

## ISOLATED TOUGHNESS FOR PATH FACTORS IN NETWORKS

SUFANG WANG<sup>1,\*</sup> AND WEI ZHANG<sup>2</sup>

**Abstract.** Let  $\mathcal{H}$  be a set of connected graphs. Then an  $\mathcal{H}$ -factor is a spanning subgraph of  $G$ , whose every connected component is isomorphic to a member of the set  $\mathcal{H}$ . An  $\mathcal{H}$ -factor is called a path factor if every member of the set  $\mathcal{H}$  is a path. Let  $k \geq 2$  be an integer. By a  $P_{\geq k}$ -factor we mean a path factor in which each component path admits at least  $k$  vertices. A graph  $G$  is called a  $(P_{\geq k}, n)$ -factor-critical covered graph if for any  $W \subseteq V(G)$  with  $|W| = n$  and any  $e \in E(G - W)$ ,  $G - W$  has a  $P_{\geq k}$ -factor covering  $e$ . In this article, we verify that (i) an  $(n + \lambda + 2)$ -connected graph  $G$  is a  $(P_{\geq 2}, n)$ -factor-critical covered graph if its isolated toughness  $I(G) > \frac{n + \lambda + 2}{2\lambda + 3}$ , where  $n$  and  $\lambda$  are two nonnegative integers; (ii) an  $(n + \lambda + 2)$ -connected graph  $G$  is a  $(P_{\geq 3}, n)$ -factor-critical covered graph if its isolated toughness  $I(G) > \frac{n + 3\lambda + 5}{2\lambda + 3}$ , where  $n$  and  $\lambda$  be two nonnegative integers.

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### 1. INTRODUCTION

We may model real-world networks by graphs. The vertices of the graph stand for the nodes of the network, and the edges of the graph act as the links between the nodes in the network. Henceforth, we replace “network” by the term “graph”. Thus we may use several graphic parameters to characterize the robustness and vulnerability of the network, for instance, isolated toughness, independence number and minimum degree, and so on. In data transmission networks, the data transmission between two sites of a network goes through a path between two corresponding vertices of a corresponding graph. Therefore, the availability of data transmission in the network is equal to the existence of path factor in the corresponding graph which is generated by the network. When some nodes are damaged and a special channel is assigned, the possibility of data transmission in a data transmission network is equivalent to the existence of path factor critical covered graph. Research on the existence of path factors or path factor critical covered graphs under specific network structures can help scientists to design and construct networks with high data transmission rates. In this article, we study the existence of path factor critical covered graphs which play a key role in investigating data transmissions of data transmission networks. We find that there is strong essential connection between isolated toughness and the existence of path factor critical covered graphs, and hence investigations on isolated toughness, which play an irreplaceable role in the vulnerability of the network and the feasibility of data transmission, can yield theoretical guidance to meet data transmission and network security requirements.

*Keywords.* Graph, isolated toughness,  $P_{\geq k}$ -factor,  $P_{\geq k}$ -factor covered graph,  $(P_{\geq k}, n)$ -factor-critical covered graph.

<sup>1</sup> School of Public Management, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, P.R. China.

<sup>2</sup> School of Economics and Management, Wenzhou University of Technology, Wenzhou, Zhejiang 325000, P.R. China.

\*Corresponding author: [wangsufangjust@163.com](mailto:wangsufangjust@163.com)

In this article, we discuss only finite undirected graphs without loops or multiple edges. We denote a graph by  $G = (V(G), E(G))$ , where  $V(G)$  is the vertex set of  $G$  and  $E(G)$  is the edge set of  $G$ . Let  $i(G)$  denote the number of isolated vertices of  $G$ , and let  $\omega(G)$  denote the number of connected components of  $G$ . For  $x \in V(G)$ , we denote by  $d_G(x)$  the degree of  $x$  in  $G$ . Let  $X$  be a subset of  $V(G)$ . We call that  $X$  is independent if no two elements in  $X$  are adjacent, and we denote by  $G[V(G) \setminus X]$  the subgraph of  $G$  induced by  $V(G) \setminus X$ . Yang *et al.* [13] defined a graphic parameter, isolated toughness of  $G$ , denoted by  $I(G)$ , namely,

$$I(G) = \min \left\{ \frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) \geq 2 \right\}$$

if  $G$  is not a complete graph; otherwise,  $I(G) = +\infty$ . Write  $K_n$  and  $P_n$  for the complete graph and the path of order  $n$ , respectively.

Let  $\mathcal{H}$  be a set of connected graphs. Then an  $\mathcal{H}$ -factor is a spanning subgraph of  $G$ , whose every connected component is isomorphic to a member of the set  $\mathcal{H}$ . An  $\mathcal{H}$ -factor is called a path factor if every member of the set  $\mathcal{H}$  is a path. Let  $k \geq 2$  be an integer. By a  $P_{\geq k}$ -factor we mean a path factor in which each component path admits at least  $k$  vertices. A graph  $G$  is called a  $P_{\geq k}$ -factor covered graph if for any  $e \in E(G)$ ,  $G$  has a  $P_{\geq k}$ -factor containing  $e$ . A graph  $G$  is called a  $(P_{\geq k}, n)$ -factor-critical covered graph if for any  $Q \subseteq V(G)$  with  $|Q| = n$ ,  $G - Q$  is a  $P_{\geq k}$ -factor covered graph.

To characterize a graph with a  $P_{\geq 3}$ -factor, Kaneko [5] introduced the concept of a sun. A graph  $R$  is called a factor-critical graph if for any  $x \in V(R)$ ,  $R - x$  has a perfect matching. Assume that  $R$  is a factor-critical graph with vertex set  $V(R) = \{x_1, x_2, \dots, x_n\}$ . By adding new vertices  $y_1, y_2, \dots, y_n$  together with new edges  $x_1y_1, x_2y_2, \dots, x_ny_n$  to  $R$ , we acquire a new graph, which is called a sun. In particular,  $K_1$  and  $K_2$  are also called suns. A sun with at least six vertices is called a big sun. A component of  $G - X$  is called a sun component if it is isomorphic to a sun. Let  $\text{sun}(G - X)$  be the number of sun components in  $G - X$ .

A criterion for a graph to admit a  $P_{\geq 2}$ -factor was derived by Las Vergnas [9].

**Theorem 1.1** ([9]). *A graph  $G$  has a  $P_{\geq 2}$ -factor if and only if*

$$i(G - X) \leq 2|X|$$

for all  $X \subseteq V(G)$ .

A characterization for a graph with a  $P_{\geq 3}$ -factor was provided by Kaneko [5].

**Theorem 1.2** ([5]). *A graph  $G$  has a  $P_{\geq 3}$ -factor if and only if*

$$\text{sun}(G - X) \leq 2|X|$$

for all  $X \subseteq V(G)$ .

Zhang and Zhou [15] extended Theorems 1.1 and 1.2, and got two characterizations for graphs to be  $P_{\geq 2}$ -factor and  $P_{\geq 3}$ -factor covered graphs.

**Theorem 1.3** ([15]). *A connected graph  $G$  is a  $P_{\geq 2}$ -factor covered graph if and only if*

$$i(G - X) \leq 2|X| - \varepsilon_1(X)$$

for all  $X \subseteq V(G)$ , where  $\varepsilon_1(X)$  is defined by

$$\varepsilon_1(X) = \begin{cases} 2, & \text{if } X \text{ is not an independent set;} \\ 1, & \text{if } X \text{ is a nonempty independent set, and } G - X \text{ admits} \\ & \text{a nontrivial component;} \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 1.4** ([15]). *A connected graph  $G$  is a  $P_{\geq 3}$ -factor covered graph if and only if*

$$\text{sun}(G - X) \leq 2|X| - \varepsilon_2(X)$$

for all  $X \subseteq V(G)$ , where  $\varepsilon_2(X)$  is defined by

$$\varepsilon_2(X) = \begin{cases} 2, & \text{if } X \text{ is not an independent set;} \\ 1, & \text{if } X \text{ is a nonempty independent set, and } G - X \text{ admits} \\ & \text{a non-sun component;} \\ 0, & \text{otherwise.} \end{cases}$$

Kano *et al.* [7] claimed that a graph has a  $P_{\geq 3}$ -factor if  $i(G - X) \leq \frac{2}{3}|X|$  for any  $X \subseteq V(G)$ . Kawarabayashi *et al.* [8] showed that a 2-connected cubic graph with at least six vertices admits a  $P_{\geq 6}$ -factor. Kano *et al.* [6] verified that a connected cubic bipartite graph with at least eight vertices has a  $P_{\geq 8}$ -factor. Zhou [19], Zhou *et al.* [22], Zhou *et al.* [25], Gao and Wang [3], Zhou, Sun and Bian [24] discussed the existence of  $P_{\geq 2}$ -factors and  $P_{\geq 3}$ -factors with given properties in graphs. Zhou *et al.* [21] posed two sufficient conditions for a graph to be a  $(P_{\geq 3}, n)$ -factor-critical covered graph. Gao *et al.* [4] showed a binding number condition for a graph to be a  $(P_{\geq 3}, n)$ -factor-critical covered graph. Zhou *et al.* [26] obtained a result on the existence of  $(P_{\geq 3}, n)$ -factor-critical covered graphs. Some recent advances on graph factors can be found in Zhou *et al.* [20, 23], Zhou [16–18], Wang and Zhang [10–12], Yuan and Hao [14], Anstee and Nam [1] and Egawa *et al.* [2]. In this article, we study the existence of  $(P_{\geq 2}, n)$ -factor-critical covered graphs and  $(P_{\geq 3}, n)$ -factor-critical covered graphs, and derive the following two results.

**Theorem 1.5.** *Let  $n$  and  $\lambda$  be two nonnegative integers. Then an  $(n + \lambda + 2)$ -connected graph  $G$  is a  $(P_{\geq 2}, n)$ -factor-critical covered graph if its isolated toughness  $I(G) > \frac{n+\lambda+2}{2\lambda+3}$ .*

**Theorem 1.6.** *Let  $n$  and  $\lambda$  be two nonnegative integers. Then an  $(n + \lambda + 2)$ -connected graph  $G$  is a  $(P_{\geq 3}, n)$ -factor-critical covered graph if its isolated toughness  $I(G) > \frac{n+3\lambda+5}{2\lambda+3}$ .*

## 2. THE PROOF OF THEOREM 1.5

*Proof of Theorem 1.5.* If  $G$  is a complete graph, then it is obvious that  $G$  is a  $(P_{\geq 2}, n)$ -factor-critical covered graph by  $G$  being  $(n + \lambda + 2)$ -connected. In what follows, we consider that  $G$  is not a complete graph.

Let  $G' = G - W$  for any  $W \subseteq V(G)$  with  $|W| = n$ . To prove Theorem 1.5, it suffices to verify that  $G'$  is a  $P_{\geq 2}$ -factor covered graph. By means of contrary, we assume that  $G'$  is not a  $P_{\geq 2}$ -factor covered graph. Then it follows from Theorem 1.3 that

$$i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \tag{2.1}$$

for some vertex subset  $X$  of  $G'$ .

**Claim 1.**  $|X| \geq \lambda + 2$ .

*Proof.* Let  $|X| \leq \lambda + 1$ . Note that  $G$  is  $(n + \lambda + 2)$ -connected and  $G' = G - W$ . We see that  $G' - X$  is connected, and so  $i(G' - X) = 0$ . Combining this with (2.1) and  $\varepsilon_1(X) \leq |X|$ , we derive

$$0 = i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq |X| + 1 \geq 1,$$

which is a contradiction. This completes the proof of Claim 1.  $\square$

By virtue of (2.1),  $\varepsilon_1(X) \leq 2$  and Claim 1, we admit

$$i(G - W - X) = i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1 \geq 2(\lambda + 2) - 1 = 2\lambda + 3 > 2. \tag{2.2}$$

Using (2.2),  $\varepsilon_1(X) \leq 2$ , Claim 1 and the definition of  $I(G)$ , we obtain

$$\begin{aligned} I(G) &\leq \frac{|W \cup X|}{i(G - W - X)} \leq \frac{n + |X|}{2|X| - \varepsilon_1(X) + 1} \\ &\leq \frac{n + |X|}{2|X| - 1} = \frac{1}{2} + \frac{n + \frac{1}{2}}{2|X| - 1} \\ &\leq \frac{1}{2} + \frac{n + \frac{1}{2}}{2(\lambda + 2) - 1} = \frac{n + \lambda + 2}{2\lambda + 3}, \end{aligned}$$

which contradicts that  $I(G) > \frac{n + \lambda + 2}{2\lambda + 3}$ . Theorem 1.5 is verified.  $\square$

**Remark 2.1.** Next, we claim that the condition  $I(G) > \frac{n + \lambda + 2}{2\lambda + 3}$  in Theorem 1.5 is sharp. We construct a graph  $G = K_{n + \lambda + 2} \vee ((2\lambda + 3)K_1)$ , where  $n$  and  $\lambda$  are two nonnegative integers,  $\vee$  means “join”. Obviously,  $G$  is  $(n + \lambda + 2)$ -connected and  $I(G) = \frac{n + \lambda + 2}{2\lambda + 3}$ . For  $W \subseteq V(K_{n + \lambda + 2})$  with  $|W| = n$ , let  $G' = G - W = K_{\lambda + 2} \vee ((2\lambda + 3)K_1)$ . Select  $X = V(K_{\lambda + 2})$  in  $G'$ . Then  $\varepsilon_1(X) = 2$ , and we obtain

$$i(G' - X) = 2\lambda + 3 > 2\lambda + 2 = 2(\lambda + 2) - 2 = 2|X| - \varepsilon_1(X).$$

In terms of Theorem 1.3,  $G'$  is not a  $P_{\geq 2}$ -factor covered graph, and so  $G$  is not a  $(p_{\geq 2}, n)$ -factor-critical covered graph.

**Remark 2.2.** In what follows, we claim that the condition  $(n + \lambda + 2)$ -connectivity in Theorem 1.5 is best possible. We construct a graph  $G = K_{n + \lambda + 1} \vee ((2\lambda + 1)K_1)$ , where  $n \geq 0$  and  $\lambda \geq 1$  are two integers,  $\vee$  means “join”. Clearly,  $I(G) = \frac{n + \lambda + 1}{2\lambda + 1} > \frac{n + \lambda + 2}{2\lambda + 3}$  and  $G$  is  $(n + \lambda + 1)$ -connected. Let  $G' = G - W = K_{\lambda + 1} \vee ((2\lambda + 1)K_1)$ , where  $W \subseteq V(K_{n + \lambda + 1})$  with  $|W| = n$ . Choose  $X = V(K_{\lambda + 1})$  in  $G'$ . Then  $\varepsilon_1(X) = 2$ , and we admit

$$i(G' - X) = 2\lambda + 1 > 2\lambda = 2(\lambda + 1) - 2 = 2|X| - \varepsilon_1(X).$$

By means of Theorem 1.3,  $G'$  is not a  $P_{\geq 2}$ -factor covered graph, and so  $G$  is not a  $(p_{\geq 2}, n)$ -factor-critical covered graph.

### 3. THE PROOF OF THEOREM 1.6

*Proof of Theorem 1.6.* If  $G$  is a complete graph, then it is clear that  $G$  is a  $(P_{\geq 3}, n)$ -factor-critical covered graph by  $G$  being  $(n + \lambda + 2)$ -connected. In what follows, we consider that  $G$  is not a complete graph.

Let  $G' = G - W$  for any  $W \subseteq V(G)$  with  $|W| = n$ . To verify Theorem 1.6, it suffices to justify that  $G'$  is a  $P_{\geq 3}$ -factor covered graph. By means of contrary, we assume that  $G'$  is not a  $P_{\geq 3}$ -factor covered graph. Then by Theorem 1.4, we have

$$\text{sun}(G' - X) \geq 2|X| - \varepsilon_2(X) + 1 \tag{3.1}$$

for some vertex subset  $X$  of  $G'$ .

Next, we shall discuss two cases by the value of  $|X|$ , and derive a contradiction in each case.

**Case 1.**  $0 \leq |X| \leq \lambda + 1$ .

Since  $G' = G - W$  and  $G$  is  $(n + \lambda + 2)$ -connected,  $G'$  is  $(\lambda + 2)$ -connected. Combining this with the definition of sun component, we know that  $\text{sun}(G') = 0$ . If  $|X| = 0$ , then by (3.1) and  $\varepsilon_2(X) \leq |X|$ , we derive

$$0 = \text{sun}(G') = \text{sun}(G' - X) \geq 2|X| - \varepsilon_2(X) + 1 \geq |X| + 1 = 1,$$

which is a contradiction.

If  $1 \leq |X| \leq \lambda + 1$ , then we have

$$\omega(G' - X) = 1 \quad (3.2)$$

by  $G'$  being  $(\lambda + 2)$ -connected. By virtue of (3.1), (3.2) and  $\varepsilon_2(X) \leq |X|$ , we get

$$1 = \omega(G' - X) \geq \text{sun}(G' - X) \geq 2|X| - \varepsilon_2(X) + 1 \geq |X| + 1 \geq 2,$$

which is a contradiction.

**Case 2.**  $|X| \geq \lambda + 2$ .

In this case, we assume that there exist  $a$  isolated vertices,  $b$   $K_2$ 's and  $c$  big sun components  $Q_1, Q_2, \dots, Q_c$ , where  $|V(Q_i)| \geq 6$  for  $1 \leq i \leq c$ , in  $G' - X$ , and so

$$\text{sun}(G' - X) = a + b + c. \quad (3.3)$$

We choose one vertex from every  $K_2$  component of  $G' - X$ , and use  $Y$  to denote the set of such vertices. Let  $R_i$  be the factor-critical subgraph of  $Q_i$  for  $1 \leq i \leq c$ . Obviously, we have  $|Y| = b$  and  $i(Q_i - V(R_i)) = |V(R_i)|$ . Thus, we derive

$$\begin{aligned} i(G - (W \cup X \cup Y \cup V(R_1) \cup \dots \cup V(R_c))) &= i(G' - (X \cup Y \cup V(R_1) \cup \dots \cup V(R_c))) \\ &= a + b + i(Q_1 - V(R_1)) + \dots + i(Q_c - V(R_c)) \\ &= a + b + |V(R_1)| + \dots + |V(R_c)| \\ &= a + b + \sum_{i=1}^c |V(R_i)|. \end{aligned} \quad (3.4)$$

It follows from (3.1), (3.3), (3.4),  $\varepsilon_2(X) \leq 2$  and  $|V(R_i)| \geq 3$  that

$$\begin{aligned} i(G - (W \cup X \cup Y \cup V(R_1) \cup \dots \cup V(R_c))) &= a + b + \sum_{i=1}^c |V(R_i)| \\ &\geq a + b + 3c \geq a + b + c = \text{sun}(G' - X) \\ &\geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 \\ &\geq 2(\lambda + 2) - 1 = 2\lambda + 3 \geq 3. \end{aligned} \quad (3.5)$$

In terms of (3.5),  $I(G) > \frac{n+3\lambda+5}{2\lambda+3}$  and the definition of  $I(G)$ , we deduce

$$\begin{aligned} \frac{n+3\lambda+5}{2\lambda+3} < I(G) &\leq \frac{|W \cup X \cup Y \cup V(R_1) \cup \dots \cup V(R_c)|}{i(G - (W \cup X \cup Y \cup V(R_1) \cup \dots \cup V(R_c)))} \\ &= \frac{n + |X| + |Y| + \sum_{i=1}^c |V(R_i)|}{a + b + \sum_{i=1}^c |V(R_i)|} \\ &= \frac{n + |X| + b + \sum_{i=1}^c |V(R_i)|}{a + b + \sum_{i=1}^c |V(R_i)|}, \end{aligned}$$

which implies

$$(n + 3\lambda + 5)a + (n + \lambda + 2)b + (n + \lambda + 2) \sum_{i=1}^c |V(R_i)| - (2\lambda + 3)|X| - (2\lambda + 3)n < 0. \quad (3.6)$$

By means of (3.1), (3.3), (3.6),  $\varepsilon_2(X) \leq 2$  and  $|V(R_i)| \geq 3$ , we derive

$$\begin{aligned} 0 &> (n + 3\lambda + 5)a + (n + \lambda + 2)b + (n + \lambda + 2) \sum_{i=1}^c |V(R_i)| - (2\lambda + 3)|X| - (2\lambda + 3)n \\ &\geq (n + \lambda + 2)a + (n + \lambda + 2)b + (n + \lambda + 2)c - (2\lambda + 3)|X| - (2\lambda + 3)n \\ &= (n + \lambda + 2)(a + b + c) - (2\lambda + 3)|X| - (2\lambda + 3)n \\ &= (n + \lambda + 2)\text{sun}(G' - X) - (2\lambda + 3)|X| - (2\lambda + 3)n \\ &\geq (n + \lambda + 2)(2|X| - \varepsilon_2(X) + 1) - (2\lambda + 3)|X| - (2\lambda + 3)n \\ &\geq (n + \lambda + 2)(2|X| - 1) - (2\lambda + 3)|X| - (2\lambda + 3)n \\ &= (2n + 1)|X| - (\lambda + 2)(2n + 1), \end{aligned}$$

which implies that  $|X| < \lambda + 2$ , which contradicts that  $|X| \geq \lambda + 2$ . This completes the proof of Theorem 1.6.  $\square$

**Remark 3.1.** Next, we claim that the condition  $I(G) > \frac{n+3\lambda+5}{2\lambda+3}$  in Theorem 1.6 is sharp. We construct a graph  $G = K_{n+\lambda+2} \vee ((2\lambda + 3)K_2)$ , where  $n$  and  $\lambda$  are two nonnegative integers,  $\vee$  means “join”. Obviously,  $G$  is  $(n + \lambda + 2)$ -connected and  $I(G) = \frac{n+3\lambda+5}{2\lambda+3}$ . For  $W \subseteq V(K_{n+\lambda+2})$  with  $|W| = n$ , let  $G' = G - W = K_{\lambda+2} \vee ((2\lambda + 3)K_2)$ . Select  $X = V(K_{\lambda+2})$  in  $G'$ . Then  $\varepsilon_2(X) = 2$ , and we derive

$$\text{sun}(G' - X) = 2\lambda + 3 > 2\lambda + 2 = 2(\lambda + 2) - 2 = 2|X| - \varepsilon_2(X).$$

According to Theorem 1.4,  $G'$  is not a  $P_{\geq 3}$ -factor covered graph, and so  $G$  is not a  $(p_{\geq 3}, n)$ -factor-critical covered graph.

**Remark 3.2.** In what follows, we claim that the condition  $(n + \lambda + 2)$ -connectivity in Theorem 1.6 is best possible. We construct a graph  $G = K_{n+\lambda+1} \vee ((2\lambda + 1)K_2)$ , where  $n \geq 0$  and  $\lambda \geq 1$  are two integers,  $\vee$  means “join”. Clearly,  $I(G) = \frac{n+3\lambda+2}{2\lambda+1} > \frac{n+3\lambda+5}{2\lambda+3}$  and  $G$  is  $(n + \lambda + 1)$ -connected. Let  $G' = G - W = K_{\lambda+1} \vee ((2\lambda + 1)K_2)$ , where  $W \subseteq V(K_{n+\lambda+1})$  with  $|W| = n$ . Choose  $X = V(K_{\lambda+1})$  in  $G'$ . Then  $\varepsilon_2(X) = 2$ , and we have

$$\text{sun}(G' - X) = 2\lambda + 1 > 2\lambda = 2(\lambda + 1) - 2 = 2|X| - \varepsilon_2(X).$$

In light of Theorem 1.4,  $G'$  is not a  $P_{\geq 3}$ -factor covered graph, and so  $G$  is not a  $(p_{\geq 3}, n)$ -factor-critical covered graph.

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