RAIRO-Oper. Res. 56 (2022) 2895–2901
https://doi.org/10.1051/ro/2022112

RAIRO Operations Research
www.rairo-ro.org

SUFFICIENT CONDITIONS FOR GRAPHS WITH $\{P_2, P_5\}$-FACTORS

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Abstract. For a graph G, a spanning subgraph F of G is called an $\{P_2, P_5\}$-factor if every component of F is isomorphic to P_2 or P_5, where P_k denotes the path of order k. It was proved by Egawa and Furuya that if G satisfies $3c_1(G-S) + 2c_3(G-S) \le 4|S| + 1$ for all $S \subseteq V(G)$, then G has a $\{P_2, P_5\}$-factor, where $c_k(G-S)$ denotes the number of components of $G-S$ with order k. By this result, we give some other sufficient conditions for a graph to have a $\{P_2, P_5\}$-factor by various graphic parameters such as toughness, binding number, degree sums, etc. Moreover, we obtain some regular graphs and some $K_{1,r}$-free graphs having $\{P_2, P_5\}$-factors.

Mathematics Subject Classification. 05C70, 05C38.

Received September 6, 2021. Accepted June 24, 2022.

1. Introduction

In this paper, we consider only finite and undirected graph without loops or multiple edges. Other basic graph-theoretic terminologies not defined here can be found in [4]. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A spanning subgraph of G is a subgraph H of G such that $V(H) = V(G)$ and $E(H) \subseteq E(G)$. For $X \subseteq V(G)$, $G - X$ denotes the graph obtained from G by deleting all the vertices of X and $G[X]$ denotes the subgraph of G induced by X. For $v \in V(G)$, we use $d_G(v)$ and $N_G(v)$ to denote the degree of v and the set of vertices adjacent to v in G, respectively. For $S \subseteq V(G)$, we write $N_G(S) = \cup_{v \in S} N_G(v)$. A graph G is said to be r-regular if every vertex of G has degree r. We denote the minimum degree and the number of connected components of a graph G by $\delta(G)$ and $\omega(G)$, respectively. Define $\sigma_2(G) = \min\{d_G(u) + d_G(v) : \{u, v\} \subseteq V(G) \text{ is an independent set of } G\}$.

For a connected graph G, its *toughness*, denoted by $\tau(G)$, was first introduced by Chvátal [5] as follows. If G is complete, then $\tau(G) = +\infty$; otherwise,

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \ge 2\right\}.$$

Keywords. $\{P_2, P_5\}$-factor, degree sum, binding number, toughness, regular graph, $K_{1,r}$-free graph.

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The *binding number* is introduced by Woodall [21] and defined as

$$bind(G) = \min\left\{ \frac{|N_G(S)|}{|S|} : \emptyset \neq S \subseteq V(G), N_G(S) \neq V(G) \right\}.$$

The complete bipartite graph $K_{1,r}$ is called the *star* of order $r+1$. We call a graph G is $K_{1,r}$-free if G does not contain an induced subgraph isomorphic to $K_{1,r}$. In particular, a graph is said to be claw-free if it is $K_{1,3}$-free.

For a family of connected graphs $\mathcal{F}$, a spanning subgraph H of a graph G is called an $\mathcal{F}$-factor of G if each component of H is isomorphic to some graph in $\mathcal{F}$. Let P_k denote the path of order k. A spanning subgraph of a graph G is called a $\{P_2, P_5\}$-factor of G if its each component is isomorphic to P_2 or P_5. Similarly, $\{P_2, P_3\}$-factor means a graph factor in which every component is a path of order exactly two or three.

Since Tutte proposed the well-known Tutte 1-factor theorem [20], path-factors of graphs [3, 7, 8, 14, 17] and path-factor covered graphs [6, 9, 23] have attracted a great deal of attention. More results on graph factors are referred to the survey papers and books [1, 19, 22].

Akiyama *et al.* [2] demonstrated the following classical result, which is a criterion for graphs with $\{P_2, P_3\}$-factors. We denote by $i(G)$ the number of isolated vertices of a graph G.

Theorem 1.1 ([2]). *A graph G has a $\{P_2, P_3\}$-factor if and only if $i(G-S) \leq 2|S|$ for all $S \subseteq V(G)$.*

For an integer $k \geq 2$, a $\{P_i : i \geq k\}$-factor is briefly denoted by $\mathcal{P}_{\geq k}$-factor. Note that a graph has $\mathcal{P}_{\geq 2}$-factors if and only if it has $\{P_2, P_3\}$-factors. Kaneko [13] gave a necessary and sufficient condition for the existence of $\mathcal{P}_{\geq 3}$-factors. For $k \geq 4$, it is not known that whether the existence problem of $\mathcal{P}_{\geq k}$-factors is polynomially solvable or not, though some results about such factors on special classes of graphs have been obtained (see, for example, Kano *et al.* [16], Ando *et al.* [3], and Kawarabayashi *et al.* [17]).

A graph H is *hypomatchable* if $H - x$ has a perfect matching for every $x \in V(H)$. A graph is a *propeller* if it is obtained from a hypomatchable graph H by adding new vertices u, v and edge uv, and joining u to some vertices of H. Loebal and Poljak [18] proved the following theorem.

Theorem 1.2 ([18]). *Let H be a connected graph. If H has a perfect matching, H is hypomatchable, or H is a propeller, then the existence problem of a $\{P_2, H\}$-factor is polynomially solvable. The problem is* ***NP****-complete for all other graphs H.*

In particular, the existence problem of a $\{P_2, P_{2k+1}\}$-factor is **NP**-complete for $k \geq 2$. As $\{P_2, P_{2k+1}\}$-factor is a useful tool for finding large matchings, Egawa *et al.* [12] investigated the existence of $\{P_2, P_{2k+1}\}$-factors and obtained the following theorem.

For $S \subseteq V(G)$, let $\mathcal{C}_i(G-S)$ be the set of components of order i in $G-S$, where integer $i \geq 1$. Write $c_i(G-S) = |\mathcal{C}_i(G-S)|$. For $0 \leq i \leq k-1$, we use $c^o_{<2k}(G-S)$ to denote the number of odd components of $G-S$ with order less than $2k$, that is, $c^o_{<2k}(G-S) = \sum_{1\leq i\leq k} c_{2i-1}(G-S)$.

Theorem 1.3 ([12]). *Let $k \geq 3$ be an integer, and let G be a graph. If $c^o_{<2k}(G-S) \leq \frac{5}{6k^2}|S|$ for all $S \subseteq V(G)$, then G has a $\{P_2, P_{2k+1}\}$-factor.*

Recently, Egawa and Furuya [10, 11] obtained stronger sufficient conditions for $\{P_2, P_{2k+1}\}$-factors with $k = 2, 3, 4$. In particular, they proved the following theorem.

Theorem 1.4 ([10]). *A graph G has a $\{P_2, P_5\}$-factor if $3c_1(G-S) + 2c_3(G-S) \leq 4|S| + 1$ for all $S \subseteq V(G)$.*

Although a sufficient condition for the existence of $\{P_2, P_5\}$-factors was proposed by Egawa and Furuya, to check the condition in Theorem 1.4 is a non-trivial task. This paper is attempted to find more sufficient conditions for the existence of $\{P_2, P_5\}$-factors using various graphic parameters, or to determine special classes of graphs to have $\{P_2, P_5\}$-factors such as r-regular graphs, planar graphs and $K_{1,r}$-free graphs. The graphic parameters been studied in this paper include minimum degree, toughness, binding number, etc.

Theorem 1.5. *Let G be a connected graph of order $n \geq 4$. Then G has a $\{P_2, P_5\}$-factor if one of the following statements holds:* (i) $\tau(G) \geq \frac{3}{4}$; (ii) $bind(G) \geq \frac{5}{2}$; (iii) $n \geq 9$ *and* $\sigma_2(G) \geq \frac{6n}{7}$; (iv) $i(G-S) \leq \frac{2}{5}|S|$ *for all* $S \subseteq V(G)$.

Theorem 1.6. *A connected graph G has a $\{P_2, P_5\}$-factor if G is one of the following two special classes of graphs:* (i) *r-regular graphs with $r \geq 3$;* (ii) *$K_{1,r}$-free graphs with $\delta(G) \geq \frac{3r+5}{4}$.*

2. Proof of Theorem 1.5

Suppose, to the contrary, that G is a connected graph of order $n \geq 4$ and contains no $\{P_2, P_5\}$-factor. By Theorem 1.4, there exists $S \subseteq V(G)$ such that $3c_1(G-S) + 2c_3(G-S) > 4|S| + 1$. Due to the integrality, we obtain

$$c_1(G-S) + c_3(G-S) \geq c_1(G-S) + \frac{2}{3}c_3(G-S) \geq \frac{4}{3}|S| + \frac{2}{3}. \tag{2.1}$$

Claim 2.1. *$S \neq \emptyset$.*

Proof. Suppose that $S = \emptyset$, then by (2.1), we have $c_1(G)+c_3(G) = c_1(G-S)+c_3(G-S) \geq \frac{2}{3}$. According to the integrality, $c_1(G)+c_3(G) \geq 1$. Note that $c_1(G)+c_3(G) \leq \omega(G) = 1$ since G is connected. Then, $G \in \{K_1, K_3, P_3\}$ and thus $|G| \leq 3$, a contradiction. □

(i) If G is complete, then G has a Hamilton path P and $|P| \geq 4$. Obviously, P has a $\{P_2, P_5\}$-factor which is also a $\{P_2, P_5\}$-factor of G, a contradiction. In the following, we assume that G is not complete.
By Claim 2.1 and (2.1), we have that

$$\begin{aligned}|S| &\leq \frac{3 \times (c_1(G-S) + c_3(G-S))}{4} - \frac{1}{2} \\ &< \frac{3}{4} \times (c_1(G-S) + c_3(G-S)).\end{aligned}$$

By the definition of $\tau(G)$, it follows that

$$\tau(G) \leq \frac{|S|}{\omega(G-S)} < \frac{\frac{3}{4} \times (c_1(G-S) + c_3(G-S))}{c_1(G-S) + c_3(G-S)} = \frac{3}{4}.$$

This contradiction completes the proof of Statement (i) of Theorem 1.5.

(ii) We choose one vertex from each component of $G-S$ with order 3, and denote by S' the set of such vertices. Let S'' be the set of isolated vertices of $G-S$. By (2.1), we have that

$$\begin{aligned}|S| &\leq \frac{3}{4} \times \left(c_1(G-S) + \frac{2}{3}c_3(G-S) - \frac{2}{3}\right) \\ &= \frac{3}{4}c_1(G-S) + \frac{1}{2}c_3(G-S) - \frac{1}{2}.\end{aligned}$$

Then,

$$\begin{aligned}|N_G(S' \cup S'')| &\leq |S| + 2 \times c_3(G-S) \\ &\leq \frac{3}{4}c_1(G-S) + \frac{1}{2}c_3(G-S) - \frac{1}{2} + 2 \times c_3(G-S) \\ &= \frac{3}{4}c_1(G-S) + \frac{5}{2}c_3(G-S) - \frac{1}{2} \\ &< \frac{3}{4}c_1(G-S) + \frac{5}{2}c_3(G-S).\end{aligned}$$

It follows that
$$\frac{5}{2} \le bind(G) \le \frac{|N_G(S' \cup S'')|}{|S' \cup S''|} < \frac{\frac{3}{4}c_1(G-S) + \frac{5}{2}c_3(G-S)}{c_1(G-S) + c_3(G-S)} \le \frac{5}{2}.$$

This contradiction completes the proof of Statement (ii) of Theorem 1.5.

(iii) By Claim 2.1 and (2.1), we have that

$$c_1(G-S) + c_3(G-S) \ge \frac{4}{3}|S| + \frac{2}{3} \ge 2. \tag{2.2}$$

Case 1. $c_1(G-S) \ge 2$.

Let $\{x, y\}$ be two distinct isolated vertices of $G-S$. Since $\sigma_2(G) \ge \frac{6n}{7}$ and $N_G(x) \cup N_G(y) \subseteq S$, we have that
$$|S| \ge \frac{1}{2}\sigma_2(G) \ge \frac{3n}{7}.$$

It follows from (2.2) that

$$c_1(G-S) + c_3(G-S) \ge \frac{4}{3} \times \frac{3n}{7} + \frac{2}{3} = \frac{4n}{7} + \frac{2}{3}$$

and thus
$$n \ge |S| + c_1(G-S) + 3 \times c_3(G-S) \ge \frac{3n}{7} + \frac{4n}{7} + \frac{2}{3} > n,$$

a contradiction.

Case 2. $c_1(G-S) \le 1$.

In this case, by (2.2), we have $c_3(G-S) \ge 1$. Let $C_1, C_2, \ldots, C_t$ be the components of $G-S$ such that $|C_1| = 1$ or 3 and $|C_i| = 3$ for $2 \le i \le t$. We take a vertex $c_i \in V(C_i)$ for every $1 \le i \le t$. Obviously, $c_1c_2 \notin E(G)$. Then $d_G(c_1) + d_G(c_2) \ge \sigma_2(G) \ge \frac{6n}{7}$. Here we assume $d_G(c_2) \ge \frac{d_G(c_1)+d_G(c_2)}{2} \ge \frac{3n}{7}$. Note that in the case were $d_G(c_2) \ge \frac{3n}{7}$, the following argument can be applied. Then $d_{C_2}(c_2) \le 2$ and so

$$|S| \ge d_G(c_2) - d_{C_2}(c_2) \ge \frac{3n}{7} - 2.$$

Since $n \ge 9$ and (2.2),

$$\begin{aligned}
n &\ge |S| + c_1(G-S) + 3 \times c_3(G-S) \\
&= |S| + 3 \times (c_1(G-S) + c_3(G-S)) - 2 \times c_1(G-S) \\
&\ge |S| + 3 \times \left(\frac{4}{3}|S| + \frac{2}{3}\right) - 2 \\
&= 5|S| \\
&\ge \frac{15n}{7} - 10 > n.
\end{aligned}$$

This contradiction completes the proof of Statement (iii) of Theorem 1.5.

(iv) We choose two vertex from each nontrivial component of $G-S$ with order 3, and denote the set of such vertices by X. Let $S' = S \cup X$, then

$$i(G-S') = c_1(G-S) + c_3(G-S).$$

It follows from (2.1) that $3c_1(G-S) + 2c_3(G-S) \ge 4|S| + 2 > 4|S|$. Thus we have

$$|S| < \frac{3}{4}c_1(G-S) + \frac{1}{2}c_3(G-S).$$

Furthermore, it follows that

$$\begin{aligned}|S'| &= |S| + |X| \\ &< \frac{3}{4}c_1(G-S) + \frac{1}{2}c_3(G-S) + 2c_3(G-S) \\ &= \frac{3}{4}c_1(G-S) + \frac{5}{2}c_3(G-S) \\ &\le \frac{5}{2}(c_1(G-S) + c_3(G-S)) \\ &= \frac{5}{2}i(G-S').\end{aligned}$$

This contradicts the condition that $i(G - S') \le \frac{2}{5}|S'|$ for all $S' \subseteq V(G)$. This completes the proof of Statement (iv) of Theorem 1.5.

3. Proof of Theorem 1.6

Suppose, to the contrary, that G is a connected graph and contains no $\{P_2, P_5\}$-factor. By Theorem 1.4, there exists $S \subseteq V(G)$ such that $3c_1(G-S) + 2c_3(G-S) > 4|S| + 1$. Due to the integrality, we obtain

$$c_1(G-S) + c_3(G-S) \ge c_1(G-S) + \frac{2}{3}c_3(G-S) \ge \frac{4}{3}|S| + \frac{2}{3}\cdot \tag{3.1}$$

It follows immediately that

$$|S| \le \frac{3}{4}(c_1(G-S) + c_3(G-S)) - \frac{1}{2}\cdot \tag{3.2}$$

(i) We first argue that $|S| \ge 1$.

Claim 3.1. $S \neq \emptyset$.

Proof. Suppose that $S = \emptyset$, then by (3.1), we have $c_1(G) + c_3(G) = c_1(G-S) + c_3(G-S) \ge \frac{2}{3}$. According to the integrality, $c_1(G) + c_3(G) \ge 1$. Note that $c_1(G) + c_3(G) \le \omega(G) = 1$ since G is connected. Then, $G \in \{K_1, K_3, P_3\}$, which contradicts that G is r-regular where $r \ge 3$. □

Let $\mathcal{C}$ be the set of component of $G - S$ with order 1 or 3, and let $X := \cup_{C\in\mathcal{C}} V(C)$. Let $H := [X, S]$ be a bipartite graph such that $V(H) = X \cup S$ and $xs \in E(H)$ if and only if $xs \in E(G)$ for any $x \in X$ and $s \in S$. Let $a = c_1(G-S)$ and $b = c_3(G-S)$. Since X is an independent set of H and $N_H(X) \subseteq S$, we have that $|E(H)| \le r|S|$. Then, by (3.2),

$$r \times a + 3(r-2) \times b \le |E(H)| \le r|S| \le r \times \left(\frac{3}{4}(a+b) - \frac{1}{2}\right)\cdot$$

That is

$$\frac{1}{4}ra + \frac{9}{4}rb + \frac{1}{2}r \le 6b. \tag{3.3}$$

It follows from (3.3) and $r \ge 3$ that

$$6b \ge \frac{1}{4}ra + \frac{9}{4}rb + \frac{1}{2}r \ge \frac{3}{4}a + \frac{27}{4}b + \frac{3}{2} > 6b + 1,$$

a contradiction. This completes the proof of Statement (i) of Theorem 1.6.

(ii) We distinguish two cases below to show that G has a $\{P_2, P_5\}$-factor, which is a contradiction.

Case 1. $S = \emptyset$.

In this case, by (3.1), we have $c_1(G) + c_3(G) = c_1(G - S) + c_3(G - S) \geq \frac{2}{3}$. According to the integrality, $c_1(G) + c_3(G) \geq 1$. On the other hand, $c_1(G) + c_3(G) \leq \omega(G) = 1$ since G is connected. So, we obtain that $G \in \{K_1, K_3, P_3\}$, which contradicts the minimum degree of G.

Case 2. $|S| \geq 1$.

Let $S = \{x_1, x_2, \ldots, x_k\}$, where $|S| = k \geq 1$. Let $\{C_1, C_2, \ldots, C_t\}$ be the set of components of $G - S$ with order 1 or 3. Then, by (3.1), we have that

$$t = c_1(G - S) + c_3(G - S) \geq \frac{4k + 2}{3}\cdot \tag{3.4}$$

For any $i \in [1, t]$, there exists $y_i \in V(C_i)$ such that $d_{C_i}(y_i) \leq 2$. Let $Y = \{y_1, y_2, \ldots, y_t\}$. Then we construct a bipartite subgraph $H \subseteq G$ such that $V(H) = S \cup Y$ and $x_i y_j \in E(H)$ if and only if $x_i y_j \in E(G)$ for any $i \in [1, k],\ j \in [1, t]$. Since for any $j \in [1, t]$, $d_{C_j}(y_j) \leq 2$, we have

$$d_H(y_j) = d_G(y_j) - d_{C_j}(y_j) \geq \frac{3r + 5}{4} - 2 = \frac{3r - 3}{4}\cdot \tag{3.5}$$

It follows from (3.5) that

$$|X| \geq d_H(y_j) \geq \frac{3r - 3}{4}\cdot$$

Then, by (3.4) and (3.5), we have that

$$\begin{aligned} |E(H)| &= \sum_{j=1}^{t} d_H(y_j) \\ &\geq t \times \frac{3r - 3}{4} \\ &\geq \frac{4k + 2}{3} \times \frac{3r - 3}{4} \\ &= k(r - 1) + \frac{r - 1}{2}\cdot \end{aligned}$$

Since $k(r - 1) + \frac{r-1}{2} \leq |E(H)| = \sum_{i=1}^{k} d_H(x_i)$, there exists $x_b \in S$ such that $d_H(x_b) \geq r$. Then $H[\{x_b\} \cup N_H(x_b)] = G[\{x_b\} \cup N_H(x_b)]$ includes $K_{1,r}$. This is a contradiction and completes the proof of Statement (ii) of Theorem 1.6.

Acknowledgements. This work is supported by the National Natural Science Foundation of China (Grant Nos. 11871239, 11971196, 11871280 and U1811461), the Natural Science Foundation of Guangdong Province (Grant No. 2020B1515310009).

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