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THE EXISTENCE OF PATH-FACTOR UNIFORM GRAPHS WITH LARGE CONNECTIVITY

Sizhong Zhou* and Qiuxiang Bian

Abstract. A path-factor is a spanning subgraph F of G such that every component of F is a path with at least two vertices. Let $k \geq 2$ be an integer. A $P_{\geq k}$-factor of G means a path factor in which each component is a path with at least k vertices. A graph G is a $P_{\geq k}$-factor covered graph if for any $e \in E(G)$, G has a $P_{\geq k}$-factor covering e. A graph G is called a $P_{\geq k}$-factor uniform graph if for any $e_1, e_2 \in E(G)$ with $e_1 \neq e_2$, G has a $P_{\geq k}$-factor covering e_1 and avoiding e_2. In other words, a graph G is called a $P_{\geq k}$-factor uniform graph if for any $e \in E(G)$, $G - e$ is a $P_{\geq k}$-factor covered graph. In this paper, we present two sufficient conditions for graphs to be $P_{\geq 3}$-factor uniform graphs depending on binding number and degree conditions. Furthermore, we show that two results are best possible in some sense.

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1. Introduction

In our daily life many physical structures can conveniently be simulated by networks. The core issue of network security is the ruggedness and vulnerability of the network, which is also one of the key topics that researchers must consider during the network designing phase. To study the properties of the network, we use a graph to simulate the network. Vertices of the graph correspond nodes of the network and edges of the graph represent links between the nodes of the network. In data transmission networks, the data transmission between two nodes stands for a path between two corresponding vertices. Therefore, the availability of data transmission in the network is equivalent to the existence of path-factor of the corresponding graph which is generated by the network. Obviously, research on the existence of path-factors under specific network structures can help scientists design and construct networks with high data transmission rates. Furthermore, the existence of a path-factor uniform graph also plays a key role in data transmission of a network. If a link is assigned and a link is damaged in the process of data transmission at the moment, the possibility of data transmission between nodes is characterized by whether the corresponding graph of the network is a path-factor uniform graph. We find that there are strong essential connection between some graphic parameters (for instance, degree and binding number, and so on) and the existence of path-factors in graphs (or path-factor uniform graphs), and

Keywords. Graph, degree condition, binding number, $P_{\geq 3}$-factor, $P_{\geq 3}$-factor uniform graph.

School of Science, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, P.R. China.
*Corresponding author: zsz_cumt@163.com

hence investigations on degree and binding number can yield theoretical guidance to meet data transmission and network security requirements.

Throughout this paper we discuss only finite undirected graphs which admit neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x in G, denoted by $d_G(x)$, is the number of edges incident with x. The neighborhood of a vertex x in G, denoted by $N_G(x)$, is the set of vertices adjacent to x. We denote by xy an edge joining vertices x and y. The number of isolated vertices in G is denoted by $i(G)$ and the number of connected components of G is denoted by $\omega(G)$. For a vertex subset X of G, we write $N_G(X)$ for $\bigcup_{x\in X} N_G(x)$ and denote by $G-X$ the subgraph derived from G by removing the vertices in X together with the edges adjacent to vertices in X. A vertex subset X of G is said to be independent if $X\cap N_G(X)=\emptyset$. For an edge subset E' of G, we denote by $G-E'$ the subgraph derived from G by removing edges of E'. The binding number of G, denoted by $bind(G)$, was first introduced by Woodall [14] and is defined by

$$bind(G)=\min\left\{\frac{|N_G(X)|}{|X|}:\emptyset\neq X\subseteq V(G), N_G(X)\neq V(G)\right\}.$$

We use P_n and K_n to denote the path and the complete graph of order n, respectively. Let G_1 and G_2 be two vertex-disjoint graphs. We denote the join and the union of G_1 and G_2 by $G_1\vee G_2$ and $G_1\cup G_2$, respectively. Let r be a real number. Recall that $\lceil r\rceil$ is the smallest integer such that $\lceil r\rceil\geq r$.

A subgraph of a graph G is spanning if the subgraph includes all vertices of G. Let $\mathcal{H}$ be a family of connected graphs. A spanning subgraph F of a graph G is called an $\mathcal{H}$-factor if each component of F is isomorphic to an element in $\mathcal{H}$. A path-factor is a spanning subgraph F of G such that every component of F is a path with at least two vertices. Let $k\geq 2$ be an integer. A $P_{\geq k}$-factor of G means a path factor in which each component is a path with at least k vertices.

A 1-factor of G is a spanning subgraph F of G satisfying $d_F(x)=1$ for any $x\in V(G)$. A graph H is factor-critical if $H-x$ contains a 1-factor for any $x\in V(H)$. To characterize a graph with a $P_{\geq 3}$-factor, Kaneko [7] introduced the concept of a sun. A sun is a graph derived from a factor-critical graph H by adding n new vertices $x_1,x_2,\cdots,x_n$ and n new edges $y_1x_1,y_2x_2,\cdots,y_nx_n$, where $V(H)=\{y_1,y_2,\cdots,y_n\}$. According to Kaneko, K_1 and K_2 are also suns. A sun with at least six vertices is called a big sun. There is no sun with at most five vertices except for K_1 and K_2. A component of G is called a sun component if it is isomorphic to a sun. We use $sun(G)$ to denote the number of sun components of G.

Kaneko [7] showed a criterion for a graph with a $P_{\geq 3}$-factor. Kano *et al.* [8] put forward a simpler proof.

Theorem 1.1 ([7,8])**.** *A graph G has a $P_{\geq 3}$-factor if and only if*

$$sun(G-X)\leq 2|X|$$

for any vertex subset X of G.

Kaneko [7] verified that a regular graph with degree at least 2 admits a $P_{\geq 3}$-factor. Bazgan *et al.* [1] posed a sufficient condition for a graph to admit a $P_{\geq 3}$-factor. Kano *et al.* [10] claimed that a graph with $i(G-X)\leq\frac{2}{3}|X|$ for any $X\subseteq V(G)$ has a $P_{\geq 3}$-factor. Zhou *et al.* [18,23,24,27] investigated the existence of $P_{\geq 3}$-factors in graphs with given properties. Gao *et al.* [4] obtained some results on the existence of $P_{\geq 3}$-factors in graphs. Wang and Zhang [13] posed an isolated toughness condition for graphs to have $P_{\geq 3}$-factors. Kano *et al.* [9] proved that every connected cubic bipartite graph with at least eight vertices admits a $P_{\geq 8}$-factor. Johnson *et al.* [6] presented a sufficient condition for a graph having a path-factor. Egawa and Furuya [2] derived a result on path-factors in graphs. Zhou *et al.* [16,19,20,25,28] and Wang and Zhang [12] established some relationships between degree conditions and graph factors. Wang and Zhang [11] and Zhou [17] obtained some binding number conditions for the existence of graph factors.

For a subgraph H and an edge e of G, we call that H covers e if $e\in E(H)$. Later, Zhang and Zhou [15] defined a graph G to be a $P_{\geq k}$-factor covered graph if for any $e\in E(G)$, G has a $P_{\geq k}$-factor covering e, and showed a criterion for a graph to be a $P_{\geq 3}$-factor covered graph.

Theorem 1.2 ([15]). *Let G be a connected graph. Then G is a $P_{\geq 3}$-factor covered graph if and only if*

$$sun(G-X) \leq 2|X| - \varepsilon(X)$$

for any vertex subset X of G, where $\varepsilon(X)$ is defined by

$$\varepsilon(X) = \begin{cases} 2, & \text{if } X \text{ is not an independent set;} \\ 1, & \text{if } X \text{ is a nonempty independent set and } G-X \text{ admits a non-sun component;} \\ 0, & \text{otherwise.} \end{cases}$$

More recently, Zhou and Sun [21] posed the concept of a $P_{\geq k}$-factor uniform graph, that is, a graph G is called a $P_{\geq k}$-factor uniform graph if for any $e_1, e_2 \in E(G)$ with $e_1 \neq e_2$, G has a $P_{\geq k}$-factor covering e_1 and avoiding e_2. In other words, a graph G is called a $P_{\geq k}$-factor uniform graph if for any $e \in E(G)$, $G - e$ is a $P_{\geq k}$-factor covered graph. Furthermore, they put forward a binding number condition for the existence of a $P_{\geq 3}$-factor uniform graph.

Theorem 1.3 ([21]). *A 2-edge-connected graph G is a $P_{\geq 3}$-factor uniform graph if its binding number $bind(G) > \frac{9}{4}$.*

Gao and Wang [3] improved the binding number condition in Theorem 1.3, and derived the following theorem.

Theorem 1.4 ([3]). *A 2-edge-connected graph G is a $P_{\geq 3}$-factor uniform graph if its binding number $bind(G) > \frac{5}{3}$.*

Some other results on $P_{\geq 3}$-factor uniform graphs can be found in Zhou *et al.* [22, 26] and Hua [5]. In this paper, we characterize $P_{\geq 3}$-factor uniform graphs with respect to the vertex degree or the binding number, and obtain the following two results.

Theorem 1.5. *Let G be a $\left(\lceil \frac{t-1}{2} \rceil + 3\right)$-connected graph of order n with $n \geq 2t+7$, where $t \geq 2$ is an integer. If*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_t)\} \geq \frac{n+2}{3}$$

for any independent set $\{x_1, x_2, \cdots, x_t\}$ of G, then G is a $P_{\geq 3}$-factor uniform graph.

Theorem 1.6. *An $(r+1)$-connected graph G is a $P_{\geq 3}$-factor uniform graph if its binding number $bind(G) > \frac{4r+1}{3r}$, where $r \geq 1$ is an integer.*

2. The proof of Theorem 1.5

Proof of Theorem 1.5. Suppose that there exists an edge $e = xy$ in G such that $G' = G - e$ is not a $P_{\geq 3}$-factor covered graph. Then from Theorem 1.2, there exists a vertex subset X of G' such that

$$sun(G'-X) \geq 2|X| - \varepsilon(X) + 1. \tag{2.1}$$

In what follows, we verify two claims.

Claim 2.1. $i(G-X) \leq t-1$.

Proof. If $i(G-X) \geq t$, then there exist at least t isolated vertices $x_1, x_2, \cdots, x_t$ in $G-X$. Clearly, $d_{G-X}(x_i) = 0$ for $1 \leq i \leq t$. Thus, we acquire

$$d_G(x_i) \leq d_{G-X}(x_i) + |X| = |X| \tag{2.2}$$

for $1 \leq i \leq t$. It follows from (2.2) and the degree condition of Theorem 1.5 that

$$\frac{n+2}{3} \leq \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_t)\} \leq |X|. \tag{2.3}$$

According to (2.1), (2.3) and $\varepsilon(X) \leq 2$, we infer

$$n \geq |X| + sun(G' - X) \geq |X| + 2|X| - \varepsilon(X) + 1 \geq 3|X| - 1 \geq 3 \cdot \frac{n+2}{3} - 1 = n + 1,$$

which is a contradiction. We finish the proof of Claim 2.1. $\square$

Claim 2.2. $|X| \geq \lceil \frac{t-1}{2} \rceil + 2$.

Proof. Let $|X| \leq \lceil \frac{t-1}{2} \rceil + 1$. Combining this with G being $\big(\lceil \frac{t-1}{2} \rceil + 3\big)$-connected, we easily see that $G - X$ is 2-connected and $G' - X$ is connected. Hence, $\omega(G' - X) = 1$. Then by (1), $|X| \geq 0$ and $\varepsilon(X) \leq |X|$, we derive

$$1 = \omega(G' - X) \geq sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq |X| + 1 \geq 1,$$

which implies that $|X| = 0$ and $sun(G') = 1$. Therefore, G' is a big sun by $n \geq 2t+7$. Let H be the factor-critical subgraph of G'. Then $d_{G'}(x) = 1$ for any $x \in V(G') \setminus V(H)$. Combining this with $G' = G - e$ and $n \geq 2t + 7$, there exists an independent set $\{x_1, x_2, \cdots, x_t\} \subseteq V(G) \setminus V(H)$ of G such that $d_G(x_i) = 1$ for $1 \leq i \leq t$. Thus, we obtain by $n \geq 2t + 7$

$$1 = \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_t)\} \geq \frac{n+2}{3} > 1,$$

which is a contradiction. Claim 2.2 is proved. $\square$

By virtue of (2.1), $\varepsilon(X) \leq 2$ and Claim 2.2, we get

$$\begin{aligned} sun(G' - X) &\geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 \\ &\geq 2\left(\left\lceil \frac{t-1}{2} \right\rceil + 2\right) - 1 = 2 \cdot \left\lceil \frac{t-1}{2} \right\rceil + 3 \\ &\geq 2 \cdot \frac{t-1}{2} + 3 = t + 2, \end{aligned}$$

and so

$$sun(G - X) \geq sun(G' - X) - 2 \geq (t + 2) - 2 = t,$$

which implies that $G - X$ admits at least t sun components, denoted by $R_1, R_2, \cdots, R_t$. Choose $x_i \in V(R_i)$ such that $d_{R_i}(x_i) \leq 1$ for $1 \leq i \leq t$. Obviously, $\{x_1, x_2, \cdots, x_t\}$ is an independent set of G. Hence, we admit

$$\begin{aligned} \frac{n+2}{3} &\leq \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_t)\} \\ &\leq \max\{d_{R_1}(x_1), d_{R_2}(x_2), \cdots, d_{R_t}(x_t)\} + |X| \\ &\leq 1 + |X|, \end{aligned}$$

that is,

$$|X| \geq \frac{n-1}{3}. \tag{2.4}$$

According to (2.1), (2.4), Claim 2.1, $\varepsilon(X) \leq 2$, $t \geq 2$, $n \geq 2t + 7$ and $i(G' - X) \leq i(G - X) + 2$, we have

$$\begin{aligned} n &\geq |X| + 2 \cdot sun(G' - X) - i(G' - X) \\ &\geq |X| + 2(2|X| - \varepsilon(X) + 1) - i(G - X) - 2 \\ &\geq |X| + 2(2|X| - 1) - i(G - X) - 2 \\ &= 5|X| - i(G - X) - 4 \\ &\geq 5 \cdot \frac{n-1}{3} - (t - 1) - 4 \end{aligned}$$

$$
\begin{aligned}
&= n + \frac{2n-5}{3} - t - 3 \\
&\geq n + \frac{2(2t+7)-5}{3} - t - 3 \\
&= n + \frac{t}{3} \\
&> n,
\end{aligned}
$$

which is a contradiction. This completes the proof of Theorem 1.5. □

Remark 2.3. In what follows, we show that

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_t)\} \geq \frac{n+2}{3}$$

in Theorem 1.5 cannot be replaced by

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_t)\} \geq \frac{n+2}{3} - 1.$$

Let $t \geq 2$ be an integer and r be a sufficiently large integer. We establish a graph $G = K_{rt} \vee ((2rt-2)K_1 \cup K_2)$. Then $n = 3rt$ and

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_t)\} = rt = \frac{n}{3} > \frac{n+2}{3} - 1$$

for any independent set $\{x_1, x_2, \cdots, x_t\} \subseteq V((2rt-2)K_1)$ of G. And so

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_t)\} \geq rt = \frac{n}{3} > \frac{n+2}{3} - 1$$

for any independent set $\{x_1, x_2, \cdots, x_t\}$ of G. Write $X = V(K_{rt})$, and so $\varepsilon(X) = 2$. Let $G' = G - e$ for $e \in E(K_2)$. Thus, we infer

$$sun(G' - X) = 2rt = 2|X| > 2|X| - 2 = 2|X| - \varepsilon(X).$$

By virtue of Theorem 1.2, G' is not a $P_{\geq 3}$-factor covered graph. Hence, G is not a $P_{\geq 3}$-factor uniform graph.

3. The proof of Theorem 1.6

Proof of Theorem 1.6. Theorem 1.6 holds for $r = 1$ by Theorem 1.4. In what follows, we assume that $r \geq 2$. We proceed by contradiction.

Suppose that there exists an edge $e = xy$ in G such that $G' = G - e$ is not a $P_{\geq 3}$-factor covered graph. Then it follows from Theorem 1.2 that

$$sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \tag{3.1}$$

for some vertex subset X of G'.

Claim 3.1. $|X| \geq r + 1$.

Proof. If $0 \leq |X| \leq r - 1$, then it follows from G being $(r+1)$-connected that $G - X$ is 2-connected. And so $G' - X$ is connected, that is, $\omega(G' - X) = 1$. Combining this with (5) and $\varepsilon(X) \leq |X|$, we obtain

$$1 = \omega(G' - X) \geq sun(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq |X| + 1 \geq 1,$$

which implies that $|X| = 0$ and $sun(G') = 1$, and so G' is a sun. Note that $|V(G')| = |V(G)| \geq r + 2 \geq 4$. Hence, G' is a big sun, which implies that there exists $v \in V(G')$ with $d_{G'}(v) = 1$. Thus, we deduce

$$d_G(v) \leq d_{G'}(v) + 1 = 2 \leq r,$$

which contradicts that G is $(r+1)$-connected.

If $|X| = r$, then by G being $(r+1)$-connected, we have $\omega(G-X) = 1$. Thus, we obtain $\omega(G'-X) = \omega(G-e-X) \leq \omega(G-X)+1 = 2$. Combining this with (5), $\varepsilon(X) \leq 2$ and $r \geq 2$, we get

$$2 \geq \omega(G'-X) \geq sun(G'-X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 = 2r - 1 \geq 3,$$

which is a contradiction. Hence, $|X| \geq r+1$. We finish the proof of Claim 3.1. □

Suppose that there exist a isolated vertices, b K_2's and c big sun components $R_1, R_2, \cdots, R_c$, where $|V(R_i)| \geq 6$, in $G'-X$. Thus, we obtain

$$sun(G'-X) = a + b + c. \tag{3.2}$$

In terms of (3.1), (3.2), $\varepsilon(X) \leq 2$, $r \geq 2$ and Claim 3.1, we get

$$a + b + c = sun(G'-X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 \geq 2(r+1) - 1 = 2r + 1 \geq 5. \tag{3.3}$$

Write $R = R_1 \cup R_2 \cup \cdots \cup R_c$. Then $|V(R)| = \sum_{i=1}^{c} |V(R_i)| \geq 6c$.

Claim 3.2. $x \notin V(aK_1)$ and $y \notin V(aK_1)$.

Proof. Suppose that $x \in V(aK_1)$ or $y \in V(aK_1)$. Without loss of generality, let $x \in V(aK_1)$.

If $y \in V(aK_1)$, then $a \geq 2$. Select $Q = V(aK_1) \cup V(bK_2) \cup V(R) \setminus \{x\}$. Then $Q \neq \emptyset$ and $N_G(Q) \neq V(G)$. Thus, we admit

$$\frac{4r+1}{3r} < bind(G) \leq \frac{|N_G(Q)|}{|Q|} \leq \frac{|X| + 2b + |V(R)| + 1}{a + 2b + |V(R)| - 1},$$

which implies

$$3r|X| > (4r+1)a + (2r+2)b + (r+1)|V(R)| - 7r - 1. \tag{3.4}$$

By virtue of (3.3), (3.4), $a \geq 2$ and $|V(R)| \geq 6c$, we deduce

$$\begin{aligned}
3r|X| &> (4r+1)a + (2r+2)b + (r+1)|V(R)| - 7r - 1 \\
&\geq (4r+1)a + (2r+2)b + (2r+2)c - 7r - 1 \\
&= (2r+2)(a+b+c) + (2r-1)a - 7r - 1 \\
&\geq (2r+2)(2|X|-1) + 2(2r-1) - 7r - 1 \\
&= (4r+4)|X| - 5r - 5,
\end{aligned}$$

which implies

$$|X| < \frac{5r+5}{r+4} \leq r+1,$$

which contradicts Claim 3.1.

If $y \in V(bK_2) \cup V(R)$, then $a \geq 1$ and we choose $Q = V(aK_1) \cup V(bK_2) \cup V(R) \setminus \{y\}$. Then $Q \neq \emptyset$ and $N_G(Q) \neq V(G)$. Thus, we derive

$$\frac{4r+1}{3r} < bind(G) \leq \frac{|N_G(Q)|}{|Q|} \leq \frac{|X| + 2b + |V(R)|}{a + 2b + |V(R)| - 1},$$

which implies

$$3r|X| > (4r+1)a + (2r+2)b + (r+1)|V(R)| - 4r - 1. \tag{3.5}$$

It follows from (3.3), (3.5), $a \geq 1$ and $|V(R)| \geq 6c$ that

$$3r|X| > (4r+1)a + (2r+2)b + (r+1)|V(R)| - 4r - 1$$

$$\begin{aligned}
&\geq (4r+1)a + (2r+2)b + (2r+2)c - 4r - 1\\
&= (2r+2)(a+b+c) + (2r-1)a - 4r - 1\\
&\geq (2r+2)(2|X|-1) + (2r-1) - 4r - 1\\
&= (4r+4)|X| - 4r - 4,
\end{aligned}$$

which implies

$$(r+4)|X| < 4r+4. \tag{3.6}$$

According to Claim 3.1, we deduce

$$(r+4)|X| \geq (r+4)(r+1) = r^2 + 5r + 4 > 4r + 4,$$

which contradicts (3.6).

If $y \in V(G) \setminus (V(aK_1) \cup V(bK_2) \cup V(R))$, then $a \geq 1$ and we select $Q = V(aK_1) \cup V(bK_2) \cup V(R)$. Then $Q \neq \emptyset$ and $N_G(Q) \neq V(G)$. Thus, we admit

$$\frac{4r+1}{3r} < bind(G) \leq \frac{|N_G(Q)|}{|Q|} \leq \frac{|X| + 2b + |V(R)| + 1}{a + 2b + |V(R)|},$$

which implies

$$3r|X| > (4r+1)a + (2r+2)b + (r+1)|V(R)| - 3r. \tag{3.7}$$

By means of (3.3), (3.7), $a \geq 1$, $|V(R)| \geq 6c$ and Claim 3.1, we infer

$$\begin{aligned}
3r|X| &> (4r+1)a + (2r+2)b + (r+1)|V(R)| - 3r\\
&\geq (4r+1)a + (2r+2)b + (2r+2)c - 3r\\
&= (2r+2)(a+b+c) + (2r-1)a - 3r\\
&\geq (2r+2)(2|X|-1) + (2r-1) - 3r\\
&= (4r+4)|X| - 3(r+1)\\
&\geq (4r+4)|X| - 3|X|\\
&= (4r+1)|X|,
\end{aligned}$$

which is a contradiction. Claim 3.2 is proved. □

Claim 3.3. $a \geq 1$.

Proof. If $a = 0$, then $b + c \geq 5$ by (3.3). Hence, there exist two vertices v and w in $(bK_2) \cup R$ such that the degree of w in $(bK_2) \cup R$ is 1, $vw \in E((bK_2) \cup R)$ and $w \notin \{x, y\}$. Select $Q = V(bK_2) \cup V(R) \setminus \{v\}$. Then $Q \neq \emptyset$ and $N_G(Q) \neq V(G)$. Thus, we get

$$\frac{4r+1}{3r} < bind(G) \leq \frac{|N_G(Q)|}{|Q|} \leq \frac{|X| + 2b + |V(R)|}{2b + |V(R)| - 1},$$

namely,

$$0 > (2r+2)b + (r+1)|V(R)| - 3r|X| - 4r - 1. \tag{3.8}$$

In light of (3.3), (3.8), $a = 0$, $|V(R)| \geq 6c$ and Claim 3.1, we obtain

$$\begin{aligned}
0 &> (2r+2)b + (r+1)|V(R)| - 3r|X| - 4r - 1\\
&\geq (2r+2)b + (2r+2)c - 3r|X| - 4r - 1\\
&= (2r+2)(a+b+c) - 3r|X| - 4r - 1
\end{aligned}$$

$$\begin{aligned} &\geq (2r+2)(2|X|-1) - 3r|X| - 4r - 1 \\ &= (r+4)|X| - 6r - 3 \\ &\geq (r+4)(r+1) - 6r - 3 \\ &= r^2 - r + 1 > 0, \end{aligned}$$

which is a contradiction. Hence, $a \geq 1$. This completes the proof of Claim 3.3. □

Set $Q = V(aK_1) \cup V(bK_2) \cup V(R)$. By Claims 3.2 and 3.3, we see that $Q \neq \emptyset$ and $N_G(Q) \neq V(G)$. Thus, we derive

$$\frac{4r+1}{3r} < bind(G) \leq \frac{|N_G(Q)|}{|Q|} \leq \frac{|X| + 2b + |V(R)| + 1}{a + 2b + |V(R)|},$$

that is,

$$3r|X| > (4r+1)a + (2r+2)b + (r+1)|V(R)| - 3r. \tag{3.9}$$

Equation (3.9) is completely the same as equation (3.7), and so we can also deduce a contradiction. We finish the proof of Theorem 1.6. □

Remark 3.4. Next, we show that the condition $bind(G) > \frac{4r+1}{3r}$ in Theorem 1.6 cannot be placed by $bind(G) > \frac{4r+1}{3r+1}$.

We construct a graph $G = K_{r+1} \vee \left(\frac{3r+2}{2}K_2\right)$, where $r = 2$. Then we easily see that $bind(G) = \frac{4r+2}{3r+1} > \frac{4r+1}{3r+1}$ and G is $(r+1)$-connected. For any $e \in E\left(\frac{3r+2}{2}K_2\right)$, we let $G' = G - e$. Write $X = V(K_{r+1})$. Then $|X| = r+1$ and $\varepsilon(X) = 2$. Thus, we infer

$$sun(G' - X) = \frac{3r}{2} + 2 > 2(r+1) - 2 = 2|X| - \varepsilon(X).$$

By virtue of Theorem 1.2, G' is not a $P_{\geq 3}$-factor covered graph. Therefore, G is not a $P_{\geq 3}$-factor uniform graph.

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