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A TWO-STAGE STRUCTURE WITH UNDESIRABLE OUTPUTS: SLACKS-BASED AND ADDITIVE SLACKS-BASED MEASURES DEA MODELS

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Abstract. The slacks-based measure (SBM) and additive SBM (ASBM) models are two widely used DEA models acting based on inputs and outputs slacks and giving efficiency scores between zero and unity. In this paper, we use both models with the application of the weak disposability axiom for outputs to evaluate efficiency in a two-stage structure in the presence of undesirable outputs. In the external evaluation, the SBM model is reformulated as a linear program and the ASBM model is reformulated as a second-order cone program (SOCP) that is a convex programming problem. In the internal evaluation, the SBM model for a specific choice of weights is linearized while the ASBM model is presented as an SOCP for arbitrary choice of weights. Finally, the proposed models are applied on a real dataset for which efficiency comparison and Pearson correlation coefficients analysis show advantages of the ASBM model to the SBM model.

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1. Introduction

Data envelopment analysis (DEA) is a widely used tool for efficiency analysis, performance evaluation, and benchmarking. It identifies a set of best units from a given set of decision- making units (DMUs) with multiple inputs and outputs [6]. There are several DEA models measuring the efficiency of DMUs based on inputs and outputs slacks. Charnes *et al.* developed the first additive DEA model in which the objective function is defined as the summation of all inputs and outputs slacks [4]. While the additive model can identify efficient DMUs, it fails to produce a comparable DEA score or composite index. To overcome this drawback, Green *et al.* improved this model with an additive slacks-based measure (ASBM) that assigns an efficiency score between zero and unity to each DMU and unit invariance property is also assured [14]. Chen and Zhu have shown that ASBM is equivalent to Russell graph measure and thus reformulated it as a second order cone program (SOCP) [5]. Further, they have extended it for network DEA structures. Gerami *et al.* developed a novel geometric interpretation of value efficiency to ASBM modeling under the hypothesis of variable returns to scale [12]. The

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authors solved the ASBM model without changing its objective function and constraints. They used convenient transformations to linearize the ASBM model and then they extended the ASBM model to compute value efficiency.

In classical production theory, the aim is to minimize inputs and maximize produced outputs, also, in most DEA models, the resource assignment is in such a way that all inputs are applied entirely to produce good (desirable) outputs and bad (undesirable) outputs. In specific cases, when there are undesirable outputs in the production system, an increase in the production of this type of outputs is not desirable. In such situations, one of the best solutions is to apply weak disposability axiom for outputs in the productivity possibility set (PPS). Weak disposability means that in order to decrease the amount of undesirable outputs, the desirable outputs have to decrease commensurately, too. While Färe *et al.* utilized one abatement factor for all DMUs, Kuosmanen used a distinct abatement factor for each DMU [10,19]. Later, Kuosmanen *et al.* demonstrated that using a uniform abatement factor may result in non-convex PPS. They also asserted that their introduced PPS is the correct minimum extrapolation technology that satisfies the strong disposability of desirable outputs and inputs; weak disposability of all outputs and convexity axioms [20]. In recent decades, scientists have focused on the performance evaluation of the production units by taking into account internal structure and relations of sub-processes. In the sequel, we review some of the studies related to the network SBM models in DEA that takes into account undesirable outputs as well.

Fukuyama *et al.* proposed a slacks-based inefficiency measure for a two-stage system with bad outputs and applied it for the evaluation of banks [11]. Bian *et al.* presented the efficiency evaluation of Chinese regional industrial systems with undesirable factors using a two-stage SBM approach [2]. They provided analytical suggestions for improving the performance of a regional industrial system by identifying its inefficient internal stages; that helped to find out the main sources of the inefficiency arises from the internal stages within the system. This analysis cannot be done using conventional environmental DEA models. Song *et al.* carried out systematic research on efficiency evaluation using the SBM model considering undesirable outputs, and expanded it to analysis of network structures [27]. Their model also calculated desirable and undesirable outputs separately. Zhu *et al.* utilized an SBM approach on the network DEA model to evaluate the eco-efficiency of products [35]. Moreover, Lozano proposed an efficiency model based on the SBM model for a general network with undesirable outputs based on variable returns to scale (VRS), non-decreasing returns to scale (NDRS), and non-increasing returns to scale (NIRS) technologies [21]. Cui *et al.* designed a network epsilon-based model with managerial disposability to evaluate efficiency of the airlines under CNG2020 approach[7]. In another study, Cui *et al.* presented a network range adjusted model with weak-G disposability model to compute the environmental efficiency of 29 global airlines [8]. Zhou *et al.* extended a mixed network structure two-stage SBM model to measure the detailed efficiencies and find out the weak link of the system and applied it successfully to the provincial dataset of China during 2006–2015 [34]. Hu *et al.* proposed an extended two-stage SBM model with undesirable output and a feedback variable and then applied it to a real example from oil industry [16]. Cui *et al.* presented a novel model based on the modified SBM model in a network structure to evaluate the environmental efficiencies of airlines when the inputs or outputs of airlines are negative [9]. The authors dealt with undesirable outputs by turning them into negative data for efficiency scales. Shi *et al.* proposed a network SBM model with undesirable outputs to evaluate the performance of production processes having complex structures containing both series and parallel processes [26]. Yang *et al.* enlarged a dynamic network SBM model to recognize the proficiency of regional industrial water systems that wastewater is considered as an undesirable intermediate output and total volume of industrial wastewater discharged is supposed as an undesirable output, which prepared effective information for managers to handle the inefficiencies of the subsystems and periods [32].

In this paper, we study the performance evaluation of DMUs in a two-stage structure with undesirable outputs by the SBM model of Tone and the ASBM model of Green *et al.* from both external and internal perspectives [14, 29]. The SBM model is defined in a ratio form of aggregated output slacks to input slacks. To deal with undesirable outputs, we apply the weak disposability of Kuosmanen to reduce pollutant outputs that should be decreased [19]. For internal evaluation, we use the weighted average of stages' efficiencies for the overall efficiency of DMUs as in Kao [18]. The unknown weights are functions of slack variables and may vary from one

DMU to another. The value of the variables is determined after solving the proposed SBM model. On the other hand, the proposed ASBM models for both internal and external evaluations are presented as SOCP problems for arbitrary weights. Some advantages of the proposed SBM and ASBM models are as follows:

- Both models apply weak disposability of Kuosmanen [19], to reduce undesirable outputs.
- The SBM model in the external evaluation is linear while the ASBM model is an SOCP that is also a convex program.
- The SBM model in the internal evaluation is linear for a specific choice of weights while the ASBM model is again an SOCP.
- In the internal evaluation of the ASBM model, weights can be determined by the decision-maker preference.
- The proposed models are consistent, unit invariant, and translation invariant.
- Using different abatement factors in disposability set improves the discrimination power of models.
- Since there is no need to determine the weights for combining stage efficiencies, one can locate different stage efficiencies of Pareto optimal equivalence by varying combining weights from zero to unity.

The remainder of the paper is organized as follows. In Section 2, we define the weak disposability principle of outputs. In Section 3, the external and internal evaluations of DMUs *via* the SBM and the ASBM models in the presence of undesirable outputs are introduced. Section 4 compares the proposed models on a real numerical example and our conclusions follow in Section 5.

2. Two-stage structure in the presence of undesirable outputs

In this section, first we discuss the weak disposability of outputs due Kuosmanen [19], for the classical DEA models, then we present its extension for the two-stage structure in this paper.

2.1. Weak disposability of outputs

The classical DEA approaches rely on minimizing inputs and maximizing outputs to improve the efficiency score. However, in some applications, there are undesirable measures (for instance emission of harmful substances in air, energy wasted in power plant) that should be minimized. Hailu and Veeman in [15] have developed non-parametric productivity analysis approach to comprise undesirable outputs. They presented a monotonicity condition on their technology and asserted it is premier to "weak disposability" concept in DEA. Fare and Grosskopf showed that using monotonicity conditions in the steal of weak disposability is inconvenient with natural law [10]. Assume that there are n DMUs to be evaluated, indexed by $j = 1, \ldots, n$ and the vectors of inputs, desirable outputs and undesirable outputs are $x_{ij}\ \ (i = 1, \ldots, m)$, $y_{pj}\ \ (p = 1, \ldots, P)$ and $w_{bj}\ \ (b = 1, \ldots, B)$, respectively. The production technology can be illustrated by:

$$P(X) = \{(y, w) |\, X \text{ can produce} (y, w),\ X \in \mathbb{R}^m\}.$$

Definition 2.1. Desirable and undesirable outputs are weakly disposable if and only if $(y, w) \in P(X)$ and $0 \leq \theta \leq 1$ imply $(\theta y, \theta w) \in P(X), X \in \mathbb{R}^m$ (for more details see [25]).

Fare and Grosskopf [10] presented a technology under VRS satisfying weak disposability supposition that the contraction parameter θ in their approach corresponds to Definition 2.1. This parameter permits simultaneous contraction of good and bad outputs. Kuosmanen mentioned that Fare's approach applies a uniform abatement factor to all firms [19]. Applying different abatement factor for each DMU can have two main advantages. First, the corresponding models are linear. Second, the presented performance measurement approaches have more discrimination power. To allow non-uniform abatement factors of the individual firms, Kuosmanen suggested

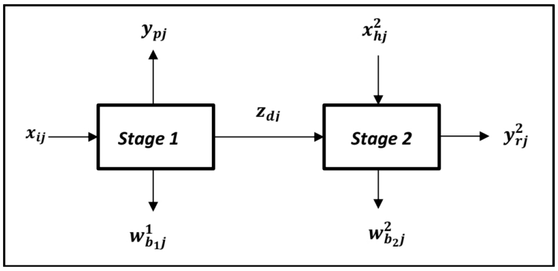

FIGURE 1. A two-stage structure in the presence of undesirable outputs.

the following production technology:

$$T_K = \Bigg\{ (y,w,x) \mid y_p \le \sum_{j=1}^{n} \theta_j y_{pj} \lambda_j,\ p = 1,\dots,P,$$
$$x_i \ge \sum_{j=1}^{n} x_{ij}\lambda_j,\ i = 1,\dots,m,\ w_b = \sum_{j=1}^{n} \theta_j w_{bj}\lambda_j,\ \ b = 1,\dots,B, \qquad (PPS_1)$$
$$\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \ge 0,\ \ 0 \le \theta_j \le 1,\ \ j = 1,\dots n \Bigg\}.$$

Free disposability of the inputs and good outputs is formulated through the use of inequality restrictions regarding y and x. The nonlinear technology T_K is further transformed into an equivalent linear form by Kuosmanen letting $\lambda_j = \mu_j + \nu_j$ and $\mu_j = \theta_j\lambda_j$ [19]. We use this technology to model undesirable measures in a two-stage production process in the next subsection.

2.2. Weak disposability in a two-stage structure

Consider a two-stage structure in which each DMU is composed of two sub-DMUs (stages) sequentially as follows:

Again assume that there are n DMUs to be evaluated, indexed by $j = 1,\dots,n$ and each DMU has a two-stage structure as given in Figure 1. In the first stage, by consuming m inputs x_{ij} $(i = 1,\dots,m)$, P desirable outputs y_{pj} $(p = 1,\dots,P)$ and B_1 undesirable outputs $w^1_{b_1j}$ $(b_1 = 1,\dots,w^1_{B_1})$ are produced. In addition to these, D desirable outputs $z_{dj}(d = 1,\dots,D)$ also are produced, called intermediate products that are inputs to the second stage. In stage two, in addition to the intermediate products, H other inputs x^2_{hj} $(h = 1,\dots,H)$ also are consumed to produce S desirable outputs y^2_{rj} $(r = 1,\dots,S)$ and B_2 undesirable outputs $w^2_{b_2j}$ $(b_2 = 1,\dots,B_2)$. Based on (PPS_1), the PPS for the two-stage structure given in Figure 1 for modeling undesirable outputs under VRS is as follows:

$$T_\nu = \Bigg\{ (X,Y,X^2,W^1,Z,Y^2,W^2) \mid X \ge \sum_{j=1}^{n} x_{ij}\lambda^1_j,\ X^2 \ge \sum_{j=1}^{n} x^2_{hj}\lambda^2_j,$$
$$Z \le \sum_{j=1}^{n} \theta^1_j z_{dj}\lambda^1_j,\ W^1 = \sum_{j=1}^{n} \theta^1_j w^1_{b_1j}\lambda^1_j,\ Z \ge \sum_{j=1}^{n} z_{dj}\lambda^2_j,$$
$$Y \le \sum_{j=1}^{n} \theta^1_j y_{pj}\lambda^1_j,\ Y^2 \le \sum_{j=1}^{n} \theta^2_j y^2_{rj}\lambda^2_j,\ W^2 = \sum_{j=1}^{n} \theta^2_j w^2_{b_2j}\lambda^2_j,$$

$$\left.\sum_{j=1}^{n}\lambda_j^1 = 1,\ \sum_{j=1}^{n}\lambda_j^2 = 1,\ \lambda_j^1, \lambda_j^2 \geq 0,\ 0 \leq \theta_j^1, \theta_j^2 \leq 1,\ \forall j\right\}, \tag{2.1}$$

where, non-uniform abatement factor θ_j^1, θ_j^2 allows simultaneous restriction of desirable and undesirable outputs. Desirable inputs and desirable outputs are strongly disposable, but outputs (bad and good) are weakly disposable. Now, based on the T_ν and by adding slacks, we have the following constraints:

$$x_{io} = \sum_{j=1}^{n} x_{ij}\lambda_j^1 + s_i^-, \qquad \forall i, \tag{2.2a}$$

$$y_{po} = \sum_{j=1}^{n} \theta_j^1 y_{pj}\lambda_j^1 - s_p^+, \qquad \forall p, \tag{2.2b}$$

$$w_{b_1 o}^1 = \sum_{j=1}^{n} \theta_j^1 w_{b_1 j}^1 \lambda_j^1, \qquad \forall b_1, \tag{2.2c}$$

$$x_{ho}^2 = \sum_{j=1}^{n} x_{hj}^2 \lambda_j^2 + s_h^-, \qquad \forall h, \tag{2.2d}$$

$$y_{ro}^2 = \sum_{j=1}^{n} \theta_j^2 y_{rj}^2 \lambda_j^2 - s_r^+, \qquad \forall r, \tag{2.2e}$$

$$w_{b_2 o}^2 = \sum_{j=1}^{n} \theta_j^2 w_{b_1 j}^2 \lambda_j^2, \qquad \forall b_2, \tag{2.2f}$$

$$\sum_{j=1}^{n} \lambda_j^1 = 1, \tag{2.2g}$$

$$\sum_{j=1}^{n} \lambda_j^2 = 1, \tag{2.2h}$$

$$0 \leq \theta_j^1, \theta_j^2 \leq 1, \qquad \forall j, \tag{2.2i}$$

$$s_i^-, s_r^+, s_h^-, s_p^+, \lambda_j^1, \lambda_j^2 \geq 0, \qquad \forall i, r, h, p, j. \tag{2.2j}$$

Note that (2.2) is the union of restrictions of slack variables for each stage without considering the connection between the stages. For this reason, (2.2) can be let as the restrictions for solving VRS-SBM and ASBM models for each stage. In addition, z_{dj}'s are intermediate products of the two-stage network and can be hidden from the external viewers. Now, considering the continuity of activities between the stages at the two-stage production system, without loss of generality, we suppose that the intermediate products (*i.e.*, z_{dj}) consumed by a DMU are not completely generated within the system. Also, as the first stage produces and second consumes, the amount of the first stage is either larger or equal to the second stage amount *i.e.*, the following constraint:

$$\sum_{j=1}^{n} \theta_j^1 z_{dj}\lambda_j^1 \geq \sum_{j=1}^{n} z_{dj}\lambda_j^2, \qquad \forall d. \tag{2.3}$$

3. Efficiency evaluation

In this section, we evaluate the efficiency of the two-stage structure of Figure 1 from both external and internal perspectives following Kao's framework [17, 18], for defining the overall efficiency of the structure.

3.1. External evaluation

In external evaluation approach, the overall efficiency is considered without slack variables of intermediates which are only visible by the insiders. In this case, first we calculate the overall efficiency and then the efficiencies of the first and second stages are obtained as in [18,31]. Thus the SBM model based on (2.2) is as follows:

$$\begin{aligned}
\min \quad & \varphi = \frac{1 - \frac{1}{m+H}\left(\sum_{i=1}^{m} \frac{s_i^-}{x_{io}} + \sum_{h=1}^{H} \frac{s_h^-}{x_{ho}^2}\right)}{1 + \frac{1}{P+s}\left(\sum_{p=1}^{P} \frac{s_p^+}{y_{po}} + \sum_{r=1}^{s} \frac{s_r^+}{y_{ro}^2}\right)} \\
s.t. \quad & \sum_{j=1}^{n} \theta_j^1 z_{dj} \lambda_j^1 \geq \sum_{j=1}^{n} z_{dj} \lambda_j^2, \qquad \forall d, \\
& \text{Constraint sets (2.2a) to (2.2j)}
\end{aligned} \tag{3.1}$$

Model (3.1) has nonlinear constraints and nonlinear objective function. One of the advantages of using distinct abatement factors is that model (3.1) can be linearized by the following change of variables due to [19]:

$$\begin{aligned}
\mu_j^1 = \theta_j^1 \lambda_j^1, \quad \lambda_j^1 = \mu_j^1 + \nu_j^1, \quad \forall j, \\
\mu_j^2 = \theta_j^2 \lambda_j^2, \quad \lambda_j^2 = \mu_j^2 + \nu_j^2, \quad \forall j,
\end{aligned} \tag{3.2}$$

Further let

$$1 + \frac{1}{s+P}\left(\sum_{p=1}^{P} \frac{s_p^+}{y_{po}} + \sum_{r=1}^{s} \frac{s_r^+}{y_{ro}^2}\right) = \frac{1}{t},$$

and

$$\begin{aligned}
&s_i'^- = ts_i^-, && s_p'^+ = ts_p^+, && s_h'^- = ts_h^-, && s_r'^+ = ts_r^+, \\
&\mu_j'^1 = t\mu_j^1, && \mu_j'^2 = t\mu_j^2, && \nu_j'^1 = t\nu_j^1, && \nu_j'^2 = t\nu_j^2,
\end{aligned}$$

then, model (3.1) becomes

$$\begin{aligned}
\min \quad & \varphi = t - \frac{1}{m+H}\left(\sum_{i=1}^{m} \frac{s_i'^-}{x_{io}} + \sum_{h=1}^{H} \frac{s_h'^-}{x_{ho}^2}\right) \\
s.t. \quad & tx_{io} = \sum_{j=1}^{n} x_{ij}\left(\mu_j'^1 + \nu_j'^1\right) + s_i'^-, && \forall i, \\
& ty_{po} = \sum_{j=1}^{n} y_{pj} \mu_j'^1 - s_p'^+, && \forall p, \\
& tw_{b_1 o}^1 = \sum_{j=1}^{n} w_{b_1 j}^1 \mu_j'^1, && \forall b_1, \\
& tx_{ho}^2 = \sum_{j=1}^{n} x_{hj}^2 \left(\mu_j'^2 + \nu_j'^2\right) + s_h'^-, && \forall h, \\
& ty_{ro}^2 = \sum_{j=1}^{n} y_{rj}^2 \mu_j'^2 - s_r'^+, && \forall r, \\
& tw_{b_2 o}^2 = \sum_{j=1}^{n} w_{b_2 j}^2 \mu_j'^2, && \forall b_2,
\end{aligned} \tag{3.3}$$

$$\sum_{j=1}^{n} z_{dj}\mu_j'^1 \geq \sum_{j=1}^{n} z_{dj}\left(\mu_j'^2+\nu_j'^2\right), \qquad \forall d,$$
$$t+\frac{1}{p+s}\left(\sum_{p=1}^{P}\frac{s_p'^+}{y_{po}}+\sum_{r=1}^{s}\frac{s_r'^+}{y_{ro}^2}\right)=1,$$
$$\sum_{j=1}^{n}\left(\mu_j'^1+\nu_j'^1\right)=t,$$
$$\sum_{j=1}^{n}\left(\mu_j'^2+\nu_j'^2\right)=t,$$
$$t\geq\varepsilon,$$
$$s_i'^-, s_r'^+, s_h'^-, s_p'^+, \mu_j'^1, \nu_j'^1, \mu_j'^2, \nu_j'^2 \geq 0, \qquad \forall i,r,h,p,j,$$

where ϵ is an infinitesimal positive real number. Now, let $\left(s_i^{-*}, s_r^{+*}, s_p^{+*}, s_h^{-*}, \lambda_j^{1*}, \lambda_j^{2*}, \theta_j^{1*}, \theta_j^{2*}\right)$ and $(t^*, s_i'^{-*}$, $s_r'^{+*}$, $s_p'^{+*}$, $s_h'^{-*}$, $\mu_j'^{1*}$, $\nu_j'^{1*}$, $\mu_j'^{2*}$, $\nu_j'^{2*})$ be the optimal solutions of models (3.1) and (3.3), respectively. Then we have

$$s_i^{-*}=\frac{s_i'^{-*}}{t^*}, \qquad s_p^{+*}=\frac{s_p'^{+*}}{t^*}, \qquad s_h^{-*}=\frac{s_h'^{-*}}{t^*}, \qquad s_r^{+*}=\frac{s_r'^{+*}}{t^*},$$
$$\mu_j^{1*}=\frac{\mu_j'^{1*}}{t^*}, \qquad \mu_j^{2*}=\frac{\mu_j'^{2*}}{t^*}, \qquad \nu_j^{1*}=\frac{\nu_j'^{1*}}{t^*}, \qquad \nu_j^{2*}=\frac{\nu_j'^{2*}}{t^*}.$$

After solving model (3.3), to obtain the efficiencies of the first and second stages, the slack variables of the intermediate products are calculated as follows [5]:

$$t_d'^{+*}=\sum_{j=1}^{n}\mu_j'^{1*}z_{dj}-tz_{do}, \qquad \forall d,$$
$$t_d'^{-*}=tz_{do}-\sum_{j=1}^{n}\left(\mu_j'^{2*}+\nu_j'^{2*}\right)z_{dj}, \quad \forall d,$$

where, $t_d^{-*}=\frac{t_d'^{-*}}{t^*}$, $t_d^{+*}=\frac{t_d'^{+*}}{t^*}$. Then, the efficiencies of the first and second stages are obtained as follows:

$$\rho^1=\frac{1-\frac{1}{m}\left(\sum_{i=1}^{m}\frac{s_i^{-*}}{x_{io}}\right)}{1+\frac{1}{P+D}\left(\sum_{p=1}^{P}\frac{s_p'^{+*}}{y_{po}}+\sum_{d=1}^{D}\frac{t_d'^{+*}}{z_{do}}\right)}, \tag{3.4}$$
$$\rho^2=\frac{1-\frac{1}{D+H}\left(\sum_{h=1}^{H}\frac{s_h^{-*}}{x_{ho}^2}+\sum_{d=1}^{D}\frac{t_d^-}{z_{do}}\right)}{1+\frac{1}{s}\left(\sum_{r=1}^{s}\frac{s_r^{+*}}{y_{ro}^2}\right)}. \tag{3.5}$$

The overall efficiency of the ASBM model in external evaluation is as follows:

$$\min\ \frac{1}{(m+s+H+P)}\left(\sum_{i=1}^{m}\frac{x_{io}-s_i^-}{x_{io}}+\sum_{p=1}^{P}\frac{y_{po}}{y_{po}+s_p^+}+\sum_{h=1}^{H}\frac{x_{ho}^2-s_h^-}{x_{ho}^2}+\sum_{r=1}^{s}\frac{y_{ro}^2}{y_{ro}^2+s_r^+}\right)$$

$$s.t. \sum_{j=1}^{n} \theta_j^1 z_{dj} \lambda_j^1 \geq \sum_{j=1}^{n} z_{dj} \lambda_j^2, \qquad \forall d, \tag{3.6}$$

Constraint sets (2.2a) to (2.2j)

In the next theorem, we reformulate model (3.6) as an SOCP that is a convex program.

Theorem 3.1. *Problem (3.6) is equivalent to the following SOCP:*

$$\begin{aligned}
\min \quad & E = \mathcal{T}^1 + \sum_{p=1}^{P} \mathcal{T}_p^2 + \sum_{r=1}^{s} \mathcal{T}_r^3 \\
s.t. \quad & x_{io} = \sum_{j=1}^{n} x_{ij} \left(\mu_j^1 + \nu_j^1\right) + s_i^-, \quad \forall i, \\
& y_{po} = \sum_{j=1}^{n} y_{pj} \mu_j^1 - s_p^+, \quad \forall p, \\
& w_{b_1 o}^1 = \sum_{j=1}^{n} w_{b_1 j}^1 \mu_j^1, \quad \forall b_1, \\
& x_{ho}^2 = \sum_{j=1}^{n} x_{hj}^2 \left(\mu_j^2 + \nu_j^2\right) + s_h^-, \quad \forall h, \\
& y_{ro}^2 = \sum_{j=1}^{n} y_{rj}^2 \mu_j^2 - s_r^+, \quad \forall r, \\
& w_{b_2 o}^2 = \sum_{j=1}^{n} w_{b_2 j}^2 \mu_j^2, \quad \forall b_2, \\
& \sum_{j=1}^{n} z_{dj} \mu_j^1 \geq \sum_{j=1}^{n} z_{dj} \left(\mu_j^2 + \nu_j^2\right), \quad \forall d, \\
& \sum_{j=1}^{n} \left(\mu_j^1 + \nu_j^1\right) = 1, \\
& \sum_{j=1}^{n} \left(\mu_j^2 + \nu_j^2\right) = 1, \\
& \frac{1}{(m+s+H+P)} \left(\sum_{i=1}^{m} \frac{x_{io} - s_i^-}{x_{io}} + \sum_{h=1}^{H} \frac{x_{ho}^2 - s_h^-}{x_{ho}^2} \right) \leq \mathcal{T}^1, \\
& \left\| \begin{matrix} \sqrt{y_{po}} \\ \frac{1}{2}\left((m+s+H+P)\left(y_{po} + s_p^+\right) - \mathcal{T}_p^2\right) \end{matrix} \right\|_2 \leq \frac{1}{2}\left((m+s+H+P)\left(y_{po} + s_p^+\right) + \mathcal{T}_p^2\right), \quad \forall p \\
& \left\| \begin{matrix} \sqrt{y_{ro}^2} \\ \frac{1}{2}\left((m+s+H+P)\left(y_{ro}^2 + s_r^+\right) - \mathcal{T}_r^3\right) \end{matrix} \right\|_2 \leq \frac{1}{2}\left((m+s+H+P)\left(y_{ro}^2 + s_r^+\right) + \mathcal{T}_r^3\right), \quad \forall r \\
& s_i^-, s_r^+, s_h^-, s_p^+, \mu_j^1, \nu_j^1, \mu_j^2, \nu_j^2 \geq 0, \quad \forall i, r, h, p, j.
\end{aligned} \tag{3.7}$$

Proof. Let $\mathcal{T}_p^2$, $\mathcal{T}_r^3$, and $\mathcal{T}^1$ such that

$$\frac{1}{(m+s+H+P)} \frac{y_{po}}{y_{po} + s_p^+} \leq \mathcal{T}_p^2, \qquad \forall p, \tag{3.8}$$

$$\frac{1}{(m+s+H+P)} \frac{y_{ro}^2}{y_{ro}^2+s_r^+} \leq \mathcal{T}_r^3, \qquad \forall r, \tag{3.9}$$

$$\frac{1}{(m+s+H+P)} \left(\sum_{i=1}^{m} \frac{x_{io}-s_i^-}{x_{io}} + \sum_{h=1}^{H} \frac{x_{ho}^2-s_h^-}{x_{ho}^2} \right) \leq \mathcal{T}^1. \tag{3.10}$$

Inequality (3.8) is equivalent to

$$\begin{aligned}
&(m+s+H+P)\left(y_{po}+s_p^+\right)\mathcal{T}_p^2 \geq \left(\sqrt{y_{po}}\right)^2 && \forall p\\
\Leftrightarrow &\left(\left(\frac{(m+s+H+P)\left(y_{po}+s_p^+\right)+\mathcal{T}_p^2}{2} \right)^2 - \left(\frac{(m+s+H+P)\left(y_{po}+s_p^+\right)-\mathcal{T}_p^2}{2} \right)^2 \right)\\
&\qquad \geq \left(\sqrt{y_{po}}\right)^2 && \forall p\\
\Leftrightarrow &\sqrt{\left(\sqrt{y_{po}}\right)^2 + \left(\frac{1}{2}\left((m+s+H+P)(y_{po}+s_p^+)-\mathcal{T}_p^2\right) \right)^2}\\
&\qquad \leq \frac{1}{2}\left((m+s+H+P)(y_{po}+s_p^+)+\mathcal{T}_p^2\right), && \forall p
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow &\left\| \begin{matrix} \sqrt{y_{po}} \\ \left(\frac{1}{2}\left((m+s+H+P)\left(y_{po}+s_p^+\right)-\mathcal{T}_p^2\right)\right) \end{matrix} \right\|_2\\
&\qquad \leq \frac{1}{2}\left((m+s+H+P)\left(y_{po}+s_p^+\right)+\mathcal{T}_p^2\right), \quad \forall p
\end{aligned} \tag{3.11}$$

Similarly, inequality (3.9) can be written as follows:

$$\begin{aligned}
&\left\| \begin{matrix} \sqrt{y_{ro}^2} \\ \left(\frac{1}{2}\left((m+s+H+P)\left(y_{ro}^2+s_r^+\right)-\mathcal{T}_r^3\right)\right) \end{matrix} \right\|_2\\
&\qquad \leq \frac{1}{2}\left((m+s+H+P)\left(y_{ro}^2+s_r^+\right)+\mathcal{T}_r^3\right), \quad \forall r
\end{aligned} \tag{3.12}$$

where (3.11) and (3.12) are second-order cone inequalities and (3.10) is a linear inequality. For the other constraints we apply (3.2), thus (3.7) is obtained. □

Now suppose that $\left(\mathcal{T}^{1^*}, \mathcal{T}_p^{2^*}, \mathcal{T}_r^{3^*}, s_i^{-^*}, s_r^{+^*}, s_p^{+^*}, s_h^{-^*}, \mu_j^{1^*}, \nu_j^{1^*}, \mu_j^{2^*}, \nu_j^{2^*}\right)$ is an optimal solution of model (3.7). To obtain the efficiencies of the first and second stages, first we calculate the slack variables of the intermediate products as follows:

$$t_d^{+^*} = \sum_{j=1}^{n} \mu_j^{1^*} z_{dj} - z_{do}, \qquad \forall d, \qquad t_d^{-^*} = z_{do} - \sum_{j=1}^{n} \left(\mu_j^{2^*} + \nu_j^{2^*}\right) z_{dj}, \qquad \forall d.$$

Then, the efficiencies of the first and second stages are obtained as follows:

$$E^1 = \frac{1}{(D+P+m)} \left(\sum_{i=1}^{m} \frac{x_{io}-s_i^{-^*}}{x_{io}} + \sum_{p=1}^{P} \frac{y_{po}}{y_{po}+s_p^{+^*}} + \sum_{d=1}^{D} \frac{z_{do}}{z_{do}+t_d^{+^*}} \right), \tag{3.13}$$

$$E^2 = \frac{1}{(D+H+s)} \left(\sum_{h=1}^{H} \frac{x_{ho}^2-s_h^{-^*}}{x_{ho}^2} + \sum_{r=1}^{s} \frac{y_{ro}^2}{y_{ro}^2+s_r^{+^*}} + \sum_{d=1}^{D} \frac{z_{do}-t_d^{-^*}}{z_{do}} \right). \tag{3.14}$$

In general, external evaluation can be seen as a non-cooperative process, where overall efficiency and stages efficiencies are computed in a serial way. In the sequel, we discuss internal evaluation that is able to gain overall efficiency and stages efficiencies together.

3.2. Internal evaluation

Internal evaluation is a process of efficiency aggregation that can be considered as a contrary method of external evaluation, where the overall efficiency and stage efficiencies are calculated in a sequential procedure [18]. However, in internal evaluation, overall efficiency, and stages' efficiencies are obtained simultaneously. In the internal evaluation, the overall efficiency also includes the intermediate product, thus we have the following two constraints:

$$\begin{aligned} z_{do} &= \sum_{j=1}^{n} \theta_j^1 z_{dj} \lambda_j^1 - t_d^+, \quad \forall d,\\ z_{do} &= \sum_{j=1}^{n} z_{dj} \lambda_j^2 + t_d^-, \quad \forall d,\\ t_d^-, t_d^+ &\geq 0 \quad \forall d. \end{aligned} \tag{3.15}$$

Based on (2.2), (2.3) and (3.15), the overall efficiency of the SBM model in internal evaluation is as follows [18,30]:

$$\begin{aligned} \min\ \rho = &\left(w_1 \frac{1 - \frac{1}{m}\left(\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}\right)}{1 + \frac{1}{P+D}\left(\sum_{p=1}^{P} \frac{s_p^+}{y_{po}} + \sum_{d=1}^{D} \frac{t_d^+}{z_{do}}\right)} + w_2 \frac{1 - \frac{1}{D+H}\left(\sum_{h=1}^{H} \frac{s_h^-}{x_{ho}^2} + \sum_{d=1}^{D} \frac{t_d^-}{z_{do}}\right)}{1 + \frac{1}{s}\left(\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}^2}\right)} \right)\\ s.t.\ &\text{Constraint sets (2.2a) to (2.2j)}\\ &z_{do} = \sum_{j=1}^{n} \theta_j^1 z_{dj} \lambda_j^1 - t_d^+, \quad \forall d,\\ &z_{do} = \sum_{j=1}^{n} z_{dj} \lambda_j^2 + t_d^-, \quad \forall d,\\ &\sum_{j=1}^{n} \theta_j^1 z_{dj} \lambda_j^1 \geq \sum_{j=1}^{n} z_{dj} \lambda_j^2, \quad \forall d,\\ &t_d^-, t_d^+ \geq 0, \quad \forall d, \end{aligned} \tag{3.16}$$

where w_1 and w_2 are weights of the first and second stages, respectively with $w_1 + w_2 = 1$. If we let

$$w_1 = \frac{1 + \frac{1}{P+D}\left(\sum_{p=1}^{P} \frac{s_p^+}{y_{po}} + \sum_{d=1}^{D} \frac{t_d^+}{z_{do}}\right)}{2 + \frac{1}{P+D}\left(\sum_{p=1}^{P} \frac{s_p^+}{y_{po}} + \sum_{d=1}^{D} \frac{t_d^+}{z_{do}}\right) + \frac{1}{s}\left(\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}^2}\right)}, \tag{3.17}$$

$$w_2 = \frac{1 + \frac{1}{s}\left(\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}^1}\right)}{2 + \frac{1}{P+D}\left(\sum_{p=1}^{P} \frac{s_p^+}{y_{po}} + \sum_{d=1}^{D} \frac{t_d^+}{z_{do}}\right) + \frac{1}{s}\left(\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}^2}\right)}, \tag{3.18}$$

and use Charnes-Cooper transformation [3], we get the following equivalent LP as in the external case:

$$\begin{aligned} \min\quad & \rho = 1 - \frac{1}{2}\left(\frac{1}{m}\left(\sum_{i=1}^{m} \frac{s_i'^-}{x_{io}}\right) + \frac{1}{D+H}\left(\sum_{h=1}^{H} \frac{s_h'^-}{x_{ho}^2} + \sum_{d=1}^{D} \frac{t_d'^-}{z_{do}}\right)\right)\\ s.t.\quad & t x_{io} = \sum_{j=1}^{n} x_{ij}\left(\mu_j'^1 + \nu_j'^1\right) + s_i'^-, \quad \forall i,\\ & t y_{po} = \sum_{j=1}^{n} y_{pj} \mu_j'^1 - s_p'^+, \quad \forall p, \end{aligned}$$

$$
\begin{aligned}
& tw^1_{b_1 o} = \sum_{j=1}^{n} w^1_{b_1 j} \mu'^1_j, && \forall b_1,\\
& tz_{do} = \sum_{j=1}^{n} z_{dj} \mu'^1_j - t'^+_d, && \forall d,\\
& tz_{do} = \sum_{j=1}^{n} z_{dj} \left(\mu'^2_j + \nu'^2_j\right) + t'^-_d, && \forall d,\\
& tx^2_{ho} = \sum_{j=1}^{n} x^2_{hj} \left(\mu'^2_j + \nu'^2_j\right) + s'^-_h, && \forall h,\\
& ty^2_{ro} = \sum_{j=1}^{n} y^2_{rj} \mu'^2_j - s'^+_r, && \forall r, \qquad (3.19)\\
& tw^2_{b_2 o} = \sum_{j=1}^{n} w^2_{b_2 j} \mu'^2_j, && \forall b_2,\\
& \sum_{j=1}^{n} z_{dj} \mu'^1_j \geq \sum_{j=1}^{n} z_{dj} \left(\mu'^2_j + \nu'^2_j\right), && \forall d,\\
& t + \frac{1}{2}\left(\frac{1}{P+D}\left(\sum_{p=1}^{P} \frac{s'^+_p}{y_{po}} + \sum_{d=1}^{D} \frac{t^+_t}{z_{do}}\right) + \frac{1}{s}\left(\sum_{r=1}^{s} \frac{s'^+_r}{y^2_{ro}}\right)\right) = 1,\\
& \sum_{j=1}^{n} \left(\mu'^1_j + \nu'^1_j\right) = t,\\
& \sum_{j=1}^{n} \left(\mu'^2_j + \nu'^2_j\right) = t,\\
& t \geq \varepsilon,\\
& s'^-_i, s'^+_r, s'^-_h, s'^+_p, t'^-_d, t'^+_d, \mu'^1_j, \nu'^1_j, \mu'^2_j, \nu'^2_j \geq 0, && \forall i, r, h, p, j, d,
\end{aligned}
$$

where ε is an infinitesimal positive real number. Let $\left(s^{-*}_i, s^{+*}_r, s^{+*}_p, s^{-*}_h, t^{+*}_d, t^{-*}_d, \lambda^{1*}_j, \lambda^{2*}_j\right)$ and $\left(t^*, s'^{-*}_i, s'^{+*}_r, s'^{+*}_p, s'^{-*}_h, t'^{+*}_d, t'^{-*}_d, \mu'^{1*}_j, \nu'^{1*}_j, \mu'^{2*}_j, \nu'^{2*}_j\right)$ be the optimal solutions of model (3.16) and (3.19), respectively. Then we have

$$
\begin{aligned}
& s^{-*}_i = \frac{s'^{-*}_i}{t^*}, \quad s^{+*}_r = \frac{s'^{+*}_r}{t^*}, \quad s^{+*}_p = \frac{s'^{+*}_p}{t^*}, \quad s^{-*}_h = \frac{s'^{-*}_h}{t^*}, \quad t^{-*}_d = \frac{t'^{-*}_d}{t^*},\\
& t^{+*}_d = \frac{t'^{+*}_d}{t^*}, \quad \mu^{1*}_j = \frac{\mu'^{1*}_j}{t^*}, \quad \nu^{1*}_j = \frac{\nu'^{1*}_j}{t^*}, \quad \mu^{2*}_j = \frac{\mu'^{2*}_j}{t^*}, \quad \nu^{2*}_j = \frac{\nu'^{2*}_j}{t^*}.
\end{aligned}
$$

After solving model (3.19), the efficiency of the subunits can be also observed.

Similarly, based on (2.2), (2.3) and (3.15), the ASBM model for the internal evaluation is as follows:

$$\begin{aligned}
\min \quad & E' = \frac{w_1}{(P+D+m)}\left(\sum_{i=1}^{m}\frac{x_{io}-s_i^-}{x_{io}} + \sum_{p=1}^{P}\frac{y_{po}}{y_{po}+s_p^+} + \sum_{d=1}^{D}\frac{z_{do}}{z_{do}+t_d^+}\right)\\
& + \frac{1-w_1}{(D+H+s)}\left(\sum_{h=1}^{H}\frac{x_{ho}^2-s_h^-}{x_{ho}^2} + \sum_{r=1}^{s}\frac{y_{ro}^2}{y_{ro}^2+s_r^+} + \sum_{d=1}^{D}\frac{z_{do}-t_d^-}{z_{do}}\right)\\
s.t. \quad & \text{Constraint sets (2.2a) to (2.2j)}\\
& z_{do} = \sum_{j=1}^{n}\theta_j^1 z_{dj}\lambda_j^1 - t_d^+, && \forall d, \qquad (3.20)\\
& z_{do} = \sum_{j=1}^{n} z_{dj}\lambda_j^2 + t_d^-, && \forall d,\\
& \sum_{j=1}^{n}\theta_j^1 z_{dj}\lambda_j^1 \geq \sum_{j=1}^{n} z_{dj}\lambda_j^2, && \forall d,\\
& t_d^-, t_d^+ \geq 0, && \forall d,
\end{aligned}$$

where weights are selected by the decision maker such that $0 < w_1 \leq 1$. In the next theorem, similar to the external evaluation, equivalent SOCP formulation of model (3.20) is presented.

Theorem 3.2. *Model (3.20) is equivalent to the following SOCP:*

$$\begin{aligned}
\min \quad & E' = \mathcal{T}^1 + \sum_{d=1}^{D}\mathcal{T}_d^2 + \sum_{p=1}^{P}\mathcal{T}_p^3 + \sum_{r=1}^{s}\mathcal{T}_r^4\\
s.t. \quad & x_{io} = \sum_{j=1}^{n} x_{ij}\left(\mu_j^1+\nu_j^1\right) + s_i^-, \quad \forall i,\\
& y_{po} = \sum_{j=1}^{n} y_{pj}\mu_j^1 - s_p^+, \quad \forall p,\\
& w_{b_1o}^1 = \sum_{j=1}^{n} w_{b_1j}^1\mu_j^1, \quad \forall b_1,\\
& z_{do} = \sum_{j=1}^{n} z_{dj}\mu_j^1 - t_d^+, \quad \forall d,\\
& z_{do} = \sum_{j=1}^{n} z_{dj}\left(\mu_j^2+\nu_j^2\right) + t_d^-, \quad \forall d,\\
& x_{ho}^2 = \sum_{j=1}^{n} x_{hj}^2\left(\mu_j^2+\nu_j^2\right) + s_h^-, \quad \forall h,\\
& y_{ro}^2 = \sum_{j=1}^{n} y_{rj}^2\mu_j^2 - s_r^+, \quad \forall r,\\
& w_{b_2o}^2 = \sum_{j=1}^{n} w_{b_2j}^2\mu_j^2, \quad \forall b_2,
\end{aligned}$$

$$
\begin{aligned}
&\sum_{j=1}^{n} z_{dj}\mu_j^1 \geq \sum_{j=1}^{n} z_{dj}\left(\mu_j^2+\nu_j^2\right), \quad \forall d, \\
&\sum_{j=1}^{n}\left(\mu_j^1+\nu_j^1\right)=1, \\
&\sum_{j=1}^{n}\left(\mu_j^2+\nu_j^2\right)=1, \\
&\frac{w_1}{m+D+P}\left(\sum_{i=1}^{m}\frac{x_{io}-s_i^-}{x_{io}}\right)+\frac{1}{D+H+s}\left(\sum_{h=1}^{H}\frac{x_{ho}^2-s_h^-}{x_{ho}^2}+\sum_{d=1}^{D}\frac{z_{zo}-t_d^-}{z_{zo}}\right)\leq \mathcal{T}^1, \\
&\left\|\begin{matrix}\sqrt{w_1 z_{do}}\\ \frac{1}{2}\left((m+D+P)\left(z_{do}+t_d^+\right)-\mathcal{T}_d^2\right)\end{matrix}\right\|_2 \leq \frac{1}{2}\left((m+D+P)\left(z_{do}+t_d^+\right)+\mathcal{T}_d^2\right), \quad \forall d \\
&\left\|\begin{matrix}\sqrt{w_1 y_{po}}\\ \frac{1}{2}\left((m+D+P)\left(y_{po}+s_p^+\right)-\mathcal{T}_p^3\right)\end{matrix}\right\|_2 \leq \frac{1}{2}\left((m+D+P)\left(y_{po}+s_p^+\right)+\mathcal{T}_p^3\right), \quad \forall p \\
&\left\|\begin{matrix}\sqrt{(1-w_1)y_{ro}^2}\\ \frac{1}{2}\left((H+D+s)\left(y_{ro}^2+s_r^+\right)-\mathcal{T}_r^4\right)\end{matrix}\right\|_2 \leq \frac{1}{2}\left((H+D+s)\left(y_{ro}^2+s_r^+\right)+\mathcal{T}_r^4\right), \quad \forall r \\
&s_i^-, s_r^+, s_h^-, s_p^+, t_d^-, t_d^+, \mu_j^1, \nu_j^1, \mu_j^2, \nu_j^2 \geq 0, \quad \forall i, r, h, p, j, d.
\end{aligned}
\tag{3.21}
$$

Proof. The proof is similar to the proof of Theorem 3.1. □

After solving model (3.21), the overall efficiency and efficiency of subunits can be viewed simultaneously. In this section, we discussed SBM and ASBM models from external and internal perspectives. The SBM models are in the form LP for both cases, while using specific weights for the internal evaluation. However, the nonlinear ASBM models are reformulated as SOCPs for both external and internal evaluations independent of weights. In the next section, we use a real dataset from the literature to evaluate both models.

4. Case study and discussion

In this section, the presented models are applied on a real dataset to show advantages of the ASBM model. All models are solved using CVX software package Grant *et al.* [13]. The data for this example, given in Table 1, is taken from Lozano *et al.* [22], to evaluate 39 Spanish airports in the presence of adverse outputs. Airport processes are divided into two stages: the aircraft movement process and the aircraft loading process. In the first stage, it has three inputs, Total runway area (x_1), Apron capacity (x_2), Number of boarding gates (x_3) and two undesirable outputs, Number of delayed flights (w_1^1), Accumulated flight delays (w_2^1), and an intermediate product, Aircraft traffic movements (z_1). In the second stage, in addition to the intermediate product, it also has two other inputs, Number of baggage belts (x_1^2), Number of check-in counters (x_2^2), and two final desirable outputs, Annual passenger movements (y_1^2) and Cargo handled (y_2^2), (Fig. 2).

4.1. External evaluation

The results of the external evaluation of the SBM and ASBM models are reported in Tables 2 and 3, respectively. The results show that efficient DMUs are the same in both models and overall efficiencies of the ASBM model are always greater than or equal of those of the SBM model. However, it can be seen that the obtained rankings from the SBM and the ASBM are different for some DMUs.

Moreover, from the results shown in Table 2 it can be seen that 51.2% and 48.7% of DMUs in overall system and stage 2, respectively, have efficiency equivalent to zero. Also, in stage 2, the efficiency scores for DMU_{16} and DMU_{27} are greater than one. This means that the efficiencies for the most DMUs in stage 2 and overall system are not reasonable. In Table 3, DMU_2, DMU_7, DMU_{12}, DMU_{20}, DMU_{21} and DMU_{25} are all efficient

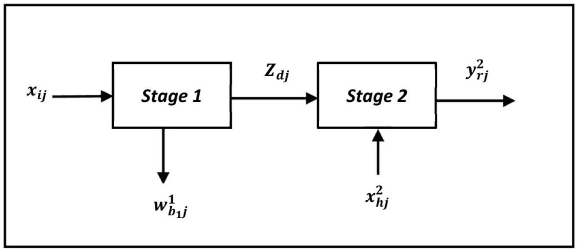

FIGURE 2. The structure of Spanish airports.

in the overall system and the efficiencies for the other DMUs in both stages are reasonable. Comparing the efficiency scores of Tables 2 and 3 illustrate that external ASBM model is more preferable than the external SBM model. Furthermore, Pearson correlation coefficients test results for both models at significant level 0.01 are given in Tables 4–5. As can be seen from Table 4, for the external SBM model the overall efficiency is significantly correlated with only stage 2, which also confirmed the overall system inefficiency sources. However, Table 5 shows that the overall efficiency of the ASBM model is correlated with both stages. In other words, comparing the efficiency scores of Tables 2 and 3 illustrate that ASBM model is a powerful alternative to SBM in the case of external estimation due to the consistency and similarities in ranking found as discussed.

In the sequel, we apply internal evaluations models to the dataset in order to detect the overall system's inefficiency sources. In summary, the decomposition of the overall system into two stages can help managers to recognize the real causes of inefficiencies in the network structures.

4.2. Internal evaluation

The results of the internal evaluation for both models are given in Tables 6 and 7, respectively. Note that in the ASBM model, the weight of each stage is considered 0.5. As can be seen, in internal evaluation, as in external evaluation, efficient units are the same in both models, and also the overall efficiency of the ASBM model is always greater than or equal to the SBM model. Moreover, the rankings of the two models differ for several DMUs. Here unlike the external evaluation, the efficiency is calculated in terms of the weighted average subunits. Thus, it is possible to get different rankings for the ASBM model for different weights, while the weights of the SBM model are fixed. This can be considered as an advantage of the ASBM model to the SBM model. As well, Table 6 displays that 38.4% and 46.1% of DMUs in the overall system and stage 2, respectively, have an efficiency equivalent to zero which is not the case in Table 7 for the ASBM model. Figure 3 shows that the overall efficiency of the ASBM model is between efficiencies of stage 1 and stage 2 which is not the case in the external evaluation (for example see DMUs 4 and 5 in Tab. 3). Although this observation holds for the SBM model as can be seen in Table 6 for fixed weights, but for the ASBM it holds for arbitrary choice of weights. Also, Figure 4 shows the radar chart of the ranking changes of the ASBM model for three different weights compared to the SBM ranking. It can be seen that internal efficiency scores of the most DMUs in ASBM model increases with the weight.

Furthermore, Pearson correlation coefficients between the two stages and the overall system in the internal evaluation are reported on Tables 8–9. For the SBM model, the overall efficiency is significantly correlated only with stage 2, while for ASBM it is correlated with both stages. This might be due to specific choice of weights for the SBM model, where the weights of the second stage in most cases are higher than those of the first stage. Indeed, in the SBM model these weights present virtual restrictions that changes the feasible region and thus the resulting SBM models stages efficiencies may not be one of the Pareto solutions. However, the ASBM model allows managers to specify constant weights, and can locate different stage efficiencies of Pareto optimal equivalence by varying combining weight from zero to unity. Therefore, ASBM is more than a reliable alternative to SBM in the case of internal evaluation.

TABLE 1. Inputs and outputs data of 39 Spanish airports.

DMU	x_1	x_2	x_3	z_1	w_1^1	w_2^1	x_1^2	x_2^2	y_1^2	y_2^2
1	87300	5	4	17.719	1218	23783.4	10	3	1174.97	283.571
2	162000	2	2	2.113	58	1376.5	4	1	19.254	8.924
3	135000	31	16	81.097	7642	142445.8	42	9	9678.304	5982.313
4	144000	15	5	18.28	1114	20149.1	17	4	1024.303	21.322
5	99000	7	9	18.371	1310	23893.5	11	3	1530.245	139.465
6	171000	1	2	4.033	137	2365.4	4	1	81.01	0.0001
7	475000	121	65	321.693	33036	645924.6	143	19	30272.08	103996.489
8	207000	21	12	61.682	4592	80848.2	36	7	4172.903	3178.758
9	62100	23	1	9.604	14	254.4	1	0.000001	22.23	0.000001
10	37500	3	2	4.775	27	641.6	5	1	195.425	171.717
11	153000	34	10	44.552	3920	72179.7	34	8	4492.003	2722.661
12	108000	17	7	49.927	4992	100305.6	18	3	5510.97	184.127
13	139500	55	38	116.252	7463	136380.7	86	19	10212.12	33695.248
14	134550	11	3	19.279	951	17868.8	12	3	1422.014	66.889
15	126000	25	12	57.233	6193	152840.1	48	8	4647.36	3928.387
16	103500	9	5	50.551	1174	19292.2	13	3	1303.817	90.428
17	45000	3	2	3.393	17	420.7	5	1	41.89	7.863
18	99000	5	5	20.109	423	8286	13	2	1151.357	1277.264
19	108000	24	16	53.375	5104	101685.6	49	8	5438.178	5429.589
20	94500	5	2	5.705	442	7197.5	3	1	123.183	15.579
21	927000	263	230	469.746	52526	908360	484	53	50846.49	329186.631
22	144000	43	30	119.821	15.548	277663.8	85	16	12813.47	4800.271
23	64260	5	2	10.959	218	2979.6	4	1	314.643	3863.34
24	138000	5	5	19.379	1344	24103.1	18	4	1876.255	2.73
25	295650	86	68	193.379	26038	501486	204	16	22832.86	21395.791
26	99135	7	2	12.971	666	11691.8	4	1	434.477	52.942
27	110475	5	5	26.676	943	18240.8	8	3	1278.074	119.848
28	150000	6	2	12.45	427	6626.1	4	2	60.103	0.000001
29	78930	6	3	12.282	713	11184	6	2	403.191	63.791
30	104400	8	5	19.198	1004	17842	8	2	856.606	37.482
31	14400	16	12	21.945	2007	34322.3	19	5	1917.466	2418.894
32	302310	12	3	14.584	1095	19547.6	6	2	594.952	2143.894
33	151200	23	10	65.067	2567	51084.9	42	6	4392.148	6102.264
34	153000	16	16	67.8	1783	32637	37	5	4236.615	2781.674
35	144000	44	22	60.779	5254	110818.9	87	14	8251.989	8567.093
36	144000	35	18	96.795	4998	102719.2	42	8	5779.343	13325.799
37	180000	7	5	13.002	843	14760.6	8	2	479.689	65.34
38	108000	8	6	17.934	1535	25593.6	12	3	1278.762	1481.393
39	157500	18	3	12.225	669	11585.8	7	2	67.818	34989.272

5. CONCLUSIONS

In this paper, we studied the SBM and ASBM models to evaluate the efficiency of decision-making units in a two-stage structure in the presence of undesirable outputs under the weak disposability of Kuosmanen [19]. The weak disposability axiom for all outputs is beneficial for the performance measurement of the production units. In order to perform a comprehensive analysis, we evaluated the efficiency from two managerial perspectives, namely the external and internal ones. We proposed two equivalent LPs for the SBM in both evaluations, where in the internal case this is achieved for specific weights. However, the ASBM models, in both evaluations, are reformulated as SOCPs that are convex programs independent of weights choice, which is an important

Table 2. Results of external evaluation in the SBM model.

DMU	Overall efficiency	Rank	Stage 1 efficiency	Stage 2 efficiency
1	0.0242	25	1	0.0236
2	1	1	1	1
3	0.6978	9	0.9182	0.7196
4	0.0007	39	0.4581	0.0010
5	0.0075	28	0.5505	0.0098
6	0.0010	36	1	0.0010
7	1	1	1	1
8	0.0898	21	0.8514	0.0996
9	0.0011	35	1	0.0008
10	0.2190	13	1	0.2145
11	0.0744	24	0.5250	0.0993
12	1	1	1	1
13	0.4918	10	0.9567	0.4833
14	0.0036	32	0.7177	0.0041
15	0.1179	20	1	0.1123
16	0.0043	31	1.8221	0.0037
17	0.0071	29	1	0.0069
18	0.0895	22	0.9840	0.0868
19	0.1480	18	0.7465	0.1661
20	1	1	1	1
21	1	1	1	1
22	0.1568	15	1	0.1518
23	0.0752	23	1	0.0752
24	0.0008	37	1	0.0007
25	1	1	1	1
26	0.0099	27	1	0.0099
27	0.0101	26	1.1457	0.0097
28	0.0008	37	1	0.0004
29	0.0059	30	1	0.0058
30	0.0023	33	0.7450	0.0027
31	0.1987	14	1	0.1922
32	0.7814	7	0.6357	1
33	0.1525	17	0.7868	0.1822
34	0.4290	11	0.7423	0.5119
35	0.1531	16	0.5584	0.1952
36	0.3019	12	0.9890	0.3217
37	0.0017	34	0.4996	0.0024
38	0.1265	19	1	0.1253
39	0.7285	8	0.5474	1

computational achievement. Moreover, the flexibility of weighs enable decision-makers to incorporate managerial preferences in the evaluation process. The application of the proposed models for a real dataset showed that in the external and internal cases the SBM model gives unreasonable efficiency scores for about 50% and 40% of DMUs in overall and stage 2. Moreover, Pearson correlation coefficients showed that in both external and internal evaluations, the SBM model relies more on the second stage which is not the case in the ASBM model where it relies on both stages almost equally. In addition, the proposed ASBM model in the internal evaluation can identify the desirable outputs from the undesirable outputs, thereby avoiding the need for weighting with managers priorities.

Table 3. Results of external evaluation in the ASBM model.

DMU	Overall efficiency	Rank	Stage 1 efficiency	Stage 2 efficiency
1	0.7016	20	1	0.5823
2	1	1	1	0.7422
3	0.9124	7	0.9599	0.9098
4	0.4121	39	0.6296	0.4759
5	0.5661	34	0.7297	0.6118
6	0.7842	13	1	0.6979
7	1	1	1	1
8	0.5983	31	0.8748	0.5634
9	0.8604	8	1	0.8023
10	0.8509	9	1	0.7912
11	0.5628	35	0.7029	0.6265
12	1	1	1	1
13	0.7453	15	0.8314	0.7881
14	0.5678	33	0.7842	0.5690
15	0.7163	19	1	0.6028
16	0.6155	30	1	0.4164
17	0.7438	16	1	0.6413
18	0.6307	29	0.8355	0.6246
19	0.6554	25	0.8239	0.6586
20	1	1	1	1
21	1	1	1	1
22	0.7872	12	1	0.7021
23	0.7486	14	1	0.6480
24	0.6940	22	1	0.5717
25	1	1	1	1
26	0.7009	21	1	0.5812
27	0.6378	27	0.9742	0.5373
28	0.5327	36	1	0.3256
29	0.6554	25	1	0.5175
30	0.5318	37	0.7847	0.5179
31	0.7311	18	1	0.6235
32	0.8439	10	0.7268	1
33	0.5830	32	0.7914	0.6082
34	0.6661	23	0.7546	0.7498
35	0.6366	28	0.7223	0.7231
36	0.6604	24	0.8761	0.6643
37	0.4479	38	0.6668	0.4987
38	0.7399	17	1	0.6359
39	0.8060	11	0.6606	1

Table 4. Pearson correlation coefficients between overall and two stages efficiency in the SBM model.

	Overall efficiency	Stage 1	Stage 2
Overall efficiency	1.000		
Stage 1	0.0275	1.000	
Stage 2	0.9905	−0.0305	1.000

Table 5. Pearson correlation coefficients between overall and two stages efficiency in the ASBM model.

	Overall efficiency	Stage 1	Stage 2
Overall efficiency	1.000		
Stage 1	0.5617	1.000	
Stage 2	0.8494	0.0841	1.000

Table 6. Results of internal evaluation of the SBM model.

DMU	Overall efficiency	Rank	w_1	Stag 1 efficiency	w_2	Stag 2 efficiency
1	0.0474	25	0.0245	1	0.9755	0.0236
2	1	1	0.5	1	0.5	1
3	0.9550	7	0.5100	0.9117	0.4900	1
4	0.0019	35	0.0015	0.5288	0.9985	0.0011
5	0.0294	26	0.183	0.6547	0.9817	0.0178
6	0.0010	36	0.0005	1	0.9995	0.0005
7	1	1	0.5	1	0.5	1
8	0.1657	23	0.1078	0.7127	0.8922	0.0996
9	0.0010	36	0.0005	1	0.9995	0.0005
10	0.3592	14	0.1842	1	0.8158	0.2145
11	0.2191	20	0.1803	0.5109	0.8191	0.1549
12	1	1	0.5	1	0.5	1
13	0.6734	11	0.3679	1	0.6321	0.4833
14	0.0102	31	0.0061	0.7103	0.9939	0.0060
15	0.2094	21	0.1093	1	0.8907	0.1123
16	0.0085	32	0.0047	1	0.9953	0.0037
17	0.0143	29	0.0074	1	0.9926	0.0069
18	0.1681	22	0.0888	1	0.9112	0.0871
19	0.3384	15	0.1944	0.7566	0.8056	0.2375
20	1	1	0.5	1	0.5	1
21	1	1	0.5	1	0.5	1
22	0.2715	18	0.1411	1	0.8589	0.1518
23	0.1399	24	0.0699	1	0.9301	0.0752
24	0.0009	38	0.0005	1	0.9995	0.0004
25	1	1	0.5	1	0.5	1
26	0.0196	28	0.0098	1	0.9902	0.0099
27	0.0213	27	0.0117	1	0.9883	0.0097
28	0.0008	39	0.0005	1	0.9995	0.0003
29	0.0117	30	0.0059	1	0.9941	0.0058
30	0.0047	33	0.0028	0.6922	0.9972	0.0027
31	0.3316	16	0.1725	1	0.8275	0.1922
32	0.6777	10	0.5943	0.4577	0.4057	1
33	0.2918	17	0.1642	0.8497	0.8358	0.1822
34	0.6970	10	0.3890	1	0.6110	0.5042
35	0.7329	8	0.6051	0.5586	0.3949	1
36	0.5126	13	0.2815	1	0.7185	0.3217
37	0.0043	34	0.0027	0.5920	0.9973	0.0027
38	0.2248	19	0.1138	1	0.8862	0.1253
39	0.5325	12	0.7062	0.3380	0.2938	1

TABLE 7. Results of internal evaluation of the ASBM model.

DMU	Overall efficiency	Rank	Stage 1 efficiency	Stage 2 efficiency
1	0.7911	23	1	0.5823
2	1	1	1	1
3	0.9690	7	0.9380	1
4	0.5349	39	0.6030	0.4668
5	0.6754	34	0.7259	0.6249
6	0.8489	13	1	0.6979
7	1	1	1	1
8	0.6797	32	0.7976	0.5617
9	0.9016	8	1	0.8014
10	0.8956	9	1	0.7912
11	0.6590	36	0.6926	0.6253
12	1	1	1	1
13	0.8597	11	1	0.7194
14	0.6758	33	0.7824	0.5692
15	0.7892	25	1	0.5783
16	0.7080	31	1	0.4159
17	0.8201	18	1	0.6401
18	0.8019	21	1	0.6038
19	0.7415	30	0.8174	0.6656
20	1	1	1	1
21	1	1	1	1
22	0.8510	12	1	0.7021
23	0.8240	17	1	0.6480
24	0.7858	26	1	0.5717
25	1	1	1	1
26	0.7906	24	1	0.5812
27	0.7686	27	1	0.5373
28	0.6612	35	1	0.3225
29	0.7588	28	1	0.5175
30	0.6335	37	0.7511	0.5160
31	0.8117	20	1	0.6235
32	0.8367	15	0.6735	1
33	0.7477	29	0.8873	0.6081
34	0.8748	10	1	0.7495
35	0.8446	14	0.6891	1
36	0.8312	16	1	0.6625
37	0.5649	38	0.6318	0.4979
38	0.8179	19	1	0.6359
39	0.7961	22	0.5922	1

TABLE 8. Pearson correlation coefficients between overall and two stages efficiency in the SBM model.

	Overall efficiency	Stage 1	Stage 2
Overall efficiency	1.000		
Stage 1	0.0831	1.000	
Stage 2	0.9531	−0.1153	1.000

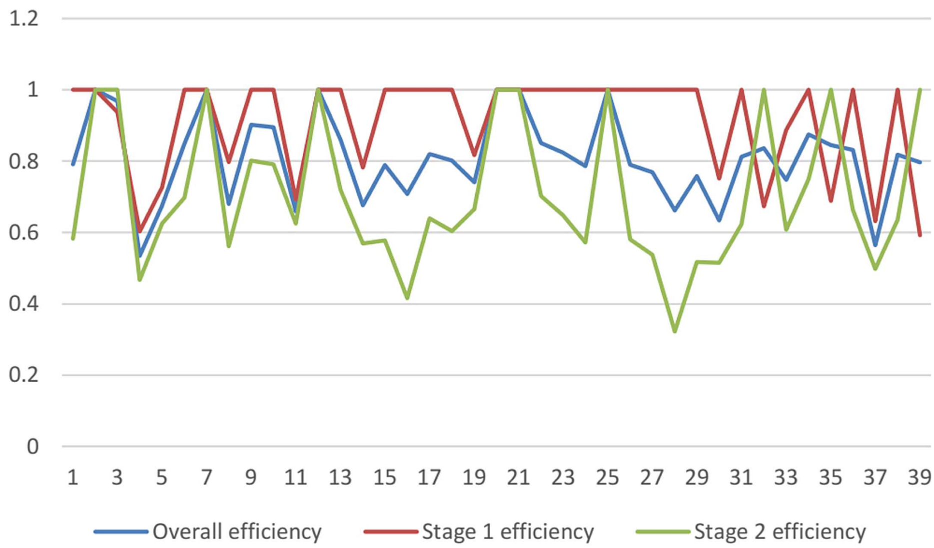

Figure 3. Overall and stages efficiencies of the ASBM model for internal evaluation.

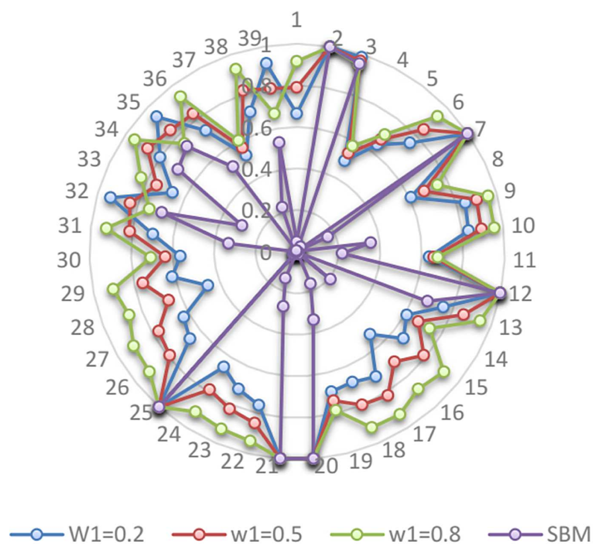

Figure 4. The internal Efficiency scores for different weights of ASBM and SBM model.

Table 9. Pearson correlation coefficients between overall and two stages efficiency in the ASBM model.

	Overall efficiency	Stage 1	Stage 2
Overall efficiency	1.000		
Stage 1	0.5422	1.000	
Stage 2	0.8204	0.0157	1.000

In recent studies a secondary goal has been added to the DEA models in order to consider DMUs fairness mentality that plays an important role in behavioral economics [36]. Thus one may combine it with our proposed ASBM models in the presence of undesirable outputs. Moreover, the idea of uncontrollable inputs [33], non-discretionary factors [24] and uncertainty in dataset [1, 23, 28], have also pervasive applications, so including them in the proposed models would be an absorbing future research directions.

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