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# INDEPENDENCE NUMBER AND CONNECTIVITY FOR FRACTIONAL $(a,b,k)$-CRITICAL COVERED GRAPHS

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**Abstract.** A graph $G$ is a fractional $(a,b,k)$-critical covered graph if $G-U$ is a fractional $[a,b]$-covered graph for every $U \subseteq V(G)$ with $|U|=k$, which is first defined by (Zhou, Xu and Sun, *Inf. Process. Lett.* **152** (2019) 105838). Furthermore, they derived a degree condition for a graph to be a fractional $(a,b,k)$-critical covered graph. In this paper, we gain an independence number and connectivity condition for a graph to be a fractional $(a,b,k)$-critical covered graph and verify that $G$ is a fractional $(a,b,k)$-critical covered graph if

$$\kappa(G) \geq \max\left\{\frac{2b(a+1)(b+1)+4bk+5}{4b}, \frac{(a+1)^2\alpha(G)+4bk+5}{4b}\right\}.$$

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## 1. Introduction

The fractional factor problems in graphs can be regarded as a relaxation of the well-known cardinality matching problems. In a data transmission net work, the feasible assignment of data packets can be seen as a fractional flow problem, that is, the existence of a fractional $[a,b]$-factor [10].

The existence of a fractional $(a,b,k)$-critical covered graph plays a key role in the data transmission of networks. If a channel is assigned, and some nodes are damaged in the process of the data transmission at the moment, the possibility of transmission between data is characterized by whether the corresponding graph of the network is a fractional $(a,b,k)$-critical covered graph [21]. The independence number and connectivity conditions are often applied to measure the vulnerability and robustness of a network, which are two important parameters in network design and data transmission.

We discuss finite graphs which have neither multiple edges nor loops. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $X$ and $Y$ be disjoint vertex subsets of $G$. The number of edges of $G$ joining $X$ to $Y$ is denoted by $e_G(X,Y)$. We denote by $G[X]$ and $G-X$ the subgraph of $G$ induced by $X$ and $V(G)\setminus X$, respectively. A vertex subset $X$ of $G$ is called independent if $G[X]$ does not contain edges. We use $\alpha(G)$ and $\kappa(G)$

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to denote the independence number and the connectivity of $G$, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of $x$ in $G$ and by $N_G(x)$ the set of vertices adjacent to $x$ in $G$. We set $\delta(G) = \min\{d_G(x) : x \in V(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$. For any $X \subseteq V(G)$, we write $d_G(X) = \sum_{x \in X} d_G(x)$. We denote by $K_n$ the complete graph of order $n$.

Let $a \le b$ be two positive integers. *A spanning subgraph $F$ of $G$ is called an $[a,b]$-factor if $a \le d_F(x) \le b$ for every $x \in V(G)$.* Let $h : E(G) \to [0,1]$ be a function. *If $a \le \sum_{e \ni x} h(e) \le b$ holds for all $x \in V(G)$, then we call a graph $F$ with vertex set $V(G)$ and edge set $E_h$ a fractional $[a,b]$-factor of $G$ with indicator function $h$, where $E_h = \{e : e \in E(G), h(e) > 0\}$. A fractional $[a,b]$-factor of $G$ is called a fractional $r$-factor of $G$ if $a = b = r$. A graph $G$ is called a fractional $[a,b]$-covered graph if for any $e \in E(G)$, $G$ admits a fractional $[a,b]$-factor $F$, with indicator function $h_F$, such that $h_F(e) = 1$.* Bian and Zhou posed an independence number and connectivity condition for the existence of fractional $r$-factors in graphs, which is a special case of a more general result by Bian and Zhou [4].

**Theorem 1.1.** *([4]). Let $G$ be a graph, and $r \ge 1$ be an integer. Then $G$ possesses a fractional $r$-factor if*

$$\kappa(G) \ge \max\left\{\frac{(r+1)^2}{2}, \frac{(r+1)^2\alpha(G)}{4r}\right\}.$$

Some other results related to factors [1–3, 12, 14, 16, 19, 20, 22, 23, 25–27] and fractional factors [5–8, 11, 13, 15, 17, 18, 24] of graphs were obtained by many authors. *A graph $G$ is called a fractional $(a,b,k)$-critical covered graph if after removing any $k$ vertices of $G$, the resulting graph of $G$ is a fractional $[a,b]$-covered graph*, which is first defined by Zhou, Xu and Sun [21]. Furthermore, Zhou, Xu and Sun [21] obtained a degree condition for graphs being fractional $(a,b,k)$-critical covered.

**Theorem 1.2.** *([21]). Let $a, b$ and $k$ be three integers with $a \ge 1$, $b \ge \max\{2, a\}$ and $k \ge 0$, and let $G$ be a graph of order $n$ with $n \ge \frac{(a+b)(a+b-1)+bk+3}{b}$ and $\delta(G) \ge a+k+1$. Then $G$ is a fractional $(a,b,k)$-critical covered graph if*

$$\max\{d_G(x), d_G(y)\} \ge \frac{an+bk+2}{a+b}$$

*for every pair of nonadjacent vertices $x$ and $y$ of $G$.*

In this paper, we study the relationship between the independence number, connectivity and fractional $(a,b,k)$-critical covered graphs, and gain a new result on the existence of fractional $(a,b,k)$-critical covered graphs, which is an extension of Theorem 1.1.

**Theorem 1.3.** *Let $a, b$ and $k$ be three integers with $b \ge a \ge 1$ and $k \ge 0$, and let $G$ be a graph. Then $G$ is a fractional $(a,b,k)$-critical covered graph if*

$$\kappa(G) \ge \max\left\{\frac{2b(a+1)(b+1)+4bk+5}{4b}, \frac{(a+1)^2\alpha(G)+4bk+5}{4b}\right\}.$$

We immediately derive the following result from Theorem 1.3.

**Corollary 1.4.** *Let $a$ and $b$ be integers with $b \ge a \ge 1$, and let $G$ be a graph. Then $G$ is a fractional $[a,b]$-covered graph if*

$$\kappa(G) \ge \max\left\{\frac{2b(a+1)(b+1)+5}{4b}, \frac{(a+1)^2\alpha(G)+5}{4b}\right\}.$$

If $a = b = r$ in Theorem 1.3, then we have the following corollary. Obviously, Corollary 1.5 is a generalization of Theorem 1.1 by the definition of a fractional $r$-covered graph.

**Corollary 1.5.** *Let $r$ be an integer with $r \geq 1$, and let $G$ be a graph. Then $G$ is a fractional $r$-covered graph if*

$$\kappa(G) \geq \max\left\{\frac{2r(r+1)^2+5}{4r}, \frac{(r+1)^2\alpha(G)+5}{4r}\right\}.$$

## 2. Proof of Theorem 3

We use the following lemma to verify Theorem 1.3.

**Lemma 2.1.** *([9]). Let $G$ be a graph, and let $a, b$ be two integers with $b \geq a \geq 0$. Then $G$ is a fractional $[a, b]$-covered graph if and only if for any vertex subset $X$ of $G$,*

$$\theta_G(X, Y) = b|X| + d_{G-X}(Y) - a|Y| \geq \varepsilon(X, Y)$$

*where $Y = \{y : y \in V(G) \setminus X, d_{G-X}(y) \leq a\}$ and $\varepsilon(X, Y)$ is defined by*

$$\varepsilon(X,Y) = \begin{cases} 2, \textit{ if } X \textit{ is not independent}, \\ 1, \textit{ if } X \textit{ is independent and there is an edge joining } X \textit{ and } V(G) \setminus (X \cup Y), \textit{ or} \\ \quad \textit{there is an edge } e = xy \textit{ joining } X \textit{ and } Y \textit{ with } d_{G-X}(y) = a \textit{ for } y \in Y, \\ 0, \textit{ otherwise}. \end{cases}$$

*Proof of Theorem 3.* Let $U \subseteq V(G)$ with $|U| = k$, and let $H = G - U$. It suffices to verify that $H$ is a fractional $[a, b]$-covered graph. We shall prove this by contradiction. Suppose that $H$ is not a fractional $[a, b]$-covered graph. Then using Lemma 2.1, we have

$$\theta_H(X, Y) = b|X| + d_{H-X}(Y) - a|Y| \leq \varepsilon(X, Y) - 1 \tag{2.1}$$

for some vertex subset $X$ of $H$ and $Y = \{y : y \in V(H) \setminus X, d_{H-X}(y) \leq a\}$.

**Claim 2.2.** *$Y \neq \emptyset$.*

*Proof.* Let $Y = \emptyset$. Then by (2.1) and $\varepsilon(X, \emptyset) \leq |X|$, we get

$$\varepsilon(X, \emptyset) - 1 \geq \theta_H(X, \emptyset) = b|X| \geq |X| \geq \varepsilon(X, \emptyset),$$

this is a contradiction. Claim 2.2 is proved. □

**Claim 2.3.** *$X \neq \emptyset$.*

*Proof.* Let $X = \emptyset$. Then $\varepsilon(\emptyset, Y) = 0$. Note that $\delta(H) = \delta(G - U) \geq \delta(G) - k \geq \kappa(G) - k \geq \frac{2b(a+1)(b+1)+4bk+5}{4b} - k = \frac{(a+1)(b+1)}{2} + \frac{5}{4b} > a + 1$. Then using (2.1) and Claim 2.2 we obtain

$$\begin{aligned} -1 \geq \theta_H(\emptyset, Y) &= d_H(Y) - a|Y| \geq (\delta(H) - a)|Y| \\ &> ((a+1) - a)|Y| = |Y| \geq 1, \end{aligned}$$

which is a contradiction. We prove Claim 2.3. □

Since $Y \neq \emptyset$ (by Claim 2.2), we take $y_1 \in Y$ such that $y_1$ is the vertex with the minimum degree in $H[Y]$. We set $N_1 = N_H[y_1] \cap Y$ and $Y_1 = Y$. If $Y - \bigcup_{1\le j<i} N_j \neq \emptyset$ for $i \geq 2$, write $Y_i = Y - \bigcup_{1\le j<i} N_j$. Then take $y_i \in Y_i$ such that $y_i$ is the vertex with the minimum degree in $H[Y_i]$, and $N_i = N_H[y_i] \cap Y_i$. We go on these procedures until we arrive at the situation in which $Y_i = \emptyset$ for some $i$, say for $i = m+1$. Then from the definition above we see that $\{y_1, y_2, \cdots, y_m\}$ is an independent set of $H$, and is also an independent set of $G$. Obviously, $m \geq 1$ by Claim 2.2.

If we write $|N_i| = n_i$, then $|Y| = \sum_{1\le i\le m} n_i$. We set $W = V(H) \setminus (X \cup Y)$ and $\kappa(H - X) = t$.

**Claim 2.4.** *$m \neq 1$ or $W \neq \emptyset$.*

*Proof.* Let $m = 1$ and $W = \emptyset$. Note that $W = \emptyset$ is equivalent to saying that there is no vertex $y$ in $V(H) \setminus X$ such that $d_{H-X}(y) > a$. Then we easily see that

$$|V(G)| = |U| + |X| + n_1 = k + |X| + n_1 > \kappa(G) \geq \frac{2b(a+1)(b+1) + 4bk + 5}{4b},$$

namely,

$$|X| > \frac{2b(a+1)(b+1)+5}{4b} - n_1. \tag{2.2}$$

According to (2.1), (2.2), $\varepsilon(X, Y) \leq 2$ and the choice of $y_1$, we get

$$\begin{aligned}
1 \geq \varepsilon(X,Y) - 1 \geq \theta_H(X,Y) &= b|X| + d_{H-X}(Y) - a|Y| \\
&= b|X| + n_1(n_1 - 1) - an_1 \\
&> b\Big(\frac{2b(a+1)(b+1)+5}{4b} - n_1\Big) + n_1(n_1-1) - an_1 \\
&= \frac{2b(a+1)(b+1)+5}{4} + \Big(n_1 - \frac{a+b+1}{2}\Big)^2 - \frac{(a+b+1)^2}{4} \\
&\geq \frac{2b(a+1)(b+1)+5}{4} - \frac{(a+b+1)^2}{4} \\
&= \frac{b^2(2a+1) - (a+1)^2 + 5}{4} \\
&\geq \frac{a^2(2a+1) - (a+1)^2 + 5}{4} \\
&= \frac{a(a^2-1)+2}{2} \geq 1,
\end{aligned}$$

which is a contradiction. We verify Claim 2.4. □

**Claim 2.5.** $d_{H-X}(Y) \geq \sum_{1\le i\le m} n_i(n_i - 1) + \frac{mt}{2}$.

*Proof.* In terms of the choice of $y_i$, we derive

$$\sum_{1\le i\le m} \Big(\sum_{y\in N_i} d_{Y_i}(y)\Big) \geq \sum_{1\le i\le m} n_i(n_i-1). \tag{2.3}$$

For the left-hand side of (2.3), an edge joining a vertex $x$ in $N_i$ and a vertex $y$ in $N_j$ $(i < j)$ is counted only once, namely, it is counted in $d_{Y_i}(x)$ but not in $d_{Y_j}(y)$. The edges of type $xy$ ($x \in N_i$ and $y \in N_j$, for $i < j$) are not counted on the RHS of (2.3). Hence, we admit

$$d_{H-X}(Y) \geq \sum_{1\le i\le m} n_i(n_i-1) + \sum_{1\le i<j\le m} e_H(N_i, N_j) + e_H(Y, W). \tag{2.4}$$

In light of $\kappa(H-X)=t$ and Claim 2.4, we get

$$e_H(N_i, \bigcup_{j\neq i} N_j) + e_H(N_i, W) \geq t \tag{2.5}$$

for every $N_i$ $(1 \leq i \leq m)$. Using (2.5), we obtain

$$\sum_{1\leq i\leq m} (e_H(N_i, \bigcup_{j\neq i} N_j) + e_H(N_i, W)) = 2 \sum_{1\leq i<j\leq m} e_H(N_i, N_j) + e_H(Y, W) \geq mt.$$

We easily see that

$$\sum_{1\leq i<j\leq m} e_H(N_i, N_j) + e_H(Y, W) \geq \frac{mt}{2}. \tag{2.6}$$

It follows from (2.4) and (2.6) that

$$d_{H-X}(Y) \geq \sum_{1\leq i\leq m} n_i(n_i-1) + \frac{mt}{2}.$$

We finish the proof of Claim 2.5. □

Note that $|Y| = \sum_{1\leq i\leq m} n_i$. It follows from (2.1), $\varepsilon(X, Y) \leq 2$ and Claim 2.5 that

$$\begin{aligned}
1 &\geq \varepsilon(X, Y) - 1 \geq \theta_H(X, Y) = b|X| + d_{H-X}(Y) - a|Y| \\
&\geq b|X| + \sum_{1\leq i\leq m} n_i(n_i-1) + \frac{mt}{2} - a \sum_{1\leq i\leq m} n_i \\
&= b|X| + \sum_{1\leq i\leq m} \left(\left(n_i - \frac{a+1}{2}\right)^2 - \frac{(a+1)^2}{4}\right) + \frac{mt}{2} \\
&\geq b|X| - \frac{m(a+1)^2}{4} + \frac{mt}{2} \\
&= b|X| + \left(\frac{t}{2} - \frac{(a+1)^2}{4}\right)m,
\end{aligned}$$

namely,

$$1 \geq b|X| + \left(\frac{t}{2} - \frac{(a+1)^2}{4}\right)m. \tag{2.7}$$

Combining (2.7), Claim 2.3 and $m \geq 1$, we admit

$$\begin{aligned}
1 &\geq b|X| + \left(\frac{t}{2} - \frac{(a+1)^2}{4}\right)m \\
&\geq b + \left(\frac{t}{2} - \frac{(a+1)^2}{4}\right)m \\
&\geq 1 + \left(\frac{t}{2} - \frac{(a+1)^2}{4}\right)m,
\end{aligned}$$

which implies

$$\frac{t}{2} - \frac{(a+1)^2}{4} \leq 0. \tag{2.8}$$

We easily see that $\alpha(G) \geq \alpha(G[Y]) = \alpha(H[Y]) \geq m$ and $\kappa(G) \leq \kappa(G-U)+k = \kappa(H)+k \leq \kappa(H-X)+|X|+k = t+|X|+k$. In light of $\kappa(G) \geq \max\left\{\frac{2b(a+1)(b+1)+4bk+5}{4b}, \frac{(a+1)^2\alpha(G)+4bk+5}{4b}\right\}$, (2.7) and (2.8), we derive

$$\begin{aligned}
1 &\geq b|X| + \Big(\frac{t}{2} - \frac{(a+1)^2}{4}\Big)m \\
&\geq b(\kappa(G)-k-t) + \Big(\frac{t}{2} - \frac{(a+1)^2}{4}\Big)\alpha(G) \\
&\geq b(\kappa(G)-k-t) + \Big(\frac{t}{2} - \frac{(a+1)^2}{4}\Big)\cdot\frac{4b\kappa(G)-4bk-5}{(a+1)^2} \\
&= \frac{5}{4} + \Big(\frac{4b\kappa(G)-4bk-5}{2(a+1)^2} - b\Big)t \\
&\geq \frac{5}{4} + \Big(\frac{b(a+1)(b+1)}{(a+1)^2} - b\Big)t \\
&\geq \frac{5}{4},
\end{aligned}$$

which is a contradiction. Theorem 1.3 is verified. $\square$

## 3. Remarks

**Remark 3.1.** Now we discuss a sharpness of the connectivity condition in Theorem 1.3. This condition is best possible in the sense that we cannot replace it by $\kappa(G) \geq \frac{2b(a+1)(b+1)+4bk+5}{4b} - 1$.

Let $k \geq 0$ be an integer, and $b \geq a \geq 4$ be two even integers such that $\frac{b(a+1)(b+1)-b}{2a}$ is an integer. We construct a graph $G = K_{\frac{(a+1)(b+1)-1}{2}+k} \vee (\frac{b(a+1)(b+1)-b}{2a}K_1)$, where $\vee$ means "join". Then it is obvious that $\frac{2b(a+1)(b+1)+4bk+5}{4b} > \kappa(G) = \frac{(a+1)(b+1)-1}{2}+k = \frac{2b(a+1)(b+1)+4bk+2b}{4b}-1 > \frac{2b(a+1)(b+1)+4bk+5}{4b}-1$ by the definition of $\kappa(G)$. We select $U \subseteq V(K_{\frac{(a+1)(b+1)-1}{2}+k})$ with $|U| = k$, $X = V(K_{\frac{(a+1)(b+1)-1}{2}+k}) \setminus U$, $Y = V(\frac{b(a+1)(b+1)-b}{2a}K_1)$ and $H = G-U$. Clearly, $\varepsilon(X,Y) = 2$. Thus, we derive

$$\begin{aligned}
\theta_H(X,Y) &= b|X| + d_{H-X}(Y) - a|Y| \\
&= b\cdot\frac{(a+1)(b+1)-1}{2} - a\cdot\frac{b(a+1)(b+1)-b}{2a} \\
&= 0 < 2 = \varepsilon(X,Y).
\end{aligned}$$

By Lemma 2.1, $H$ is not a fractional $[a,b]$-covered graph, namely, $G$ is not a fractional $(a,b,k)$-critical covered graph.

**Remark 3.2.** Now we discuss a sharpness of the connectivity condition in Theorem 1.3. This condition is best possible in the sense that we cannot replace it by $\kappa(G) \geq \frac{(a+1)^2\alpha(G)+4bk+4}{4b}$.

Let $b \geq a \geq 1$, $m \geq 1$ and $k \geq 0$ be integers such that $a$ is odd and $\frac{(a+1)^2m+4}{4b}$ is an integer. We construct a graph $G = K_{p+k} \vee (mK_{\frac{a+1}{2}})$, where $p = \frac{(a+1)^2m+4}{4b}$. From the definitions of $\alpha(G)$ and $\kappa(G)$, we easily see that $\alpha(G) = m$ and $\kappa(G) = p+k = \frac{(a+1)^2m+4bk+4}{4b} = \frac{(a+1)^2\alpha(G)+4bk+4}{4b}$. We select $U \subseteq V(K_{p+k})$ with $|U| = k$, $X = V(K_{p+k}) \setminus U$, $Y = V(mK_{\frac{a+1}{2}})$ and $H = G-U$. Obviously, $\varepsilon(X,Y) = 2$. Thus, we gain

$$\begin{aligned}\theta_H(X,Y) &= b|X| + d_{H-X}(Y) - a|Y| \\ &= b \cdot \frac{(a+1)^2 m + 4}{4b} + m \cdot \frac{a+1}{2} \cdot \Big(\frac{a+1}{2} - 1\Big) - am \cdot \frac{a+1}{2} \\ &= 1 < 2 = \varepsilon(X,Y).\end{aligned}$$

Using Lemma 2.1, $H$ is not a fractional $[a, b]$-covered graph, and so $G$ is not a fractional $(a, b, k)$-critical covered graph.

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