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RESTRAINED DOUBLE ROMAN DOMINATION OF A GRAPH

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Abstract. For a graph $G = (V, E)$, a restrained double Roman dominating function is a function $f : V \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then the vertex v must have at least one neighbor w with $f(w) \geq 2$, and at the same time, the subgraph $G[V_0]$ which includes vertices with zero labels has no isolated vertex. The weight of a restrained double Roman dominating function f is the sum $f(V) = \sum_{v\in V} f(v)$, and the minimum weight of a restrained double Roman dominating function on G is the restrained double Roman domination number of G. We initiate the study of restrained double Roman domination with proving that the problem of computing this parameter is NP-hard. Then we present an upper bound on the restrained double Roman domination number of a connected graph G in terms of the order of G and characterize the graphs attaining this bound. We study the restrained double Roman domination *versus* the restrained Roman domination. Finally, we investigate the bounds for the restrained double Roman domination of trees and determine trees T attaining the exhibited bounds.

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1. Introduction

Dominations have become one of the major areas in graph theory. Their steady and rapid growth during the past thirty years may be due to the diversity of their applications to both theoretical and real-world problems, such as facility location problems, strategy of defence of cities and etc. Among the domination-type parameters that have been studied, are the Roman [18], double Roman [3], Italian [4] and double Italian [13] domination numbers in graphs.

The initial studies of Roman domination [14, 18] have been motivated by a historical application. In the 4th century, Emperor Constantine was faced with a difficult problem of how to defend the Roman Empire with limited resources. He decreed that for all cities in the Roman Empire, at most two legions should be stationed. Further, if a location having no legions was attacked, then it must be within the vicinity of at least one city at which two legions were stationed, so that one of the two legions could be sent to defend the attacked city. Beeler *et al.* [3] have defined double Roman domination. What they propose is a stronger version of Roman

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domination that doubles the protection by ensuring that any attack can be defended by at least two legions. In Roman domination at most two Roman legions are deployed at any one location. But as we will see in what follows, the ability to deploy three legions at a given location provides a level of defense that is both stronger and more flexible, at less than the anticipated additional cost. Here we define the double restrained domination (double Roman dominating set for which the set of vertices with label 0 has a neighbor with label 0). In terms of the double Roman Empire, this defense strategy requires that every location with no legion has at least also a neighboring location with no legion for lessing the cost of expenses of empire.

Throughout this paper, we consider G as a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. All definitions, symbols and terms used in this article are taken from the reference [20]. The open neighborhood of a vertex v is denoted by $N(v)$, and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. The minimum (maximum) degrees of G are denoted by $\delta(G)$ $(\Delta(G))$, respectively. Given subsets $A, B \subseteq V(G)$, we denote by $[A, B]$ the set of all edges with one end point in A and the other in B. For a given subset $S \subseteq V(G)$, $G[S]$ represents the subgraph induced by S in G. A tree T is a double star if it contains exactly two vertices that are not leaves. A double star with p and q leaves attached to each support vertex, respectively, is denoted by $S_{p,q}$. A wounded spider is a tree obtained from subdividing at most $n-1$ edges of a star $K_{1,n}$. A wounded spider obtained by subdividing $t \leq n-1$ edges of $K_{1,n}$, is denoted by $ws(1, n, t)$.

A set $S \subseteq V(G)$ is called a dominating set if every vertex not in S has a neighbor in S. The domination number $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G. A restrained dominating set (RD set) in a graph G is a dominating set S in G for which every vertex in $V(G) - S$ is adjacent to another vertex in $V(G) - S$. The restrained domination number (RD number) of G, denoted by $\gamma_r(G)$, is the smallest cardinality of an RD set of G. This concept was formally introduced in [6] (albeit, it was indirectly introduced in [7, 8]).

The variants of restrained domination have been considered in the literature. For instance, a total restrained domination of a graph G is an RD set of G for which the subgraph induced by the dominating set of G has no isolated vertex [5, 19]. Secure restrained dominating set (SRDS) is a set $S \subseteq V(G)$ for which S is restrained dominating and for all $u \in V \setminus S$ there exists $v \in S \cap N(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is restrained dominating set [15].

The restrained Roman dominating function is a Roman dominating function $f : V(G) \to \{0, 1, 2\}$ such that the subgraph induced by the set $\{v \in V(G) : f(v) = 0\}$ has no isolated vertex [16]. The restrained Italian dominating function (RIDF) is an Italian dominating function $f : V(G) \to \{0, 1, 2\}$ such that the subgraph induced by the set $\{v \in V(G) : f(v) = 0\}$ has no isolated vertex [17].

Now we consider a variant of double Roman dominating functions f (restrained double Roman dominating functions) where the subgraph induced by V_0^f has no isolated vertex. This new parameter, namely restrained double Roman domination, is the subject of this paper.

Beeler *et al.* [3] introduced the concept of double Roman domination number of a graph. This parameter has been studied further in [1, 11, 21].
Let $f : V(G) \to \{0, 1, 2, 3\}$ be a function, and (V_0, V_1, V_2, V_3) an ordered subsets of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2, 3$ and there is a 1–1 correspondence between the function f and the ordered partition (V_0, V_1, V_2, V_3). That is, $f = (V_0, V_1, V_2, V_3)$. A double Roman dominating (DRD) function of a graph G is a function f for which the following conditions are satisfied.

(a) If $f(v) = 0$, then the vertex v must have at least two neighbors in V_2 or one neighbor in V_3.
(b) If $f(v) = 1$, then the vertex v must have at least one neighbor in $V_2 \cup V_3$.

Accordingly, a restrained double Roman dominating (RDRD) function is a DRD function f such that the subgraph induced by V_0 (the vertices with zero labels under f) $G[V_0]$ has no isolated vertex. The RDRD number $\gamma_{rdR}(G)$ is the minimum weight of an RDRD function f of G. For the sake of convenience, an RDRD function f of a graph G with weight $\gamma_{rdR}(G)$ is called a $\gamma_{rdR}(G)$-function.

This paper is organized as follows. In Section 2, we prove that the RDRD problem is NP-hard even for general graphs and we obtain RDRD number of some custom graphs in Section 3. Then in Section 4, we present an upper bound on the RDRD number of a connected graph G in terms of the order of G and characterize the graphs attaining this bound. We study the restrained double Roman domination *versus* the restrained Roman domination in Section 5. Finally in Section 6, we characterize trees T by the given restrained double Roman domination number of T.

2. Complexity and computational issues

We consider the problem of deciding whether a graph G has an RDRD function whose weight is at most a given integer $p \leq |V(G)|$.
We prove its NP-completeness by reducing the following vertex cover decision:

> Consider a graph $G = (V, E)$ and a positive integer $p \leq |V(G)|$. Does there exist a subset $C \subseteq V(G)$ of size at most p such that for each edge $xy \in E(G)$ we have $x \in C$ or $y \in C$?

This vertex cover decision problem is NP-complete for general graphs [12].

Theorem 2.1. *RDRD is NP-complete for general graphs.*

Proof. We transform the vertex cover decision problem for general graphs to the restrained double Roman domination decision problem for general graphs. For a given graph $G = (V(G), E(G))$, let $m = 3|V(G)| + 4$ and construct a graph $H = (V(H), E(H))$ as follows. Let $V(H) = \{x_i : 1 \leq i \leq m\} \cup \{y\} \cup V(G) \cup \{u_{j_i} : 1 \leq i \leq m$ for each $e_j \in E(G)\}$, and let

$$\begin{gathered} E(H) = \{x_i x_{i+1} : (\text{mod } m)\ 1 \leq i \leq m\} \\ \cup \{x_i y : 1 \leq i \leq m\} \cup \{vy : v \in V(G)\} \\ \cup \{v u_{j_i} : v \text{ is the vertex of edge } e_j \in E(G) \text{ and } 1 \leq i \leq m\} \\ \cup \{u_{j_i} u_{j_{(i+1)}}\ (\text{mod } m) : 1 \leq i \leq m\}. \end{gathered}$$

Figure 1 shows the graph H obtained from $G = P_4 = a_1a_2a_3a_4$ by the above procedure. Because $m = 3|V(G)| + 4 = 16$ for this example, and G has three edges e_1, e_2, e_3, $H[\{x_i : 1 \leq i \leq 16\}] \cong H[\{u_{1_i} : 1 \leq i \leq 16\}] \cong H[\{u_{2_i} : 1 \leq i \leq 16\}] \cong H[\{u_{3_i} : 1 \leq i \leq 16\}] \cong C_{16}$, y is adjacent to x_i for $1 \leq i \leq 16$ and a_l for $1 \leq l \leq 4$; u_{j_i} is adjacent to both a_j, a_{j+1} for $1 \leq j \leq 3$, $1 \leq i \leq 16$.

We claim that G has a vertex cover of size at most k if and only if H has an RDRD function with weight at most $3k + 3$. Hence the NP-completeness of the restrained double Roman domination problem in general graphs will be equivalent to the NP-completeness of vertex cover problem. First, if G has a vertex cover C of size at most k, then the function f defined on $V(G)$ by $f(v) = 3$ for $v \in C \cup \{y\}$ and $f(v) = 0$ otherwise, is an RDRDF with weight at most $3k + 3$. On the other hand, suppose that g is an RDRDF on H with weight at most $3k + 3$. If $g(y) \neq 3$, then there exist two cases.

Case 1. Let $g(y) \in \{0, 1\}$. Then

$$\sum_{i=1}^{m} g(x_i) \geq \gamma_{rdR}(C_m) \geq \gamma_{dR}(C_m) \geq m > 3|V(G)| + 3 \geq 3k + 3$$

which is a contradiction.

Case 2. Let $g(y) = 2$ and $C_m = \{x_i x_{i+1} : (\mod m)\ 1 \leq i \leq m\}$. Then $g(C_m) \geq 2m/3$ and $g(H) \geq 2m/3 + 2k + 2 = 2(3|V(G)| + 4)/3 + 2k + 2 \geq 4k + 14/3 > 3k + 3$ which is a contradiction. Thus $g(y) = 3$. Similarly, we have $g(u) = 3$ or $g(v) = 3$ for any $e = uv \in E(G)$. Therefore $C = \{v \in V : g(v) = 3\}$ is a vertex cover of G and $3|C| + 3 \leq w(g) \leq 3k + 3$. Consequently, $|C| \leq k$. □

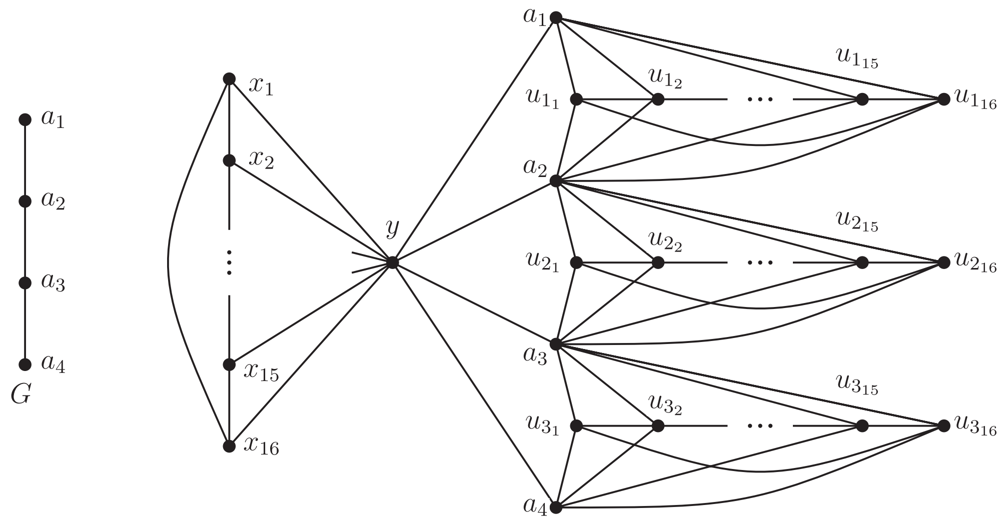

FIGURE 1. The graphs $G = P_4$ and H.

3. RDRD NUMBER OF SOME GRAPHS

In this section we investigate the exact value of the RDRD number of some graphs.

Observation 3.1. *For a complete graph K_n and a complete bipartite graph $K_{m,n}$,*

(i) $\gamma_{rdR}(K_n) = 3$ *for* $n \geq 2$.
(ii) $\gamma_{rdR}(K_{n,m}) = 6$ *for* $m \geq 2,\ n \geq 2$.
(iii) $\gamma_{rdR}(K_{1,m}) = m + 2$.
(iv) $\gamma_{rdR}(K_{n_1,n_2,\cdots,n_m}) = \begin{cases} 3, \textit{ if} \min\{n_1, n_2, \cdots, n_m\} = 1, \\ 6, \textit{ otherwise.} \end{cases}$.

Theorem 3.2. *For a path P_n $(n \geq 4)$, $\gamma_{rdR}(P_n) = n + 2$.*

Proof. Assume that $n \geq 4$ and $P_n = v_1 v_2 \cdots v_n$.
Let $n \equiv 0 \,(\text{mod } 3)$. Define $h : V(P_n) \to \{0, 1, 2, 3\}$ by $h(v_k) = 3$ for $k \equiv 2 \pmod 3$, $h(v_1) = h(v_n) = 1$ and $h(v) = 0$ otherwise.
Let $n \equiv 1 \pmod 3$. Define $h : V(P_n) \to \{0, 1, 2, 3\}$ by $h(v_k) = 3$ for $k \equiv 1 \pmod 3$ and $h(v) = 0$ otherwise.
Let $n \equiv 2 \pmod 3$. Define $h : V(P_n) \to \{0, 1, 2, 3\}$ by $h(v_k = 3$ for $k \equiv 1 \pmod 3$, $h(v_1) = 1$ and $h(v) = 0$ otherwise. Therefore $\gamma_{rdR}(P_n) \leq n + 2$ for $n \geq 4$.

Now we prove the inverse inequality. We have $\gamma_{rdR}(P_n) = n + 2$ for $4 \leq n \leq 6$. For $n \geq 7$ we proceed by induction on n. Let $n \geq 7$ and let the inverse inequality be true for every path of order less than n. Assume that $f = (V_0, V_1, V_2, V_3)$ is a γ_{rdR}-function of P_n. We have $f(v_n) \neq 0$. If $f(v_n) = 1$, then $f(v_{n-1}) \geq 2$. Define $g : P_{n-1} \to \{0, 1, 2, 3\}$, $g(v_i) = f(v_i)$ for $1 \leq i \leq n-1$. But, g is an RDRD function of P_{n-1}. It follows from the induction hypothesis that

$$\gamma_{rdR}(P_n) = w(f) = w(g) + 1 \geq \gamma_{rdR}(P_{n-1}) + 1 \geq (n-1) + 2 + 1 \geq n + 2.$$

If $f(v_n) = 2$, then $f(v_{n-1}) = 1$ and $f(v_{n-2}) \geq 1$. Define $g : P_{n-2} \to \{0, 1, 2, 3\}$, $g(v_i) = f(v_i)$ for $1 \leq i \leq n-2$. g is an RDRD function of P_{n-2}. As above we obtain,

$$\gamma_{rdR}(P_n) = w(f) = w(g) + 3 \geq \gamma_{rdR}(P_{n-2}) + 3 \geq (n-2) + 2 + 3 = n + 3.$$

If $f(v_n) = 3$, then $f(v_{n-1}) = 0$, $f(v_{n-2}) = 0$ and $f(v_{n-3}) = 3$. Define $g : P_{n-3} \to \{0, 1, 2, 3\}$, $g(v_i) = f(v_i)$ for $1 \le i \le n-3$. g is an RDRD function of P_{n-3}. It also follows from the induction hypothesis that

$$\gamma_{rdR}(P_n) = w(f) = w(g) + 3 \ge \gamma_{rdR}(P_{n-3}) + 3 \ge (n-3) + 2 + 3 = n + 2.$$

Thus the proof is complete. □

Theorem 3.3. *For a cycle C_n, $(n \ge 3)$,* $\gamma_{rdR}(C_n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 3, \\ n+2, & \text{otherwise.} \end{cases}$

Proof. Assume that $n \ge 3$ and $C_n = v_1 v_2 \cdots v_n v_1$.
Let $n \equiv 0 \pmod 3$. Define $h : V(C_n) \to \{0, 1, 2, 3\}$ by $h(v_k) = 3$ for $k \equiv 0 \pmod 3$ and $h(v) = 0$ otherwise.
Let $n \equiv 1 \pmod 3$. Define $h : V(C_n) \to \{0, 1, 2, 3\}$ by $h(v_k) = 3$ for $k \equiv 1 \pmod 3$ and $h(v) = 0$ otherwise.
Let $n \equiv 2 \pmod 3$. Define $h : V(C_n) \to \{0, 1, 2, 3\}$ by $h(v_k) = 3$ for $k \equiv 2 \pmod 3$, $h(v_1) = 1$ and $h(v) = 0$ otherwise,

$$\gamma_{rdR}(C_n) \le \begin{cases} n, & \text{if } n \equiv 0 \pmod 3, \\ n+2, & \text{otherwise.} \end{cases}$$

Now we prove the inverse inequality. For $n \equiv 0 \pmod 3$, since $\gamma_{rdR}(C_n) \ge \gamma_{dR}(C_n) = n$, (see [1,3]), the result holds. Let $n \not\equiv 0 \pmod 3$ and let $f = (V_0, V_1, V_2, V_3)$ be a γ_{rdR}-function of C_n. Since the neighbor of vertex of weight 0 is a vertex of weight 3 and a vertex of weight 0, if $n \not\equiv 0 \pmod 3$, there are two adjacent vertices v_i, v_{i+1} in C_n such that their weights are positive. Now, if $f(v_i) \ge 2$ and $f(v_{i+1}) \ge 2$, then by removing the edge $v_i v_{i+1}$, the resulted graph is P_n. Define $g : P_n \to \{0, 1, 2, 3\}$, $g(v_i) = f(v_i)$ for $1 \le i \le n$. But, g is an RDRD function of P_n with $w(g) = w(f)$. Since $w(g) \ge n + 2$ then $w(f) \ge n + 2$.
Let $f(v_i) \ge 2$ and $f(v_{i+1}) = 1$. Then $f(v_{i+2}) \ge 1$. Now remove the edge $v_{i+1} v_{i+2}$ and obtain a P_n. Define $g : P_n \to \{0, 1, 2, 3\}$, $g(v_i) = f(v_i)$ for $1 \le i \le n$. But, g is an RDRD function of P_n with $w(g) = w(f)$. Thus $w(f) \ge n + 2$.
Let $f(v_i) = f(v_{i+1}) = 1$. As above, we remove the edge $v_i v_{i+1}$ and the resulted graph P_n has an RDRD function g of weight at least $w(f)$. That is $w(f) \ge n + 2$. Therefore the proof is complete. □

4. Upper bounds on the RDRD number

In this section we obtain sharp upper bounds on the restrained double Roman domination number of a graph.

Proposition 4.1. *Let G be a connected graph of order $n \ge 2$. Then $\gamma_{rdR}(G) \le 2n - 1$, with equality if and only if $n = 2$.*

Proof. If w is a vertex of G, then define the function f by $f(w) = 1$ and $f(x) = 2$ for $x \in V(G) \setminus \{w\}$. Since G is connected of order $n \ge 2$, we observe that f is an RDRD function of G of weight $2n - 1$ and thus $\gamma_{rdR}(G) \le 2n - 1$. If $n \ge 3$, then G contains a vertex w with at least two neighbors u, v. Now define the function g by $g(u) = g(v) = 1$, $g(x) = 2$ for $x \in V(G) \setminus \{u, v\}$. Then g is an RDRD function of G of weight $2n - 2$ and so $\gamma_{rdR}(G) \le 2n - 2$ in this case. Since $\gamma_{rdR}(K_2) = 3 = 2 \cdot 2 - 1$, the proof is complete. □

Proposition 4.2. *Let G be a connected graph of order $n \ge 2$. Then $\gamma_{rdR}(G) \le 2n + 1 - diam(G)$ and this bound is sharp for the path P_n $(n \ge 4)$.*

Proof. By Theorem 3.2, $\gamma_{rdR}(P_n) \le n + 2$. Let $P = v_1 v_2 \cdots v_{diam(G)+1}$ be a diametrical path in G. Let g be a γ_{rdR}-function of P. Then $w(g) \le diam(G) + 3$. Now we define an RDRD function f as:

$$f(x) = \begin{cases} 2, & x \notin V(P), \\ g(x), & \text{otherwise.} \end{cases}$$

f is an RDRD function of G of weight $w(f) \leq 2(n - (diam(G) + 1)) + diam(G) + 3$. Therefore $\gamma_{rdR}(G) \leq 2n + 1 - diam(G)$.
Theorem 3.2 shows this bound is sharp. □

Proposition 4.3. *Let G be a connected graph of order n and circumference $c(G) < \infty$. Then $\gamma_{rdR}(G) \leq 2n + 2 - c(G)$, and this bound is sharp for each cycle C_n with $n \not\equiv 0 \ (mod\ 3)$.*

Proof. Let C be a longest cycle of G, that means $|V(C)| = c(G)$. By Theorem 3.3, $\gamma_{rdR}(C) \leq c(G) + 2$. Let h be a γ_{rdR}-function on C. Then $w(h) \leq c(G) + 2$. Now we define an RDRD function f as:

$$f(x) = \begin{cases} 2, & x \notin V(C), \\ h(x), & \text{otherwise.} \end{cases}$$

f is an RDRD function of G of weight $w(f) \leq 2(n - c(G)) + c(G) + 2$. Therefore $\gamma_{rdR}(G) \leq 2n + 2 - c(G)$. For sharpness, if $G = C_n$ and $n \not\equiv 0 \pmod 3$, then $\gamma_{rdR}(C_n) = n + 2 = 2n + 2 - n = 2n + 2 - c(G)$. □

Observation 4.4. *Let G be a graph and $f = (V_0, V_1, V_2)$ a γ_{rR}-function of G. Then $\gamma_{rdR}(G) \leq 2|V_1| + 3|V_2|$.*

Proof. Let G be a graph and $f = (V_0, V_1, V_2)$ a γ_{rR}-function of G. We define a function $g = (V_0', V_2', V_3')$ as follows: $V_0' = V_0$, $V_2' = V_1$, $V_3' = V_2$. Note that under g, every vertex with a label 0 has a neighbor assigned 3 and each vertex with label 1 becomes a vertex with label 2 and also $G[V_0']$ has no isolated vertex. Hence, g is a restrained double Roman dominating function. Thus, $\gamma_{rdR}(G) \leq 2|V_2'| + 3|V_3'| = 2|V_1| + 3|V_2|$. □

The bound of observation 4.4 is sharp, as can be seen with the path $G = P_4$, where $\gamma_{rR}(G) = 4$ and $\gamma_{rdR}(G) = 6$. The strict inequality in the bound can be achieved by the subdivided star $G = S(K_{1,k})$ which formed by subdividing each edge of the star $K_{1,k}$, for $k \geq 3$, exactly once. Thus, we must have $\gamma_{rR}(G) = 2k + 1$ and $\gamma_{rdR}(G) = 3k$. Hence, $|V_1| = 1$ and $|V_2| = k$, and so, $3k = \gamma_{rdR}(G) < 2|V_1| + 3|V_2| = 2 + 3k$.

Lemma 4.5. *If a graph G has a non-pendant edge, then there exists a $\gamma_{rdR}(G)$-function $f = (V_0, V_1, V_2, V_3)$ such that $V_0 \cup V_1 \neq \emptyset$.*

Proof. If $\gamma_{rdR}(G) < 2n$, then $V_0 \cup V_1 \neq \emptyset$. Now we show that $\gamma_{rdR}(G) < 2n$. Let uw be a non-pendant edge with $\deg(u)$ and $\deg(w)$ be at least 2.
Assume that $N_G(u) \cap N_G(w) \neq \emptyset$, and let v be a vertex in $N_G(u) \cap N_G(w)$. Then the function $f = (V_0 = \{u, w\}, V_1 = \emptyset, V_2 = V(G) \setminus \{u, w, v\}, V_3 = \{v\})$ is an RDRD function of G with $w(f) \leq 2n - 3$.
Assume that $N_G(u) \cap N_G(w) = \emptyset$, and let $a \in N_G(u) \setminus \{w\}$, $b \in N_G(w) \setminus \{u\}$. Then the function $f = (V_0 = \{u, w\}, V_1 = \emptyset, V_2 = V(G) \setminus \{u, w, a, b\}, V_3 = \{a, b\})$ is an RDRD function of G with $w(f) \leq 2n - 2$. This completes the proof.
□

For any integer $n \geq 3$, let H_n be the graph obtained from $(n-2)/2$ copies of K_2 and a copy of K_1 by adding a new vertex and joining it to both leaves of each K_2 and the given K_1, and let F_n be the graph obtained from $(n-2)/2$ copies of K_2 by adding a new vertex and joining it to both leaves of each K_2. Thus for $n \geq 4$, H_n have a vertex of degree $n-1$, a vertex of degree 1 and other vertices of degree two and for $n \geq 3$, F_n have a vertex of degree $n-1$ and other vertices of degree two. Figure 2 shows the graphs H_{10}, F_9. Let $\mathcal{H} = \{H_n : n \geq 4 \text{ is even}\}$, $\mathcal{F} = \{F_n : n \geq 3 \text{ is odd}\}$.

Theorem 4.6. *For every connected graph G of order $n \geq 3$ with m edges, $\gamma_{rdR}(G) \geq 2n + 1 - \lceil (4m - 1)/3 \rceil$, with equality if and only if $G \in \mathcal{H} \cup \mathcal{F}$ or $G \in \{K_{1,2}, K_{1,3}, K_{1,4}\}$.*

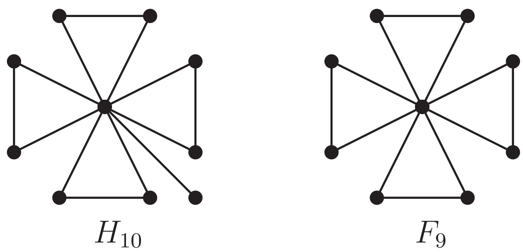

FIGURE 2. The graphs H_{10}, F_9.

Proof. If $G = K_{1,n-1}$ is a star, then $\gamma_{rdR}(G) = n+1$ and $m = n-1$. Now we have, $\gamma_{rdR}(K_{1,n-1}) = 2n+1-\lceil(4m-1)/3\rceil$ for $3 \le n \le 5$ and $\gamma_{rdR}(K_{1,n-1}) > 2n+1-\lceil(4m-1)/3\rceil$ for $n \ge 6$. Next assume that G is not a star. By Lemma 4.5 there is a $\gamma_{rdR}(G)$-function of $f = (V_0, V_1, V_2, V_3)$ such that $V_0 \cup V_1 \ne \emptyset$. It is well known that, the induced subgraph $G[V_0]$ has no isolated vertex. Therefore, $|E(G[V_0])| \ge |V_0|/2$. Let $V_0' = \{v \in V_0 : N(v) \subseteq V_2\}$ and $V_0'' = \{v \in V_0 : v \textit{ has a neighbor in } V_3\}$. Then $|E(V_0, V_2)| \ge 2|V_0'|$, $|E(V_0, V_3)| \ge |V_0''|$ and $|E(V_1, V_2 \cup V_3)| \ge |V_1|$. Therefore

$$|E(G)| = m \ge |V_0|/2 + 2|V_0'| + |V_0''| + |V_1|.$$

Since $|V_0| = |V_0'| + |V_0''|$, we deduce that

$$(4m-1)/3 \ge 2|V_0| + 4/3|V_0'| + 4/3|V_1| - 1/3 \tag{4.1}$$

and thus

$$2n+1-\lceil(4m-1)/3\rceil \le 2n+1-(4m-1)/3 \le 2n+1-2|V_0| - 4/3|V_0'| - 4/3|V_1| + 1/3. \tag{4.2}$$

Because $\gamma_{rdR}(G) = |V_1|+2|V_2|+3|V_3|$, $|V_0|+|V_1|+|V_2|+|V_3| = n$ and $2n+1 = 2|V_0|+2|V_1|+2|V_2|+2|V_3|+1$, we obtain

$$\begin{aligned} 2n+1-2|V_0| - 4/3|V_0'| - 4/3|V_1| + 1/3 &= -4/3|V_0'| + 2/3|V_1| + 2|V_2| + 2|V_3| + 4/3 \\ &= \gamma_{rdR}(G) - 4/3|V_0'| - 1/3|V_1| - |V_3| + 4/3. \end{aligned}$$

Next we show that

$$\gamma_{rdR}(G) - 4/3|V_0'| - 1/3|V_1| - |V_3| + 4/3 \le \gamma_{rdR}(G) \tag{4.3}$$

or $\gamma_{rdR}(G) \ge 2n+1-\lceil(4m-1)/3\rceil$. If $|V_0'| \ge 1$, then $-4/3|V_0'| - 1/3|V_1| - |V_3| + 4/3 \le 0$ and so $\gamma_{rdR}(G) - 4/3|V_0'| - 1/3|V_1| - |V_3| + 4/3 \le \gamma_{rdR}(G)$.
Let now $|V_0'| = 0$. The condition $V_0 \cup V_1 \ne \emptyset$ implies $V_0'' \cup V_1 \ne \emptyset$.

We now distinguish the two cases: $V_1 = \emptyset$, and $V_1 \ne \emptyset$. When $V_1 = \emptyset$, we deduce that $|V_0''| \ge 1$ and therefore $|V_3| \ge 1$. If there are at least two vertices of weight 3, then $\gamma_{rdR}(G) - 4/3|V_0'| - 1/3|V_1| - |V_3| + 4/3 < \gamma_{rdR}(G)$. If there is only one vertex of weight 3, then $m \ge n-1+\frac{n-1}{2} = \frac{3(n-1)}{2}$. We deduce that $\gamma_{rdR}(G) \ge 3 \ge 2n+1-\left\lceil\frac{6(n-1)-1}{3}\right\rceil \ge 2n+1-\left\lceil\frac{4m-1}{3}\right\rceil$, with equality if and only if $|V_2| = 0$, n is odd and $m = \frac{3(n-1)}{2}$, that means $G \in \mathcal{F}$.

When $V_1 \ne \emptyset$; *i.e.*, when $|V_1| \ge 1$, we further distinguish two cases. If $|V_0''| \ge 1$, then $|V_3| \ge 1$ and thus $\gamma_{rdR}(G) - 4/3|V_0'| - 1/3|V_1| - |V_3| + 4/3 \le \gamma_{rdR}(G)$. Next let $|V_0''| = 0$. If $|V_3| \ge 1$, then $\gamma_{rdR}(G) - 4/3|V_0'| - 1/3|V_1| - |V_3| + 4/3 \le \gamma_{rdR}(G)$. Now assume that $|V_3| = 0$. This implies that all vertices have weight 1 or

2. If $3 \leq n \leq 5$, then it is easy to see that $\gamma_{rdR}(G) > 2n+1-\lceil\frac{4m-1}{3}\rceil$. Let now $n \geq 6$. If $|V_1| \geq 5$, then $\gamma_{rdR}(G) - 4/3|V_0'| - 1/3|V_1| - |V_3| + 4/3 < \gamma_{rdR}(G)$. Otherwise $|V_1| \leq 4$, $|V_2| \geq n-4$ and $m \geq n-1$. This implies

$$\gamma_{rdR}(G) \geq 2(n-4)+4 = 2n-4 > 2n+1-\left\lceil\frac{4(n-1)-1}{3}\right\rceil \geq 2n+1-\left\lceil\frac{4m-1}{3}\right\rceil.$$

Thus $\gamma_{rdR}(G) \geq 2n+1-(4m-1)/3 \geq 2n+1-\lceil(4m-1)/3\rceil$.
For equality: If $G \in \mathcal{H}$, then $G = H_n$ for $n \geq 4$ even and $|E(H_n)| = 3(n-2)/2+1$. Thus $2n+1-(4(3(n-2)/2+1)-1)/3 = 2n+1-\lceil(4(3(n-2)/2+1)-1)/3\rceil = 2n+1-2(n-2)-1 = 4 = \gamma_{rdR}(H_n)$. If $G \in \mathcal{F}$, then $G = F_n$ for $n \geq 3$ odd and $|E(F_n)| = 3(n-1)/2$. Thus $2n+1-\lceil(4(3(n-1)/2))-1/3\rceil = 2n+1-2(n-1) = 3 = \gamma_{rdR}(F_n)$.

Conversely, assume that $\gamma_{rdR}(G) = 2n+1-\lceil(4m-1)/3\rceil$. Then all inequalities occurring in the proof become equalities. When $|V_1| = 0$, the equality holds if and only if $G \in \mathcal{F}$. When $|V_1| \geq 1$, $|V_3| \geq 1$. Therefore the equality in Inequality (4.3) leads to $|V_3| = |V_1| = 1$ and $|V_0'| = 0$. Hence $V_0 = V_0''$. Thus equality in (4.1) or equivalently, in the inequality $|E(G)| = m \geq |V_0|/2+2|V_0'|+|V_0''|+|V_1|$ leads to $m = 3/2|V_0''|+1$. Now let the vertices v, u be of weight $3, 1$ respectively. Then $m = |E(G)| \geq |E(v, V_0'')|+G[V_0'']+1 \geq |V_0''|+1/2|V_0''|+1 = 3/2|V_0''|+1$. If $|V_2| \neq 0$, then the connectivity of G leads to the contradiction $m \geq 3/2|V_0''|+2$. Consequently, $|V_2| = 0, |V_0| = (2m-2)/3$ and u and v are adjacent. Since G is connected, $G \in \mathcal{H}$. □

5. RDRD-set *versus* RRD-set

In this section we study the comparability between the RDRD-set and RRD-set.

Proposition 5.1. *For any graph G, $\gamma_{rdR}(G) \leq 2\gamma_{rR}(G)$ with equality if and only if $G = \overline{K_n}$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{rR}-function of G. Since $\gamma_{rR}(G) = |V_1| + 2|V_2|$, by Observation 4.4, we have that $\gamma_{rdR}(G) \leq 2|V_1| + 3|V_2| = \gamma_{rR}(G) + |V_1| + |V_2| \leq 2\gamma_{rR}(G)$. If $\gamma_{rdR}(G) = 2\gamma_{rR}(G) = 2|V_1| + 4|V_2|$, then since $\gamma_{rdR}(G) \leq 2|V_1| + 3|V_2|$, we must have $V_2 = \emptyset$. Hence, $V_0 = \emptyset$ must hold, and so $V = V_1$. By definition of γ_{rR}-function, we deduce that no two vertices in G are adjacent, for otherwise, if u and v are adjacent, then only one of them in every γ_{rdR}-function on G has a label of 2 which contradicts with $\gamma_{rdR}(G) = 2\gamma_{rR}(G)$. □

As an immediate consequence of Proposition 5.1, we have the following Corollary and Theorem.

Corollary 5.2. *For any nontrivial connected graph G, $\gamma_{rdR}(G) < 2\gamma_{rR}(G)$.*

The proof of Lemma 4.5 shows the next proposition.

Proposition 5.3. *If G contains a triangle, then $\gamma_{rdR}(G) \leq 2n-3$.*

Theorem 5.4. *For every graph G, $\gamma_{rR}(G) < \gamma_{rdR}(G)$.*

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{rdR}(G)$-function. If $V_3 \neq \emptyset$, then $(V_0' = V_0, V_1' = V_1, V_2' = V_2 \cup V_3)$ is an RRD function g such that $w(g) < w(f)$. Let $V_3 = \emptyset$. If $V_0 = \emptyset$, then, since $V_2 \neq \emptyset$, $g = (\emptyset, V_1' = V_1 \cup V_2, \emptyset)$ is an RRD function such that $w(g) < w(f)$. If $V_0 \neq \emptyset$, then $|V_2| \geq 2$. Let $f(v) = 2$ for a vertex v. Then $g = (V_0' = V_0, V_1' = V_1 \cup \{v\}, V_2' = V_2 - \{v\})$ is an RRD function g for which $w(g) < w(f)$. Therefore $\gamma_{rR}(G) < \gamma_{rdR}(G)$. □

Theorem 5.5. *Let G be a graph of order n. Then $\gamma_{rdR}(G) = \gamma_{rR}(G)+1$ if and only if G is one of the following graphs.*

1. *G has a vertex of degree $n-1$.*
2. *There exists a subset S of $V(G)$ such that:*

2.1. *every vertex of $V-S$ is adjacent to a vertex in S,* 2.2. *there are two subsets A_0 and A_1 of $V-S$ with $A_0 \cup A_1 = V-S$ such that A_0 is the set of non-isolated vertices in $N(S)$ and each vertex in A_0 has at least two neighbors in S,*

2.3. *for any 2-subset $\{a,b\}$ of S, $N(\{a,b\}) \cup A_0 \neq \emptyset$ and for a 3-subset $\{x,y,z\}$ of S, if $\{x,y,z\} \cap A_0 \neq \emptyset$, then there are three vertices u,v,w in A_0 such that $N(u) \cup S = \{x,y\}$, $N(v) \cup S = \{x,z\}$ and $N(w) \cup S = \{y,z\}$.*

Proof. Let $\gamma_{rdR}(G) = \gamma_{rR}(G) + 1$ with a $\gamma_{rdR}(G)$-function $f = (V_0, V_1, V_2, V_3)$ and a $\gamma_{rR}(G)$-function $g = (U_0, U_1, U_2)$. If $V_3 \neq \emptyset$, then $|V_3| = 1$. If $|V_3| \geq 2$ then by changing 3 to 2 we obtain a RRD function h with $w(h) < w(g)$, a contradiction. Let $V_3 = \{v\}$. In addition, we must have $|V_2| = 0$. If we suppose that $|V_2| \geq 1$, then let $u \in V_2$. Then $h = (V_0' = V_0, V_1' = V_1 \cup \{u\}, V_2' = V_2 \cup \{v\})$ is an RRD function g for which $w(h) < w(g)$, a contradiction. Thus all vertices different from v are adjacent to the vertex v such that the non-isolated vertices in $N(v)$ are attributed to 0 and the isolated vertices in $N(v)$ are attributed to 1. In this case $U_0 = V_0, U_1 = V_1$ and $U_2 = V_3$.

If $V_3 = \emptyset$, then $V_2 \neq \emptyset$ and $|V_2| \geq 2$. In this case, there must exist a vertex $v \in V_2$ such that $U_0 = V_0, U_1 = V_1 \cup \{v\}$ and $U_2 = V_2 - \{v\}$. There is such a function f if we guarantee a subset S of $V(G)$ with each vertex of weight 2 for which every other vertex in $V - S$ has to adjacent to a vertex of S, that is the condition 2.1 holds.
Since we can only change one of vertices of weight 2 in f to a vertex of weight 1 in g, there must be existed two subsets A_0 and A_1 in $V - S$ such that the conditions 2.2 and 2.3 hold.

Conversely, if the condition 1 holds, then $f = (V_0, V_1, \emptyset, V_3 = \{v\})$ and $g = (U_0 = V_0, U_1 = V_1, U_2 = \{v\})$ are $\gamma_{rdR}(G)$-function and $\gamma_{rR}(G)$-function respectively where V_0 is the set of non-isolated vertices in $N(v)$ and V_1 is the set of isolated vertices in $N(v)$. Thus $\gamma_{rdR}(G) = \gamma_{rR}(G) + 1$.
If the condition 2 holds, then we can have only one vertex of weight 2 in G under f such that it changes to the weight 1 in G under g. Thus $\gamma_{rdR}(G) = \gamma_{rR}(G) + 1$. □

We showed that for any graph G, $\gamma_{rdR}(G) \leq 2\gamma_{rR}(G)$ and the equality holds if and on if G is a trivial graph $\overline{K_n}$. Hence, for any nontrivial graph G, $\gamma_{rdR}(G) \leq 2\gamma_{rR}(G) - 1$. Now we characterise graph G with this property $\gamma_{rdR}(G) = 2\gamma_{rR}(G) - 1$.

Theorem 5.6. *If G is a nontrivial graph, then $\gamma_{rdR}(G) \leq 2\gamma_{rR}(G) - 1$. If $\gamma_{rdR}(G) = 2\gamma_{rR}(G) - 1$, then G consists of a K_2 and $n - 2$ isolated vertices or G consists of a vertex h and two disjoint vertex sets H and R such that $H = N(h)$, $G[H]$ does not have isolated vertices, $G[R]$ is trivial, there is no edge between h and R and $N(h) \cap N(R) \neq N(h)$.*

Proof. Since G is a nontrivial graph, Proposition 5.1 implies $\gamma_{rdR}(G) \leq 2\gamma_{rR}(G) - 1$. Now we investigate the equality.
Let $\gamma_{rdR}(G) = 2\gamma_{rR}(G) - 1$, where $f = (V_0, V_1, V_2, V_3)$ is a $\gamma_{rdR}(G)$-function and $g = (U_0, U_1, U_2)$ is a $\gamma_{rR}(G)$-function. Then $2|U_1| + 4|U_2| - 1 = |V_1| + 2|V_2| + 3|V_3|$. On the other hand, since $2|U_1| + 4|U_2| - 1 = |V_1| + 2|V_2| + 3|V_3| = \gamma_{rdR}(G) \leq 2|U_1| + 3|U_2|$, it follows that $|U_2| \leq 1$.

If $U_2 = \emptyset$, then $|U_0| = 0$ and therefore $|U_1| = n$. Using the inequality above, we obtain

$$2n - 1 = 2|U_1| - 1 \leq \gamma_{rdR}(G) \leq 2|U_1| = 2n.$$

If $\gamma_{rdR}(G) = 2n$, then G is trivial, a contradiction. If $\gamma_{rdR}(G) = 2n - 1$, then Proposition 4.1 shows that G consists of a K_2 and $n - 2$ isolated vertices.

Let now $|U_2| = 1$ such that $U_2 = \{h\}$, $H = N(h)$, $R = V(G) \setminus N[h] = \{u_1, u_1, \ldots, u_p\}$. Then, $U_0 \subseteq H$ and $R \subseteq U_1$.

If H contains exactly $s \geq 1$ isolated vertices, then $\gamma_{rR}(G) = 2 + s + p$ and therefore $\gamma_{rdR}(G) \leq 3 + s + 2p \leq 2\gamma_{rR}(G) - 2$, a contradiction. Hence $H = N(h)$ does not contain isolated vertices and thus $\gamma_{rR}(G) = p + 2$.

If $G[R]$ contains an edge, then we obtain the contradiction $\gamma_{rdR}(G) \leq 3 + 2p - 1 = 2p + 2 \leq 2\gamma_{rR}(G) - 2$. Thus $G[R]$ is trivial.

If there is an edge between h and R, then we also obtain the contradiction $\gamma_{rdR}(G) \leq 3 + 2p - 1 = 2p + 2 \leq 2\gamma_{rR}(G) - 2$.

If $N(h) \cap N(R) = N(h)$, then $f = (H, \emptyset, \{h\} \cup R, \emptyset)$ is an RDRD function of G, and hence $\gamma_{rdR}(G) \leq 2p + 2 \leq 2\gamma_{rR}(G) - 2$, a contradiction. □

6. Trees

In this section we study the restrained double Roman domination of trees.

Theorem 6.1. *If T is a tree of order $n \geq 2$, then $\gamma_{rdR}(T) \leq \lceil \frac{3n-1}{2} \rceil$. The equality holds if $T \in \{P_2, P_3, P_4, P_5, S_{1,2}, ws(1,n,n-1), ws(1,n,n-2)\}$.*

Proof. Let T be a tree of order $n \geq 2$. We will proceed by induction on n. If $n = 2$, then $\gamma_{rdR}(T) = 3 = \lceil \frac{3n-1}{2} \rceil$. If $n \geq 3$, then $diam(T) \geq 2$. If $diam(T) = 2$, then T is the star $K_{1,n-1}$ for $n \geq 3$ and $\gamma_{rdR}(T) = n+1 \leq \lceil \frac{3n-1}{2} \rceil$. If $diam(T) = 3$, then T is a double star $S_{r,s}$ for $1 \leq r \leq s$. Hence, $n = r+s+2 \geq 4$. If $r = 1 = s$, then $T = P_4$ and $\gamma_{rdR}(T) = 6 \leq \lceil \frac{12-1}{2} \rceil$. If $r = 1, s \geq 2$, then $n = s+3$ and $\gamma_{rdR}(T) = s+5 \leq \lceil \frac{3(s+3)-1}{2} \rceil$. If $r \geq 2, s \geq 2$, then $n = r+s+2$ and $\gamma_{rdR}(T) = r+s+4 \leq \lceil \frac{3(r+s+2)-1}{2} \rceil$.
Hence, we may assume that $diam(T) \geq 4$. This implies that $n \geq 5$. Assume that any tree T' with order $2 \leq n' < n$ has $\gamma_{rdR}(T') \leq \lceil \dfrac{3n'-1}{2} \rceil$. Among all longest paths in T, choose P to be one that maximizes the degree of its next-to-last vertex v, and let w be a leaf neighbor of v. Note that by our choice of v, every child of v is a leaf. Since $deg(v) \geq 2$, the vertex v has at least one leaf as a child. Now we put $T' = T - T_v$ where the order of the substar T_v is $k+1$ with $k \geq 1$. Note that since $diam(T) \geq 4$, T' has at least three vertices, that is, $n' \geq 3$. Let f' be a γ_{rdR}-function of T'. Form f from f' by letting $f(x) = f'(x)$ for all $x \in V(T')$, $f(v) = 2$, and $f(z) = 1$ for all leaf neighbors of v. Thus f is a restrained double Roman dominating function of T, implying that $\gamma_{rdR}(T) \leq \gamma_{rdR}(T')+k+2 \leq \lceil \dfrac{3(n-k-1)-1}{2} \rceil + k + 2 = \lceil \dfrac{3n-k}{2} \rceil \leq \lceil \dfrac{3n-1}{2} \rceil$.

If $T \in \{P_2, P_3, P_4, P_5, S_{1,2},, ws(1,n,n-1), ws(1,n,n-2)\}$, then we must have $\gamma_{rdR}(T) = \lceil \dfrac{3n-1}{2} \rceil$. □

Theorem 6.2. *For every tree T of order $n \geq 3$, with l leaves and s support vertices, we have $\gamma_{rdR}(T) \leq \dfrac{4n+2s-l}{3}$, and this bound is sharp for the family of stars ($K_{1,n-1}$ $n \geq 3$), double stars, caterpillars for which each vertex is a leaf or a support vertex and all support vertices have even degree $2m$ or at most two end support vertices has degree $2m-1$ and the other support vertices has degree $2m$, wounded spiders in which the central vertex is adjacent with at least two leaves.*

Proof. Let T be a tree with order $n \geq 3$. Since $n \geq 3$, $diam(T) \geq 2$. If $diam(T) = 2$, then T is the star $K_{1,n-1}$ for $n \geq 3$ and $\gamma_{rdR}(T) = n+1 \leq \dfrac{4n+2-(n-1)}{3} = \dfrac{3n+3}{3} = n+1$. If $diam(T) = 3$, then T is a double star $S_{r,t}$ for $1 \leq r \leq t$. We have $\gamma_{rdR}(T) = n+2 = \dfrac{4n+2s-l}{3}$. Hence, we may assume $diam(T) \geq 4$. Thus, $n \geq 5$. Assume that any tree T' with order $3 \leq n' < n$, l' leaves and s' support vertices has $\gamma_{rdR}(T') \leq \dfrac{4n'+2s'-l'}{2}$. Among all longest paths in T, choose P to be one that maximizes the degree of its next-to-last vertex u, and let x be a leaf neighbor of u, w be a parent vertex of v and v be a parent vertex of u. Note that by our choice of u, every child of u is a leaf. Since $t = deg(u) \geq 2$, the vertex u has at least one leaf children. We now consider the two cases are as follows:
Case 1. $deg(v) \geq 3$. In this case, we put $T' = T - T_u$, where the order of the star T_u is t with $t \geq 2$. Note that since $diam(T) \geq 4$, T' has at least three vertices, that is, $n' \geq 3$. Let f' be a γ_{rdR}-function of T'. Thus we have $n' = n-t$, $l' = l-(t-1)$ and $s' = s-1$. But, $\gamma_{rdR}(T) \leq \gamma_{rdR}(T')+t+1 \leq \dfrac{4(n-t)+2(s-1)-(l-(t-1))}{3} + t+1 = \dfrac{4n+2s-l}{3}$.
Case 2. $deg(v) = 2$. We now consider the following two subcases.
i. $deg(w) > 2$. Then we put $T' = T - T_v$ where order of subtree T_v is $t+1$. But, we have $n' = n-(t+1)$,

$s' = s-1$ and $l' = l-(t-1)$. Thus, $\gamma_{rdR}(T) \leq \gamma_{rdR}(T')+t+2 \leq \dfrac{4(n-t-1)+2(s-1)-(l-(t-1))}{3}+t+2 = \dfrac{4n+2s-l-1}{3} \leq \dfrac{4n+2s-l}{3}$.

ii. $deg(w) = 2$. Then we put $T' = T - T_v$, where the order of the subtree T_v is $t+1$. Thus in this case, w in the subtree T' becomes a leaf and we have $n' = n-(t+1)$, $s' \leq s$ and $l' = l-(t-1)+1$. Thus, $\gamma_{rdR}(T) \leq \gamma_{rdR}(T')+t+2 \leq \dfrac{4(n-t-1)+2(s)-(l-(t-1)+1)}{3}+t+2 = \dfrac{4n+2s-l}{3}$. □

Theorem 6.3. *If T is a tree, then $\gamma_r(T)+1 \leq \gamma_{rdR}(T) \leq 3\gamma_r(T)$, and equality for the lower bound holds if and only if T is a star. The upper bound is sharp for the paths P_m ($m \equiv 1$ mod 3).*

Proof. Let T be a tree. Since at least one vertex has value 2 under any RDRD function of T, we see that $\gamma_r(T)+1 \leq \gamma_{rdR}(T)$. If we assign the value 3 to the vertices in a $\gamma_r(T)$-set, then we obtain an RDRD function of T. Therefore $\gamma_{rdR}(T) \leq 3\gamma_r(T)$.
The sharpness of the upper bound is deuced from Propositions 1-7 of [6] and Observation 3.1, Theorem 3.2 and Theorem 3.3.
To prove equality at the lower bound, if $T = K_{1,n-1}$ is a star, then we have $\gamma_{rdR}(T) = n+1$ and $\gamma_r(T) = n$. If T is a tree and $\gamma_{rdR}(T) = \gamma_r(T)+1$, then we have only one vertex of value 2 in any $\gamma_{rdR}(T)$-function and the other vertices of positive weight have value 1. In addition, the vertices of value 1 are adjacent to the vertex of value 2, and therefore T is a star. □

The following result gives us the RDRD of G in terms of the size of $E(G)$, and order of G.

Proposition 6.4. *Let G be a connected graph G of order $n \geq 2$ with m edges. Then $\gamma_{rdR}(G) \leq 4m-2n+3$, with equality if and only if G is a tree with $\gamma_{rdR}(G) = 2n-1$.*

Proof. For the given connected graph, $m \geq n-1$ and according to Proposition 4.2 $\gamma_{rdR}(G) \leq 2n-1 = 4n-4-2n+3 \leq 4m-2n+3$.
If $\gamma_{rdR}(G) = 4m-2n+3$, then $m = n-1$ and G is a tree with $\gamma_{rdR}(G) = 2n-1$.
Conversely, assume that G is a tree with $\gamma_{rdR}(G) = 2n-1$. Hence $\gamma_{rdR}(G) = 4m-2n+3$. □

7. Conclusions and open problems

In this paper, we investigated the concept of restrained double Roman domination in graphs. We studied the computational complexity of this concept and proved some bounds on the RDRD number of graphs. In the case of trees, we characterized all trees attaining the exhibited bound. We end the paper with four open problems emerge from this research.

1. Characterize the graphs G with small or large RDRD numbers.
2. It is also worthwhile proving some other nontrivial sharp bounds on $\gamma_{rdR}(G)$ for general graphs G or some well-known families such as, chordal, planar, triangle-free, or claw-free graphs.
3. The decision problem RDRD is NP-complete for general graphs, as proved in Theorem 2.1. By the way, there might be some families of graphs such that RDRD is NP-complete for them or there might be some polynomial-time algorithms for computing the RDRD number of some well-known families of graphs, for instance, trees. Can you provide these families?
4. In Theorems 6.1 and 6.2 we showed upper bounds for $\gamma_{rdR}(T)$. The sufficient and necessity conditions for equality can be posed as open problems.

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