j) + c_\ell(s_j - x), k_\ell + U(x)\}.\end{aligned}$$

Now if $\ell \le j$, Lemma 5.6 implies that $\big(M_\ell y_j\big)(s_j) + c_\ell(s_j - x) = v_\ell(x) \ge v(x)$. On the other hand if $\ell > j$, then by what we initially deduced, $\big(M_\ell y_j\big)(s_j) \ge y_j(s_j)$. Furthermore, by Lemma 5.8, $y_j(s_j) = v(s_j) = v_j(s_j)$. So recalling (2.4), $\big(M_\ell y_j\big)(s_j) + c_\ell(s_j - x) \ge v_j(s_j) + c_\ell(s_j - x) \ge v_j(s_j) + c_j(s_j - x) = v_j(x) \ge v(x)$. Hence, whether $\ell \le j$ or not,

$$\big(M_\ell U\big)(x) \ge \min\{v(x), k_\ell + U(x)\}. \tag{6.2}$$

Since $v(x) \ge U(x)$ and $k_\ell \ge 0$, this delivers $\big(M_\ell U\big)(x) \ge U(x)$, which, in view of the arbitrariness of $x$, leads to $U \le M_\ell U$ in $(-\infty, s_j)$. Thus $U \le M_\ell U$ in $\mathbb{R}$ for every $\ell \in \mathcal{J}$. The conclusion $U \le MU$ follows. □

**Lemma 6.4.** *Let $\ell \in \mathcal{J}$, $x < s_j$ and $z > x$. Then (6.1) holds. Furthermore, it holds with equality if $\ell \le j$, $U(x) = v_\ell(x)$ and $z = S_{j,\ell}$, and only if $x \in \Omega_j$.*

*Proof.* The inequality (6.1) in itself is a corollary of Lemma 6.3. That it holds with equality if $\ell \le j$, $U(x) = v_\ell(x)$ and $z = S_{j,\ell}$, follows from the construction of $v$. To show that (6.1) is strict if $x \in \mathcal{S}_j$, we distinguish the cases $k_\ell > 0$ and $k_\ell = 0$. When $x \in \mathcal{S}_j$, $v(x) > U(x)$. So if $k_\ell > 0$, inequality (6.2) yields $\big(M_\ell U\big)(x) > U(x)$. Whereupon, Lemma C.1 in Appendix C tells us that (6.1) is strict. The proof for $k_\ell = 0$ is less straightforward. Let $(a_\nu, b_\nu)$ be the component of $\mathcal{S}_j$ in which $x$ lies. By Lemma 6.1, $U'(\eta) > \big(D_- U\big)(b_\nu) \ge \big(D_- U\big)(s_j)$ for all $\eta \in (a_\nu, b_\nu)$. Invoking Lemma 5.8, this gives $U'(\eta) > -c_j$ for all such $\eta$. However, as $k_\ell = 0$, necessarily $c_j \le c_\ell$. Therefore, $U' + c_\ell > 0$ in $(a_\nu, b_\nu)$. Meanwhile, by Lemmas 6.3 and B.1 in Appendix B, $\eta \mapsto U(\eta) + c_\ell \eta$ is nondecreasing in $[b_\nu, \infty)$. Taken together, these two conclusions imply that $U(z) + c_\ell z > U(x) + c_\ell x$, whether $z \le b_\nu$ or $z > b_\nu$. The deduced inequality is equivalent to (6.1) with strictness. ☐

**Lemma 6.5.** *Let $\theta$ be the least minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. Then there is a $w < s_\theta$ such that $(w, s_\theta] \subset \Omega_j$.*

*Proof.* By Lemma 5.12, $s_\theta \le s_j$. Subsequently, by Lemmas 5.8, 5.10 and 5.11, $U$ and $v$ are differentiable at $s_\theta$, $U(s_\theta) = v(s_\theta) = y_j(s_\theta)$ and $U'(s_\theta) = v'(s_\theta) = y_j'(s_\theta)$. Therefore, should the lemma be false, by Lemmas 5.10 and 5.11 there would be an $a_\nu < s_\theta$ and a function (5.20) such that $U$ and $v$ are differentiable at $a_\nu$, $U = Y_\nu < v$ in $(a_\nu, s_\theta)$, $Y_\nu(a_\nu) = v(a_\nu)$, and $Y_\nu(s_\theta) = v(s_\theta) = y_j(s_\theta)$. As solutions of the initial-value problem for (4.1) are unique, the latter would imply that $Y_\nu = y_j$. Consequently, by Lemma 5.8, there would be an $\ell \in \{1, 2, \dots, j\}$ such that $y_j = y_\theta = y_\ell$ and $a_\nu = s_\ell < s_\theta$. By Lemma 5.12, this necessitates $\ell < \theta$. Therewith, we have arrived at a contradiction of $\theta$ being the least minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. So the lemma must be true. ☐

Building upon the analysis in Section 5, Lemmas 6.2–6.5 lead to the desired existence result.

**Theorem 6.6.** *Suppose that (2.4) and (2.5) hold and $f_\ell$ satisfies Hypothesis 4.2 for every $\ell \in \mathcal{J}$. Then (2.10) has a unique solution satisfying Ansatz 3.3.*

*Proof.* Let $j$ be the greatest minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$, and $u$ be the function $U_j$ defined by Lemma 5.11. Then by Lemma 6.3, $u \le Mu$ in $\mathbb{R}$. Furthermore, by Lemmas 5.10 and 5.11, $Au \le f$ in $\mathbb{R}$. If now, $u < Mu$ at some $x \in \mathbb{R}$, Lemmas 6.2 and 6.4 imply that either $x > s_j$ or $x \in \mathcal{S}_j$. Hence, by Lemmas 5.10 and 5.11, $u$ is differentiable and $Au = f$ at $x$. This confirms that $(Au - f)(u - Mu) = 0$ in $\mathbb{R}$. Thus we have proven that $u$ solves (2.10). With regard to satisfaction of Ansatz 3.3, let $\Omega$ denote the set of $x \in \mathbb{R}$ for which (3.1) holds. By Lemmas 6.2 and 6.4,

$$\Omega = \begin{cases} \Omega_j & \text{if } k_j > 0 \\ \Omega_j \setminus \{s_j\} & \text{if } k_j = 0. \end{cases}$$

This implies that $s_j$ is the least upper bound of $\Omega$ when $k_j > 0$. In the light of Lemma 6.5, it likewise implies that $s_j$ is the least upper bound of $\Omega$ when $j$ is the unique minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. However when $k_j = 0$ and there is more than one such minimizer, Lemmas 5.8 and 5.14 say that the least upper bound of $\Omega$ is $s_i$, where $i$ is the second greatest minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. Whatever, $\Omega$ has the structure set out in Ansatz 3.3. Lemma 5.10 affirms that $u$ has the required regularity. ☐

Theorem 6.6 spawns a number of corollaries. The proof of the first is contained in that of the theorem. That of the second, third and fourth is to be found in Section 5. The fifth is given by Lemmas 5.8, 5.14 and 6.5, and the theorem.

**Corollary 6.7.** *If $k_J = 0$, $J$ is a minimizer of $y_\ell$ with respect to $\ell \in \mathcal{J}$, and there is at least one other such minimizer, then $j$ in Theorem 4.1 is the second greatest of these minimizers. Otherwise, $j$ is the greatest.*

**Corollary 6.8.** *Except in a finite subset of* $\mathbb{R}$*, the unique solution* $u$ *of* (2.10) *satisfying Ansatz* 3.3 *is continuously differentiable. Let* $j$ *be the greatest minimizer of* $y_\ell$ *with respect to* $\ell \in \mathcal{J}$*. Define* $v$ *by* (5.8)*,* (5.12) *and* (5.13)*, the partition* (5.14) *such that* $v$ *is affine in each of the intervals* (5.15) *with* $\mathsf{N}$ *the smallest natural number for which such a partition exists,* $\mathcal{M}$ *by* (5.16)*,* $T_m$ *for* $m \in \mathcal{M}$ *by* (5.17)*,* $L$ *and* (5.19) *by Lemma* 5.10*, and* $\Omega_j$ *by* (5.21) *and* (5.22)*. Then the finite subset referred to,* $\Xi$ *say, comprises those* $\sigma_m \in \Omega_j$ *for which* $2 \leq m \leq \mathsf{N}$ *and those* $b_\nu$ *for which* $b_\nu < a_{\nu+1}$ *and* $1 \leq \nu \leq L$*. The derivative of* $u$ *has a jump discontinuity at every* $x \in \Xi$*.*

**Corollary 6.9.** *There holds* $D_+u + \alpha u < D_-u + \alpha u \leq f$ *at every* $x \in \Xi$*,* $Au < f$ *in the interior of* $\Omega_j \setminus \Xi$*, and* $Au = f$ *elsewhere.*

**Corollary 6.10.** *If* $x \in \Omega_j \setminus \Xi$ *then* $u(x) = v_\ell(x)$ *for a unique* $\ell \in \{1, 2 \ldots, j\}$*.*

**Corollary 6.11.** *Necessarily,* $L \geq \mathfrak{n} - 1$*,* $b_{L+1-\mathfrak{n}} < s_\theta$ *when* $L \geq \mathfrak{n}$*,* $a_{L+2-\mathfrak{n}} = s_\theta$*, and* $b_\nu = a_{\nu+1} = s_{\lambda(L-\nu+1)}$ *for* $L + 2 - \mathfrak{n} \leq \nu \leq L$*, where* $\mathfrak{n}$ *is the total number of minimizers* $\lambda(1) = j > \lambda(2) > \cdots > \lambda(\mathfrak{n}) = \theta$ *of* $y_\ell$ *with respect to* $\ell \in \mathcal{J}$*.*

## 7. Supplementary properties

The purpose of this section is to ascertain that the unique solution $u$ of (2.10) satisfying Ansatz 3.3 possesses certain properties that can intuitively be expected of an optimal inventory control policy.

The first of the afore-mentioned properties is the stability of $u$ in the sense that small perturbations of (2.5) do not engender large changes in $u$. In other words, under the predominant supposition that every $f_\ell$ satisfies Hypothesis 4.2, $u$ depends continuously on the set-up costs (2.5). Taken together with the existence and uniqueness results, this establishes that the problem of solving (2.10) under Ansatz 3.3 is mathematically well posed in the sense ascribed to Hadamard. The precise stability statement is the following and is proven in Appendix E.

**Theorem 7.1.** *Let* $u$ *be the unique solution of* (2.10) *given by Theorem* 6.6*, and* $u^{(i)}$ *the corresponding solution with* (2.5) *replaced by a like set of numbers furnished with a superscript* $(i)$ *for* $i \in \mathbb{N}$*. If* $k_\ell^{(i)} \to k_\ell$ *as* $i \to \infty$ *for every* $\ell \in \mathcal{J}$ *then* $u^{(i)} \to u$ *uniformly in* $\mathbb{R}$*.*

It might further be expected that, given a selection of available suppliers, should a further supplier appear on the scene, the total future cost of the optimal inventory control policy could fall, at least for some inventory levels if not all. The cost would certainly not be expected to rise. As this property is equivalent to the property that reducing the selection of available suppliers leads to the solution $u$ of (2.10) satisfying Ansatz 3.3 increasing or remaining the same, and the latter property is easier to formulate, we shall view the phenomenon from this perspective. The theorem below summarizes it and is proven in Appendix F.

**Theorem 7.2.** *Suppose that* $J \geq 2$ *and* $\ell \in \mathcal{J}$*. Let* $u$ *be the solution of* (2.10) *given by Theorem* 6.6 *and* $u^*$ *the corresponding solution whereby* $c_\ell$ *and* $k_\ell$ *are omitted from* (2.4) *and* (2.5) *and* $J$ *is lessened by* 1*. Then* $u^* \geq u$*.*

In the same vein as the previous property, should the cost per item or the set-up cost of one or more suppliers within a set of available suppliers be increased, then the total future cost of the optimal inventory control policy could be expected to rise for some if not all inventory levels. It is inconceivable that this will provoke a fall in the cost. This monotonicity property is captured by the coming theorem, whose proof is presented in Appendix G.

**Theorem 7.3.** *Let* $u^\pm$ *be solutions of* (2.10) *given by Theorem* 6.6 *corresponding to* (2.4) *and* (2.5) *furnished with a superscript* $\pm$*. If* $c_\ell^+ \geq c_\ell^-$ *and* $k_\ell^+ \geq k_\ell^-$ *for every* $\ell \in \mathcal{J}$ *then* $u^+ \geq u^-$*.*

Finally, in the introduction, it was argued that suppliers, that incur both a cost per item and a set-up cost greater than or equal to those of another supplier, do not have to be taken into account. It could equally as well be argued that, as long as no two suppliers have exactly the same costs, it should not be necessary to dismiss such suppliers *a priori*. If the model were robust, then these suppliers would be excluded from an optimal policy as an outcome. A sift through the theory developed in the preceding sections, verifies that the latter is indeed the case. The theorem below abrogates the previously stated existence, uniqueness and stability results.

**Theorem 7.4.** *Suppose that*

$$c_1 \leq c_2 \leq \cdots \leq c_J,$$
$$k_1 \geq 0, \quad k_2 \geq 0, \quad \ldots, \quad k_J \geq 0,$$
$$k_\ell > k_{\ell+1} \quad \textit{if} \quad c_\ell = c_{\ell+1}$$

*for* $\ell \in \mathcal{J} \setminus \{J\}$, *and* $f_\ell$ *satisfies Hypothesis* 4.2 *for every* $\ell \in \mathcal{J}$. *Then* (2.10) *has a stable unique solution satisfying Ansatz* 3.3. *Moreover, it is the same as that with*

$$c_{(N)} < c_{(N-1)} < \cdots < c_{(1)} \quad \textit{and} \quad k_{(N)} > k_{(N-1)} > \cdots > k_{(1)} \geq 0$$

*for some* $\{(1), (2), \ldots, (N)\} \subseteq \mathcal{J}$.

## 8. Type

In the prototypical circumstance that there is a single supplier, *i.e.* $J = 1$, with a set-up cost $k_1 > 0$, the stable unique solution $u$ of (2.10) satisfying Ansatz 3.3 corresponds to a standard $(s, S)$ policy. The numbers $s$ and $S$ are $s_1$ and $S_1$ respectively. Confronted with an inventory level $x \leq s$, the policy is to place an order to bring the inventory level up to $S$. Faced with an inventory level $x > s$, the policy is not to intervene. When the set-up cost $k_1 = 0$, and therefore $s_1 = S_1$, Theorem 6.6 delivers a degenerate $(s, S)$ policy with $S = s$. If $x < s$ then the policy is to order up to the level $S$. If $x > s$ then one does not intervene. However, if $x = s = S$, then one maintains the inventory at this level. This is feasible as ordering incurs only the cost per item.

When there are several suppliers, *i.e.* $J \geq 2$, the exact type of the optimal policy propagated by $u$ depends on three facets. Taking it as read that (2.4) and (2.5) apply, the foremost is the greatest minimizer $j$ of $y_\ell$ with respect to $\ell \in \mathcal{J}$. In layman's terms, this is the supplier for which the single-supplier $(s, S)$ policy involves the least cost when there is a large stock in hand, or, if there is more than one such supplier, that one of these suppliers with the least set-up cost. Of all these suppliers, that is also the supplier for which the single-supplier policy has the least value of $S$ and the greatest value of $s$. When $j = 1$, *i.e.* the supplier with the most favourable single-supplier $(s, S)$ policy is also the supplier with the least cost per item overall, $u$ corresponds to an $(s, S)$ policy. This is indistinguishable from the $(s, S)$ policy with supplier 1 as the sole available supplier. Otherwise, $u$ does not represent an $(s, S)$ policy.

When the pivotal supplier $j$ is not the supplier with the least cost per item overall, then, in the second instance, the type of policy is determined by the numbers $T_m$ for $m \in \mathcal{M} \setminus \{1\}$. These numbers are associated with an inventory level $\sigma_m$, which has the property that one would ostensibly order from one supplier if one had a slightly smaller backlog and from another if one had a slightly greater backlog. The nonnegativity of $T_m$ is a way of testing that it is more economical to order from the second supplier than not to place an order at all. The number $T_1 = 0$ irrespective of any other considerations.

If $j \geq 2$, $T_m \geq 0$ for every $m \in \mathcal{M}$, and $k_j > 0$, then $u$ corresponds to a generalized $(s, S)$ policy involving $N$ suppliers and numbers (1.1). In terms of the theory leading up to Theorem 6.6, $N = \mathsf{N}$, supplier (1) is supplier $j$, $s_{(1)} = s_j$, and $S_{(1)} = S_j$. For $2 \leq n \leq N$, supplier $(n)$ is supplier $\ell$, where $\ell$ is that number in $\{1, 2 \ldots, j\}$ for which $v = v_\ell$ in $I_n$, $s_{(n)} = \sigma_n$ and $S_{(n)} = S_{j,\ell}$. Irrevocably, supplier $(N)$ is supplier 1.

The strategy espoused by the generalized $(s, S)$ policy is that if the inventory level $x$ is such that $x < s_{(N)}$ one orders from supplier $(N)$ to bring the inventory level up to $S_{(N)}$. If $s_{(n+1)} < x < s_{(n)}$ for $2 \leq n \leq N$ or $x = s_{(n)}$ for $n = 1$, one orders from supplier $(n)$ to bring the inventory level up to $S_{(n)}$. If $x > s_{(1)}$ one does not intervene. With regard to an inventory level $x = s_{(n)}$ for $2 \leq n \leq N$, one may order from supplier $(n)$ or supplier $(n-1)$ to bring the level of inventory up to the appropriate target level. Possibly there are further suppliers, otherwise excluded from the policy, with which one could place an order from this level.

When $j \geq 2$, $T_m \geq 0$ for every $m \in \mathcal{M}$, and $k_j = 0$, then $u$ betokens a degenerate generalized $(s, S)$ policy. It is degenerate in the sense that (1.1) is replaced by

$$s_{(N)} < s_{(N-1)} < \cdots < s_{(1)} = S_{(1)} < S_{(2)} < \cdots < S_{(N)}. \tag{8.1}$$

Except in one detail, the policy has the same characteristics as the generalized $(s, S)$ policy just described. The difference is that when $x = s_{(1)} = S_{(1)}$, the policy is to maintain the inventory at this level.

Should there be more than one minimizer of $y_\ell$ with respect to $\ell \in \mathcal{J}$, then necessarily $T_m < 0$ for some $m \in \mathcal{M}$. Thus, without further ado, it can be stated that the solution $u$ of (2.10) satisfying Ansatz 3.3 will not deliver a generalized $(s, S)$ policy let alone an $(s, S)$ policy.

Given that $j \geq 2$, $T_m < 0$ for some $m \in \mathcal{M}$, and $k_j > 0$, the function $u$ given by Theorem 6.6 represents a hyper-generalized $(s, S)$ policy that is not a generalized $(s, S)$ policy and certainly not an $(s, S)$ policy. This involves numbers (1.1)–(1.3), with $s_{(n+1)} < r_{(n)}$ for at least one $n \in \{1, 2, \ldots, N-1\}$. Supplier (1) is supplier $j$, $s_{(1)} = s_j$, and $S_{(1)} = S_j$ as with a generalized $(s, S)$ policy. The numbers $s_{(n+1)}$ and $r_{(n)}$ for $1 \leq n \leq N-1$ are collectively the numbers $a_\nu$ and $b_\nu$ for $1 \leq \nu \leq L$, and those $\sigma_m$ for $2 \leq m \leq \mathsf{N}$ within the set $\Omega_j$ defined in Section 5. Supplier $(n)$ for $n$ from 2 to $N$ can be identified in that order as the supplier $\ell$ for which $v(x) = v_\ell(x)$ for some $x \in \Omega_j$ at which $v$ is differentiable as $x$ decreases from $\sigma_1$. Thereupon, $S_{(n)} = S_{j,\ell}$, $s_{(n)}$ is the least upper bound of such $x$, and $r_{(n)}$ for $n \leq N-1$ is the greatest lower bound of these $x$. As with a generalized $(s, S)$ policy, supplier $(N)$ is unavoidably supplier 1.

The strategy embodied in a hyper-generalized $(s, S)$ policy is as follows. If the inventory level $x$ is such that $x < s_{(N)}$ then one orders from supplier $(N)$ to bring the inventory level up to $S_{(N)}$. If $s_{(n+1)} < x < r_{(n)}$ for $1 \leq n \leq N-1$ then one does not intervene. If $r_{(n)} < x < s_{(n)}$ for $1 \leq n \leq N-1$ one orders from supplier $(n)$ to bring the inventory level up to $S_{(n)}$. Finally, if $x > s_{(1)}$ one does not intervene. Regarding the watershed levels, if $s_{(n)} < r_{(n-1)}$ for $n = N$, $r_{(n)} < s_{(n)} < r_{(n-1)}$ for $2 \leq n \leq N-1$, or $r_{(n)} < s_{(n)}$ for $n = 1$, then at the inventory level $x = s_{(n)}$ one should order from supplier $(n)$ to bring the inventory level up to $S_{(n)}$. If $s_{(n+1)} < r_{(n)} = s_{(n)} < r_{(n-1)}$ for $2 \leq n \leq N-1$, or $s_{(n+1)} < r_{(n)} = s_{(n)}$ for $n = 1$, then at the inventory level $x = r_{(n)} = s_{(n)}$ one may choose either to order from supplier $(n)$ to bring the inventory level up to $S_{(n)}$ or not to intervene. There are no other options. At an inventory level $x = s_{(n)}$, where $s_{(n)} = r_{(n-1)} < s_{(n-1)}$ and $n = N$, or $r_{(n)} < s_{(n)} = r_{(n-1)} < s_{(n-1)}$ and $2 \leq n \leq N-1$, one has the same ambiguity as with a generalized $(s, S)$ policy. At an inventory level $x = r_{(n)}$, where $s_{(n+1)} < r_{(n)} < s_{(n)}$ and $1 \leq n \leq N-1$, one may choose to order from supplier $(n)$ to bring the inventory level up to $S_{(n)}$ or not to intervene. There could also be suppliers otherwise excluded from the policy, with which one could place an order from this level.

When $j \geq 2$, $T_m < 0$ for some $m \in \mathcal{M}$, and $k_j = 0$, then (1.1) is replaced by (8.1). Except for $x = s_{(1)}$, the advocated strategy is the same as that of the hyper-generalized $(s, S)$ policy just described. However, for $x = s_{(1)}$ one should maintain the inventory level at $x$ when $s_{(1)} > r_{(1)}$. For $x = s_{(1)}$ when $s_{(1)} = r_{(1)}$ one has the choice of either maintaining the inventory level at $x$ or not intervening.

In a nutshell, the following has emerged.

**Theorem 8.1.** *Relaxing* (1.1) *to*

$$s_{(N)} < s_{(N-1)} < \cdots < s_{(1)} \leq S_{(1)} < S_{(2)} < \cdots < S_{(N)}, \tag{8.2}$$

*and adopting the convention that hyper-generalized $(s, S)$ policies include generalized $(s, S)$ policies, and generalized $(s, S)$ policies include $(s, S)$ policies, the solution $u$ of* (2.10) *given by Theorem* 6.6 *represents a hyper-generalized $(s, S)$ policy. With reference to the notation set out in Corollary* 6.8*, it is a generalized $(s, S)$ policy if and only if $T_m \geq 0$ for every $m \in \mathcal{M}$, and an $(s, S)$ policy if and only if $j = 1$. The weak inequality in* (8.2) *is strict if and only if $k_j > 0$.*

## 9. The case of two suppliers

As much as Theorem 8.1 establishes the necessary and sufficient conditions for the occurrence of a generalized $(s, S)$ policy and an $(s, S)$ policy, it provides little insight into the way in which this occurrence is regulated by the costs of the suppliers. The present section addresses this lacuna by examining the particular case of two suppliers in more detail. The next three theorems furnish a conspectus. Their proof can be found in Appendices H–J respectively.

**Theorem 9.1.** *Suppose that* $J = 2$*, hypotheses* (2.4) *and* (2.5) *hold, and the functions* $f_1$ *and* $f_2$ *satisfy Hypothesis* 4.2*. Then there are strictly increasing continuous functions* $K_*$ *and* $K_\dagger$ *with domain* $[0, \infty)$ *such that*

$$K_*(k) > K_\dagger(k) > k \quad \text{for all} \quad k \geq 0 \tag{9.1}$$

*and the following holds. If* $k_1 < K_\dagger(k_2)$ *the solution of* (2.10) *satisfying Ansatz* 3.3 *corresponds to an* $(s, S)$ *policy involving only supplier* 1*. If* $k_1 \geq K_*(k_2)$ *it corresponds to a generalized* $(s, S)$ *policy involving both suppliers. If* $K_\dagger(k_2) \leq k_1 < K_*(k_2)$ *it corresponds to a hyper-generalized* $(s, S)$ *policy involving both suppliers that is not a generalized* $(s, S)$ *policy.*

**Theorem 9.2.** *Suppose that the hypotheses of Theorem* 9.1 *are met by* $f_1^\pm$ *with numbers* $c_1^\pm$*, delivering* $K_\dagger^\pm$ *and* $K_*^\pm$ *respectively. If* $c_1^+ > c_1^-$ *then* $K_\dagger^+ < K_\dagger^-$ *and* $K_*^+ < K_*^-$ *on* $[0, \infty)$*.*

**Theorem 9.3.** *Suppose that the hypotheses of Theorem* 9.1 *are met by* $f_2^\pm$ *with numbers* $c_2^\pm$*, delivering* $K_\dagger^\pm$ *and* $K_*^\pm$ *respectively. If* $c_2^+ > c_2^-$ *then* $K_\dagger^+ > K_\dagger^-$ *and* $K_*^+ > K_*^-$ *on* $[0, \infty)$*.*

**Remark 9.4.** When $f$ takes on the classical expression (2.6) and (4.7) holds,

$$\begin{aligned} K_\dagger(k_2) = {} & (p - \alpha c_1)\big[\ln\big\{(p - \alpha c_1)/(p - \alpha c_2)\big\} - \alpha s_2\big]/\alpha^2 \\ & - (q + \alpha c_1)\ln\big\{[p + q - (p - \alpha c_2)\mathrm{e}^{\alpha s_2}]/(q + \alpha c_1)\big\}/\alpha^2 \end{aligned}$$

and

$$\begin{aligned} K_*(k_2) = {} & (p - \alpha c_1)\{c_2 - c_1 - (p - \alpha c_2)s_2\}/\{\alpha(p - \alpha c_2)\} \\ & - (q + \alpha c_1)\ln\big\{[p + q - (p - \alpha c_2)\mathrm{e}^{\alpha s_2}]/(q + \alpha c_1)\big\}/\alpha^2 \end{aligned}$$

where $s_2$ is given by (5.6).

Theorem 9.1 establishes that in the case of two available suppliers ordered in terms of increasing cost per item, there are two critical levels for the set-up cost of the first supplier relative to that of the second. When the set-up cost of the first supplier is below the lesser critical level, the second supplier is excluded from the optimal policy. When the set-up cost of the first supplier is at this level or above, both suppliers are involved. When in addition the set-up cost of the first supplier is below the greater critical level, the optimal policy contains shortage levels, between ones where a manager would order from one supplier or the other, for which the policy is to let a backlog grow. When the set-up cost of the first supplier is at or above this greater critical level, the optimal policy entails ordering from one supplier or the other for all inventory levels below the greatest for which this is an option. This is perhaps counterintuitive. An explanation can be sought in looking at what happens to the greatest inventory level, $s_{(1)}$, from which an order may be placed, as the set-up cost of the first supplier increases. When the set-up cost of the first supplier cost is below the lesser critical level, $s_{(1)}$ has the value of $s$ in the $(s, S)$ policy of that supplier. When the cost reaches the critical level, it jumps to the value of $s$ in the single-supplier $(s, S)$ policy of the other supplier. Thus, as it were, an interval of inventory levels for which the policy was not to intervene is instantly annihilated. What one observes in the optimal policy is a remnant of this interval in which it still pays not to intervene. Increasing the set-up cost of the first supplier further, traces of the interval disappear. Theorem 9.1 also shows that as the set-up cost of the second supplier increases, the critical levels for the first supplier adjust. The lesser the set-up cost of the first supplier or the greater that of the second, the greater the likelihood that the second supplier is excluded from the policy. Theorems 9.2 and 9.3 confirm that the same applies with regard to the cost per item. These conclusions are as one would expect. Indeed, they have been drawn on the basis of a numerical experiment in [6]. When more than two suppliers are available, Example 4.29 of [6] illustrates that the prediction of the interrelation between the costs of several suppliers becomes considerably more problematic.

## 10. Computation

The stable unique solution $u$ of problem (2.10) satisfying Ansatz 3.3 is characterized in sufficient detail in Section 5 to inform how to compute it. The algorithm below summarizes the procedure. This is a polished version of Algorithm 4.27 of [6] taking advantage of the new insights. The most noteworthy alteration is that the greatest minimizer of $y_\ell$ with respect to $\ell \in \mathcal{J}$ is replaced by the least. Since it has been shown that whichever minimizer is used, the end result is the same, this reduces the work when there is more than one. The worst-case complexity of the algorithm is $O(J^2)$ with regard to the total number of steps that will be executed for a large number $J$ of available suppliers.

**Algorithm 10.1.** Step 1. For every $\ell \in \mathcal{J}$, find the unique solution of the simultaneous equations (5.1).
Step 2. Pick a convenient $\zeta \in \mathbb{R}$, and compute

$$\Upsilon_\ell = c_\ell \mathrm{e}^{\alpha\zeta} - \int_{s_\ell}^{\zeta} \mathrm{e}^{\alpha\eta}\,\mathrm{d}f_\ell(\eta) \quad \text{ for } \ell \in \mathcal{J}.$$

Step 3. Define $\Lambda = \{\ell \in \mathcal{J} : \Upsilon_\ell \le \Upsilon_i \text{ for every } i \in \mathcal{J}\}$, $j = \min \Lambda$, and

$$B_j = \{f_j(s_j) + c_j\}/\alpha. \tag{10.1}$$

Step 4. If $j = 1$, set $\mathsf{N} = 1$, $\sigma_{\mathsf{N}} = s_j$ and $L = 0$. Then proceed to Step 8. Otherwise, continue to Step 5.
Step 5. For $\ell = 1, 2, \dots, j-1$, determine $S_{j,\ell} > S_j$ from equation (5.9), and set

$$B_\ell = k_\ell + \{f_\ell(S_{j,\ell}) + c_\ell\}/\alpha. \tag{10.2}$$

Step 6. Let $\sigma_1 = s_j$, $\kappa(1) = j$ and $m = 2$.
- (a) Define $\mathcal{K} = \{1, 2 \dots, \kappa(m-1) - 1\}$.
- (b) For every $\ell \in \mathcal{K}$, compute $\sigma_m^{(\ell)} = \big(B_{\kappa(m-1)} - B_\ell\big)/\big(c_{\kappa(m-1)} - c_\ell\big)$.
- (c) Define $\sigma_m = \max\{\sigma_m^{(\ell)} : \ell \in \mathcal{K}\}$ and $\kappa(m) = \min\{\ell \in \mathcal{K} : \sigma_m^{(\ell)} = \sigma_m\}$.
- (d) Let $T_m = f_{\kappa(m)}(\sigma_m) + c_{\kappa(m)} - \alpha B_{\kappa(m)}$.
- (e) If $\kappa(m) = 1$, set $\mathsf{N} = m$ and proceed to Step 7. Otherwise, increase $m$ by 1, and return to Step 6(a).

Step 7. Let $\mathsf{n} = \mathsf{N}$ and $\nu = 0$.
- (a) Define $\mathcal{N} = \{2, 3, \dots, \mathsf{n}\}$.
- (b) If $T_m \ge 0$ for every $m \in \mathcal{N}$, set $L = \nu$, and proceed to Step 8. Otherwise, increase $\nu$ by 1, and continue to Step 7(c).
- (c) Define $\mu(\nu) = \max\{m \in \mathcal{N} : T_m < 0\}$ and $\ell = \kappa(\mu(\nu))$.
- (d) Determine $a_\nu < \sigma_{\mu(\nu)}$ from the equation

$$f_\ell(a_\nu) = \alpha B_\ell - c_\ell. \tag{10.3}$$

- (e) Set

$$Y_\nu(x) = \mathrm{e}^{-\alpha x}\left\{(B_\ell - c_\ell a_\nu)\mathrm{e}^{\alpha a_\nu} + \int_{a_\nu}^{x} \mathrm{e}^{\alpha\eta} f(\eta)\,\mathrm{d}\eta\right\}. \tag{10.4}$$

- (f) Working through the sequence $\mu(\nu)-1, \mu(\nu)-2, \dots, 1$ in that order, let $\mathsf{n}$ be first such number encountered for which the equation

$$Y_\nu(x) = B_{\kappa(\mathsf{n})} - c_{\kappa(\mathsf{n})} x \tag{10.5}$$

  has a solution in $(\sigma_{\mathsf{n}+1}, \sigma_{\mathsf{n}}]$.
- (g) Let $b_\nu$ be the least solution of (10.5) in $(\sigma_{\mathsf{n}+1}, \sigma_{\mathsf{n}}]$.
- (h) If $\mathsf{n} = 1$, set $L = \nu$, and proceed to Step 8. Otherwise, return to Step 7(a).

Step 8. Output $u(x) = Y_\nu(x)$ for $a_\nu < x < b_\nu$ and $1 \le \nu \le L$. For all other $x < s_j$, $u(x) = B_1 - c_1 x$ if $x \le \sigma_{\mathsf{N}}$, and $u(x) = B_{\kappa(m)} - c_{\kappa(m)} x$ if $\sigma_{m+1} < x \le \sigma_m$ and $1 \le m \le \mathsf{N} - 1$. For $x \ge s_j$, $u(x)$ is given by the right-hand side of (10.4) with $\ell = j$ and $a_\nu = s_j$. **End.**

A few words regarding Algorithm 10.1 may be helpful. In Step 2, $\Lambda$ is the set of minimizers of $y_\ell$ with respect to $\ell \in \mathcal{J}$ because, by (5.3), $\Upsilon_\ell = \mathrm{e}^{\alpha\zeta}\big(\alpha y_\ell - f\big)(\zeta)$ for $\ell \in \mathcal{J}$. Hence, $j = \min\Lambda$ identifies $j$ as the least minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. The number $B_\ell$ defined for $\ell = j$ in Step 3 and for $\ell = 1, 2, \ldots, j-1$ in Step 5 is introduced to succinctly express (5.13) as $v_\ell(x) = B_\ell - c_\ell x$. Step 6 extracts the partition (5.14) with the property that $v$ defined by (5.12) is affine in each of the intervals (5.15) and $\mathsf{N}$ is the smallest number for which such a partition exists. The function $\kappa : \mathcal{M} = \{1, 2, \ldots, \mathsf{N}\} \to \{1, 2, \ldots, j\}$ is a device for recording that $\ell$ for which $v = v_\ell$ in $I_m$. Step 7 determines the number $L$, the numbers (5.19), and the functions (5.20) signalled in Lemma 5.10. The function $\mu : \{1, 2, \ldots, L\} \to \mathcal{M}$ uncovered in this step keeps a tab of that $m$ for which $a_\nu \in I_m$. In general, equation (10.5) has a unique solution, a pair of solutions, or no solution in an interval $(\sigma_{\mathfrak{n}+1}, \sigma_{\mathfrak{n}}]$. So 'least' in Step 7(g) should be understood as 'unique' or 'lesser'.

**Remark 10.2.** Regarding Algorithm 10.1 when $f$ is given by (2.6) and (4.7) holds, $s_\ell$ can be computed at Step 1 *via* (5.6), whereupon $S_j$ can be found at Step 5 *via* (5.5). In Step 2, it is convenient to take $\zeta = 0$, leading to

$$\Upsilon_\ell = \{p - (p - \alpha c_\ell)\mathrm{e}^{\alpha s_\ell}\}/\alpha. \tag{10.6}$$

The expression (10.1) simplifies to

$$B_j = \{c_j - (p - \alpha c_j)s_j\}/\alpha, \tag{10.7}$$

and (10.2) to

$$B_\ell = k_\ell + \{(q + \alpha c_\ell)S_{j,\ell} + c_\ell\}/\alpha. \tag{10.8}$$

Where needed, one can also explicitly solve (10.3) as

$$a_\nu = (c_\ell - \alpha B_\ell)/(p - \alpha c_\ell). \tag{10.9}$$

The thrust of Algorithm 10.1 is the calculation of the solution $u$ of (2.10) satisfying Ansatz 3.3. The corresponding hyper-generalized $(s, S)$ policy, and its identification as a generalized $(s, S)$ policy or an $(s, S)$ policy, is distilled from this. The following sequel, employing the data acquired, outlines the procedure. The worst-case complexity of the algorithm with regard to the total number of computational steps for a large number $J$ of available suppliers is $O(J)$.

**Algorithm 10.3.** Step 1. Count the number $\mathfrak{n}$ of elements of $\Lambda$.

Step 2. If $\mathfrak{n} = 1$ proceed to Step 4. Otherwise continue to Step 3.

Step 3. Sort the elements of $\Lambda$ into the order $\lambda(1) > \lambda(2) > \cdots > \lambda(\mathfrak{n}-1) > j$, and, for $n$ from 1 to $\mathfrak{n}-1$, define supplier $(n)$ to be supplier $\lambda(n)$, $r_{(n)} = s_{(n)} = s_{\lambda(n)}$, and $S_{(n)} = S_{\lambda(n)}$.

Step 4. Set $n = \mathfrak{n}$, $s_{(n)} = s_j$ and $S_{(n)} = S_j$.

Step 5. If $\mathsf{N} = 1$ proceed to Step 7. Otherwise, continue to Step 6.

Step 6. Let $m = 1$ and $\nu = L$.

- (a) If $\nu \geq 1$ and $b_\nu \geq \sigma_{m+1}$, define $r_{(n-1)} = b_\nu$ and $s_{(n)} = a_\nu$. Set $m = \mu(\nu)$. Then decrease $\nu$ by 1. Otherwise, increase $m$ by 1, and define $r_{(n-1)} = s_{(n)} = \sigma_m$.
- (b) If $m = \mathsf{N}$ proceed to Step 7. Otherwise, increase $n$ by 1, define supplier $(n)$ to be supplier $\kappa(m)$ and $S_{(n)} = S_{j,\kappa(m)}$. Then return to Step 6(a).

Step 7. Set $N = n$. The output is a hyper-generalized $(s, S)$ policy with numbers (8.2), (1.2) and (1.3). This is a generalized $(s, S)$ policy, *i.e.* such that $r_{(n)} = s_{(n+1)}$ for every $1 \leq n \leq N-1$, if and only if $\mathfrak{n} + L = 1$. It is an $(s, S)$ policy if and only if $N = 1$. **End.**

Application of Algorithms 10.1 and 10.3 demonstrates that an $(s, S)$ policy, a generalized $(s, S)$ policy that is not an $(s, S)$ policy, and a hyper-generalized $(s, S)$ policy that is not a generalized $(s, S)$ policy may each occur when there are two suppliers available and one incurs no set-up cost. Each of the following three examples concerns one of these mutually exclusive alternatives. Where transcendental equations have been encountered, they have been solved using readily available propriety software.

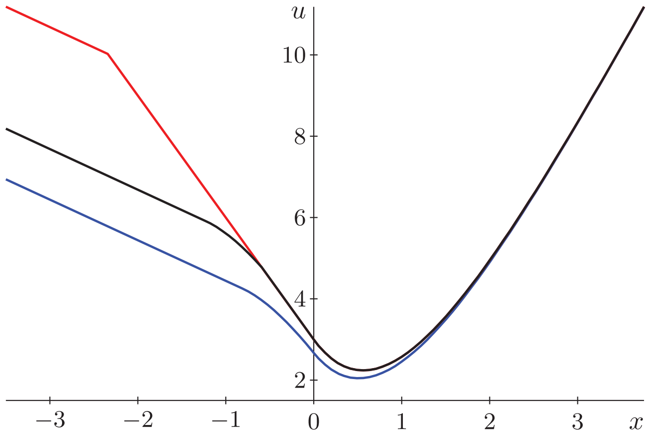


FIGURE 1. Cost function for Examples 10.4–10.6. That for Example 10.4 is the lowermost curve, for Example 10.5 the uppermost curve, and for Example 10.6 the intermediate curve.

**Example 10.4.** Let $f$ be given by (2.6) with $p = q = 4$, $\alpha = 1$, $J = 2$, $c_1 = k_1 = 1$, $c_2 = 3$, and $k_2 = 0$. So (4.7) is satisfied. Following Algorithm 10.1, solution of (5.6) yields $s_1 \approx -0.814$, while it can be verified that $s_2 = 0$. Formula (10.6) subsequently gives $\Upsilon_1 \approx 2.670$ and $\Upsilon_2 = 3$. Therefore $\Lambda = \{1\}$, $j = 1$, $\sigma_1 = s_1$, $\mathsf{N} = 1$ and $L = 0$. Formula (5.5) leads to $S_1 \approx 0.288$. Proceeding to Algorithm 10.3, $N = 1$. Therefore the solution of (2.10) satisfying Ansatz 3.3 corresponds to an $(s, S)$ policy involving only supplier 1 with $s_{(1)} = s_1 < S_{(1)} = S_1$.

**Example 10.5.** Substitution of $k_1 = 5$ in Example 10.4 gives similarly $s_1 \approx -2.392$ and $\Upsilon_1 \approx 3.726$, while the values of $s_2$ and $\Upsilon_2$ remain the same. This means that $\Lambda = \{2\}$, $j = 2$, and $\sigma_1 = s_2$. By (10.7) $B_2 = 3$, by (5.5) $S_2 = 0$, by (5.11) $S_{2,1} \approx 0.336$, and, by (10.8) $B_1 \approx 7.682$. Step 6 of Algorithm 10.1 delivers $\kappa(1) = 2$, $\sigma_2 = (B_2 - B_1)/(c_2 - c_1) \approx -2.341$, $\kappa(2) = 1$, $T_2 = f_1(\sigma_2) + c_1 - \alpha B_1 \approx 0.341$, and $\mathsf{N} = 2$. Thereafter, Step 7 gives $L = 0$. Proceeding to Algorithm 10.3, $\mathfrak{n} + L = 1$ and $N = 2$. The outcome is a generalized $(s, S)$ policy involving both suppliers with $s_{(2)} = \sigma_2 < s_{(1)} = S_{(1)} = s_2 = S_2 < S_{(2)} = S_{2,1}$, supplier (1) is supplier 2, and supplier (2) is supplier 1.

**Example 10.6.** Substitution of $k_1 = 2$ in Example 10.4 leads to $s_1 \approx -1.263$ and $\Upsilon_1 \approx 3.152$, while the values of $s_2$ and $\Upsilon_2$ are unchanged. This results in $\Lambda = \{2\}$. Subsequently $j$, $\sigma_1 = s_2$, $B_2$, $S_2$ and $S_{2,1}$ are as calculated in Example 10.5. However (10.8) gives $B_1 \approx 4.682$. Step 6 of Algorithm 10.1 then delivers $\kappa(1) = 2$, $\sigma_2 = (B_2 - B_1)/(c_2 - c_1) \approx -0.841$, $\kappa(2) = 1$, $T_2 = f_1(\sigma_2) + c_1 - \alpha B_1 \approx -1.159$, and $\mathsf{N} = 2$. Thereafter, Step 7 leads to $\mu(1) = 2$, formula (10.9) with $\ell = \kappa(2) = 1$ gives $a_1 \approx -1.227$, formula (10.4) with $\ell = 1$ gives $Y_1(x) \approx -0.879\mathrm{e}^{-x} + 4 - 4x$, $b_1 \approx -0.597$, and $L = 1$. Proceeding to Algorithm 10.3, $\mathfrak{n} + L = 1$ and $N = 2$. The outcome is a hyper-generalized $(s, S)$ policy that is not a generalized $(s, S)$ policy involving both suppliers with $s_{(2)} = a_1 < r_{(1)} = b_1 < s_{(1)} = S_{(1)} = s_2 = S_2 < S_{(2)} = S_{2,1}$, supplier (1) is supplier 2, and *vice versa*.

Figure 1 displays the cost function $u$ given by Examples 10.4–10.6. The lowermost curve is that for Example 10.4 and corresponds to an $(s, S)$ policy, the uppermost curve is that for Example 10.5 corresponding to a generalized $(s, S)$ policy, while the intermediate curve is that for Example 10.6 corresponding to a hyper-generalized $(s, S)$ policy that is not a generalized $(s, S)$ policy. Note the discontinuity in the derivative of $u$ appearing in the uppermost curve at $x \approx -2.341$, and in the intermediate curve at $x \approx -0.597$, reflecting the existence and uniqueness theory of Sections 5 and 6. All three examples have two available suppliers with the same cost per item incurred using supplier 1, the same cost per item incurred using supplier 2, and the same set-up cost incurred using supplier 2. The distinction is in the set-up cost $k_1$ incurred using supplier 1. From top to bottom, the curves in Figure 1 are those for progressively decreasing $k_1$, as anticipated by Theorem 7.3.

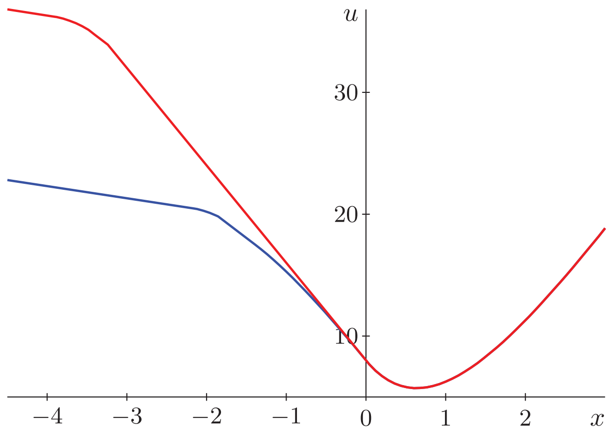


FIGURE 2. Cost function for Examples 10.7 and 10.8. That for Example 10.7 is the lower curve.

Examples where both suppliers have a positive set-up cost, complementary to Examples 10.4–10.6, have been presented in [6]. The occurrence of each type of policy is in line with Theorem 9.1. We refer to Example 4.29 of [6] for an illustration of how the occurrence of the different types of policy becomes more opaque when more than a couple of suppliers are available.

In the development of the theory of hyper-generalized $(s,S)$ policies in the preceding sections, marginal possibilities arise. The objective of the coming examples is to show that, however unlikely it is that these may appear in practice, they cannot be dismissed.

**Example 10.7.** Let $f$ be given by (2.6) with $p=q=9$, $\alpha=1$, $J=8$, $c_\ell=\ell$ for $\ell\in\{1,4,5,\dots,8\}$, $k_1=12$, $k_4=6$, and $k_8=0$. Noting that (5.6) gives $s_8=0$, define $s_\ell=s_8-\ln\{(p-\alpha c_8)/(p-\alpha c_\ell)\}/\alpha$ and $k_\ell$ *via* (5.6) for $\ell\in\{5,6,7\}$. This yields $k_5\approx 2.827$, $k_6\approx 1.418$, and $k_7\approx 0.416$. Next, define $S_5$ by (5.5), $S_{5,1}$ by (5.11), $B_5$ by (10.7), $B_1$ by (10.8) with $S_{j,\ell}=S_{5,1}$, $a_1$ by (10.9) with $\ell=1$, and $Y_1$ by (10.4) with $\ell=1$. It can be discerned that $Y_1(x)=B_5-c_5x$ has a unique solution, $b_3$ say, in $(a_1,s_5)$. Subsequently, set $a_2=(2a_1+b_3)/3$ and $a_3=(a_1+2b_3)/3$. Then for $\ell\in\{2,3\}$, take $c_\ell=Y_1'(a_\ell)$, $B_\ell=Y_1(a_\ell)+c_\ell a_\ell$, $S_{5,\ell}$ from (5.11), and $k_\ell$ from (10.8). This yields $c_2\approx 1.785$, $c_3\approx 2.493$, $k_2\approx 9.955$ and $k_3\approx 8.234$. Herewith we have a full complement of numbers (2.4) and (2.5). By design, $\Upsilon_5=\Upsilon_6=\Upsilon_7=\Upsilon_8$, and by computation using (5.6) and (10.6), $\Upsilon_\ell>\Upsilon_8$ for $\ell\in\{1,2,3,4\}$. Thus Step 3 of Algorithm 10.1 gives $\Lambda=\{5,6,7,8\}$ and $j=5$. Step 6 subsequently delivers $\kappa(1)=5$, $\kappa(2)=3$, $T_2<0$, $\kappa(3)=2$, $T_3<0$, $\kappa(4)=1$, $T_4<0$, and $\mathsf{N}=4$. By contrivance, Step 7 yields $\mu(1)=4$, $a_1$ as previously stated, $b_1=a_2$, $\mu(2)=3$, $b_2=a_3$, $\mu(3)=2$, $b_3$ as previously mentioned, and $L=3$. Algorithm 10.3 then leads to $N=7$ and $\mathfrak{n}+L=7$. The output is a hyper-generalized $(s,S)$ policy involving seven suppliers, suppliers (1)–(7) being 8, 7, 6, 5, 3, 2 and 1 in the original ordering, for which (8.1) with $N=7$ holds. Furthermore, $s_{(5)}<r_{(4)}<s_{(4)}$, and, $r_{(n)}=s_{(n)}$ for $n\in\{1,2,3,5,6\}$, in agreement with (1.2) and (1.3). The cost function is illustrated in Figure 2.

The remarkable feature of Example 10.7 is that in terms of the statement of Lemma 5.10 there are $\nu\in\{1,2,\dots,L\}$ for which $b_\nu=a_{\nu+1}$. For the final hyper-generalized $(s,S)$ policy this means that there are $n\in\{1,2,\dots,N-1\}$ for which $r_{(n)}=s_{(n)}$. In Example 10.7, this occurs for $n\in\{1,2,3,5,6\}$. The set $\Omega_j$, where $j$ is the greatest minimizer of $y_\ell$ with respect to $\ell\in\mathcal{J}$, is

$$\Omega_j=\big(-\infty,s_{(7)}\big]\cup\big\{s_{(6)},s_{(5)}\big\}\cup\big[r_{(4)},s_{(4)}\big]\cup\big\{s_{(3)},s_{(2)},s_{(1)}\big\}.$$

The mechanism that leads to the occurrence of $r_{(n)}=s_{(n)}$ for $n\in\{1,2,3\}$ is the non-uniqueness of a minimizer of $y_\ell$ with respect to $\ell\in\mathcal{J}$. In contrast, the occurrence for $n\in\{5,6\}$ is attributable to functions (5.20), besides having the default properties $Y_\nu < v$ in $(a_\nu, b_\nu)$ and $Y_\nu = v$ at $b_\nu$, being such that $v$ is differentiable and $Y_\nu' = v'$ at $b_\nu$.

An isolated element $s_{(n)}$ for a supplier $(n)$ with a positive set-up cost has implications for the dynamic situation in which, as advocated by the policy, a manager with a greater initial inventory level is not intervening. When the level reaches $s_{(n)}$, the manager has the option of ordering up to the level $S_{(n)}$ or not intervening. However, this freedom of choice is momentary. Should the moment be missed, the optimal strategy returns to that of allowing the backlog to grow. Related considerations apply to the inventory level $s_{(1)}$ in Example 10.7. As supplier (1) incurs no set-up cost, at the inventory level $s_{(1)}$, a manager momentarily has the choice of not intervening or maintaining the inventory at the level $s_{(1)} = S_{(1)}$. Failing the latter, the strategy reverts to not intervening.

Supplementary to Example 10.7, consider the following.

**Example 10.8.** Let $f$ be given by (2.6) with $p = q = 9$, $\alpha = 1$, $J = 8$, $c_\ell = \ell$ for $\ell \in \{1, 2, 5, 6, 7, 8\}$, $k_1 = 26$, $k_2 = 23$, $k_5 = 10$ and $k_8 = 0$. Next, let $s_5$ and $s_8$ be the respective solutions of (5.6). Note that $s_8 = 0$, whereupon (5.5) gives $S_8 = 0$. Define $S_{8,\ell}$ for $\ell \in \{1, 5, 6, 7\}$ by (5.11), $B_8$ by (10.7), $B_1$ by (10.8) with $S_{j,\ell} = S_{8,1}$, and $B_5$ by (10.8) with $S_{j,\ell} = S_{8,5}$. Subsequently define $B_\ell = B_8 - (B_8 - B_5)(c_8 - c_\ell)/(c_8 - c_5)$, and $k_\ell$ *via* (10.8) with $S_{j,\ell} = S_{8,\ell}$ for $\ell \in \{6, 7\}$. This yields $k_6 \approx 6.601$ and $k_7 \approx 3.269$. Next, define $a_1$ by (10.9) with $\ell = 1$, and $Y_1$ by (10.4) with $\ell = 1$. It can be ascertained that the equation $Y_1(x) = B_5 - c_5 x$ has a unique solution, $b_1$ say, in $(a_1, s_5)$. Thereafter, set $c_3 = \{c_5 - 2y_1'(b_1)\}/3$ and $c_4 = \{2c_5 - y_1'(b_1)\}/3$. Then for $\ell \in \{3, 4\}$, take $S_{8,\ell}$ from (5.11), $B_\ell = Y_1(b_1) + c_\ell b_1$, and $k_\ell$ from (10.8). This yields $c_3 \approx 4.166$, $c_4 \approx 4.583$, $k_3 \approx 13.098$ and $k_4 \approx 11.543$. Herewith we have completed (2.4) and (2.5). Computation using (5.6) and (10.6) verifies that $\Upsilon_\ell > \Upsilon_8$ for $\ell \in \{1, 2, \ldots, 7\}$. Thus in Algorithm 10.1, $\Lambda = \{8\}$ and $j = 8$. Step 6 then delivers $\kappa(1) = 8$, $\kappa(2) = 5$, $T_2 > 0$, $\kappa(3) = 3$, $T_3 > 0$, $\kappa(4) = 1$, $T_4 < 0$, and $\mathsf{N} = 4$. Thereafter, Step 7 leads to $\mu(1) = 4$, $a_1$ and $b_1$ as stated, and $L = 1$. Finally, Algorithm 10.3 delivers $N = 3$ and $\mathfrak{n} + L = 2$. The output is a hyper-generalized $(s, S)$ policy involving three suppliers for which (8.1) holds. Suppliers (1)–(3) are respectively 8, 5, and 1 according to the original ranking. With regard to (1.2) and (1.3), $s_{(3)} < r_{(2)} < s_{(2)} = r_{(1)} < s_{(1)}$.

The extraordinary aspect of Example 10.8 is the options entertained for the levels $s_{(2)}$ and $r_{(2)}$. Referring back to Lemma 5.10, $s_{(2)} = \sigma_2$ lies in the interior of the interval $[b_1, \sigma_1]$ throughout which $u = v$. On the other hand, $r_{(2)} = b_1$ separates an interval $(a_1, b_1)$ in which $u < v$ from a proper interval $[b_1, \sigma_2]$ in which $u = v$. By the way in which (2.4) and (2.5) have been rigged, $u = v_5 = v_6 = v_7 = v_8$ at $\sigma_2$, and, $u = v_3 = v_4 = v_5$ at $b_1$. This means that at the inventory level $s_{(2)} = r_{(1)}$, an inventory manager not only has the option of ordering from supplier (1) – originally 8 – to the level $S_{(1)} = S_8$ or from supplier (2) – originally 5 – to the level $S_{(2)} = S_{8,5}$, but also to the level $S_{8,\ell}$ from supplier $\ell$ for $\ell \in \{6, 7\}$, albeit that suppliers 6 and 7 are otherwise excluded from the policy. Likewise, at the inventory level $r_{(2)}$, an inventory manager not only has the option of ordering from supplier (2) to the level $S_{(2)}$ or not intervening, but also that of ordering to the level $S_{8,\ell}$ from supplier $\ell$ for $\ell \in \{3, 4\}$, albeit that suppliers 3 and 4 are otherwise excluded from the policy. The aforesaid notwithstanding, there is no way in which an order would be placed with supplier 2, inasmuch $u \le v = \min\{v_1, v_3, v_5, v_8\} = \min\{v_1, v_3, v_4, \ldots, v_8\} < v_2$ in $(-\infty, s_{(1)}]$. The cost function given by Example 10.8 is shown in Figure 2.

Incidentally, Examples 10.7 and 10.8 show that the sets $\{\sigma_m : m \in \mathcal{M}\}$ and $\{b_\nu : 1 \le \nu \le L\}$, introduced in Section 5, can contain a common element. Reverted to the convention throughout Section 6 that $j$ is the greatest minimizer of $y_\ell$ with respect to $\ell \in \mathcal{J}$, Example 10.7 has $j = 8$, $\mathsf{N} = \max \mathcal{M} = 7$, $L = 6$ and $b_6 = \sigma_1$. In agreement with Corollary 6.8, $u$ and $v$ are differentiable and $u' = v'$ at $b_6 = \sigma_1$. In Example 10.8, $\mathsf{N} = 4$, $L = 1$, and $b_1 = \sigma_3$. There holds $D_- u > D_- v > D_+ v = D_+ u$ at $b_1 = \sigma_3$, likewise agreeing with Corollary 6.8.

## 11. Relation to antecedent results

As mentioned in the introduction, the corresponding problem with a stochastic demand and two suppliers, one of which incurs a negligible set-up cost, has been investigated heretofore by Fox *et al.* [11]. Related problems have been studied more recently by Benjaafar *et al.* [4] and Helal *et al.* [15]. This section explores the connection between the conclusions of [4, 11, 15] and Theorems 6.6 and 9.1.

Apart from the demand being deterministic and not stochastic, the most striking difference between the present investigation and its predecessors [4, 11, 15] is the formulation of the problem as a QVI. The approach in each of the earlier papers is more pragmatic, and can be embedded in the deterministic continuous-time continuous-state setting as follows.

**Ansatz 11.1.** The function sought is a continuous real function $u$ such that $u \leq Mu$ in $\mathbb{R}$. The set $\Omega$ of $x \in \mathbb{R}$ for which (3.1) holds is an interval with a finite least upper bound $s$. Furthermore, $u$ is differentiable and $Au = f$ in $[s, \infty)$. Finally, given any function $v$ with these properties, $u \leq v$ in $\mathbb{R}$.

The merit of Ansatz 11.1 is reflected in the next theorem.

**Theorem 11.2.** *Suppose that $J \geq 2$, hypotheses (2.4) and (2.5) hold, and $f_\ell$ satisfies Hypothesis 4.2 for every $\ell \in \mathcal{J}$. Then there is a unique function satisfying Ansatz 11.1, which corresponds to an $(s, S)$ policy involving only supplier 1, or to a generalized $(s, S)$ policy involving $N \geq 2$ suppliers with levels (8.2) whereby supplier $(N)$ is supplier 1. When $k_J = 0$ and $N = J$, property (8.2) narrows to (8.1).*

*Proof.* The proof of Theorem 4.1 is independent of the inequality $Au \leq f$ in (2.10). So it carries through. Barring that of Lemma 5.11, so too does the proof of every lemma in Section 5. As a result, it can be concluded that a function $\upsilon$ satisfies Ansatz 11.1 only if $\upsilon = y_j$ in $[s_j, \infty)$ and $\upsilon = v$ in $(-\infty, s_j)$, where $j$ is a minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$, and $v$ is defined by (5.8), (5.12) and (5.13). Furthermore, if $k_J = 0$, $J$ is a minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$, and there is at least one other such minimizer, then $j$ is the second greatest minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. Otherwise, it is the greatest. Appealing to the arguments used to prove Theorem 6.6, it can be verified that $\upsilon$ meets the requirements of Ansatz 11.1 with $s = s_j$. The considerations leading from Theorem 6.6 to Theorem 8.1, lead to the deductions regarding $\upsilon$ representing an $(s, S)$ or a generalized $(s, S)$ policy. ☐

Theorem 11.2 corroborates the conclusions reached for the model with a stochastic demand by Fox *et al.* [11]. When $J = 2$ and $k_2 = 0$, a function satisfying Ansatz 11.1 corresponds to an $(s, S)$ policy involving only supplier 1, or a mixed-ordering policy involving both suppliers with levels $s_{(2)} < s_{(1)} = S_{(1)} < S_{(2)}$, whereby supplier (1) is the supplier with no set-up cost and supplier (2) is the supplier with a positive set-up cost. This should come as no surprise to those supporting the view that stochastic models are a generalization of deterministic ones, or, alternatively, that deterministic models are special or limiting cases of stochastic ones.

The above naturally raises the question of the equivalence of the function given by Theorem 11.2 and that given by Theorem 6.6. The next theorem provides the answer.

**Theorem 11.3.** *Under the conditions of Theorem 11.2, let $u$ be the solution of (2.10) satisfying Ansatz 3.3, and $\upsilon$ be the function satisfying Ansatz 11.1. Furthermore, if $k_J = 0$, $J$ is a minimizer of $y_\ell$ with respect to $\ell \in \mathcal{J}$, and there is at least one other such minimizer, let $j$ be the second greatest minimizer. Otherwise, let $j$ be the greatest. Denote the number subsequently given by Lemma 5.10 by $L$. Then*

$$u = \upsilon \quad \text{if} \;\; L = 0, \quad \text{and} \quad u \prec \upsilon \quad \text{if} \;\; L \geq 1,$$

*where $u \prec \upsilon$ means that $u \leq \upsilon$ everywhere in $\mathbb{R}$ and $u < \upsilon$ in a nonempty open subset of $\mathbb{R}$.*

*Proof.* In proving Theorem 11.2 it has been established that $\upsilon = y_j$ in $[s_j, \infty)$ and $\upsilon = v$ in $(-\infty, s_j)$, where $v$ is defined by (5.8), (5.12) and (5.13). By Lemmas 5.10 and 5.11, $u = y_j$ in $[s_j, \infty)$ and $u \leq v$ in $(-\infty, s_j)$ with equality throughout if and only if $L = 0$. ☐

The phenomenon that the function $\upsilon$ found in Theorem 11.2 need not be the solution $u$ of (2.10) provided by Theorem 6.6 has two sides. Notwithstanding that the number $s$ in Ansätze 3.3 and 11.1 fulfils a common role, one side is that $\upsilon$ does not satisfy the inequality $A\upsilon \leq f$ throughout $(-\infty, s]$. The reverse is that $u$ is not such that $u = v$, where $v$ is defined by (5.8), (5.12) and (5.13), throughout $(-\infty, s]$. What is clear though, whether one prefers the QVI or the more perfunctory approach, is that when $u$ and $v$ do not coincide, $u \prec v$ and $u$ corresponds to a hyper-generalized $(s, S)$ policy that is not a generalized $(s, S)$ policy. So, under this circumstance, the intuitional generalized $(s, S)$ policy is not the optimal inventory control policy, and, a hyper-generalized $(s, S)$ policy that is not a generalized $(s, S)$ policy is. We speculate that the same is true for the stochastic model.

For the problem with a stochastic demand, several suppliers, one of which may incur a negligible set-up cost, periodic review, and a finite planning horizon, Benjaafar *et al.* [4] concluded that for each period, except for a bounded interval of inventory levels, a generalized $(s, s)$ policy is optimal. A hyper-generalized $(s, S)$ policy would account for the exceptional interval of inventory levels.

The results of Helal *et al.* [15] for the discrete-time problem with a stochastic demand, two available suppliers, one of which may incur a negligible set-up cost, and an infinite planning horizon have a similar character to those of [4]. They affirm conditions under which an $(s, S)$ policy involving only the supplier with the greater set-up cost is optimal, and, when the demand distribution is exponential, antithetical conditions under which a generalized $(s, S)$ policy involving both suppliers is optimal. Beyond technicalities of proof, an explanation of why these two sets of conditions are not complementary is that there are circumstances under which the optimal policy is a hyper-generalized $(s, S)$ policy. Noteworthy is that the conditions under which an $(s, S)$ policy involving only the supplier with the greater set-up cost is shown to be optimal include the relative closeness of $k_1$ and $k_2$, which is a defining feature of Theorem 9.1.

A subsidiary result of Benjaafar *et al.* [4] is that if $k_J$ is large enough then a generalized $(s, S)$ policy is optimal. However, this result relies on a rather strong convexity assumption that does not apply to the model considered in the present paper. Indeed, Theorem 9.1 precludes the analogous result for the problem dealt with in the present paper.

In [4], it is further reported that extensive numerical experiments with normal, log-normal, gamma, and Poisson distributions, and thirty thousand experiments with randomly generated distributions all found that a generalized $(s, S)$ policy is optimal. Moreover, even for the examples demonstrating that a generalized $(s, S)$ policy is not optimal for a bounded interval of inventory levels, a generalized $(s, S)$ policy becomes optimal when the planning horizon is large enough. This has accordingly suggested that a generalized $(s, S)$ policy is optimal for the problem with an infinite planning horizon. A rerun with an infinite planning horizon of all the previously reported experiments has found exclusively that a generalized $(s, s)$ policy is optimal. We conjecture, as with the analysis of Fox *et al.*, that this comes about because this is what is sought. Theorems 6.6, 11.2 and 11.3 are indicative. Under those circumstances where there is a hyper-generalized $(s, S)$ policy that is not a generalized $(s, S)$ policy, this policy will turn out to be optimal.

## 12. Conclusion

The present paper continues the study of a deterministic continuous-time continuous-state inventory model with several suppliers in [5, 6]. In [5], the decision problem was formulated as a QVI, and a solution was postulated in the form of an ansatz corresponding to a generalized $(s, S)$ policy. Under the premise that every supplier incurs a significant set-up cost, it was shown that there is at most one solution satisfying the ansatz, and the necessary and sufficient conditions for the existence of such a solution were established. In [6] the scope of the ansatz was widened to admit correspondence with a refinement of a generalized $(s, S)$ policy, labelled a hyper-generalized $(s, S)$ policy. Retaining the premise that every supplier has a positive set-up cost, this led to the successful proof of the unconditional existence and uniqueness of a solution of the QVI. In the present paper, a further adaptation of the ansatz has led to the extension to the situation that a supplier may incur a negligible set-up cost. Moreover, it has been shown that the solution is stable and depends monotonically on the number of potential suppliers and the costs of each suppliers. The case of two suppliers has been scrutinized in some detail, and intuitive ideas, touched upon in the antecedent papers, about the way in which the solution is influenced by the costs of the suppliers have been substantiated.

A stochastic demand is arguably more realistic than a deterministic one. Nevertheless, some similarity between the optimal inventory control policy for the studied problem with a deterministic demand and related problems with a stochastic demand is to be expected. The stochastic problem with two suppliers has been investigated previously by Fox *et al.* [11], who concluded that the optimal policy is of one of two types. The analogous approach to the deterministic problem leads to the same conclusion. However, approaching the problem through the QVI and admitting the possibility of a hyper-generalized $(s, S)$ policy, it transpires that in those situations where the latter occurs, this supersedes the surrogate policy. Comparable models with several suppliers, one of which may incur a negligible set-up cost, have been more recently studied by Benjaafar *et al.* [4] and Helal *et al.* [15]. The conclusion of the former that, except for a bounded interval of inventory levels, a generalized $(s, s)$ policy is optimal, and of the latter that under some conditions an $(s, s)$ policy is optimal and under others which are not complementary a generalized $(s, s)$ policy is optimal, indicates that a hyper-generalized $(s, S)$ policy is appropriate in these situations too. The inference is that a hyper-generalized $(s, S)$ policy has a role to play in both of these problems with a stochastic demand and other more elaborate models with a more sophisticated demand.

Avenues for future research include the adaptation of the model to account for lost sales when demand is not met, the extension to suppliers having a limit to the quantity of stock that they can deliver, the extension to set-up costs being an increasing piecewise-constant function of the quantity supplied, and taking supplier-dependent lead-times into consideration. Moving to comparable stochastic continuous-time continuous-state inventory models with several suppliers, that with an exponential demand distribution holds promise, as do models with a diffusion demand [16,32,37], and their counterparts with a jump-diffusion demand [7,21]. Further possibilities are provided by the corresponding discrete-time models and discrete-demand models with an infinite planning horizon and with a finite planning horizon, complementing the analysis in [4,15].

## Appendix A. Notation

**Roman capitals**

| | |
|---|---|
| $A$ | Differential operator |
| $B_\ell$ | Base value of the affine function $v_\ell$ |
| $\mathcal{C}$ | Constant |
| $D_+$ | Right derivative of the function that follows |
| $D_-$ | Left derivative of the function that follows |
| $F_\ell$ | Function mapping $s_\ell$ to $k_\ell$ |
| $G$ | Function governing rate of change of inventory level with inventory level as input |
| $H$ | Arbitrary real number |
| $I_m$ | Interval demarcated by partition with index $m$ |
| $J$ | Number of available suppliers |
| $\mathcal{J}$ | Set of available suppliers |
| $K_*$ | Minimum set-up cost for which the optimal policy is of generalized $(s, S)$ type |
| $K_\dagger$ | Maximum set-up cost for which the optimal policy is of $(s, S)$ type |
| $\mathcal{K}$ | Subset of $\mathcal{J}$ used temporarily in computational algorithm |
| $L$ | Number of disjoint intervals constituting the set $\mathcal{S}$ |
| $M$ | Minimization operator |
| $M_j$ | Minimization operator when supplier $j$ is the sole supplier |
| $\mathcal{M}$ | Set of indices of elements of partition |
| $N$ | Number of suppliers involved in optimal policy |
| $\mathcal{N}$ | Set of indices of intervals used temporarily in computational algorithm |
| $\mathsf{N}$ | Number of elements in partition |
| $O$ | Asymptotic order |
| $\mathbb{R}$ | Set of real numbers |

| | |
|---|---|
| $S$ | Stock level to which inventory should be replenished |
| $S_j$ | Stock level $S$ when supplier $j$ is the sole supplier |
| $S_{j,\ell}$ | Argument at which the derivative of $y_j$ takes the value $-c_\ell$ |
| $S_{(n)}$ | Stock level in optimal policy to which inventory is replenished using supplier $(n)$ |
| $\mathcal{S}$ | Set of extraordinary inventory levels from which inventory is not replenished |
| $\mathcal{S}_j$ | Set $\mathcal{S}$ when supplier $j$ is the key supplier |
| $T_m$ | Test real number for the interval $I_m$ |
| $U$ | Necessary form of the unknown function $u$ |
| $U_j$ | Function $U$ when supplier $j$ is the key supplier |
| $V$ | Upper bound for the difference of two functions |
| $X$ | Absolute minimum |
| $Y_\nu$ | Solution of differential equation in subinterval $\nu$ of the set $\mathcal{S}$ |

**Lower case Roman alphabet**

| | |
|---|---|
| $a$ | Left endpoint of interval |
| $a_\nu$ | Left endpoint of subinterval of the set $\mathcal{S}$ with index $\nu$ |
| $b$ | Right endpoint of interval |
| $b_\nu$ | Right endpoint of subinterval of the set $\mathcal{S}$ with index $\nu$ |
| $c_j$ | Cost per item incurred using supplier $j$ |
| $\mathrm{d}$ | Differential with respect to the variable that follows |
| $\mathrm{e}$ | Base of the natural logarithms |
| $f$ | Function defining running cost with inventory level as input |
| $f_\ell$ | Auxiliary function related to $f$ associated with supplier $\ell$ |
| $g_0, g_1, g_\beta$ | Parameters in expression for the function $G$ |
| $i$ | Dummy index |
| $j$ | Sections 2–4 and Appendix D: index for available suppliers |
| | Section 5 and Appendix E: candidate key supplier in optimal policy |
| | Sections 6–11 and Appendix F: key supplier in optimal policy |
| $k_j$ | Set-up cost incurred using supplier $j$ |
| $k_*$, $k_\dagger$ | Critical set-up costs |
| $\ell$ | Index of available supplier |
| $m$ | Index of element of partition |
| $n$ | Index of supplier involved in optimal policy |
| $\mathsf{n}$ | Counter used temporarily in computational algorithm |
| $\mathfrak{n}$ | Number of minimizers |
| $p$, $q$ | Parameters in expression for the function $f$ |
| $r_{(n)}$ | Least stock level in optimal policy from which inventory is replenished using supplier $(n)$ |
| $s$ | Stock level from which inventory should be replenished |
| $s_j$ | Stock level $s$ when supplier $j$ is the sole supplier |
| $s_{(n)}$ | Greatest stock level in optimal policy from which inventory is replenished using supplier $(n)$ |
| $t$ | Time |
| $u$ | Section 2: future cost over infinite time horizon |
| | Sections 3–5 and Appendices B–D: unknown in QVI |
| | Sections 6–11 and Appendices E–G: solution of QVI |
| $u_\ell$ | Auxiliary function related to $u$ associated with supplier $\ell$ |
| $v$ | Piecewise-linear function coinciding with $u$ in the set $\Omega$ |
| $v_\ell$ | Affine function associated with supplier $\ell$ leading to the construction of $v$ |
| $w$ | Dummy left endpoint of interval |
| $x$ | Inventory level |

| | |
|---|---|
| $y$ | Solution of ordinary differential equation |
| $y_j$ | Solution $y$ when supplier $j$ is the sole supplier |
| $z$ | Dummy right endpoint of interval |

**Greek capitals**

| | |
|---|---|
| $\Lambda$ | Set of minimizers |
| $\Xi$ | Set of inventory levels at which solution of QVI is not differentiable |
| $\Sigma$ | Sum |
| $\Upsilon_\ell$ | Real number identifying $y_\ell$ |
| $\Omega$ | Set of inventory levels from which inventory is replenished |
| $\Omega_j$ | Set of inventory levels related to $\Omega$ when supplier $j$ is the key supplier |

**Lower case Greek alphabet**

| | |
|---|---|
| $\alpha$ | Discount rate |
| $\beta$ | Exponent in expression for the function $G$ |
| $\gamma_\ell$ | Idiosyncratic argument of the function $f_\ell$ |
| $\varepsilon$ | Error estimate |
| $\zeta$ | Reference inventory level |
| $\eta$ | Dummy variable |
| $\theta$ | Section 5: alternative candidate key supplier in optimal policy |
| | Section 6: least alternative key supplier in optimal policy |
| $\iota$ | Imaginary supplier |
| $\kappa$ | Function mapping index $m$ to index $\ell$ via the criterion $v = v_\ell$ in $I_m$ |
| $\lambda$ | Ordering of minimizers |
| $\mu$ | Function mapping index $\nu$ to index $m$ via the criterion $a_\nu \in I_m$ |
| $\nu$ | Index of open intervals constituting the set $\mathcal{S}$ |
| $\xi$ | Dummy variable |
| $\rho$ | Maximum of subset of $\Omega$ |
| $\rho_\ell$ | Maximum of subset of $\Omega$ with index $\ell$ |
| $\sigma_m$ | Element of partition with index $m$ |
| $\upsilon$ | Alternative to solution $u$ when the optimization problem is not formulated as a QVI |
| $\varphi_\ell$ | Function mapping $s_\ell$ to $S_\ell$ |
| $\omega_j$ | Subset of $\Omega$ with index $j$ |

**Superscripts**

| | |
|---|---|
| $\prime$ | Derivative with respect to variable other than time |
| $(i)$ | Member of a sequence with index $i$ |
| $*$ | Replica |
| $\pm$ | Comparable entities |
| $\cdot$ | Derivative with respect to time |
| $\sim$ | Transformed |

## Appendix B. Proof of Theorem 3.2

The proof of Theorem 3.2 is facilitated by the lemma below.

**Lemma B.1.** *Let $\ell \in \mathcal{J}$ and $u$ be any real function such that $M_\ell u$ is well defined and $u \le M_\ell u$ in $\mathbb{R}$. Set*

$$u_\ell(x) = u(x) + c_\ell x \quad \textit{for} \quad x \in \mathbb{R}. \tag{B.1}$$

*Then $u_\ell$ is nondecreasing in $\mathbb{R}$ when $k_\ell = 0$.*

*Proof.* By the hypotheses of the lemma, and (2.8),

$$u_\ell(x) \le \big(M_\ell u\big)(x) + c_\ell x = k_\ell + \min\{u_\ell(\eta) : \eta \ge x\} \tag{B.2}$$

for all $x \in \mathbb{R}$. This inequality immediately provides the result. □

Suppose now that $k_J = 0$ and $u$ is a real function with the property that $Mu$ is well defined in $\mathbb{R}$. If $u \le Mu$ in $\mathbb{R}$, then (2.9) and Lemma B.1 imply that $u_J$ is nondecreasing on $\mathbb{R}$. Consequently, inequality (B.2) for $\ell = J$ reduces to $\big(M_J u\big)(x) + c_J x = u_J(x)$. So, $u \le Mu \le M_J u = u$ in $\mathbb{R}$. This proves the 'only if' component of Theorem 3.2. The 'if' component is a tautology.

## Appendix C. Proof of Theorem 3.4

The following three lemmas aid the proof of the theorem.

**Lemma C.1.** *Let $\ell \in \mathcal{J}$ be such that $k_\ell > 0$, $u$ be any real function such that $M_\ell u$ is well defined and $u \le M_\ell u$ in $\mathbb{R}$, and $x \in \mathbb{R}$ be arbitrary. Then $u(x) = \big(M_\ell u\big)(x)$ if and only if*

$$u(x) = k_\ell + u(x+\xi) + c_\ell \xi \quad \textit{for some} \quad \xi > 0. \tag{C.1}$$

*Proof.* Suppose first that $u(x) = \big(M_\ell u\big)(x)$. Then, by (2.8), $u(x) = k_\ell + u(x+\xi) + c_\ell\xi$ for some $\xi \ge 0$. However, as $k_\ell > 0$, said $\xi$ must be positive. Thus, property (C.1) holds. Suppose, on the other hand, that $u(x) < \big(M_\ell u\big)(x)$. Then, by (2.8), $u(x) < k_\ell + u(x+\xi) + c_\ell\xi$ for all $\xi \ge 0$. This implies that $u(x) < k_\ell + u(x+\xi) + c_\ell\xi$ for all $\xi > 0$. Therewith, property (C.1) is negated. □

**Lemma C.2.** *Suppose that $k_J > 0$. Let $u$ be any real function such that $Mu$ is well defined and $u \le Mu$ in $\mathbb{R}$, and $x \in \mathbb{R}$ be arbitrary. Then (3.1) holds if and only if $u(x) = \big(Mu\big)(x)$.*

*Proof.* Combine Lemma C.1 with (2.9). □

**Lemma C.3.** *Suppose that $u$ is a continuous real function such that $Mu$ is well defined in $\mathbb{R}$. Then for every $\ell \in \mathcal{J}$ the set of $x \in \mathbb{R}$ for which $u(x) = \big(M_\ell u\big)(x)$ is closed. Moreover, the set of $x \in \mathbb{R}$ for which $u(x) = \big(Mu\big)(x)$ is closed.*

*Proof.* Suffice to note that the continuity of $u$ in $\mathbb{R}$ implies that of $M_\ell u$ for every $\ell \in \mathcal{J}$. □

To complete the proof of Theorem 3.4, suppose first that $u$ is a solution of (2.10) satisfying Ansatz 3.1, and $k_J > 0$. Under these conditions, Lemma C.2 implies that the set $\Omega$ in Ansatz 3.3 is equal to $(-\infty, s] \setminus \mathcal{S}$. Since $(-\infty, s]$ is unbounded and closed, while $\mathcal{S}$ is the finite union of bounded open intervals, $\Omega$ is not empty and $s \in \Omega$. Consequently, inasmuch $s \in \Omega$ and $\Omega \subseteq (-\infty, s]$, $s$ must be the least upper bound of $\Omega$. Ansatz 3.1 states furthermore that $u$ is differentiable and $Au = f$ in $\mathcal{S} \cup (s, \infty)$. By Lemma 4.7 of [6], $u$ is differentiable at any point not in $\mathcal{S} \cup (s, \infty)$ that is the left endpoint of an open subinterval of $\mathcal{S} \cup (s, \infty)$. The continuity of $u$ and $f$ subsequently implies that $Au = f$ at such a point. Thus, $u$ is differentiable and $Au = f$ at the left endpoint of any subinterval of $\mathcal{S} \cup [s, \infty)$. Herewith, $u$ satisfies Ansatz 3.3.

Suppose conversely that $u$ is a solution of (2.10) satisfying Ansatz 3.3, and $k_J > 0$. In this event, Lemma C.2 implies that $u = Mu$ in $\Omega \subseteq (-\infty, s] \setminus \mathcal{S}$, and $u < Mu$ in $\mathbb{R} \setminus \Omega \subseteq \mathcal{S} \cup [s, \infty)$. By Lemma C.3, $\Omega$ is closed. Hence, $s \in \Omega$. Therefore, $u = Mu$ in $(-\infty, s] \setminus \mathcal{S}$, and $u < Mu$ in $\mathcal{S} \cup (s, \infty)$. Finally, since $u$ is differentiable and $Au = f$ at the left endpoint of any subinterval of $\mathcal{S} \cup (s, \infty)$, and $\mathcal{S} \cup (s, \infty)$ is open, necessarily $u$ is differentiable and $Au = f$ in $\mathcal{S} \cup (s, \infty)$. Hence, $u$ satisfies Ansatz 3.1.

## Appendix D. Proof of Theorem 4.1

The proof is divided into a sequence of lemmas, in which, without further mention, $u$ is a given solution of (2.10) satisfying Ansatz 3.3.

**Lemma D.1.** *Let $x \in \mathbb{R}$ and $\ell \in \mathcal{J}$ be such that (C.1) holds. Then $u_\ell$ defined by (B.1) has an absolute minimum in $[x, \infty)$ at an $X > x$ for which*

$$u_\ell(x) = u_\ell(X) + k_\ell. \tag{D.1}$$

*Furthermore, $u_\ell \le u_\ell(x)$ in $(-\infty, X]$.*

*Proof.* If $k_\ell > 0$ the result is given by Lemma 4.3 of [5] and Lemma C.1 above. On the other hand, if $k_\ell = 0$ then (C.1) is equivalent to $u_\ell(x) = u_\ell(X)$ for some $X > x$. On top of this, Lemma B.1 says that $u_\ell \le u_\ell(X)$ in $(-\infty, X]$. ☐

**Lemma D.2.** *Let $\ell \in \mathcal{J}$ and $a < b$. Suppose that (C.1) holds for all $x \in (a, b)$. Then (C.1) holds for $x = a$. Furthermore, $u_\ell$ is constant in $[a, b]$.*

*Proof.* When $k_\ell = 0$, the continuity of $u$ and the monotonicity of $u_\ell$ given by Lemma B.1 deliver the conclusion. When $k_\ell > 0$, Lemma C.1 says that $u = M_\ell u$ in $(a, b)$. Whereupon, Lemma C.3 implies that $u = M_\ell u$ in $[a, b]$. So, by Lemma C.1, property (C.1) holds for $x = a$. With regard to the remaining assertion of the lemma, let $x$ be the greatest number in $[a, b]$ with the property that $u_\ell$ is constant in $[a, x]$. If $x < b$, then by Lemma D.1, there is an $X > x$ such that $u_\ell$ has an absolute minimum in $[x, \infty)$ at $X$. Furthermore, property (D.1) holds. Hence, for all $z \in [x, X] \cap [x, b]$, one has $u_\ell(z) = u(z) + c_\ell z = \big(M_\ell u\big)(z) + c_\ell z = k_\ell + \min\{u_\ell(\eta) : \eta \ge z\} = k_\ell + u_\ell(X) = u_\ell(x)$. Herewith, one arrives at a contradiction of the definition of $x$. Thus, $x = b$. ☐

**Lemma D.3.** *For $j \in \mathcal{J}$ define*

$$\omega_j = \big\{x \in \Omega : \text{(C.1) holds for some } \ell \in \{1, 2, \dots, j\}\big\}.$$

*Suppose that $j \in \mathcal{J} \setminus \{1\}$, and $[a, b) \subset \omega_j$ for some $a < b$. Then either $[a, b] \subset \omega_{j-1}$, there is a $\rho \in [a, b)$ such that $[a, b) \cap \omega_{j-1} = [a, \rho]$, or $[a, b) \cap \omega_{j-1} = \emptyset$.*

*Proof.* Whether $k_j > 0$ or not, $k_{j-1} > 0$. So, by Lemmas C.1 and C.3, $\omega_{j-1}$ is closed. Therefore, $(a, b) \setminus \omega_{j-1}$ is open. Consequently, either $(a, b) \subset \omega_{j-1}$, or, $(a, b) \setminus \omega_{j-1}$ contains a nonempty open interval $(w, z)$ with the property that (C.1) with $\ell = j$ holds for all $x \in (w, z)$. In the latter event, by Lemma D.2, $u_j$ is constant in $[w, z]$. Suppose now that $z \in \omega_{j-1}$. This means that there is an $\ell \in \{1, 2, \dots, j-1\}$ such that (C.1) with $x = z$ holds. Hence, by Lemma C.1, $\big(M_\ell u\big)(z) = u(z)$. Meanwhile, from the constancy of $u_j$ it follows that $u_\ell' = c_\ell - c_j < 0$ in $(w, z)$. Therefore,

$$\begin{aligned}\big(M_\ell u\big)(w) &= k_\ell - c_\ell w + \min\{u_\ell(\eta) : \eta \ge w\} = k_\ell - c_\ell w + \min\{u_\ell(\eta) : \eta \ge z\} \\ &= \big(M_\ell u\big)(z) + c_\ell(z - w) = u(z) + c_\ell(z - w).\end{aligned}$$

Thus, evoking the constancy of $u_j$,

$$\big(M_\ell u\big)(w) = u(w) - (c_j - c_\ell)(z - w) < u(w).$$

This contradicts the second component of (2.10). We are therefore forced to conclude that either $(a, b) \subset \omega_{j-1}$, $(a, b) \cap \omega_{j-1} = (a, \rho]$ for some $\rho \in (a, b)$, or, $(a, b) \cap \omega_{j-1} = \emptyset$. Recalling the closure of $\omega_{j-1}$ leads to the alternatives stated. ☐

**Lemma D.4.** *Let $a < b$ be such that $[a, b) \subset \Omega$. Then $u$ is piecewise linear and concave in $[a, b]$. Furthermore, denoting the right and left derivative of $u$ by $D_+u$ and $D_-u$ respectively, for all $x \in [a, b)$ there holds $\big(D_+u\big)(x) = -c_\ell$ where $\ell$ is the greatest number in $\mathcal{J}$ for which (C.1) holds, and, for all $x \in (a, b)$ there holds $\big(D_-u\big)(x) = -c_\ell$ where $\ell$ is the least number in $\mathcal{J}$ for which (C.1) holds.*

*Proof.* According to Lemmas D.2 and D.3, there is a sequence $\rho_0 = a \le \rho_1 \le \rho_2 \le \cdots \le \rho_J = b$ such that (C.1) holds for $x \in [a,b)$ and $\ell \in \mathcal{J}$ if $\rho_{\ell-1} \le x < \rho_\ell$ and only if $\rho_{\ell-1} \le x \le \rho_\ell$. Hence, if $x \in [\rho_{\ell-1}, \rho_\ell)$ for some $\ell \in \mathcal{J}$, then $\ell$ is the largest number in $\mathcal{J}$ for which (C.1) holds, and Lemma D.2 implies that $\big(D_+u\big)(x) = -c_\ell$. Similarly, if $x \in (\rho_{\ell-1}, \rho_\ell] \cap (a,b)$ then $\ell$ is the smallest number in $\mathcal{J}$ for which (C.1) holds, and $\big(D_-u\big)(x) = -c_\ell$. The piecewise linearity and concavity of $u$ in $[a,b]$ are a consequence. □

**Lemma D.5.** *Suppose that $x \in \overline{\Omega}$ and $(x,z) \subset \mathcal{S} \cup (s,\infty)$ for some $z > x$. Then $u$ is the restriction to $[x,z)$ of a solution $y$ of (4.1) satisfying $y'(x) = -c_\ell$ and $y(x) = \big(M_\ell u\big)(x)$ for some $\ell \in \mathcal{J}$.*

*Proof.* Because $u$ satisfies $Au = f$ in $(x,z)$, it is a solution of (4.1) there. Furthermore, as $f$ is continuous in $\mathbb{R}$, this solution can be extended to one, $y$ say, in $\mathbb{R}$. The continuity of $u$ and $y$ in $\mathbb{R}$ gives $u = y$ in $[x,z]$. Moreover, as Ansatz 3.3 states that $u$ is differentiable in $[x,z)$, $u'(x) = y'(x)$. To proceed, we distinguish between the cases $x \in \overline{\Omega} \setminus \Omega$ and $x \in \Omega$. Given that $\mathcal{S}$ is the finite union of open intervals, $x \in \overline{\Omega} \setminus \Omega$ is realized only if $[w,x) \subset \Omega$ for some $w < x$. However, by Lemmas C.1, C.3 and D.4, such necessitates $u'(x) = -c_J$ and $k_J = 0$. Theorem 3.2 gives $u(x) = \big(M_J u\big)(x)$. Turning to the case $x \in \Omega$, let $\ell \in \mathcal{J}$ be such that (C.1) holds. Then, by Lemma D.1, there is an $X > x$ such that $x$ is a maximum of $u_\ell$ in $(-\infty, X]$. Therefore, by the Fermat Theorem, $u_\ell'(x) = 0$. In other words, $u'(x) = -c_\ell$. Theorem 3.2 and Lemma C.1 give $u(x) = \big(M_\ell u\big)(x)$ for $k_\ell = 0$ and $k_\ell > 0$ respectively. □

We are now in a position to prove Theorem 4.1 adapting the proof of Theorem 4.2 of [6]. By Lemma D.5, equation (4.1) has a solution $y$ such that $u = y$ in $[s,\infty)$ and $y'(s) = -c_j$ for some $j \in \mathcal{J}$. Supplementarily, Lemma D.5 tells us that (4.4) holds. Recalling (B.2) and Lemma C.1, property (4.4) testifies that $u_j$ has a minimum in $[s,\infty)$, at $S$ say, for which $u_j(s) = u_j(S) + k_j$. Now, if $S = s$, necessarily $k_j = 0$, and (4.2) and (4.3) become tautologies. Alternatively, if $S > s$, then, inasmuch $u_j$ is differentiable at $S$, the Fermat Theorem implies $u_j'(S) = 0$. Hence, conditions (4.2) and (4.3) hold in this case too. Irrespectively, Ansatz 3.3 and Lemma D.4 imply that the only points in $(-\infty, s)$ where $u$ might not be differentiable are the right endpoints of the connected components of $\mathcal{S}$ and those points in the interior of $\Omega$ that are the endpoint of an interval in which $u$ is affine. Moreover, there are finitely many such points. Subsequently, retracing the proof of Lemma 4.9 of [5], it can be shown that $y \le u$ in $(-\infty, s]$. It remains to verify that $u \le v$ in $(-\infty, s)$, and $u < v$ in $\mathcal{S}$. To this end, let $x < s$ and $\ell \in \mathcal{J}$. Then

$$
\begin{aligned}
u(x) \le \big(M_\ell u\big)(x) &= k_\ell - c_\ell x + \min\{u_\ell(\eta) : \eta \ge x\} \le k_\ell - c_\ell x + \min\{u_\ell(\eta) : \eta \ge s\} \\
&= \big(M_\ell u\big)(s) + c_\ell(s - x).
\end{aligned} \tag{D.2}
$$

Hence, $u(x) \le v(x)$. Suppose additionally that $x \in \mathcal{S}$. Then, by Lemma C.1, the first inequality in (D.2) is strict if $k_\ell > 0$. Thus, $u(x) = v(x)$ only if $k_J = 0$ and $u(x) = \big(M_J u\big)(s) + c_J(s - x)$. By Theorem 3.2, this is equivalent to $u(x) = u(s) + c_J(s - x)$, which gives (3.1) with $\ell = J$ and $\xi = s - x$. Therefore, we have arrived at a contradiction of $x \in \mathcal{S}$. We can only conclude that when $x \in \mathcal{S}$ necessarily $u(x) < v(x)$.

Lemmas D.1–D.4 supersede Lemmas 4.3–4.6 of [5].

## Appendix E. Proof of Theorem 7.1

The proof proceeds in stages, each represented by a lemma. Unless otherwise stated, the notation is taken from Sections 4 and 5.

**Lemma E.1.** *Let $u^\pm$ be solutions of (2.10) satisfying Ansatz 3.3. Taking a generic solution $u$ where $j$ is a minimizer $j$ of $\{y_\ell : \ell \in \mathcal{J}\}$ as reference, denote by $S^\pm$ and $v^\pm$ counterparts to $S_j$ and $v$. Suppose that $S^+ \ge S^-$ and*

$$v^+(x) \ge v^-(x) \quad \text{for all} \quad x \le S^-. \tag{E.1}$$

*Then*

$$u^+(x) \ge u^-(x) \quad \text{for all} \quad x \le S^-. \tag{E.2}$$

*Moreover, if (E.1) is strict, so too is (E.2).*

*Proof.* If $u^+ = v^+$ at $x \le S^-$, inequality (E.1) states that $u^+(x) \ge v^-(x)$. Whereupon, as $v^- \ge u^-$ in $(-\infty, S^-]$ by Theorem 4.1 and Lemma 5.8, inequality (E.2) follows. On the other hand if $u^+ < v^+$ at such an $x$, there is an $a < x$ such that $u^+(a) = v^+(a)$ and $u^+ < v^+$ in $(a, x]$. Furthermore, by the theory of Sections 5 and 6, $\mathrm{d}\{\mathrm{e}^{\alpha\eta}(u^+ - u^-)(\eta)\}/\,\mathrm{d}\eta = \mathrm{e}^{\alpha\eta}(Au^+ - Au^-)(\eta) = \mathrm{e}^{\alpha\eta}(f - Au^-)(\eta) \ge 0$ at any $\eta \in (a, x)$ at which $u^-$ is differentiable. Hence, integrating piecewise from $a$ to $x$, $\mathrm{e}^{\alpha x}(u^+ - u^-)(x) \ge \mathrm{e}^{\alpha a}(u^+ - u^-)(a) = \mathrm{e}^{\alpha a}(v^+ - u^-)(a) \ge \mathrm{e}^{\alpha a}(v^+ - v^-)(a) \ge 0$. Thus (E.2) holds in this case too. Retracing the proof for each case, it can be verified that strictness in (E.1) gives strictness in (E.2). □

**Lemma E.2.** *Let $u^\pm$, $S^\pm$ and $v^\pm$ be as in Lemma E.1. If $S^+ \ge S^-$ and there is a number $V \ge 0$ such that*

$$\big(v^+ - v^-\big)(x) \le V \quad \textit{for all} \;\; x \le S^-, \tag{E.3}$$

*then*

$$\big(u^+ - u^-\big)(x) \le V \quad \textit{for all} \;\; x \le S^-. \tag{E.4}$$

*Proof.* The proof is analogous to that of Lemma E.1. If $u^- = v^-$ at $x \le S^-$, inequality (E.3) says that $\big(v^+ - u^-\big)(x) \le V$. Whereupon, because $u^+ \le v^+$ in $(-\infty, S^+]$, inequality (E.4) holds. On the other hand, if $u^- < v^-$ at $x \le S^-$, there is an $a < x$ such that $u^-(a) = v^-(a)$ and $u^- < v^-$ in $(a, x]$. Furthermore, $\mathrm{d}\{\mathrm{e}^{\alpha\eta}(u^+ - u^-)(\eta)\}/\,\mathrm{d}\eta = \mathrm{e}^{\alpha\eta}(Au^+ - Au^-)(\eta) = \mathrm{e}^{\alpha\eta}(Au^+ - f)(\eta) \le 0$ at any $\eta \in (a, x)$ at which $u^+$ is differentiable. Hence, integrating piecewise, $\mathrm{e}^{\alpha x}(u^+ - u^-)(x) \le \mathrm{e}^{\alpha a}(u^+ - u^-)(a) = \mathrm{e}^{\alpha a}(u^+ - v^-)(a) \le \mathrm{e}^{\alpha a}(v^+ - v^-)(a) \le \mathrm{e}^{\alpha a}V$. Inasmuch $\alpha > 0$, $a < x$ and $V \ge 0$, inequality (E.4) follows in this case too. □

**Lemma E.3.** *Let $j$ be a minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. Suppose that (2.5) holds with $k_j$ replaced by $k_j^- < k_j$. Denote the corresponding solution of (2.10) satisfying Ansatz 3.3 by $u^-$, and that of (4.1) satisfying (4.2) and (4.3) with $k_j$ replaced by $k_j^-$ for some $S \ge s$ by $y$. Then*

$$0 < u - u^- \le k_j - k_j^- + \big(y_j - y\big)(\gamma_j) \quad \textit{in} \;\; \mathbb{R}. \tag{E.5}$$

*Proof.* Let $s_j^-$ and $S_j^-$ be the values of $s$ and $S \ge s$ for which $y$ satisfies (4.2) and (4.3) with $k_j$ replaced by $k_j^-$. By Lemmas 5.1 and 5.3,

$$s_j < s_j^- \le \gamma_j \le S_j^- < S_j, \qquad y < y_j \quad \text{and} \quad y' > y_j' \tag{E.6}$$

in $\mathbb{R}$. Hence, $j$ is a minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$ with $y_j$ replaced by $y$. Consequently, by the theory developed in Sections 4 and 5, $u = y_j$ in $[s_j, \infty)$, and $u^- = y$ in $[s_j^-, \infty)$. Thus, $0 < u - u^- = y_j - y$ in $[s_j^-, \infty)$. Inasmuch (E.6) tells us that $y_j - y$ is strictly decreasing in $\mathbb{R}$, statement (E.5) in $[\gamma_j, \infty)$ follows. To confirm (E.5) in $(-\infty, \gamma_j)$, let $S_{j,\ell}^-$ denote the number defined by (5.8) with $y$ and $s_j^-$ in lieu of $y_j$ and $s_j$ for $1 \le \ell \le j$, $v_\ell^-$ the corresponding function (5.13) with $k_j^-$ in lieu of $k_j$ when $\ell = j$, and $v^-$ the ensuing function (5.12). For $\ell < j$, formula (5.13) gives

$$v_\ell - v_\ell^- = y_j(S_{j,\ell}) - y(S_{j,\ell}^-) + c_\ell\Big(S_{j,\ell} - S_{j,\ell}^-\Big) \tag{E.7}$$

in $\mathbb{R}$. However, by (5.8), $y_j' \ge -c_\ell$ in $[S_{j,\ell}, \infty)$. Consequently, by (E.6), $y' > -c_\ell$ in $[S_{j,\ell}, \infty)$. Hence, by (5.8), $S_{j,\ell}^- < S_{j,\ell}$ and $y_j' < -c_\ell < y'$ in $(S_{j,\ell}^-, S_{j,\ell})$. So,

$$y_j(S_{j,\ell}) - y_j(S_{j,\ell}^-) < -c_\ell\Big(S_{j,\ell} - S_{j,\ell}^-\Big) < y(S_{j,\ell}) - y(S_{j,\ell}^-). \tag{E.8}$$

Substituting (E.8) in (E.7) yields $\big(y_j - y\big)(S_{j,\ell}) < v_\ell - v_\ell^- < \big(y_j - y\big)(S_{j,\ell}^-)$. Recalling that $y_j - y$ is positive and strictly decreasing in $\mathbb{R}$, this leads to

$$0 < v_\ell - v_\ell^- < \big(y_j - y\big)(\gamma_j) \quad \text{for every } 1 \le \ell < j. \tag{E.9}$$

By the same argument,

$$0 < v_j - v_j^- < k_j - k_j^- + \big(y_j - y\big)(\gamma_j). \tag{E.10}$$

Combining (E.9) and (E.10) delivers $0 < v - v^- < k_j - k_j^- + \big(y_j - y\big)(\gamma_j)$ in $\mathbb{R}$. Therefore, by Lemmas E.1 and E.2, statement (E.5) holds in $(-\infty, \gamma_j]$. □

**Lemma E.4.** *Let $j$ be a minimizer of $\{y_\ell : \ell \in \mathcal{J}\}$. Suppose that (2.5) holds with $k_\ell$ replaced by $k_\ell^+ > k_\ell$ for some $\ell \in \mathcal{J} \setminus \{j\}$. Denote the corresponding solution of (2.10) satisfying Ansatz 3.3 by $u^+$. Then*

$$0 \le u^+ - u \le k_\ell^+ - k_\ell \quad \textit{in} \ \ \mathbb{R}. \tag{E.11}$$

*Proof.* Let $y_\ell^+$ be the solution of (4.1) satisfying (4.2) and (4.3) with $c_j = c_\ell$ and $k_j = k_\ell^+$ for some $S \ge s$. By Lemma 5.1, $y_\ell^+ > y_\ell$. Hence, $j$ is a minimizer of $\{y_i : i \in \mathcal{J}\}$ with $y_\ell$ replaced by $y_\ell^+$. Consequently, according to the theory of Sections 4 and 5, $u^+ = u$ in $\mathbb{R}$ when $\ell > j$. This gives (E.11) immediately. Otherwise, $u^+ = u = y_j$ in $[s_j, \infty)$, and the function $v^+$, fulfilling the role for $u^+$ that $v$ does for $u$, differs from $v$ solely in that $v_\ell$ is replaced by $v_\ell + k_\ell^+ - k_\ell$. Thus, $u^+ = u$ in $[s_j, \infty)$ and $0 \le v^+ - v \le k_\ell^+ - k_\ell$ in $\mathbb{R}$. Application of Lemmas E.1 and E.2 results in (E.11). □

**Lemma E.5.** *Suppose that (2.5) holds with $k_\ell$ replaced by $k_\ell^\pm$ for some $\ell \in \mathcal{J}$. Denote the corresponding solutions of (2.10) given by Theorem 6.6 by $u^\pm$, and the solution of (4.1) satisfying (4.2) and (4.3) with $c_j = c_\ell$ and $k_j = k_\ell^\pm$ for some $S \ge s$ by $y_\ell^\pm$. If $k_\ell^+ > k_\ell^-$ then*

$$0 \le u^+ - u^- \le k_\ell^+ - k_\ell^- + \big(y_\ell^+ - y_\ell^-\big)(\gamma_\ell) \quad \textit{in} \ \ \mathbb{R}. \tag{E.12}$$

*Proof.* Let $j$ be a minimizer of $y_i$ with respect to $i \in \mathcal{J} \setminus \{\ell\}$. By Lemma 5.1, $y_\ell^+ > y_\ell^-$. Hence, if $y_\ell^+ \le y_j$, $\ell$ is a minimizer of $\{y_i : i \in \mathcal{J}\}$ with $y_\ell$ replaced by $y_\ell^\pm$. Consequently, by Lemma E.3, statement (E.12) holds. Conversely, if $y_\ell^- \ge y_j$, $j$ is a minimizer of $\{y_i : i \in \mathcal{J}\}$ with $y_\ell$ replaced by $y_\ell^\pm$. Hence, by Lemma E.4, statement (E.12) holds in this case too. Finally, if $y_- < y_j < y_+$ then, by Lemma 5.1, there is a $k_\ell \in (k_\ell^-, k_\ell^+)$ such that $y_\ell = y_j$. Let $u$ be the corresponding solution of (2.10) satisfying Ansatz 3.3. By contrivance, $\ell$ is a minimizer of $\{y_i : i \in \mathcal{J}\}$, and of $\{y_i : i \in \mathcal{J}\}$ with $y_\ell$ replaced by $y_\ell^-$, while $j$ is a minimizer of $\{y_i : i \in \mathcal{J}\}$, and of $\{y_i : i \in \mathcal{J}\}$ with $y_\ell$ replaced by $y_\ell^+$. Thus, by Lemma E.3, statement (E.5) with $j = \ell$ holds, and, by Lemma E.4, statement (E.11) holds. Adding these statements yields (E.12) with $y_\ell$ in lieu of $y_\ell^+$. Because $y_\ell < y_\ell^+$, this gives (E.12) as stated. □

**Lemma E.6.** *Building on Lemma E.5, suppose that (2.5) holds with $k_\ell$ replaced by $k_\ell^\pm$ for every $\ell \in \mathcal{J}$, and $k_\ell^+ \ge k_\ell^-$ for every $\ell \in \mathcal{J}$. Then*

$$0 \le u^+ - u^- \le \sum_{\ell \in \mathcal{J}} \big\{k_\ell^+ - k_\ell^- + \big(y_\ell^+ - y_\ell^-\big)(\gamma_\ell)\big\} \quad \textit{in} \ \ \mathbb{R}.$$

*Proof.* Starting with $(k_1^-, k_2^-, \ldots, k_J^-)$ and replacing $k_\ell^-$ with $k_\ell^+$ when $k_\ell^+ > k_\ell^+$ in order of increasing $\ell$ creates a sequence of $J$-tuples preserving the inequalities in (2.5) and ending with $(k_1^+, k_2^+, \ldots, k_J^+)$. Synchronous successive application of Lemma E.5 yields the result. □

**Lemma E.7.** *Under the introductory assumptions of Theorem 7.1,*

$$|u^{(i)} - u| \le \varepsilon^{(i)} = \sum_{\ell \in \mathcal{J}} \big|k_\ell^{(i)} - k_\ell + \big(y_\ell^{(i)} - y_\ell\big)(\gamma_\ell)\big| \quad \textit{in} \ \ \mathbb{R}.$$

*Proof.* Define $k_\ell^+ = \max\big\{k_\ell, k_\ell^{(i)}\big\}$ and $k_\ell^- = \min\big\{k_\ell, k_\ell^{(i)}\big\}$ for $\ell \in \mathcal{J}$, apply Lemma E.6, and note that $y_\ell^\pm = y_\ell$ when $k_\ell^\pm = k_\ell$, while $y_\ell^\pm = y_\ell^{(i)}$ when $k_\ell^\pm = k_\ell^{(i)}$. This gives $u^+ - u^- \le \varepsilon^{(i)}$. However, Lemma E.6 further implies that $u^- \le u \le u^+$ and $u^- \le u^{(i)} \le u^+$. So $|u^{(i)} - u| \le u^+ - u^-$. □

Theorem 7.1 follows from Lemmas 5.1 and E.7.

## Appendix F. Proof of Theorem 7.2

The proof of Theorem 7.2 is facilitated by the next result comparing solutions of (4.1)–(4.3) assuming (2.4) and not (2.5).

**Lemma F.1.** *Suppose that $\ell \geq 2$. Let $y_{\ell-1}$ be the solution of (4.1) satisfying (4.2) and (4.3) with $c_j = c_{\ell-1}$ and $k_j = k_{\ell-1}$ for some $S \geq s$, and $y$ the corresponding solution with $c_j = c_\ell$ and $k_j = k_{\ell-1}$. Then $y > y_{\ell-1}$.*

*Proof.* Let $\varphi_\ell$ and $F_\ell$ be the functions appearing in the proof of Lemma 5.1, and $s^*$ the unique solution of $F_\ell(s^*) = k_{\ell-1}$ in $(-\infty, \gamma_\ell]$. By (5.1),

$$\int_s^{\varphi_\ell(s)} \mathrm{e}^{\alpha\eta}\, \mathrm{d}f_\ell(\eta) = 0$$

and

$$F_\ell(s) = \frac{f_\ell(s) - f_\ell(\varphi_\ell(s))}{\alpha} = \int_s^{\varphi_\ell(s)} \frac{\mathrm{e}^{\alpha(\eta-H)} - 1}{\alpha}\, \mathrm{d}f_\ell(\eta)$$

for all $s \leq \gamma_\ell$ and $H$. It follows that

$$\int_{\varphi_\ell(s)}^{\varphi_{\ell-1}(s)} \mathrm{e}^{\alpha\eta}\, \mathrm{d}f_\ell(\eta) = \int_s^{\varphi_{\ell-1}(s)} \mathrm{e}^{\alpha\eta}\, \mathrm{d}\big(f_\ell - f_{\ell-1}\big)(\eta) \tag{F.1}$$

and

$$F_\ell(s) - F_{\ell-1}(s) = -\int_s^{\varphi_\ell(s)} \frac{1 - \mathrm{e}^{\alpha(\eta-H)}}{\alpha}\, \mathrm{d}\big(f_\ell - f_{\ell-1}\big)(\eta) - \int_{\varphi_\ell(s)}^{\varphi_{\ell-1}(s)} \frac{\mathrm{e}^{\alpha(\eta-H)} - 1}{\alpha}\, \mathrm{d}f_{\ell-1}(\eta) \tag{F.2}$$

for all such $s$ and $H$. Since $f_\ell$ satisfies Hypothesis 4.2, while $\varphi_\ell(s) \geq \gamma_\ell$ and $\varphi_{\ell-1}(s) \geq \gamma_{\ell-1} \geq \gamma_\ell$ for $s \leq \gamma_\ell$, formula (F.1) gives $\varphi_\ell \leq \varphi_{\ell-1}$ in $(-\infty, \gamma_\ell]$. Whereupon, inasmuch $f_{\ell-1}$ also satisfies Hypothesis 4.2, taking $H = \varphi_\ell(s)$ in (F.2) we deduce that $F_\ell(s) \leq F_{\ell-1}(s)$ for all $s \leq \gamma_\ell$ for which $\varphi_\ell(s) < \gamma_{\ell-1}$, and taking $H = \gamma_{\ell-1}$ that $F_\ell(s) \leq F_{\ell-1}(s)$ for all $s \leq \gamma_\ell$ for which $\varphi_\ell(s) \geq \gamma_{\ell-1}$. So, $F_\ell \leq F_{\ell-1}$ in $(-\infty, \gamma_\ell]$ unconditionally. In particular, this implies that $F_{\ell-1}(s_{\ell-1}) = k_{\ell-1} = F_\ell(s^*) \leq F_{\ell-1}(s^*)$. In view of the strict monotonicity of $F_{\ell-1}$, this necessitates $s^* \leq s_{\ell-1}$. Subsequently, by (5.3),

$$\alpha \mathrm{e}^{\alpha x}\big(y - y_{\ell-1}\big)(x) = (c_\ell - c_{\ell-1})\mathrm{e}^{\alpha s^*} - \int_{s^*}^{s_{\ell-1}} \mathrm{e}^{\alpha\eta}\, \mathrm{d}f_{\ell-1}(\eta) \tag{F.3}$$

for all $x \in \mathbb{R}$. Because $c_\ell > c_{\ell-1}$ and $s^* \leq s_{\ell-1} \leq \gamma_{\ell-1}$, formula (F.3) yields $y > y_{\ell-1}$ in $\mathbb{R}$. ☐

With the above behind us, we may pick up the thread from Sections 5 and 6.

**Lemma F.2.** *Suppose that $J \geq 2$. Then $y_1 > y_2$ when $k_1$ is sufficiently large compared to $k_2$.*

*Proof.* By (5.3) and in analogy to (F.3),

$$\alpha \mathrm{e}^{\alpha x}\big(y_1 - y_2\big)(x) = (c_1 - c_2)\mathrm{e}^{\alpha s_1} - \int_{s_1}^{s_2} \mathrm{e}^{\alpha\eta}\, \mathrm{d}f_2(\eta) \tag{F.4}$$

for all $x \in \mathbb{R}$. By Lemma 5.1, $s_2 \leq \gamma_2$ and $s_1 \to -\infty$ as $k_1 \to \infty$. By Hypothesis 4.2, $f_2$ is strictly decreasing in $(-\infty, \gamma_2]$. Thus in the limit $k_1 \to \infty$, the right-hand side of (F.4), which does not depend on $x$, is either a positive real number or $+\infty$. ☐

**Lemma F.3.** *Under the assumptions of Theorem 7.2, $u \leq u^*$ when $\ell = 1$ and $k_1$ is large.*

*Proof.* Let $j$ be the greatest minimizer of $y_i$ with respect to $i \in \mathcal{J}^* = \mathcal{J} \setminus \{1\}$, and $v^*$ the counterpart of $v$ for $u^*$. By Lemma F.2, $j$ is the greatest minimizer of $\{y_i : i \in \mathcal{J}\}$ for large $k_1$. Hence, for such $k_1$, $u = u^* = y_j$ in $[s_j, \infty)$. Consequently, $v = \min\{v_i : i \in \mathcal{J}\} \le \min\{v_i : i \in \mathcal{J}^*\} = v^*$ in $\mathbb{R}$. Whence, by Lemma E.1, $u \le u^*$ in $(-\infty, S_j]$. Thus, $u \le u^*$ everywhere. □

**Lemma F.4.** *Under the assumptions of Theorem 7.2, $u = u^*$ when $\ell \ge 2$ and $k_\ell$ is close to $k_{\ell-1}$.*

*Proof.* By Lemmas 5.1 and F.1, $y_\ell > y_{\ell-1}$ when $k_\ell > k_{\ell+1}$ is sufficiently close to $k_{\ell-1}$ for $\ell < J$, and when $k_\ell \ge 0$ is sufficiently close to $k_{\ell-1}$ for $\ell = J$. For such $k_\ell$ it follows that the greatest minimizer $j$ of $\{y_i : i \in \mathcal{J}\}$ is the greatest minimizer of $\{y_i : i \in \mathcal{J}^*\}$ where $\mathcal{J}^* = \mathcal{J} \setminus \{\ell\}$. According to the theory in Section 5, this necessitates $u = u^*$ throughout $\mathbb{R}$ when $j < \ell$, and, $u = u^* = y_j$ in $[s_j, \infty)$ when $j > \ell$. In the latter event, since (5.8) implies that $y_j' < -c_{\ell-1}$ in $(S_{j,\ell}, S_{j,\ell-1})$,

$$y_j(S_{j,\ell-1}) - y_j(S_{j,\ell}) < -c_{\ell-1}(S_{j,\ell-1} - S_{j,\ell}).$$

Subsequently, by (5.13),

$$v_\ell(x) - v_{\ell-1}(x) > k_\ell - k_{\ell-1} + (c_\ell - c_{\ell-1})(S_{j,\ell} - x).$$

Consequently, if $k_\ell > k_{\ell-1} - (c_\ell - c_{\ell-1})(S_{j,\ell} - s_j)$ in addition to the afore-mentioned closeness of $k_\ell$ to $k_{\ell-1}$, we have $v_\ell > v_{\ell-1}$ in $(-\infty, s_j]$. Thus, denoting by $v^*$ the function for $u^*$ fulfilling the role that $v$ does for $u$, necessarily $v = v^*$ in $(-\infty, s_j]$ for such $k_\ell$. Therefore, by Lemmas 5.10 and 5.11, $u = u^*$ in $(-\infty, s_j)$. Whence, $u = u^*$ everywhere. □

To complete the proof of Theorem 7.2, we note that Lemma E.5 tells us that $u$ does not increase as $k_\ell$ decreases. Hence, if the theorem is true for suitably large $k_\ell$, then it is true for all $k_\ell$ preserving the inequalities in (2.5). Lemmas F.3 and F.4 confirm that the former is indeed so.

## Appendix G. Proof of Theorem 7.3

Let us first consider a select case.

**Lemma G.1.** *Let $u$ be the solution of (2.10) given by Theorem 6.6 and $u^+$ the corresponding solution with $c_\ell$ replaced by $c_\ell^+$ for some $\ell \in \mathcal{J}$. If $c_\ell^+ > c_\ell$ then $u^+ \ge u$ in $\mathbb{R}$.*

*Proof.* Suppose to begin with that $k_\ell > 0$. Denote by $u^{(1)}$ the solution of (2.10) given by Theorem 6.6 when $c_\ell^+$ is inserted in the sequence (2.4), $k_\ell^+$ is inserted in the corresponding position in (2.5), and the structure of (2.4) and (2.5) is otherwise preserved. Denote by $u^{(2)}$ the corresponding solution when $c_\ell$ and $k_\ell$ are subsequently removed from the just-constructed sequences (2.4) and (2.5). By Lemma F.4, $u^{(1)} = u$ if $k_\ell^+$ is sufficiently close to $k_\ell$. By Theorem 7.2, $u^{(2)} \ge u^{(1)}$. Hence, $u^{(2)} \ge u$. Passage to the limit $k_\ell^+ \to k_\ell$, which is justified by Theorem 7.1, yields $u^+ \ge u$. If $k_\ell = 0$, a further passage to the limit, likewise justified by Theorem 7.1, delivers the result for this case. □

In the light of Lemma E.6, it is enough to prove Theorem 7.3 assuming that $k_\ell^+ = k_\ell^-$ for every $\ell \in \mathcal{J}$. Starting with $\left(c_1^-, c_2^-, \ldots, c_J^-\right)$ and changing $c_\ell^-$ to $c_\ell^+$ if $c_\ell^+ > c_\ell^-$ in decreasing order of $\ell \in \mathcal{J}$ leads to a sequence of $J$-tuples preserving the ordering of (2.4). Simultaneous application of Lemma G.1 shows that the corresponding sequence of solutions of (2.10) is nondecreasing. As the first of these is $u^-$ and the last is $u^+$, this provides the theorem.

## Appendix H. Proof of Theorem 9.1

Let $F_1$ and $F_2$ be the functions from the proof of Lemma 5.1 associated with $f_1$ and $f_2$ respectively. By Lemmas F.1 and F.2, the solution $y$ of equation (4.1) satisfying (4.2) and (4.3) with $c_j = c_1$ and $k_j = k$ for some $S \geq s$ is such that $y < y_2$ when $k = k_2$, and $y > y_2$ when $k$ is large. Concurrently, by Lemma 5.1, $y$ depends continuously and strictly monotonically on $k \geq 0$. Therefore, there is a number $k_\dagger > k_2$ for which $y < y_2$ when $k < k_\dagger$, $y = y_2$ when $k = k_\dagger$, and $y > y_2$ when $k > k_\dagger$. It follows that 1 is the unique minimizer of $\{y_1, y_2\}$ when $k_1 < k_\dagger$, that 2 is the unique minimizer when $k_1 > k_\dagger$, and that 1 and 2 are both minimizers when $k_1 = k_\dagger$. This gives rise to the function $K_\dagger$. By the proof of Lemma 5.1 and (F.3), it can be expressed

$$K_\dagger(k_2) = F_1(s_1), \tag{H.1}$$

where $s_1$ is the solution of

$$-\int_{s_1}^{s_2} \mathrm{e}^{\alpha\eta}\,\mathrm{d}f_1(\eta) = (c_2 - c_1)\mathrm{e}^{\alpha s_2} \tag{H.2}$$

in $(-\infty, s_2)$, and $s_2$ is the solution of

$$F_2(s_2) = k_2 \tag{H.3}$$

in $(-\infty, \gamma_2]$. The continuous and monotonic dependence of $K_\dagger$ on $k_2$ are a consequence of (H.1)–(H.3).

For $k_1 \geq K_\dagger(k_2)$, whether $\{y_1, y_2\}$ has one or two minimizers, 2 is the greatest. Therefore, tracing the procedure pinpointing the solution of (2.10) satisfying Ansatz 3.3 in Section 5,

$$\sigma_2 = \{y_2(S_2) + k_2 + c_2 S_2 - y_2(S_{2,1}) - k_1 - c_1 S_{2,1}\}/(c_2 - c_1) \tag{H.4}$$

and

$$T_2 = f(\sigma_2) + c_1 - \alpha v_1(\sigma_2) = f(\sigma_2) + c_1 - \alpha\{y_2(S_{2,1}) + k_1 + c_1(S_{2,1} - \sigma_2)\}. \tag{H.5}$$

Taking (H.4), using (4.3) to eliminate $k_2$, equation (4.1) to eliminate $y_2$ in favour of $y_2'$, condition (4.2) to eliminate $y_2'(s_2)$, and substituting $y_2'(S_{2,1}) = -c_1$ yields

$$\sigma_2 = \{f_2(s_2) - f_1(S_{2,1}) - c_1 + c_2 - \alpha k_1\}/\{\alpha(c_2 - c_1)\}. \tag{H.6}$$

Similarly, taking (H.5), using (4.1) to eliminate $y_2$ in favour of $y_2'$, substituting $y_2'(S_{2,1}) = -c_1$, and using (H.6) to eliminate $k_1$ yields

$$T_2 = f_2(\sigma_2) - f_2(s_2) + c_1 - c_2. \tag{H.7}$$

By (H.6), $\sigma_2$ is a strictly decreasing continuous function of $k_1 \geq K_\dagger(k_2)$ such that $\sigma_2 \to -\infty$ as $k_1 \to \infty$. By (H.7) and Hypothesis 4.2, $T_2$ is a strictly decreasing continuous function of $\sigma_2 \leq s_2$ such that $T_2 \to \infty$ as $\sigma_2 \to -\infty$. Therefore, as a composite function, $T_2$ is continuous and strictly increasing for $k_1 \geq K_\dagger(k_2)$, and such that $T_2 \to \infty$ as $k_1 \to \infty$. However, because $y_1 = y_2$ when $k_1 = K_\dagger(k_2)$, Lemma 5.13 implies that $T_2 < 0$ when $k_1$ takes this value. Hence, there is a $k_* > K_\dagger(k_2)$ with the property that $T_2 < 0$ when $k < k_*$ and $T_2 \geq 0$ when $k \geq k_*$. This leads to the function $K_*$, which in the light of (H.6) and (H.7) can be expressed

$$K_*(k_2) = \{f_1(\sigma_2) - f_1(S_{2,1})\}/\alpha, \tag{H.8}$$

where $\sigma_2$ is the solution of

$$f_2(\sigma_2) = f_2(s_2) - c_1 + c_2 \tag{H.9}$$

in $(-\infty, s_2)$. By (H.3), $s_2$ is a continuous strictly decreasing function of $k_2 \geq 0$. We assert that $f_1(\sigma_2) - f_1(S_{2,1})$ is a strictly decreasing continuous function of $s_2 \leq \gamma_2$. Given that this assertion is true, formula (H.8) implies that $K_*$ is continuous and strictly increasing as claimed. Confirmation of the assertion is assigned to the lemma below.

**Lemma H.1.** *The function $s_2 \mapsto f_1(\sigma_2) - f_1(S_{2,1})$ is continuous and strictly decreasing in $(-\infty, \gamma_2]$.*

*Proof.* By (H.9) and Hypothesis 4.2, $\sigma_2$ is a strictly increasing continuous function of $s_2 \le \gamma_2$. Noting that $f_1(\sigma_2) = f_2(\sigma_2) - \alpha(c_2 - c_1)\sigma_2$, it subsequently suffices to show that $f_2(\sigma_2) - f_1(S_{2,1})$ is a strictly decreasing continuous function of $s_2 \le \gamma_2$. With this in mind, formula (5.9) can be rewritten

$$\int_{\gamma_1}^{S_{2,1}} \mathrm{e}^{\alpha\eta}\,\mathrm{d}f_1(\eta) = (c_2 - c_1)\mathrm{e}^{\alpha\gamma_2} - \int_{\gamma_2}^{\gamma_1} \mathrm{e}^{\alpha\eta}\,\mathrm{d}f_1(\eta) - \int_{s_2}^{\gamma_2} \mathrm{e}^{\alpha\eta}\,\mathrm{d}f_2(\eta). \tag{H.10}$$

This is most easily verified by working from (H.10) back to (5.9), to be specific by eliminating $s_2$ with the aid of the left-hand component of (5.1), substituting $f_2(\eta) = f_1(\eta) + \alpha(c_2 - c_1)\eta$, and simplifying. By the Implicit Function Theorem applied to (H.10), $f_1(S_{2,1})$ is continuously differentiable in terms of $f_2(s_2)$ for $s_2 < \gamma_2$, with

$$\frac{\mathrm{d}f_1(S_{2,1})}{\mathrm{d}f_2(s_2)} = \mathrm{e}^{-\alpha(S_{2,1} - s_2)}.$$

Hence, employing (H.9),

$$\frac{\mathrm{d}\{f_2(\sigma_2) - f_1(S_{2,1})\}}{\mathrm{d}f_2(s_2)} = 1 - \mathrm{e}^{-\alpha(S_{2,1} - s_2)} > 0 \quad \text{for } s_2 < \gamma_2.$$

As $s_2 \mapsto f_2(s_2)$ is continuous and strictly decreasing in $(-\infty, \gamma_2]$, this leads to the desired result. □

## Appendix I. Proof of Theorem 9.2

Set $k_1 = K_\dagger^-(k_2)$ and let $y_1^\pm$ be the solution of (4.1) satisfying (4.2) and (4.3) with $c_j = c_1^\pm$ and $k_j = k_1$ for some $S \ge s$. By Lemma F.1, $y_1^+ > y_1^-$. However, since $k_1 = K_\dagger^-(k_2)$, $y_1^- = y_2$. This means that 2 is the unique minimizer of $\{y_1^+, y_2\}$. Thus, $k_1 > K_\dagger^+(k_2)$. This confirms that part of the theorem regarding $K_\dagger$.

To confirm the part concerning $K_*$, denote the respective numbers induced by (5.8) with $j = 2$ and $c_\ell = c_1^\pm$ by $S_{2,1}^\pm$, and the solutions of (H.9) in $(-\infty, s_2)$ by $\sigma_2^\pm$. By Lemma 5.6, $S_{2,1}^+ < S_{2,1}^-$, while (H.9) implies that $\sigma_2^+ > \sigma_2^-$. Furthermore, by (H.8) and (H.9),

$$K_*^\pm(k_2) = \left\{f_2(s_2) - c_1^\pm + c_2 - f\left(S_{2,1}^\pm\right)\right\}/\alpha - c_1^\pm\left(S_{2,1}^\pm - \sigma_2^\pm\right) - c_2\sigma_2^\pm.$$

Substituting $f\left(S_{2,1}^\pm\right) = \left(Ay_2\right)\left(S_{2,1}^\pm\right) = -c_1^\pm + \alpha y_2\left(S_{2,1}^\pm\right)$ in the above gives

$$K_*^\pm(k_2) = \{f_2(s_2) + c_2\}/\alpha - y_2\left(S_{2,1}^\pm\right) - c_1^\pm\left(S_{2,1}^\pm - \sigma_2^\pm\right) - c_2\sigma_2^\pm.$$

Hence,

$$\begin{aligned}\left(K_*^+ - K_*^-\right)(k_2) &= y_2\left(S_{2,1}^-\right) - y_2\left(S_{2,1}^+\right) - c_2\left(\sigma_2^+ - \sigma_2^-\right) - c_1^+\left(S_{2,1}^+ - \sigma_2^+\right) + c_1^-\left(S_{2,1}^- - \sigma_2^-\right)\\ &< y_2\left(S_{2,1}^-\right) - y_2\left(S_{2,1}^+\right) + c_1^-\left(S_{2,1}^- - S_{2,1}^+\right).\end{aligned} \tag{I.1}$$

However, statement (5.8) implies that $y_2' < -c_1^-$ in $\left(S_{2,1}^+, S_{2,1}^-\right)$. So,

$$y_2\left(S_{2,1}^-\right) - y_2\left(S_{2,1}^+\right) < -c_1^-\left(S_{2,1}^- - S_{2,1}^+\right). \tag{I.2}$$

Combining (I.1) and (I.2) yields $\left(K_*^+ - K_*^-\right)(k_2) < 0$.

## Appendix J. Proof of Theorem 9.3

Let $\gamma_2^\pm$ be the respective number for which $f_2^\pm$ satisfies Hypothesis 4.2, and $y_2^\pm$, $s_2^\pm$ and $S_2^\pm$ the solution of (4.1)–(4.3) with $c_j = c_2^\pm$ and $k_j = k_2$. Note that $K_\dagger^\pm(k_2)$ is the number $k_1 \in (k_2, \infty)$ for which $y_1 = y_2^\pm$. By Lemma F.1, $y_2^+ > y_2^-$. Hence, by Lemma 5.1, $K_\dagger^+(k_2) > K_\dagger^-(k_2)$. This gives the theorem as far as $K_\dagger$ is concerned.

To obtain the conclusion concerning $K_*$, further to the notation introduced in the preceding paragraph, let $k_1 = K_*^+(k_2)$. Since $K_\dagger^-(k_2) < K_\dagger^+(k_2)$, $K_\dagger^-(K_\dagger^+(k_2)) > K_\dagger^+(k_2)$ by (9.1), and $K_\dagger^-$ is continuous and strictly increasing in $[0, \infty)$, there is a $k_\iota \in (k_2, K_\dagger^+(k_2))$ for which $K_\dagger^-(k_\iota) = K_\dagger^+(k_2)$. Subsequently, by Lemma 5.1, equation (4.1) has a solution $y_\iota$ satisfying (4.2) and (4.3) with $c_j = c_2^-$ and $k_j = k_\iota$ for a unique pair of values $S \geq s$. Moreover, these values, $S_\iota$ and $s_\iota$ say, are such that $s_\iota < \gamma_2^- < S_\iota$. Because $K_\dagger^-(k_\iota) = K_\dagger^+(k_2)$, $y_\iota \equiv y_2^+$. Hence, by Lemma 5.12, $s_\iota < s_2^+ < S_2^+ < S_\iota$. Invoking Lemma 5.6, let $S_{2,2} = S_2^+$, $S_{2,\iota} = S_\iota$, and $S_{2,1} > S_{2,\iota}$ be the unique solution of $\left(y_2^+\right)'(S_{2,1}) = -c_1$ in $(s_\iota, \infty)$. Define $v_1$ by (5.13) with $y_j = y_2^+$ and $j = 2$, $v_\iota$ by (5.13) with $y_j = y_2^+$, $j = 2$ and $c_\ell = c_2^-$, and $v_2$ by (5.13) with $y_j = y_2^+$, $j = 2$ and $c_\ell = c_2^+$. Let $\sigma_2^{(1)}$, $\sigma_2^{(2)}$ and $\sigma_2^{(3)}$ be the unique solutions of

$$v_1\big(\sigma_2^{(1)}\big) = v_\iota\big(\sigma_2^{(1)}\big), \quad v_1\big(\sigma_2^{(2)}\big) = v_2\big(\sigma_2^{(2)}\big) \quad \text{and} \quad v_\iota\big(\sigma_2^{(3)}\big) = v_2\big(\sigma_2^{(3)}\big).$$

Inasmuch (5.8) holds for $j = 2$ and $\ell = 1$, $\sigma_2^{(2)} < s_2^+ \leq \gamma_2^+$. Similarly, $\sigma_2^{(3)} < s_2^+ \leq \gamma_2^+$ and $\sigma_2^{(1)} < s_\iota < \gamma_2^-$. Set

$$T_2^{(1)} = \big(f - Av_1\big)\big(\sigma_2^{(1)}\big), \quad T_2^{(2)} = \big(f - Av_1\big)\big(\sigma_2^{(2)}\big) \quad \text{and} \quad T_2^{(3)} = \big(f - Av_\iota\big)\big(\sigma_2^{(3)}\big).$$

By Lemma 5.13, $T_2^{(3)} < 0$. Hence,

$$\big(f - Av_2\big)\big(\sigma_2^{(3)}\big) = T_2^{(3)} + c_2^+ - c_2^- < c_2^+ - c_2^-. \tag{J.1}$$

On the other hand, since $k_1 = K_*^+(k_2)$, $T_2^{(2)} = 0$. So,

$$\big(f - Av_2\big)\big(\sigma_2^{(2)}\big) = T_2^{(2)} + c_2^+ - c_1 = c_2^+ - c_1 > c_2^+ - c_2^-. \tag{J.2}$$

From (J.1), (J.2) and Lemma 5.9, we deduce that $\sigma_2^{(2)} < \sigma_2^{(3)}$. Since $v_2' = -c_2^+ < v_\iota' = -c_2^- < v_1' = -c_1$, it follows that $\sigma_2^{(1)} < \sigma_2^{(2)}$. Consequently, by Lemma 5.9, $T_2^{(1)} = \big(f - Av_1\big)\big(\sigma_2^{(1)}\big) > \big(f - Av_1\big)\big(\sigma_2^{(2)}\big) = T_2^{(2)} = 0$. This implies that $k_1 = K_*^+(k_2) > K_*^-(k_\iota)$. However, recalling that $k_\iota > k_2$ and that $K_*^-$ is strictly increasing in $[0, \infty)$, necessarily $K_*^-(k_\iota) > K_*^-(k_2)$. Thus $K_*^+(k_2) > K_*^-(k_2)$.

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