RAIRO-Oper. Res. 56 (2022) 2443–2473
https://doi.org/10.1051/ro/2022097

RAIRO Operations Research
www.rairo-ro.org

DUAL CHANNEL SUPPLY CHAIN INVENTORY POLICIES FOR CONTROLLABLE DETERIORATING ITEMS HAVING DYNAMIC DEMAND UNDER TRADE CREDIT POLICY WITH DEFAULT RISK

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Abstract. Presently in the commercial environment, because of the high level of market globalization and rapid increase in industrialization, supply chain synchronization is playing an increasingly significant role in the proper management of the whole system including several factors at the same time. In real business world, both manufacturer and retailer accept credit to make their business position strong, as credit not only strengthens their business relationships but also increases the scale of their profits. The long period of credit may increase the demand ratebut simultaneously it can also increase the credit risk. We investigate a two-layer supply chain model under dynamic demand with a manufacturer and a retailer maintaining decaying items with controllable deterioration rates under two levels of trade credit policies. For the time of trade credit granted to the retailer, the manufacturer bears opportunity costs. To promote sales and optimize sales volume, both supply chain participants give trade credit periods to downstream members and due to the credit period, both of them are facing default risk. Both members work together to invest in preservation technologies to abate the rate of degradation. The proposed models are developed for both the centralized and the decentralized scenarios. A closed form model having profit maximization problem is developed for both the centralized and the decentralized scenarios. The focus of this study is to obtain the optimal selling price, replenishment cycle time, preservation technology cost, upstream and downstream trade credit period to optimize supply chain profit. The paper's novelty lies in introducing two level trade credit with default risk considering decaying items with controllable deterioration and price and credit sensitive customer's demand in a dual channel supply chain inventory policy. It is found that joint supply chain model can be able to enhance the total profit of the whole supply chain. Lastly, sensitivity analysis highlights the influence of major model parameters using numerical examples.

Mathematics Subject Classification. 90B05, 90B06.

Received January 30, 2022. Accepted May 28, 2022.

1. Introduction

Presently, gigantic advancement of technology, highly competitive business setup, market globalization leads to the very rapid rise in industrialization. Every company intends to develop its position and try to create synchronization among its members for the fulfillment of the needs of end consumers. In order to strengthen and establish, an effective, efficient, unified, and robust platform the coordination among the supply chain

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Keywords. Inventory, supply chain, dynamic demand, deterioration, two-level Trade credit, preservation technology investment.

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members is playing a significant role. The interrelatedness among the supply chain members (SCM) has been done for a few decades. The coordination between vendor and buyer supply chain (SC) was initially proposed by Goyal [14]. Thereafter various researchers (Barman and Mahata [2], Mahata *et al.* [33]) across the world elongated this study along with various additional characteristics. Supply chain management (SCM) refers to the collection of interconnected operations including a supplier, retailer, manufacturer, and client, as well as several factors ranging from gathering raw materials to delivering finished goods to the customer's hands. In SCM, customer satisfaction and needs play a significant role. Furthermore, the client is an essential component of the system of the supply chain. In today's increasingly competitive business contexts, it's become imperative to help other supply chain participants to increase revenue and operational excellence. In India, there are many big food production industries such as AMUL India Ltd, Nestle India Ltd, Hadrian's India Ltd focus on their supply chain system such that each sector associated with this can be dealt with very efficiently, and eventually that makes a robust framework for all-round development of those companies.

The concept of items deterioration plays a prominent role in the inventory management system. The degrading goods have been researched from afew decades, and a lot of research work will be revealed. Deterioration is a regular and practical reality that will continue to worsen as time passes. For example, the meat will deteriorate after a time when stored, alongside electrical equipment, fashionable items, volatile liquids, medicines, high-tech products, and agricultural goods. The term deterioration of items is more practical in the food sector, pharma sector, electronic components manufacturing sector, and so on. The type of deterioration of degradable products may consist of the following: (1) the products which have immediate spoiling such as fresh vegetables and foods, (2) physical exhaustions such as petrol, alcohol, etc., (3) Declines such as a change in radiation, unfavorable spoilage and loss of inventory effectiveness *e.g.*, medicine and electronic items.

Recently, Mahato and Mahata [33] proposed that the increased degradation rate would lead to higher annual total costs and lesser demand. In order to lower the degradation rate and lengthen the product expiry date, some companies invested in infrastructure. In particular, cooling devices are utilized to decrease fruit, flowers, and seafood degradation in the grocery. According to Lystad *et al.* [26], around 15% of the commodities lost are disposed of before reaching the retail consumer in the US food business, when products are declined as these findings suggest, control of product deterioration in the retail food industry is becoming extremely crucial. Maihami *et al.* [36] presented that the Food and Agriculture Organization (FAO) of the United Nations calculates that one-third (approximately 1.3 billion tonnes) of human food production worldwide is either degraded or lost. Food waste happens at every level of the SC. Li *et al.* [25], Bakker *et al.* [1], and Janssen *et al.* [23] offered up to the date literature review regarding inventory problems connecting product deterioration.

Despite India's robust agriculture production base a substantial amount of food is wasted every year owing to insufficient infrastructure *e.g.* packaging systems, lack of preservation technology, the capacity of storage, poor transportation, and so on. Because of inadequate supply chain interconnections, infrastructure constraints, and a shortage of skilled workers many food processing industries have to face a significant amount of loss per annum. As per the Ministry of Food Processing Industry in India (MoFPI), post-harvest damages are a total of US$1.5 billion (Rs 92,000 crores) every year (MoFPI). In order to enhance the maximum lifetime of degradable products, preservation technology will be playing a prominent role. Preservation technology is one kind of technology that can able to reduce the time of deterioration of degradable products. Food products, pharmaceutical products will be decayed as time increases, to prevent such degradation preservation technology investment is being incorporated. This common practice can be observed in various food processing industries across the world. Because of the fierce rivalry in the corporate world, every organization aims to better its position in order to reach more customers. Hsu *et al.* [20] first explored an inventory model that included investment in preservation technology (PT) to avoid degradation. The degradation rate in this study is a function of the PT investment cost. Afterward, Dye and Hsieh [10] elongated the study considering time-dependent deterioration and shortages of items that are partially backlogged along with preservation technology investment. One year later, Dye [9] established an inventory model that takes into account degradation and preservation technology investment. Mishra *et al.* [39] presented the relationship between the rate of deterioration and PT investment parameters $\lambda(\alpha) = \lambda_0 e^{-\delta\alpha}\lambda(\alpha)$ represented as deterioration rate, λ_0 indicates degradable rate except for preservation investment and δ is investment sensitive parameter. Recently many researchers across the world elongated inventory control management along with preservation technology investment such as [7,17,38,48,48,51,58], etc. Very recently, Saha *et al.* [48] devised a dynamic demand based inventory control management with preservation technology investment by employing modified flower algorithm approach. By motivating from existing studies and viewing the benefits of it we have incorporated investment in preservation technology at the rate $f(u)$ in this study which will be beneficial for decision makers how to incorporate such investment such that both supply chain participants will be benefited and also to slow down the degradability and limit the quantity of goods that succumb and become useless while he/she attempts to establish a robust inventory management environment.

One of the most significant aspects of this study is the dynamic demand factor. In bi-level trade credit services, the influence of the selling price, as well as the credit duration supplied by the contemporary trade retailer to downstream customers, must be considered. This is especially important for fast-moving consumer goods, as a humongous amount of stock must often be eliminated to clear the way for new items, which necessitates price reductions for traditional ones.In actuality, the retailer's demand must account for both the sales price as well as the credit duration, since the credit period's marginal influence on sales is proportional to the market demand's unfulfilled potential. When inventory clearing is the norm, such as at groceries, the positive correlation between the selling price and the credit period becomes really significant. That's why in our study we have incorporated price and credit-linked demand.

Every year, fresh sorts of offers must be included in the mix to reinforce and establish a solid source of demand for selling items. One such offer is trade credit. This idea is carried on through the buy now, pay later concept. Currently, one of the most powerful supplier mechanisms is the policy of trade credit. The benefits of trade credits for both supply chain participants are (1) it can stimulate demand and entice new consumer. (2) Enhance sales, and promote commodities (3) Inventory cost reduction (4) strengthening business partnership and expands the size of their earnings. However, one main limitation of such credit policy is that the demand rate may rise as a result of the longer credit duration, but the credit risk also rises. Based on the strong decision-making authority every participant of the supply chain that desires to acquire obtains full or partial trade credit [27]. In two-level trade credit terms, suppliers provide a trade credit time to the retailer, called upstream trade credit, and the retailer, in turn, gives a trade credit term to its consumers, recognized as downstream trade credit. During the trade credit period, theretailer company can generate sales revenues for selling the products and earn interest by investing money to a bank or share market but meanwhile if the trade credit period exceeds then a high amount of penalty will have to pay by the retailer company. There are various advantages of trade credit [32]. Trade credit is a strong supplier tool for raising revenues, increasing sales, promoting products, increasing demand and attracting new clients, promoting firm's products and industries [54], boosting competitiveness [6], inventory cost reduction [55]. Considering the concept of the trade credit period in the EOQ model, Goyel [15] was the pioneer person. Thereafter last 3 decades, many researchers across the world elongated the concepts in various ways, such as [5, 8, 30, 35, 36, 38] etc. The above-listed publications are performed by a one-level trade-credit concept where the vendor will grant the trade credit to the merchant but the merchant does not apply it to its consumer.

Bi-level trade credit period concept associated with when the manufacturer provides trade credit period (known as upstream trade credit) to retailer and retailer also gives trade credit periodto the consumers (called downstream trade credit). Huang [21] is the pioneer of an EOQ model along with a two-level trade credit period policy. Later on, Teng and Chang [54] investigated an EPQ model with two-level trade credit periods. Subsequently, Mahata and Mahata [29] and Mahata [27] presented EOQ/EPQ inventory model for degrading products with a two-level trade credit policy inthe SC system with downstream partial trade credit. Next, Wu *et al.* [57] presented inventory policies considering trapezoidal type demand with a two-level trade credit policy and maximum lifetime of products. Besides, Mahato and Mahata [35] elongated an inventory model for non-instantaneously decaying products with price-sensitive demand under a two-level trade credit policy and the rate of deterioration varies with time. Besides, to reduce the deterioration rate, retailers invest some cost to prevent product degradation/decay, known as preservation technology, is also inserted.

In the following section, we have included the credit risk due to the credit period. The proposed model's first key component is two level trade credit policies. Due to entrepreneur instincts, the credit period is provided by the supplier side to the retailer, whereby the retailer then passes it to his/her end consumers. The main significance of default risk is raising credit duration increases retailer demands, and it also enhances the non-payment risk for both the supplier and the retailer.If the retailer declines to make the payment even during the credit period, the supplier intends to levy compound interest on the principal amount. The retailer distributes the items to consumers on different short-term, interest-free loans (*i.e.*, trade credit) and offers credit duration to their final consumers, but picks a credit duration that is less than the stated credit duration (given by the supplier).Some instances of default risk are as follows

(a) Consumers refuse to return the loan they incur on their credit cards monthly.
(b) Families fail to pay the specified amount for their home loans every month or year.

This study attempts to make an effect on the literature in terms of maximum profit by adding default risk.

A manufacturing process may not always be trustworthy. Its state may change "from in-control to out-of-control", resulting inimperfect production. Porteus [46] developed a model regarding improvement of quality. Lee and Rosenblatt [24] proposed several models for the imperfect production process. Later, Sana [49], Giri and Maiti [12], Mahata [28], and many others researched in the context of inventory model in "imperfect-quality" environments.

All of the preceding research presupposes that the strategies are selected by a single decision-maker in order to optimize one's own performance. However, within integrated SCM all entities collaborate to maximize the overall profit of the SC system. The very first integrated inventory model was proposed by Goyal [14]. Following that, the integrated inventory model was created by Goyal and Gupta [16], dependent on the collaboration of the vendor and the buyer. Following then, numerous researchers used it in their studies, namely Chaharsooghi and Heydari [4], Ho [18], etc. Recently, Heydari *et al.* [19] have established a coordinated and centralized two-layer inventory model featuring periodic review policies. Tiwari *et al.* [56] proposed a carbon emissions-based two-tiered supply chain model with a single vendor and a single buyer considering declining and poor quality products. Concentrating on decaying materials, Jaggi *et al.* [22] derived two-layer SCM using Nash equilibrium and Stackelberg game technique to find out optimal trade credit decisions and inventory. A very recently two-tiered production inventory model for degrading products with preservation technology investment was investigated by Shen *et al.* [52]. Table 1 summarises the significant differences between this study and previous research.

According to the aforementioned literature review, nearly all publications developed for either demand taken as constant in natureor deterministic demand rate (credit or stock dependent demand or price).

As per the best of the author's knowledge, no one has tackled investment in preservation technology, price, trade credit reliance demand, and supply chain coordination simultaneously. The main motivation and novelty in this study are as follows.

(i) The proposed work is analysed for a single manufacturer and asingle retailersupply chain model for maximizing the total profits for both participants.
(ii) The proposed model's second major component is credit risk.To offer a more realistic perspective, default risk is permitted.
(iii) The third important component of this current model is that the demand at the retailer side is reliant on both price and trade credit periods.
(iv) Deterioration of the commoditiesis considered for both supply chain participants.
(v) The fourth essential aspect of this study is the investment in preservation technology to decrease the rate of degradation of the products, where both participants are collaborating to invest in preservation technology to decrease the degradation of items.

TABLE 1. The main characteristic of the inventory model in relevant studies.

References	Demand pattern	Deterioration	Preservation technology	Trade credit policy		Default risk	Model integrated policy
				One level	Two-level		
Goyal [14]	Constant	×	×	×	×	×	√
Hsu et al. [20]	Constant			×	×	×	×
Dye [9]	Constant demand	√	√	×	×	×	×
Dye and Yang [11]	Trade credit-dependent	√	×	√		√	×
Mahata [27]	Constant	√	×	×	√	×	×
Thangam&Uthayakumar [55]	Selling price & trade credit-dependent	√	×	×	√	×	×
Wu et al. [57]	Trade credit-dependent	√	×	×	√	√	×
Mashud et al. [38]	Price dependent	√	√	√	×	×	×
Zhang et al. [59]	Price dependent	√	√		×	×	√
Giri et al. [13]	Price dependent	√	√		×	×	√
Tayal et al. [53]	Time-dependent	√	√	√	×	×	√
Mishra et al. [39]	Price & stock dependent	√	√	×	×	×	×
Mahato and Mahata [35]	Trade credit & product freshness dependent	√	×	×	√	√	×
Maihami et al. [36]	Price dependent	√	×	×	×	×	√
Pervin et al. [44]	Price and stock dependent	√	√	×	√	×	×
Pervin et al. [45]	Quadratic demand	√	√	×	×	×	√
Paul et al. [42]	Price and credit period sensitive demand	√	×	√	×	√	×
Paul et al. [43]	Price and green sensitive demand	√	×	×	×	×	×
Roy et al. [47]	Probabilistic demand	√	×	×	√	×	√
Sepehri et al. [51]	Price dependent	√	√	×	×	×	×
Mishra et al. [40]	Price dependent	√	√	×	×	×	×
Mashud et al. [38]	Price dependent	√	×	√	×	×	√
Tiwari et al. [56]	Constant	√	×	×	×	×	√
Jaggi et al. [22]	Stock dependent	√	×	√	×	×	√
Shen et al. [52]	Constant	√	√	×	×	×	√
Our model	Price & Credit dependent	√	√	×	√	√	√

2. Assumptions and notations

The proposed supply chain model isbased on the following notations and assumptions.

2.1. Notations

Decision variables

p Retailer's unit selling price
T Time for the retailer's replenishment cycle/Manufacturer's production cycle time
u Cost of preservation technology per unit time to reduce deterioration
N Credit period provided by the retailer to the customer .

Constant parameters

A_r The retailer's ordering cost per order
A_m Setup cost per a lot of the manufacturer
θ_r, θ_m Original deterioration rate of retailer and manufacturer respectively
h_r, h_m Retailer's and manufacturer's unit inventory holding cost respectively
I_e The interest rate of revenue deposited by the retailer
I_c Interest rate to be paid to the manufacturer for the remaining stock from M to T
I_v The interest rate is used to calculate the manufacturer's opportunity interest loss owing to late payment.
P Production rate, which is a given constant
u Unit cost of preservation investment
t_p Production time for the manufacturer
c Manufacturer unit production cost
α Preservation technology investment proportion for the retailer side
β, γ Coefficient of default risk, which is a positive constant
M Credit period provided by the supplier to the retailer

Functions

$I_m(t)$ The level of inventory at any time t at manufacturer side
$I_r(t)$ The level of inventory at any time t at the retailer side
$D(p, N)$ Demand rate as a function of selling price p and downstream credit period N
$f(u)$ Proportion of reduced deterioration rate due to investment in preservation technology

2.2. Assumptions

The mathematical model of the supply chain system is based on the following assumptions:

(i) The manufacturer commits to a lot-for-lot production policy in accordance with the order quantity from the retailer side.
(ii) The rate of production is deterministic and sufficiently greater than the maximum demand rate. As manufacturer's rate of production is sufficiently larger than the retailer's demand rate, then time delay $T - t_p$ is taken by the manufacturer per production run.
(iii) Throughout the production, at any arbitrary time $t \in [0, t_p]$, the process may move from an in-control state to an out-of-control state, resulting in defective items is being produced.
(iv) For generality, we consider that customers' demand rate is a non-negative, continuous, decreasing and convex function of p, say $\alpha(p)$, where $\alpha^{'}(p) < 0$ and $\alpha^{''}(p) > 0$ and the gross revenue, $p\alpha(p)$, is a strictly concave function of p (*i.e.*, If $2\alpha^{'}(p) + p\alpha^{''}(p) < 0$ or diminishing marginal revenue). This condition is common to many price-dependent demand functions and similar to the condition of profit maximization with respect to the price p found in [35] and [50]. The lower selling price increased the rate of the selling price when the maximum selling price had a backward effect [40]. Also, note that if the grow revenue is

increasing function of price p, then price and gross revenue will always move in the same direction, hence retailers can realize infinite gross revenue by setting an infinite p. It is impossible. We also assume that $c < p \leq \overline{p}$, where $\lim_{p \to \overline{p}} \alpha(p) = 0$.

(v) In addition, we observe that downstream trade credit (*i.e.*, the retailer provides trade credit to consumers.) has a positive influence on demand. Because credit trade enables consumers to enjoy the advantage of delayed payments,extending the time will increase sales. The greater the credit duration, the greater the demand. Hence, the demand rate is positive, strictly increasing in N, say $\beta(N)$, where $\beta^{'}(N) > 0$ [35].

(vi) Combining above two relations, demand rate$D(p, N)$ is dependent on both selling price and downstream trade credit period. Here we assume the functional representation of demand rate as a multiplicative form of the credit period $N$and selling price $p$which is asfollows: $D(p, N) = \alpha(p)\beta(N)$.

(vii) To reduce the deterioration of products, preservation technology was applied in this model. Both retailer and manufacturer could decrease the initial deterioration rate θ_r and θ_m to $\theta_r f(u)$ and $\theta_m f(u)$ respectively by investing in preservation technologies at the same rate of $f(u)$, in which $f(u)$ is constrained by $0 < f(u) < 1$ and governed by the level of investment$u$ is a concave, continuous and increasing function of the preservation technology investment u, *i.e.*, $f^{'}(u) > 0$ and $f^{''}(u) < 0$. This means that the investment's marginal contribution is declining [40] and [51].

(viii) The investment in preservation technology is jointly shared by the retailer and the manufacturer.$\alpha$and $1 - \alpha$ $(0 \leq \alpha \leq 1)$ represent the proportions of the capital investment which is invested by retailer and manufacturer respectively in machinery equipment.

(ix) The manufacturer delivers the goods to the retailer on a lot-by-lot basis.Particularly, the duration of the manufacturer's production cycleis equal to the length of the retailer's replenishment cycle.

(x) The manufacturer's rate of production is greater than the retailer's demand rate, making the manufacturer begin production with a delay time $(T - t_p)$ in each production cycle.

(xi) Shortages are not allowed.

(xii) The manufacturer gives a credit period M (time unit) to the retailer and the retailer gives a credit period N (time unit) to each customer.

(xiii) When $T \geq M$, at time $t = M$ the account is settled and the retailer commences paying the interest charges on the products in stock with rate I_c over the interval $[M, T]$. If $T \leq M$, at time M the retailer settles the account, and there is no interest charge on stock throughout the complete cycle. Apart from this when $M > N$ under the down-stream and up-stream trade-credit circumstances the retailer can collect revenue and earn some interest from N to M with the rate of interest I_e

(xiv) The manufacturer faces opportunity cost with rate I_v by providing trade credit to the retailer.

(xv) High credit duration generates a high possibility of default risk. The rate of the default risk for manufacturer and retailer is considered as: $F_1(M) = 1 - e^{-\chi M}$, $F_2(N) = 1 - e^{-\gamma N}$ [41] respectively, where $\chi, \gamma > 0$, are coefficients of the default risk. The default risk function's maximum value is $F_1(M) = F_2(N) = 1$ at $M = N = \infty$ and minimum value is $F_1(M) = F_2(N) = 0$ at $M = N = 0$. Hence, the default risk is represented as the increasing function with a value lie between $[0, 1]$ (see in Figs. 1 and 2).

3. Model formulation

In this research, over an infinite time horizon, weconsider a two-layer supply chain with a manufacturer and a retailer handling degrading products under two levels of trade credit policies over an infinite time horizon (Fig. 1).

Both supply chain participants provide a trade credit term to downstream participants in order to promote sales and maximize sales volume. The level of inventory of the two participants is deteriorating at a constant rate. To decrease the rate of degradation, both participants invest in preservation technologies jointly. Based on the aforementioned assumptions and notations, we first develop the retailer and manufacturer model separately.

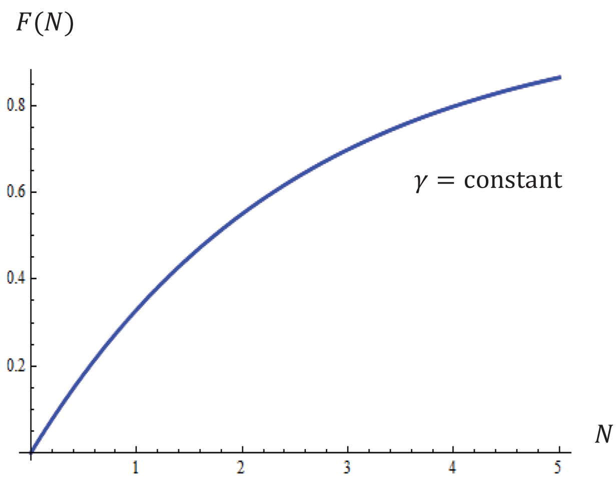

FIGURE 1. Default risk function $F(N)$.

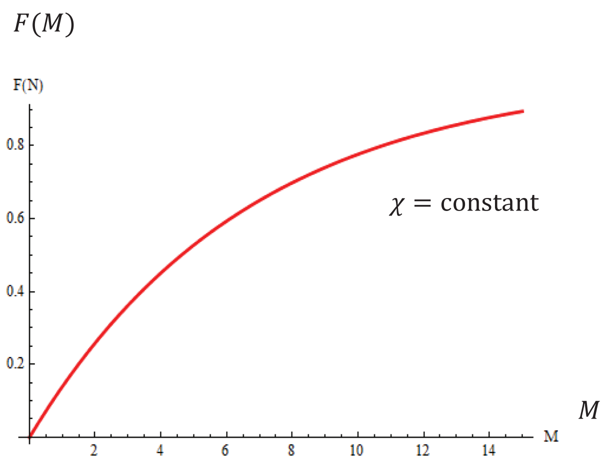

FIGURE 2. Default risk function $F(M)$.

3.1. Retailer's inventory system

The retailer receives an initial order quantity Q at time $t = 0$ and the replenishment cycle period is T. The inventory level of a retailer declines due to the combined influences of consumer demand and item degradation throughout the replenishment period $[0, T]$ and follows the pattern shown in Figure 4.

The underlying differential equation governs with the change in the retailer's inventory level:

$$\frac{dI_r(t)}{dt} + \theta_r\left(1 - f\left(u\right)\right) I_r\left(t\right) = -D\left(p, N\right), t \in \left[0, T\right]. \tag{3.1}$$

Using boundary condition $I_r\left(T\right) = 0$, the solution of the differential equation (3.1) is expressed as

$$I_r\left(t\right) = \frac{D\left(p, N\right)}{\theta_r\left(1 - f\left(u\right)\right)}\left[e^{\theta_r(1-f(u))(T-t)} - 1\right] \text{for } 0 \le t \le T. \tag{3.2}$$

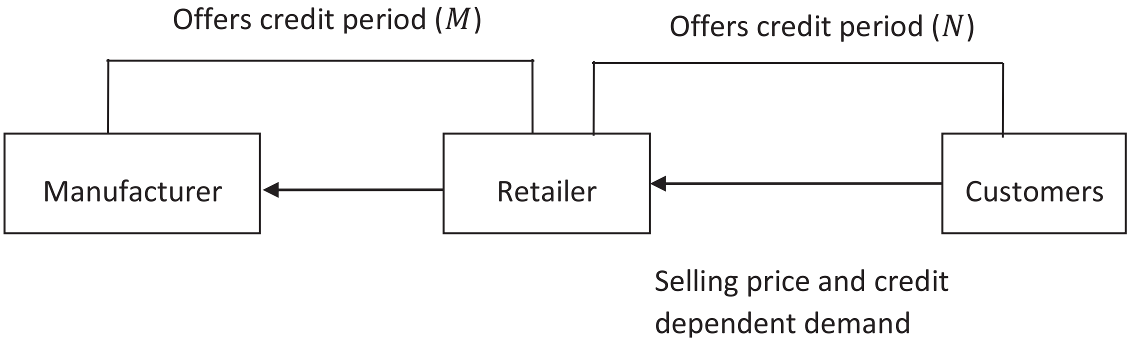

FIGURE 3. Two-level supply chain model.

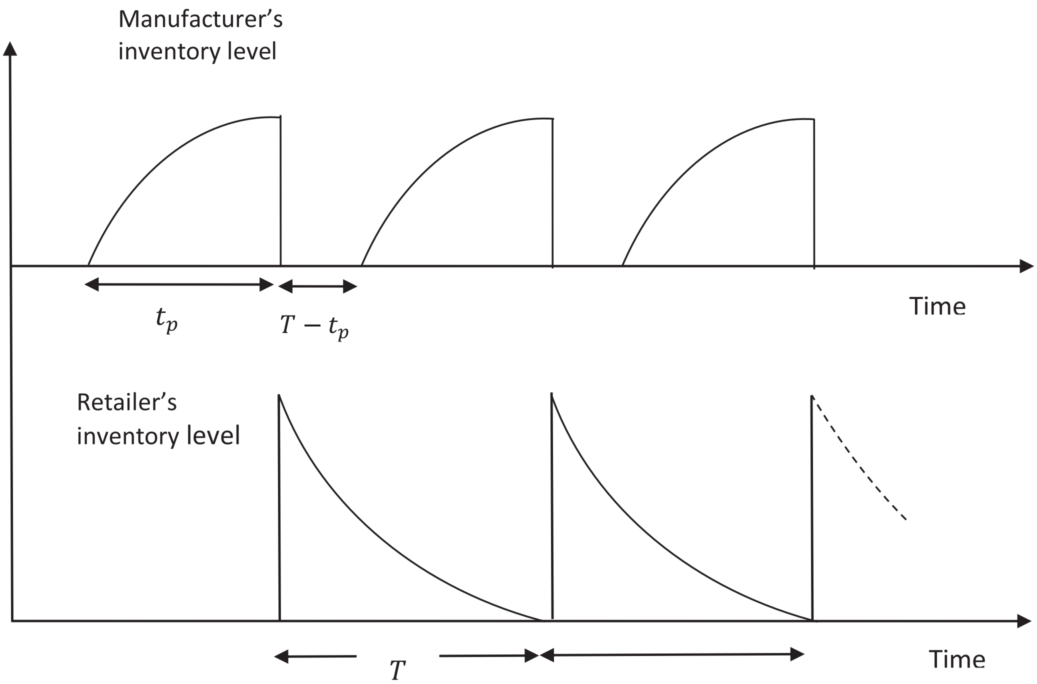

FIGURE 4. A diagrammatic representation illustrating the manufacturer and retailer's inventory level.

Consequently, the retailer's ordering volume Q can be acquired in the following ways:

$$Q = I_r(0) = \frac{D(p,N)}{\theta_r(1-f(u))}\left[e^{\theta_r T(1-f(u))} - 1\right]. \tag{3.3}$$

The total sales volume on the retailer side is calculated during the order cycle and is given by

$$Q_d = \int_0^T D(p,N)\,dt = D(p,N)\,T. \tag{3.4}$$

Throughout each replenishment cycle, the total profit includes the following components:

Ordering cost per replenishment cycle is

$$OC = A_r. \tag{3.5}$$

1. The inventory holding cost during each replenishment cycle is

$$HC_r = h_r\left(\int_0^T I_r(t)dt\right) = \frac{h_r D(p,N)}{\theta_r(1-f(u))}\left[\frac{1}{\theta_r\left(1-f\left(u\right)\right)}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right]. \tag{3.6}$$

2. As investment in preservation technology is shared between both retailer and manufacturer, where proportion α $(0 \le \alpha < 1)$ is being shared by the retailer then αguT represents the investment in preservation technology to reduce deterioration rate per cycle of replenishment.
3. The retailer's net revenue after default risk is

$$SR_r = \frac{pe^{-\gamma N}}{T}\int_0^T D\left(p,N\right)dt = pe^{-\gamma N}D\left(p,N\right). \tag{3.7}$$

4. Purchasing cost

$$PC_r = \frac{cQ}{T} = \frac{cD(p,N)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))}-1\right]. \tag{3.8}$$

Next, we need to compute the annual capital opportunity cost for the retailer in the following two cases: (i) $N < M$ and (ii) $N \ge M$

Case 1. N $\le M$

There are two possible outcomes in this case: (1) $T+N \le M$ and (2) $T+N \ge M$. Let us go through the detailed formulation of each sub-case.

Sub-case 1.1. $\mathbf{T+N \le M}$

In this case, retailer has sold all products before the allowable delay period M, therefore the interest charged is

$$IC = 0. \tag{3.9}$$

The retailer spawns revenue at the start of the cycle and at time N settles the account. So, the annual interest earned on the retailer side is

$$\begin{aligned} IE &= \frac{pI_e D(p,N)T^2}{2T} + \frac{pI_e D(p,N)T(M-T-N)}{T} \\ &= pI_e D\left(p,N\right)\left(M-N-\frac{T}{2}\right). \end{aligned} \tag{3.10}$$

In this case, in addition to the aforementioned costs, the retailer's annual total profit can be represented as

$$\begin{aligned} TP_{r1}\left(p,N,T,u\right) = pe^{-\gamma N}D\left(p,N\right) - \frac{cD(p,N)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))}-1\right] - \frac{A_r}{T} - \alpha gu \\ - \frac{h_b D(p,N)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r\left(1-f\left(u\right)\right)}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right] + pI_e D\left(p,N\right)\left(M-N-\frac{T}{2}\right). \end{aligned} \tag{3.11}$$

Sub-case 1.2. $\mathbf{M \le T+N}$

In this case, the retailer does not have enough funds to settle the account at Msince the consumer will settle the account at time $T+N$. As a result, the interest charges will be paid by the retailer which is

$$IC = \frac{cI_c D\left(p,N\right)}{T}(T+N-M)^2. \tag{3.12}$$

During the period $[N, M]$ at the rate I_e the earned interest on the generated revenue specified as

$$IE = \frac{pI_e D\,(p,N)}{2T}(M-N)^2. \tag{3.13}$$

In this case, the annual total profit for the retailer is

$$\begin{aligned} TP_{r2}\,(p,N,T,u) = pe^{-\gamma N} D\,(p,N) - \frac{cD(p,N)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))}-1\right] - \frac{A_r}{T} \\ - \frac{cI_c D\,(p,N)}{T}(T+N-M)^2 - \alpha g u \\ - \frac{h_b D(p,N)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r\,(1-f\,(u))}\left\{e^{\theta_r T(1-f(u))}-1\right\} - T\right] + \frac{pI_e D\,(p,N)}{2T}(M-N)^2. \end{aligned} \tag{3.14}$$

Case 2. $\mathbf{N \geq M}$

In this case, the retailer must finance the whole purchase cost at time M and repay the loan from time N to time $T+N$. Thus, the annual interest rate is expressed by

$$IC = \frac{cI_c D(p,N)}{T}\left[2\,(N-M)+T\right]. \tag{3.15}$$

Since $N \geq M$, there is no interest earned for the retailer, *i.e.*,

$$IE = 0. \tag{3.16}$$

In this scenario, the retailer's annual total profit is represented by

$$\begin{aligned} TP_{r3}\,(p,N,T,u) = pe^{-\gamma N} D\,(p,N) - \frac{cD(p,N)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))}-1\right] - \frac{A_r}{T} - \alpha g u \\ - \frac{cI_c D(p,N)}{T}\left[2\,(N-M)+T\right] - \frac{h_r D(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r\,(1-f\,(u))}\left\{e^{\theta_r T(1-f(u))}-1\right\} - T\right]. \end{aligned} \tag{3.17}$$

3.2. Manufacturer's inventory system

Whenever a manufacturer receives the retailer's order, he/she employs a lot-for-lot policy to fulfil the demand of theretailer. The inventory level of the manufacturer is affected by production and deterioration. At any time t the level of inventory for manufacturer's finished products is considered as $I_m\,(t)$. The inventory system onthe manufacturer side in Figure 1 is depicted by the following differential equation:

$$\frac{dI_m\,(t)}{dt} + \theta_m\,(1-f\,(u))\,I_m\,(t) = P, 0 \leq t \leq t_p, \tag{3.18}$$

where $T - t_p$ is the machine set-up time required for production.

Solving (1) and employing the boundary conditions $I_m\,(0) = 0$ and $I_m\,(t_p) = Q$ yields

$$I_m\,(t) = \frac{P}{\theta_m(1-f(u))}[1-e^{-\theta_m(1-f(u))t}]. \tag{3.19}$$

Using $I_m\,(t_p) = Q$, production cycle time is given by

$$t_p = \frac{1}{\theta_m(1-f(u))}\log\left[\frac{P}{P-Q\theta_{\mathrm{m}}(1-f(u))}\right]. \tag{3.20}$$

Costs associated with the manufacturer's inventory system are as follows:

1. The total set-up cost (SC) during a single period is

$$SC = A_m. \tag{3.21}$$

2. Holding cost (HC_v) during a single period is

$$HC_v = h_m \int_0^{t_p} I_m(t)\,dt = \frac{h_m P}{\theta_m(1-f(u))}\left[t_p + \frac{1}{\theta_m(1-f(u))}\left\{e^{-\theta_m t_p(1-f(u))} - 1\right\}\right]. \tag{3.22}$$

3. There are two cases. First, if the machine turns to an out-of-control state after the time production time t_p, then there will be no defective items, but if the machine is in the out-of-control state before t_p, then there will be defective items as given below:

$N(t_p)$ denotes the number of the defective item in a production period then

$$N(t_p) = \begin{cases} 0, & \text{when}\,\xi \geq t_p \\ \tau P(t_p - \xi), & \text{when}\,\xi < t_p \end{cases} = \max(\tau P(t-\xi), 0). \tag{3.23}$$

As a result in a production cycle, the expected number of defective products is

$$E[N(t_p)] = \int_0^{t_p} \tau P(t_p - \xi) f(\xi)\,d\xi. \tag{3.24}$$

The cost of rework can be approximated as

$$RW = sE[N(t_p)] = s\int_0^{t_p} \tau P(t_p - \xi)\mu e^{-\mu\xi} d\xi = s\tau P\mu t_p^2. \tag{3.25}$$

(4) By providing trade credit to the retailer, the vendor waives an instant cash inflow until a later date. As a result, the vendor must wait time M for cash inflows from account receivable collection. The lost-opportunity cost by providing trade credit at a finance rate i_m can be expressed as

$$OC_v = wi_m MQ. \tag{3.26}$$

(5) The retailer and manufacturer shared the investment for preservation technology where the proportion $1-\alpha$ $(0 \leq \alpha \leq 1)$ shared by manufacturer and to decrease the rate of degradation, the investment in preservation technology per production cycle for the manufacturer is

$$PTI = (1-\alpha) gu. \tag{3.27}$$

(6) Manufacturer's net revenue (RV) after default risk expressed as

$$RV = wQ(1 - F_1(M)) = wQe^{-\chi M}. \tag{3.28}$$

Themanufacturer'sannual total profit after default risk can be written as,

$$\begin{aligned} TP_m(u,T) = \tfrac{1}{T}\big[wQe^{-\chi M} - cPt_p \\ -A_m - \tfrac{h_m P}{\theta_m(1-f(u))}\left\{t_p + \tfrac{1}{\theta_m(1-f(u))}\left\{e^{-\theta_m t_p(1-f(u))} - 1\right\}\right\} - s\tau P\mu t_p^2 \\ -wi_m MQ - (1-\alpha)guT\big]. \end{aligned} \tag{3.29}$$

3.3. Joint profit of the supply chain

Next, we compute the overall integrated profit of the entire supply chain, which is the sum of the manufacturer and retailer profits. Therefore, the integrated profit $TP(p,T,N,M,u)$ can be written as follows:

$$TP(p,T,N,u) = \begin{cases} TP_1(p,T,N,u) = TP_m(u,T) + TP_{r1}(p,N,T,u), T+N \leq M \\ TP_2(p,T,N,u) = TP_m(u,T) + TP_{r2}(p,N,T,u), M \leq T+N \\ TP_3(p,T,N,u) = TP_m(u,T) + TP_{r3}(p,N,T,u), N \geq M. \end{cases} \tag{3.30}$$

The goal of this research is to maximize the profit of a supply chain that consists of single-retailer and single-manufacturer.

4. Theoretical results and optimal solutions

At first, the fundamental definition of concavity for single and multiple variable functions is provided below:

Definition 1. A function f which is defined on an open interval (a,b) is said to be concave if for $x, y \in (a,b)$ and each λ, $0 \le \lambda \le 1$, we have

$$f\left(\lambda x + (1-\lambda)\, y\right) \ge \lambda f\left(x\right) + \left(1-\lambda\right) f(y)$$

Intermediate value theorem (in real analysis)
Suppose g be a continuous function on the closed interval $[a,b]$ and let $g\left(a\right) g\left(b\right) < 0$. Then there exists a number $c \in (a,b)$ such that $g\left(c\right) = 0$.

Lemma 1. *If f is a continuous and differentiable function such that f and f' exist, then the function is said to be attained its maxima, if $f^{''} < 0$.*

Lemma 2. *(Cambini and Martein [3]) The real-valued function $h\left(x\right) = \frac{f(x)}{g(x)}$ is (strictly) pseudo-concave, if $f(x)$ is non-negative, differentiable, and (strictly) concave, and $g(x)$ is positive, differential, and convex.*

4.1. Optimality for an annual total profit of retailers

To obtain the optimal selling price p^*, cycle time T^*, credit period N^*, and investment in preservation technology u^* that maximize the retailer's annual total profit $TP_{ri}\left(p,N,T,u\right)$ for $i = 1, 2$, and 3, we first determine the optimal solutions for each case, respectively. Owing to the complexity of the problem, we are unable to prove the retailer's annual total profit $TP_{ri}\left(p,N,T,u\right)$ for $i = 1, 2$, and 3 is joint concave in p, N, T, and u. However, we can prove that $TP_{ri}\left(p,N,T,u\right)$ for $i = 1, 2$, and 3 is strictly pseudo-concave in each of the decision variables.

For any given p, N and u, by employing Lemma 4.2, it is possible that retailer's annual total profit $TP_{ri}\left(p,N,T,u\right)$ for $i = 1, 2$, and 3is strictly pseudo-concave in T. Therefore, for any given p, N and u, there exists a unique global optimal solution T_i^* such that $TP_{ri}\left(p,N,T,u\right)$ is maximized. Therefore, in two cases,the optimal solution is characterized: at first the case $N \le M$, and then the case $N \ge M$.

Case 1. When $N \le M$
Sub-case 1.1. $T + N \le M$

The following findings can be obtained by using Lemma 4.2.

Theorem 1. *For any given values of p, N and u, $TP_{r1}\left(p,N,T,u\right)$ is a strictly pseudo-concave function in T, and therefore there exists a unique maximum solution T_1^*. Then $TP_{r1}\left(p,N,T,u\right)$ subject to $N \le M$ is maximized at* $\max\left\{T_1^*, M\right\}$.

Proof. See **Appendix A**.

For any specific values p, N and u, taking the first-order derivative of $TP_{r1}\left(p,N,T,u\right)$ in (11) with respect to T, setting the outcome is zero, and simplifying terms, the necessary condition of T_1^* is as follows:

$$\frac{\partial TP_{r1}}{\partial T} = 0, i.e., \tag{4.1}$$
$$\frac{cD(p,N)}{T^2\{\theta_r(1-f(u))\}}\left[e^{\theta_r T(1-f(u))}-1\right] - \frac{cD(p,N)}{T}e^{\theta_r T(1-f(u))} + \frac{A_r}{T^2} - \alpha g u + \frac{h_r D(p,N)}{\{\theta_r T(1-f(u))\}^2}\left[e^{\theta_r T(1-f(u))}-1\right]$$
$$-\frac{h_r D(p,N)}{\{\theta_r T(1-f(u))\}}e^{\theta_r T(1-f(u))} - \frac{pI_e D(p,N)}{2} = 0.$$

It is apparent from Theorem 1 that (31) has a unique solution T_1^*. If $M \le T_1^*$, then $TP_{r1}\left(p,N,T,u\right)$ is maximized at T_1^*. Otherwise, $TP_{r1}\left(p,N,T,u\right)$ is maximized at M. □

Theorem 2. *For any given values of $p, N, u > 0$, and if $\gamma^2\beta\left(N\right) - 2\gamma\beta^{'}\left(N\right) + \beta^{''}\left(N\right) < 0$.*

Then we have

1. $TP_{r1}(p,N,T,u)$ is strictly concave function in N thus there exist a unique maximum solution.
2. If $\Delta_{N_1} = \frac{\partial TP_{r1}}{\partial N}\big|_{N=0} \le 0$, then $TP_{r1}(p,N,T,u)$ is maximized at $N_1^* = 0$.
3. If $\Delta_{N_{M_1}} = \frac{\partial TP_{r1}}{\partial N}\big|_{N=M} < 0$ but $\Delta_{N_1} = \frac{\partial TP_{r1}}{\partial N}\big|_{N=0} > 0$, thus there exist a unique solution $N_1^{**} = N_1^*$, where $0 < N_1^* < M$ such that $TP_{r1}(p,N,T,u)$ is maximized
4. If $\Delta_{N_{M_1}} = \frac{\partial TP_{r1}}{\partial N}\big|_{N=M} \ge 0$, $TP_{r1}(p,N,T,u)$ is maximized at $N_1^* = M$.

Proof. Given values of$p$, T and u, taking 1^{st} and 2^{nd} order partial derivative of $TP_{r1}(p,N,T,u)$ with respect to the independent variable N, we can get

$$\begin{aligned}\frac{\partial TP_{r1}}{\partial N} &= p\alpha(p)\,e^{-\gamma N}\left[\beta'(N) - \gamma\beta(N)\right] - \frac{c\alpha(p)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))} - 1\right]\beta'(N) \\ &- \frac{h_r\alpha(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))} - 1\right\} - T\right]\beta'(N) \\ &+ pI_e\alpha(p)\left[\beta'(N)\left\{M - N - \frac{T}{2}\right\} - \beta(N)\right],\end{aligned} \tag{4.2}$$

$$\begin{aligned}\frac{\partial^2 TP_{r1}}{\partial N^2} &= p\alpha(p)\,e^{-\gamma N}\left[\gamma^2\beta(N) - 2\gamma\beta'(N) + \beta''(N)\right] - \frac{c\alpha(p)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))} - 1\right]\beta''(N) \\ &- \frac{h_r\alpha(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))} - 1\right\} - T\right]\beta''(N) \\ &+ pI_e\alpha(p)\left[\beta''(N)\left\{-1 + M - N - \frac{T}{2}\right\} - \beta'(N)\right].\end{aligned} \tag{4.3}$$

It is obvious that the retailer's annual total profit function$TP_{r1}(p,N,T,u)$ is a continuous function of N, where $N \in [0,M]$. Therefore $TP_{r1}(p,N,T,u)$ has a maximum value for$N \in [0,M]$. To identify whether $N$is zero or positive. For our convenience, we set out the discrimination terms

$$\begin{aligned}\Delta_{N_1} &= \frac{\partial TP_{r1}}{\partial N}\bigg|_{N=0} \\ &= p\alpha(p)\left[\beta'(0) - \gamma\beta(0)\right] - \frac{c\alpha(p)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))} - 1\right]\beta'(0) \\ &- \frac{h_r\alpha(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))} - 1\right\} - T\right]\beta'(0) \\ &+ pI_e\alpha(p)\left[\beta'(0)\left\{M - \frac{T}{2}\right\} - \beta(0)\right].\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\Delta_{N_{M_1}} &= \frac{\partial TP_{r1}}{\partial N}\bigg|_{N=M} \\ &= p\alpha(p)\,e^{-\gamma M}\left[\beta'(M) - \gamma\beta(M)\right] - \frac{c\alpha(p)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))} - 1\right]\beta'(M) \\ &- \frac{h_r\alpha(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))} - 1\right\} - T\right]\beta'(M) \\ &+ pI_e\alpha(p)\left[\beta'(M)\frac{T}{2} - \beta(M)\right].\end{aligned} \tag{4.5}$$

If $\frac{\partial^2 TP_{r1}}{\partial N^2} < 0$, then $TP_{r1}(p,N,T,u)$ is a strictly concave function in N and in consequence there exist a unique maximum solution N_1^{**}. If $\Delta_{N_1} \leq 0$, then $TP_{r1}(p,N,T,u)$ is maximized at $N_1^* = 0$. If $\Delta_{N_1} > 0$ but $\Delta_{N_{M_1}} < 0$, $TP_{r1}(p,N,T,u)$ is maximized with $N_1^* = N_1^{**}$ where $0 < N_1^{**} < M$. If $\Delta_{N_{M_1}} \geq 0$, then $TP_{r1}(p,N,T,u)$ is maximized at $N_1^* = M$. □

Theorem 3. *Given values of $N, T, u > 0$, if $2\alpha'(p) + p\alpha''(p) < 0$, the total profit $TP_{r1}(p,N,T,u)$ per unit time for the retailer is a strictly concave function of p.*

Proof. Forgiven T, N, and u, taking 1^{st} order partial derivative of $TP_{r1}(p,N,T,u)$ with respect to the independent variable p, we can get

$$\begin{aligned}\frac{\partial TP_{r1}}{\partial p} &= (\alpha(p) + p\alpha'(p))\, e^{-\gamma N}\beta(N) - \frac{c}{\theta_b(1-f(u))T}\left[e^{\theta_b}T(1-f(u)) - 1\right]\alpha'(p)\beta(N) \\ &- \frac{h_b}{\theta_b(1-f(u))T}\left[\frac{1}{\theta_b(1-f(u))}\left\{e^{\theta_b}T(1-f(u)) - 1\right\} - T\right]\alpha'(p)\beta(N) \\ &+ I_e\beta(N)\left(M - N - \frac{T}{2}\right)\left[\alpha(p) + p\alpha'(p)\right] = 0.\end{aligned} \tag{4.6}$$

As $\alpha(p)$ is a decreasing function of p, so, $\alpha'(p) < 0$. Therefore, equation (4.6) has a solution only if $\alpha(p) + p\alpha'(p) < 0$, which is the first-order derivative of the revenue $p\alpha(p)$. If $p\alpha(p)$ is a strictly concave function of p, the solution of the equation $\alpha(p) + p\alpha'(p) = 0$, say, p_0, can be considered as the lower bound of the interval in equation (4.6) has a solution.

The 2^{nd} order derivative of $TP_{r1}(p,N,T,u)$ with respect to p for any given values of (N,T,u) is,

$$\begin{aligned}\frac{\partial^2 TP_{r1}}{\partial p^2} &= \left[e^{-\gamma N} + I_e\left(M - N - \frac{T}{2}\right)\right]\beta(N)\left[2\alpha'(p) + p\alpha''(p)\right] \\ &- \frac{c\beta(N)}{\theta_b(1-f(u))T}\left[e^{\theta_b}T(1-f(u)) - 1\right]\alpha''(p) \\ &- \frac{h_b\beta(N)}{\theta_b(1-f(u))T}\left[\frac{1}{\theta_b(1-f(u))}\left\{e^{\theta_b}T(1-f(u)) - 1\right\} - T\right]\alpha''(p) < 0.\end{aligned} \tag{4.7}$$

If the revenue $p\alpha(p)$ is a strictly concave function of p, then $2\alpha'(p) + p\alpha''(p) < 0$. Also, it is assumed that $\alpha(p)$ is a decreasing function of p, so $\alpha'(p) < 0$ and $\alpha''(p) > 0$. Consequently, $\frac{\partial^2 TP_{r1}}{\partial p^2} < 0$, indicating that the profit function $TP_{r1}(p|N,T,u)$ is a strictly concave function of p for any specified values of (N,T,u) and so there exists a unique value of p which maximizes $TP_{r1}(p|N,T,u)$.

Likewise, we have the following results. □

Theorem 4. *Given values of $p, N, T > 0$, the total profit $TP_{r1}(p,N,T,u)$ per unit time for the retailer is a strictly concave function ofu.*

Proof. See **Appendix B.**

Sub-case 1.2. $\mathbf{M \leq T + N}$

The following findings can be obtained using Lemma 4.2.

Theorem 5. *$TP_{r2}(p,N,T,u)$ is a strictly pseudo-concave function in T, for certain values of p, N, and u, and there exists a unique maximum solution T_2^*. Then $TP_{r2}(p,N,T,u)$ is maximized at $\max\{T_2^*, M - N\}$ subject to $N \leq M$.*

□

Proof. Proof is identical to Theorem 1.

For any specified values of p, N and u, calculating the first-order derivative of $TP_{r2}(p,N,T,u)$ in (14) with respect to T, setting the result to zero, and simplifying terms, the necessary condition of T_2^* is

$$\frac{\partial TP_{r2}}{\partial T} = 0, i.e., \tag{4.8}$$
$$\frac{\partial TP_{r2}}{\partial T} = \frac{cD(p,N)}{T^2\{\theta_r(1-f(u))\}}\left[e^{\theta_r T(1-f(u))}-1\right] - \frac{cD(p,N)}{T}e^{\theta_r T(1-f(u))} + \frac{A_r}{T^2} - \alpha gu$$
$$+ \frac{h_r D(p,N)}{\{\theta_r T(1-f(u))\}^2}[e^{\theta_r T(1-f(u))}-1] - \frac{h_r D(p,N)}{\{\theta_r T(1-f(u))\}}e^{\theta_r T(1-f(u))} - \frac{pI_e D(p,N)}{2} = 0.$$

It is obvious from Theorem 5 that (38) has a unique solution T_2^*. If $M-N \le T_2^*$, then $TP_{r2}(p,N,T,u)$ is maximized at T_2^*. Otherwise, $TP_{r2}(p,N,T,u)$ is maximized at $M-N$.

Theorem 6. *For given values of* $p, N, u > 0$, *if* $\beta''(N) - 2\gamma\beta'(N) + \gamma^2\beta(N) < 0$, *then*

□

(1.) $TP_{r2}(p,N,T,u)$ is a strictly concave function in N and hence there exist a unique maximum solution.
(2.) If $\Delta_{N_2} = \frac{\partial TP_{r2}}{\partial N}\big|_{N=0} \le 0$, then $TP_{r2}(p,N,T,u)$ is maximized at $N_2^* = 0$.
(3.) If $\Delta_{N_{M_2}} = \frac{\partial TP_{r2}}{\partial N}\big|_{N=M} < 0$ but $\Delta_{N_2} = \frac{\partial Tpr_2}{\partial N}\Big|_{N=0} > 0$, then there exists a unique solution $N_2^{**} = N_2^*$, where $0 < N_2^* < M$ such that $TP_{r2}(p,N,T,u)$ is maximized.
(4.) If $\Delta_{N_{M_2}} = \frac{\partial TP_{r2}}{\partial N}\big|_{N=M} \ge 0 TP_{r2}(p,N,T,u)$ is maximized at $N_2^* = M$.

Proof. For given values of p, T, and u, taking 1^{st} and 2^{nd} order partial derivative of $TP_{r2}(p,N,T,u)$ with respect to N, we can get

$$\frac{\partial TP_{r2}}{\partial N} = p\alpha(p)e^{-\gamma N}\left[\beta'(N) - \gamma\beta(N)\right] - \frac{c\alpha(p)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))}-1\right]\beta'(N) \tag{4.9}$$
$$- \frac{cI_c}{T}\alpha(p)\left[\beta'(N)(T+N-M)^2 + 2\beta(N)(T+N-M)\right]$$
$$- \frac{h_r\alpha(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))}-1\right\} - T\right]\beta'(N)$$
$$+ \frac{pI_e}{2T}\alpha(p)\left[\beta'(N)(M-N)^2 - 2\beta(N)(M-N)\right],$$

$$\frac{\partial^2 TP_{r2}}{\partial N^2} = p\alpha(p)e^{-\gamma N}\left[\beta''(N) - 2\gamma\beta'(N) + \gamma^2\beta(N)\right] - \frac{c\alpha(p)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))}-1\right]\beta''(N) \tag{4.10}$$
$$- \frac{cI_c}{T}\alpha(p)\left[\beta''(N)(T+N-M)^2 + 4\beta'(N)(T+N-M) + 2\beta(N)\right]$$
$$- \frac{h_r\alpha(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))}-1\right\} - T\right]\beta''(N)$$
$$+ \frac{pI_e}{2T}\alpha(p)\left[\beta''(N)(M-N)^2 - 4\beta'(N)(M-N) + 2\beta(N)\right].$$

It is obvious that the annual total profit function $TP_{r2}(p,N,T,u)$ of retailer is a continuous function of N, where $N \in [0,M]$. Therefore $TP_{r2}(p,N,T,u)$ has a maximum value for $N \in [0,M]$. For our convenience, we construct the discrimination terms to determine whether Nzero or positive is.

$$\begin{aligned}\Delta_{N_2} &= \left.\frac{\partial TP_{r2}}{\partial N}\right|_{N=0} \qquad (4.11)\\ &= p\alpha(p)\left[\beta^{'}(0)-\gamma\beta(0)\right]-\frac{c\alpha(p)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))}-1\right]\beta^{'}(0)\\ &-\frac{cI_c}{T}\alpha(p)\left[\beta^{'}(0)(T-M)^2+2\beta(N)(T-M)\right]\\ &-\frac{h_r\alpha(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right]\beta^{'}(0)\\ &+\frac{pI_e}{2T}\alpha(p)\left[\beta^{'}(0)(M)^2-2\beta(0)(M)\right]\end{aligned}$$

and

$$\begin{aligned}\Delta_{N_{M_2}} &= \left.\frac{\partial TP_{r2}}{\partial N}\right|_{N=M} \qquad (4.12)\\ &= p\alpha(p)e^{-\gamma M}\left[\beta^{''}(M)-2\gamma\beta^{'}(M)+\gamma^2\beta(M)\right]-\frac{c\alpha(p)}{\theta_r(1-f(u))T}\left[e^{\theta_r T(1-f(u))}-1\right]\beta^{''}(M)\\ &-\frac{cI_c}{T}\alpha(p)\left[\beta^{''}(M)(T)^2+4\beta^{'}(M)(T)+2\beta(M)\right]\\ &-\frac{h_r\alpha(p)}{\theta_r T(1-f(u))}\left[\frac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right]\beta^{''}(M)+\frac{pI_e}{2T}\alpha(p)2\beta(M).\end{aligned}$$

If $\frac{\partial^2 TP_{r2}}{\partial N^2}<0$, then $TP_{r2}(p,N,T,u)$ is a strictly concave function in N and hence there exist a unique maximum solution N_2^{**}. If $\Delta_{N_2}\le 0$, then $TP_{r2}(p,N,T,u)$ is maximized at $N_2^*=0$. If $\Delta_{N_2}>0$ but $\Delta_{N_{M_2}}<0$, $TP_{r2}(p,N,T,u)$ is maximized with $N_2^*=N_2^{**}$, where $0<N_2^{**}<M$. If $\Delta_{N_{M_2}}\ge 0$, then $TP_{r2}(p,N,T,u)$ is maximized at $N_2^*=M$.

Next, similar to Theorems 3 and 4, we have the following results:

Theorem 7. *The total profit $TP_{r2}(p,N,T,u)$ per unit time for the retailer is a strictly concave function of p for any specified values of $N,T,u>0$ provided that $2\alpha^{'}(p)+p\alpha^{''}(p)<0$.*

Theorem 8. *The total profit $TP_{r2}(p,N,T,u)$ per unit time for the retailer is a strictly concave function of u for any specified values of $N,T,p>0$.*

Case 3. N $\ge$ M

The following findings can be obtained utilizing Lemma 4.2.

Theorem 9. *$TP_{r3}(p,N,T,u)$ is a strictly pseudo-concave function in T, for any certain values of p, N, and u, and there exists a unique maximum solution T_3^* for $T\in(0,\infty)$. The optimal value of T corresponds to T_3^*.*

□

Proof. Proof is similar to Theorem 1.

For any specified p,N and u, calculating the first-order derivative of $TP_{r3}(p,N,T,u)$ in (17) with respect to T, equating the result as zero, and simplify the associated terms, the necessary condition of T_3^* is

$$\frac{\partial TP_{r3}}{\partial T} = 0, i.e., \tag{4.13}$$
$$\frac{cD(p,N)}{T^2\{\theta_r(1-f(u))\}}\left[e^{\theta_r T(1-f(u))}-1\right] - \frac{cD\left(p,N\right)}{T}e^{\theta_r T(1-f(u))} + \frac{A_r}{T^2} - \alpha g u$$
$$+ \frac{cI_c D\left(p,N\right)}{T^2}\left[2\left(N-M\right)+T\right] - \frac{cI_c D\left(p,N\right)}{T} + \frac{h_r D\left(p,N\right)}{\left\{\theta_r T\left(1-f\left(u\right)\right)\right\}^2}\left[e^{\theta_r T(1-f(u))}-1\right]$$
$$- \frac{h_r D\left(p,N\right)}{\left\{\theta_r T\left(1-f\left(u\right)\right)\right\}}e^{\theta_r T(1-f(u))} = 0.$$

From Theorem 7, it is clear that (43) has a unique solution T_3^* for $T \in (0,\infty)$ and the optimal value of T corresponds to T_3^*.

Likewise, for given values of p, T, and u, taking 1^{st} and 2^{nd} order partial derivatives of $TP_{r3}\left(p,N,T,u\right)$ with respect to the independent variable N, we can get

$$\frac{\partial TP_{r3}}{\partial N} = p\alpha\left(p\right)e^{-\gamma N}\left[\beta^{'}\left(N\right)-\gamma\beta\left(N\right)\right] - \frac{c\alpha\left(p\right)}{\theta_r\left(1-f\left(u\right)\right)T}\left[e^{\theta_r T(1-f(u))}-1\right]\beta^{'}\left(N\right) \tag{4.14}$$
$$- \frac{cI_c}{T}\alpha\left(p\right)\left[\beta^{'}\left(N\right)\left\{2\left(M-N\right)+T\right\}+2\beta\left(N\right)\right]$$
$$- \frac{h_r\alpha\left(p\right)}{\theta_r T\left(1-f\left(u\right)\right)}\left[\frac{1}{\theta_r\left(1-f\left(u\right)\right)}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right]\beta^{'}\left(N\right),$$

$$\frac{\partial^2 TP_{r3}}{\partial N^2} = p\alpha\left(p\right)e^{-\gamma N}\left[\gamma^2\beta\left(N\right)-2\gamma\beta^{'}\left(N\right)+\beta^{''}\left(N\right)\right]$$
$$- \frac{c\alpha\left(p\right)}{\theta_r\left(1-f\left(u\right)\right)T}\left[e^{\theta_r T(1-f(u))}-1\right]\beta^{''}\left(N\right) - \frac{cI_c}{T}\alpha\left(p\right)\left[\beta^{''}\left(N\right)\left\{2(M-N)+T\right\}+4\beta^{'}\left(N\right)\right\}\right]$$
$$- \frac{h_r\alpha\left(p\right)}{\theta_r T\left(1-f\left(u\right)\right)}\left[\frac{1}{\theta_r\left(1-f\left(u\right)\right)}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right]\beta^{''}\left(N\right). \tag{4.15}$$

Now to recognize whether $N_3^* = M$ or $N_3^* > M$, we defined here the discrimination term

$$\Delta_{N_3} = \left.\frac{\partial TP_{r3}}{\partial N}\right|_{N=M}$$
$$= p\alpha\left(p\right)e^{-\gamma M}\left[\beta^{'}\left(M\right)-\gamma\beta\left(M\right)\right] - \frac{c\alpha\left(p\right)}{\theta_r\left(1-f\left(u\right)\right)T}\left[e^{\theta_r T(1-f(u))}-1\right]\beta^{'}\left(M\right)$$
$$- \frac{cI_c}{T}\alpha\left(p\right)\left[\beta^{'}\left(M\right)\left\{2\left(M-M\right)+T\right\}+2\beta\left(M\right)\right]$$
$$- \frac{h_r\alpha\left(p\right)}{\theta_r T\left(1-f\left(u\right)\right)}\left[\frac{1}{\theta_r\left(1-f\left(u\right)\right)}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right]\beta^{'}\left(M\right)$$
$$= p\alpha\left(p\right)e^{-\gamma M}\left[\beta^{'}\left(M\right)-\gamma\beta\left(M\right)\right] - \frac{c\alpha\left(p\right)}{\theta_r\left(1-f\left(u\right)\right)T}\left[e^{\theta_r T(1-f(u))}-1\right]\beta^{'}\left(M\right)$$
$$- \frac{h_r\alpha\left(p\right)}{\theta_r T\left(1-f\left(u\right)\right)}\left[\frac{1}{\theta_r\left(1-f\left(u\right)\right)}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right]\beta^{'}\left(M\right). \tag{4.16}$$

If $\gamma^2\beta\left(N\right) - 2\gamma\beta^{'}\left(N\right) + \beta^{''}\left(N\right) < 0$, then $\left.\frac{\partial^2 Tpr_3}{\partial N^2}\right|_{N=M} < 0$. Hence $TP_{r3}\left(p,N,T,u\right)$ is a strictly concave function of N. Hence retailer's profit exists given the unique maximum solution at N_3^*. Else the optimal solution $TP_{r3}\left(p,N,T,u\right)$ is at one of the boundary points M or ∞. Now substituting $N = \infty$. But, $\lim_{N\to\infty}\frac{\partial TP_{r3}}{\partial N} \leq 0$. Hence $N = \infty$ is not an optimal solution. The optimal solution is $N_3^* = M$ with the help of the above phenomenon, it is possible to implement the following theoretical conclusions.

Theorem 10. *For any given values of* $p, N, u > 0$, *if* $\gamma^2\beta(N) - 2\gamma\beta'(N) + \beta''(N) < 0$, *then*

(i) $TP_{r3}(p, N, T, u)$ is strictly concave function in N and hence there exist a unique maximum solution.
(i) If $\Delta_{N_3} = \frac{\partial TP_{r3}}{\partial N}\big|_{N=M} < 0$, then $TP_{r3}(p, N, T, u)$ is maximized at $N_3^* = M$.
(iii) If $\Delta_{N_3} = \frac{\partial TP_{r3}}{\partial N}\big|_{N=M} > 0$, then there exists unique $N_3^* > M$ such that $TP_{r3}(p, N, T, u)$ is maximized at $N_3^{**} = N_3^* > M$.

Similar to Theorems 3 and 4, we have the following results:

Theorem 11. *The total profit* $TP_{r3}(p, N, T, u)$ *per unit time for the retailer is a strictly concave function of* p *for any specified values of* $N, T, u > 0$ *provided that* $2\alpha'(p) + p\alpha''(p) < 0$.

Theorem 12. *The total profit* $TP_{r3}(p, N, T, u)$ *per unit time is a strictly concave function ofu for any given values of* $p, N, T > 0$.

□

4.2. Optimality for an annual total profit of the manufacturer

On account of the intricacy of the problem, we are unable to prove the manufacturer's annual total profit $TP_m(u, T)$ is jointly concave in u and T. However, we can prove that $TP_m(u, T)$ is strictly pseudo-concave in each of the decision variables.

Theorem 13. *For a fixed value of* $u > 0$, *the manufacturer annual total profit function* $TP_m(u, T)$ *is concave in* T.

Proof. See **Appendix C**

Likewise, we can get the following findings.

Theorem 14. *For any given values of* $T > 0$, *the manufacturer annual total profit* $TP_m(u, T)$ *is a strictly concave function of* u.

□

Proof. See **Appendix D**

The optimal solution can be obtained explicitly by solving equations (4.2), (4.3), (4.4) and (4.5).

Algorithm for solution of the decentralized model

Step 1. Input in the parameters' values.
Step 2. Compute optimal values of p, N, T, u solving the simultaneous equations $\frac{\partial TP_{ri}}{\partial p} = 0$, $\frac{\partial TP_{ri}}{\partial N} = 0$, $\frac{\partial TP_{ri}}{\partial T} = 0$, and $\frac{\partial TP_{ri}}{\partial u} = 0$ $(i = 1, 2, 3)$.
Step 3. Calculate corresponding values of $TP_{ri}(p, N, T, u)$ $(i = 1, 2, 3)$ from equations 3.11, 3.14 and 3.17 respectively and find optimal $TP_r^* = \max\{TP_{ri}(p, N, T, u) : i = 1, 2, 3\}$.
Step 4. Compute optimal values of T, u solving the simultaneous equations $\frac{\partial TP_m}{\partial T} = 0$, and $\frac{\partial TP_m}{\partial T} = 0$.
Step 5. Calculate corresponding value of $TP_m^*(u, T)$ from equations 3.29.
Step 6. Obtain the joint profit $TP^* = TP_r^* + TP_m^*$.
Step 7. End.

□

4.3. Optimality for a joint total profit

It is considered that the manufacturer and the retailer both are integrated together as one unit and both participants invested in preservation technology and the manufacturer provides a subsidy $(1-\alpha)$ for a part of the retailer's preservation technology investment. This integrated system has a definite objective and which is, to find the optimal values of retail price (p), preservation technology investment (u), credit period (N), and replenishment cycle time (T) to maximize the profit.

The condition for determiningthe optimal values of p, u, N, and $T$to maximize the total profit function $TP\left(p,T,N,u\right)$ is $\frac{\partial TP}{\partial p}=0$, $\frac{\partial TP}{\partial T}=0$, $\frac{\partial TP}{\partial N}=0$, and $\frac{\partial TP}{\partial u}=0$ provided the determinant of the principal minors of the hessian matrix of $TP\left(p,T,N,u\right)$ is negative definite, *i.e.*, $H_1<0$, $H_2>0$, $H_3<0$, and $H_4>0$, where H_1, H_2, H_3, and H_4 are the principal minors of the hessian matrix of $TP\left(p,T,N,u\right)$.

The Hessian matrix of the profit function $TP\left(p,T,N,u\right)$ is defined as

$$H=\begin{bmatrix} \frac{\partial^2 TP}{\partial p^2} & \frac{\partial^2 TP}{\partial p\partial T} & \frac{\partial^2 TP}{\partial p\partial N} & \frac{\partial^2 TP}{\partial p\partial u} \\ \frac{\partial^2 TP}{\partial T\partial p} & \frac{\partial^2 TP}{\partial T^2} & \frac{\partial^2 TP}{\partial T\partial N} & \frac{\partial^2 TP}{\partial T\partial u} \\ \frac{\partial^2 TP}{\partial N\partial p} & \frac{\partial^2 TP}{\partial N\partial T} & \frac{\partial^2 TP}{\partial N^2} & \frac{\partial^2 TP}{\partial N\partial u} \\ \frac{\partial^2 TP}{\partial u\partial p} & \frac{\partial^2 TP}{\partial u\partial T} & \frac{\partial^2 TP}{\partial u\partial N} & \frac{\partial^2 TP}{\partial u^2} \end{bmatrix}. \tag{4.17}$$

To do so we provide the following algorithm to determine the optimal values of (p, u, N, T).

Algorithm for solution of the model

Step 1. Input in the parameters' values.
Step 2. Solve the simultaneous equations $\frac{\partial TP}{\partial p}=0$, $\frac{\partial TP}{\partial T}=0$, $\frac{\partial TP}{\partial N}=0$, and $\frac{\partial TP}{\partial u}=0$.
Step 3. Check the optimality conditions. If $H_1<0$, $H_2>0$, $H_3<0$, and $H_4>0$, then the set of solution is optimal.
Step 4. End.

The condition of optimality is verified numerically in the next section using the MATLAB software according to the previously specified algorithm, because the nature of the profit function is highly non-linear in nature.

5. Numerical analysis

5.1. Case study

The proposed model develops a two-layer supply chain model under dynamic demand with a manufacturer and a retailer maintaining decaying items with controllable deterioration rates under two levels of trade credit policies. Consider a production plant where produces some units of products (*e.g.*, juices, frozen foods, baked foods, etc.) where special handling is needed to prevent damage and decay. After producing the items, manufacturer instantly delivers the goods to the retailer on a lot-for-lot basis throughout the whole replenishment cycle period. Due to the product deterioration the level of inventory at manufacturer side will decrease significantly and to minimise such degradation manufacture invests some portion of preservation technology investment and the rest of the portion would be paid by retailer. To hold the produced items manufacture pays holding cost and set up cost per a lot to the manufacturer. Apart from this to stimulate sales and attract to retailer, manufacture offers trade credit duration to the retailer and as high credit duration generates a high possibility of default risk. Meanwhile after receiving the Initial order quantity the retailer fulfils the demand which is generated from consumer side. Due to the consumer demand and items deterioration the inventory level at the retailer side decreases drastically.To hold and ordered the items retailer will have paid holding cost and ordering cost. Apart from this, the retailer also provides some credit duration for his consumers for enhancement of the demand and makes healthy relationship to the consumer. As longer credit duration generates high possibility of credit risk the rate of default risk at the retailer side.

To represent the results, consider the following numerical example of an inventory system.

TABLE 2. Optimal Results under in centralized model.

Model	p	N	T	u	TP_r	TP_m	TP
Case 1	55.67	1.3993	0.1522	46.22	17 039.71	22 889.68	39 929.39
Case 2	**52.36**	**1.2057**	**0.3512**	**44.83**	**25 768.23**	**15 157.54**	**40 925.76**
Case 3	55.37	1.5604	0.5243	47.68	23 548.74	16 109.75	39 658.49

Notes. The bold line in Table 2 indicates the best result (Out of 3 cases) under the centralized model and which is connected with for construction of sensitivity analysis tables *i.e.* Tables 4 and 5.

TABLE 3. Optimal Results under in decentralized model.

Model	p	N	T	u	TP_r	TP_m	TP
Retailer Case 1	55.20	0.1943	0.1056	32.04	17 474.87	–	36487.09
Retailer Case 2	**55.23**	**0.1152**	**0.2848**	**37.96**	**20 161.77**	–	**39 173.99**
Retailer Case 3	56.41	0.3001	0.4557	41.43	19 627.42	–	38 639.64
Manufacturer Model	–	–	0.2571	38.37	–	19 012.22	–

Notes. The bold line in Table 3 indicates the best result for the decentralized model.

5.2. Numerical example

The following inventory situation has been setfor the benchmark case:

$A_m = 300$, $A_r = 150$, $\theta_m = 0@dot@08$, $\theta_r = 0.07$, $c = 10$, $h_m = 3$, $h_r = 3$, $I_e = 0.1, I_c = 0.12$, $i_m = 0.1$, $P = 6500$, $u = 0.3$, $\alpha = 0.08$, $\chi = 0.25$, $\gamma = 0.3$, $\alpha(p) = a - bp$, where $a = 1000$, $b = 10$, $M = 1.55$, $\beta(N) = e$, where $\lambda = 0.3$, $f(u) = 1 - e^{-\tau u}$, where $\tau = 0.05$. Based on the previous research the above-mentioned parameters are chosen on preservation technology investment for degrading items (*e.g.*, [35,59]), which enables for a detailed illustration. By considering the given data, results are obtained by utilizingMATLAB software.

From Tables 2 and 3, the optimal strategies for the centralized and decentralized model are as follows:

Centralized Case: $p^* = 52.36$, $N^* = 1.2057$, $T^* = 0.3512$, $u^* = 44.83$, and the corresponding annual total profits $TP_r^* = 25768.23$, $TP_m^* = 15157.54$, and $TP^* = 40925.76$.

Decentralized Case:$p^* = 55.23$, $N^* = 0.1152$, $T^* = 0.2848$, $u^* = 37.96$and the corresponding annual total profits $TP_r^* = 20161@dot@77$, $TP_m^* = 19012@dot@22$, and $TP^* = 39173.99$.

The above results describe that the average profit in the decentralized model is less than that of centralized model. For the centralized decision making scenario, the manufacturer and the retailer act as a single business manager and jointly make their optimal decisions in order to achieve highest whole system profit. So, the retailer can provide the product to the customers at a more frugal price than that of the decentralized case. That's why lower priced product increases theconsumer demand significantly. Then the retailers order more quantity from the manufacturer.As a result, integration between the manufacturer and the retailer increases the total systemprofit significantly. We optically canvass that the retail prices of the product decrease for the retailer. This results in an increase of profit for the entire supply chain and the wholesystem profit increased by $\$40925.76 - \$39173.99 = \$1751.77$.

6. SENSITIVITY ANALYSIS AND MANAGERIAL IMPLICATION

In this section, a sensitivity analysis has been performed to assess the robustness of the model presented above.We explored the sensitivity analysis of the optimal solutions by taking the parameters with suitable units,using the identical data as in Example 5.1. Tables 4 and 5 show the computational results.

Table 4. Sensitivity study of the parameters associated with Manufacturer Model.

Parameter	% change	p^*	N^*	T^*	u^*	TP_R^*	TP_M^*	TP^*
	−30%	52.36	1.2056	0.3512	44.83	27 450.45	13 475.32	40 925.76
	−15%	52.36	1.2056	0.3512	44.83	25 622.88	15 302.88	40 925.76
P	0%	52.36	1.2056	0.3512	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.2056	0.3512	44.83	35 830.39	50 95.371	40 925.76
	+30%	52.36	1.2056	0.3512	44.83	27 265.64	13 660.13	40 925.76
	−30%	52.01	1.3601	0.3551	43.14	29 445.01	15 692.66	45 137.67
	−15%	52.18	1.2821	0.3532	44.09	27 675.10	15 287.15	42 962.25
c	0%	52.36	1.2056	0.3512	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.53	1.1291	0.3501	45.39	25 475.52	13 541.87	39 017.38
	+30%	52.69	1.0541	0.3501	45.83	24 742.33	12 484.96	37 227.29
	−30%	52.36	1.2056	0.3512	44.83	25 768.35	15 157.42	40 925.76
	−15%	52.36	1.2056	0.3512	44.83	25 768.35	15 157.42	40 925.76
h_m	0%	52.36	1.2056	0.3512	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.2056	0.3512	44.83	25 768.35	15 157.42	40 925.76
	+30%	52.36	1.2056	0.3512	44.83	25 768.35	15 157.42	40 925.76
	−30%	52.35	1.2101	0.3131	42.95	25 582.66	15 313.92	41 196.58
	−15%	52.36	1.2081	0.3321	43.94	25 314.06	15 743.26	41 057.32
A_m	0%	52.36	1.2056	0.3512	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.37	1.2031	0.3681	45.63	22 614.53	18 186.26	40 800.79
	+30%	52.37	1.2001	0.3852	46.35	25 658.45	15 023.07	40 681.52
	−30%	55.19	1.701	0.297	66.73	31 529.34	15 157.36	46 686.71
	−15%	53.72	1.430	0.325	57.43	28 165.26	15 157.36	43 322.62
i_m	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	51.04	1.012	0.374	24.36	24008.03	15157.36	39165.39
	+30%	49.66	0.836	0.399	19.79	22 699.41	15 157.54	37 856.95
	−30%	44.82	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−15%	44.82	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
α	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	44.82	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+30%	44.82	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−30%	52.37	1.206	0.351	53.28	25 772.08	15 158.08	40 930.16
	−15%	52.37	1.206	0.351	48.66	25 770.05	15 157.82	40 927.87
g	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.35	1.205	0.351	41.55	25 766.55	15 157.28	40 923.82
	+30%	52.35	1.204	0.351	38.71	25 765.03	15 156.99	40 922.02
	−30%	52.35	1.204	0.351	52.19	27 504.00	13 416.57	40 920.57
	−15%	52.35	1.205	0.354	48.26	27 518.71	13 404.79	40 923.49
γ	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.206	0.351	41.84	26 314.07	14 613.51	40 927.58
	+30%	52.37	1.206	0.351	39.25	26 323.77	14 605.30	40 929.07
	−30%	58.07	1.622	0.317	77.18	28 395.79	10 695.13	39 090.92
	−15%	55.22	1.392	0.336	65.80	27 836.46	11 879.51	39 175.96
w	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	49.31	1.035	0.368	35.56	25 676.49	16 899.23	42 575.72
	+30%	46.44	0.906	0.388	29.69	26 589.46	17 986.76	44 576.23
	−30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−15%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
s	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−30%	52.36	1.205	0.351	44.82	25 768.40	15 157.36	40 625.76
	−15%	52.364	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
τ	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.205	0.351	44.82	25 768.40	15 157.36	40 925.76
	+30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
μ	−30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−15%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76

TABLE 4. Continued.

Parameter	% change	p^*	N^*	T^*	u^*	TP_R^*	TP_M^*	TP^*
	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
β	−30%	53.28	1.472	0.318	54.38	30 000.79	15 157.54	45 158.32
	−15%	52.83	1.331	0.335	50.07	27 640.74	15 157.54	42 798.27
	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	51.84	1.089	0.365	37.94	24 250.12	15 157.54	39 407.66
	+30%	51.24	0.978	0.380	27.88	23 001.08	15 157.54	38 158.62
θ_m	−30%	52.36	1.205	0.351	43.76	25 766.97	15 158.96	40 925.93
	−15%	52.36	1.205	0.351	44.26	25 767.55	15 158.30	40 925.85
	0%	52.36	1.205	0.351	44.83	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.205	0.351	45.43	25 769.02	15 156.65	40 925.67
	+30%	52.36	1.205	0.351	46.08	25 770.03	15155.53	40 925.56

According to the sensitivity analysis,

- We have shown that from Table 4 when manufacturer's production cost c increases with −30%, −15%, +15%, +30% then joint profit in the supply chain system TP^*, replenishment cycle time T^* and downstream trade credit N^* decreases. Therefore, the manufacturer's objective is to diminish the production cost anyhow by extending the trade credit duration for retailers and reducing order frequency to increase profit. Apart from this, an increase in production cost leads to the optimal selling price p^* of the product, preservation cost u^* increases. That means to purchase some product, the customer will have to pay more money and another side due to the increasing preservation cost buyer will not be interested to invest money for preserving the items which lead to a higher deterioration rate and higher deterioration cost.
- Sensitivity analysis of A_m indicates that If the ordering cost on the manufacturer side increases, the replenishment cycle time T^*, selling price p^* and investment cost for preservation technology u^* increased whereas joint profit in the supply chain along with downstream decreased. Thus the aim of the manufacturer will be to reduce the ordering cost anyhow to get more profit and to strengthen preservation technology for the products to reduce deterioration of items. Apart from this larger ordering cost,the retailer should order for a larger replenishment period to diminish the frequency of orders therefore retailer will have to pay less ordering cost.
- By Increasing the scaling factor $a$treating as the values of the other parameters remain constant. It is possible to deduce that optimal total profit TP^*, preservation technology investment u^*, selling price p^*, and downstream trade credit period rises, on another hand optimal replenishment cycle time T^* diminishes. This indicates that when the value of the parameter a rises, the market demand rate also rises. Furthermore, in order to maximize the profit, the organization will raise the selling price. The business organization would also invest more cash in the improvement of preservation technology in order to bring down the rate of product degradation, allowing the organization to sell more products.
- Sensitivity analysis of the price elasticity factor b rises, the optimal selling price p^*, Joint profit in supply chain TP^*, optimal Preservation technology investment u^*, and optimal downstream trade credit period decreases. This implies when price elasticity factor b rises, the organization will decrease the selling price to avoid a significant drop in demand. Furthermore, as the selling price is reduced the overall profit decreases dramatically.
- Sensitivity analysis of the parameter A_r indicates that, when the ordering cost at the retailer side increases, the joint profit in the supply chain TP^*, and downstream trade credit period decreases gradually. Apart from this increase of A_r leads to an increase of other decision variables, *i.e.*, u^*@comma@T^*, p^*. It suggests

TABLE 5. Sensitivity study of the parameters associated to Retailer Model.

Parameter	% change	p^*	N^*	T^*	u^*	TP_R^*	TP_M^*	TP^*
	−30%	31.06	0.611	0.517	32.11	12 642.99	7252.823	19 902.81
	−15%	41.64	0.917	0.418	38.21	19 174.62	9917.010	29 091.63
a	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	62.88	1.448	0.304	68.86	37 529.39	18 255.70	55 785.08
	+30%	73.37	1.668	0.267	82.22	50 578.62	23 446.19	74 024.81
	−30%	82.37	1.835	0.289	85.87	46 923.79	17 411.91	64 335.71
	−15%	64.74	1.485	0.323	69.48	36 191.21	13 523.21	49 714.66
b	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	43.03	0.960	0.379	32.09	22 363.75	12 850.00	35 217.76
	+30%	35.97	0.769	0.407	23.56	31 278.72	24.510	31 303.23
	−30%	52.37	1.205	0.351	37.74	25 554.05	15 374.08	40 928.14
	−15%	52.36	1.205	0.351	41.48	27 804.48	13 122.39	40 926.87
θ_r	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.205	0.351	47.83	35 941.56	4923.230	40 924.79
	+30%	52.36	1.205	0.351	50.55	27 172.48	13 751.45	40 923.93
	−30%	52.35	1.209	0.381	46.53	25 948.53	15 183.56	41 132.09
	−15%	52.36	1.207	0.365	45.64	25 614.69	15 412.03	41 026.72
h_r	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.203	0.338	44.07	26 341.50	14487.23	40 828.73
	+30%	52.37	1.201	0.327	43.37	27 465.30	13 269.30	40 735.20
	−30%	52.35	1.208	0.332	43.94	25 314.06	15 743.26	41 057.32
	−15%	52.36	1.206	0.342	44.39	25 404.24	15 586.41	40 990.65
A_r	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.204	0.360	45.23	26 679.64	14 182.87	40 862.52
	+30%	52.36	1.203	0.368	45.62	22 614.53	18 186.26	40 800.79
	−30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−15%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
α	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−15%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
I_e	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+30%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	−30%	52.35	1.209	0.379	47.26	25 998.08	15 119.41	41 117.49
	−15%	52.36	1.207	0.364	45.99	25 641.64	15 378.09	41 019.72
I_c	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	52.36	1.203	0.339	43.74	26 309.52	14 525.69	40 835.21
	+30%	52.37	1.202	0.328	42.74	26 464.17	14 283.55	40 747.72
	−30%	40.87	0.028	0.577	40.32	19 297.68	15 157.54	34 455.22
	−15%	47.18	0.734	0.435	36.45	21 088.56	15 157.54	36 246.10
λ	0%	52.36	1.205	0.351	44.82	25 768.23	15 157.54	40 925.76
	+15%	56.43	1.528	0.294	73.89	33 366.83	15 157.54	48 524.36
	+30%	59.84	1.771	0.246	87.80	44 397.79	15 157.36	59 555.15

that the selling price of the product will be hiked. Consequently, demand from the customer will be shrunk and as a result, total profit will shrink significantly.

- When finance rate i_m increases then replenishment period T^* increases. That means the business will be running longer durationand the rest of the decision variables p^*, N^*, u^* along with total profit decreases, which indicates that the high finance rate leads to the retailer as well as manufacturer trade credit period offer decreases.
- Sensitivity analysis of the parameter θ_m reveals that increasing original deterioration rate θ_m at the manufacturer, side leads to the increaseof selling price p^* and preservation investment technology cost u^*. This means the manufacturer will have to pay more money to preserve the products to avoid deterioration. Meanwhile total profit TP^* decreased. Whereas replenishment cycle time T^* and downstream trade-credit remain unchanged.
- In Table 5 we observed that if the parameter θ_r increases then selling price p^* along with total profit TP^* decreased. Therefore the aim of the retailer will be to decrease the original deterioration rate θ_r anyhow, such that total profit TP^* increases. Apart from this increase of θ_r leads to optimal preservation technology investment cost u^*increase and downstream trade credit period, optimal replenishment cycle time T^* remain unchanged.
- Sensitivity analysis of γ reveals that optimal selling price p^*, joint profit in supply chain TP^*, optimal downstream trade credit period increase, however optimal preservation technology investment cost u^* decreases. This means that if the retailer's investment efficiency improves, he or she can spend less on preservation in order to attain a lower deterioration rate, lowering the deterioration and preservation costs. As a result of the lower degradation cost, the retailer may charge a higher price, have a longer purchasing cycle, and keep more inventories on hand.
- If the value of the parameter h_r, I_c increases, optimal selling price p^* of the product increases whereas total profit TP^* and the rest of the decision variable N^*, u^*, T^* decreased.
- If the parameter h_m, I_e, P increases then all decision variables N^*, u^*, T^*, p^* along with total profit TP^* remain unchanged. It demonstrates that the variables insensitivity to changes on the parameter h_m, I_e, P.
- If the parameter w^* increases then all decision variables N^*, u^*, T^*, p^* along with total profit TP^*decreases.
- If the parameter h_m, I_e, P increases then all decision variables N^*, u^*, T^*, p^* along with joint profit in the supply chain TP^* remain unchanged. It demonstrates that the variables insensitivity to changes on the parameters h_m, I_e, P.
- If the parameter β increases, the replenishment period T^* increases and joint profit in supply chain and other decision variables p^*, N^*, u^* decreases.

6.1. Managerial implication

This study will provide some managerial implications to industry executives in order to improve and develop their businesses. The managerial conclusions can be formed based on the results of Tables 4 and 5:

- This model can help the business managers to meet some real-life settings, *e.g.*, the implementation of different aspects concurrently to establish a robust inventory management platform, such as items deterioration, preservation technology investment, trade credit policies and to manage default risk that may occurs due to long credit duration, which are widespread in today's business environment.
- This model will assist managers to deal appropriately the consumer demand. According to the suggested model increased demand may be accomplished by product price and duration trade credit which has been offered. Table 5 represent that the profit of the both supply chain participants is highly sensitive with respect the demand functions parameters.So, one of the main focuses for business managers to give most priority to meet the consumer demand.
- This model can cooperate decision mangers to know when product deterioration is considered in both supply chain participants then how it can affect the overall supply chain profits and decision variables such as price

of the commodities, credit duration, investment in preservation technology investment and total duration of business cycle.
- This study depicts the optimal result for both centralised and decentralised model. That can help the decision makers how proper integration of supply chain members can boost the overall profit the system.

7. Conclusions

This article addressed the two-layer supply chain model (SCM) with a manufacturer and a retailer preserving the decaying items under dynamic demand over an infinite time horizon. Both supply chain participants provide trade credit periods to downstream members to invigoratesales and optimize sales volume. For the period of trade credit granted to the retailer, the manufacturer is responsible for opportunity costs, and to slow down the rate of deterioration, both members invest some proportions in preservation technology. In this study, we have done our work through a centralized and decentralized structure and a solution procedure has been devised to ascertain optimal values of the retailprice, trade credit, preservation technology investment, and replenishment cycle time with maximization of total profit of the manufacturer and retailer. Numerical illustrations have been provided to demonstrate the model and to validate the optimality and stability of the solution. This study represents that under a centralized structure both members of the supply chain gain more profits compared to the decentralized structure.

Eventually, more realistic assumptions, such as demand which is probabilistic in nature, permitted shortages, quantity discounts, as well as time-dependent deterioration can be incorporated into the future study.

Appendix A.

Let

$$TP_{r1}\left(p,N,T,u\right)=\frac{f_1(T)}{g_1(T)}, \tag{A.1}$$

where

$$\begin{aligned} f_1\left(T\right) = & Tpe^{-\gamma N}D\left(p,N\right)-\tfrac{cD(p,N)}{\theta_r(1-f(u))}\left[e^{\theta_r T(1-f(u))}-1\right]-A_r-\alpha guT \\ & -\tfrac{h_rD(p,N)}{\theta_r(1-f(u))}\left[\tfrac{1}{\theta_r(1-f(u))}\left\{e^{\theta_r T(1-f(u))}-1\right\}-T\right]+pI_eD(p,N)T\left(M-N-\tfrac{T}{2}\right). \end{aligned} \tag{A.2}$$

Differentiating $f_1\left(T\right)$ twice with respect to T, we have

$$\begin{aligned} \frac{df_1(T)}{dT} = & pe^{-\gamma N}D(p,N) \\ & -cD(p,N)e^{\theta_r T(1-f(u))}-\alpha gu-\frac{h_rD\left(p,n\right)}{\theta_r\left[1-f\left(u\right)\right]}[e^{\theta_r T(1-f(u))}-1] \\ & +pI_eD\left(p,N\right)\left(M-N-T\right). \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} \frac{d^2f_1(T)}{dT^2} = & -cD\left(p,N\right)e^{\theta_r T(1-f(u))}\theta_r\left(1-f\left(u\right)\right) \\ & -\frac{h_rD\left(p,N\right)}{\theta_r\left[1-f\left(u\right)\right]}e^{\theta_r T(1-f(u))}\theta_r\left(1-f\left(u\right)\right)-pI_eD\left(p,N\right)<0. \end{aligned} \tag{A.4}$$

and $g_1\left(T\right)=T>0$. Therefore, by applying [3] theorem $TP_{r1}\left(p,N,T,u\right)$ is a strictly pseudo-concave function in T. This completes the proof of Theorem 1.

Appendix B.

The exponential term containing in equation 3.11 can be approximated by using the Taylor's series expansion, *i.e.*,

$$e^{\theta_b T(1-f(u))} = 1 + \theta_b T\left(1 - f\left(u\right)\right) + \frac{\left(\theta_b T\left(1 - f\left(u\right)\right)\right)^2}{2} + \mathrm{O}(\theta_b T(1 - f\left(u\right)). \tag{B.1}$$

By ignoring the fourth and higher order terms of the Taylor series expansion of $e^{\theta_b T(1-f(u))}$, equation (11) can be obtained as

$$\begin{aligned} TP_{r1}\left(p, N, T, u\right) = pe^{-\gamma N} D\left(p, N\right) - cD(p, N)\left[1 + \tfrac{\theta_b T(1-f(u))}{2}\right] - \tfrac{A_b}{T} - \alpha guT \\ -h_b D(p, N)\tfrac{T}{2} + pI_e D(p, N)\left(M - N - \tfrac{T}{2}\right). \end{aligned} \tag{B.2}$$

The 1^{st} and 2^{nd} order derivatives of $TP_{r1}\left(p, N, T, u\right)$ with respect to u can be expressed as

$$\frac{\partial TP_{r1}}{\partial u} = \frac{cD(p, N)}{2}\theta_r T f^{'}(u), \tag{B.3}$$

and

$$\frac{\partial^2 TP_{r1}}{\partial u^2} = \frac{cD\left(p, N\right)}{2}\theta_r T f^{''}\left(u\right). \tag{B.4}$$

As $f^{'}\left(u\right) > 0$ and $f^{''}\left(u\right) < 0$, $\frac{\partial^2 TP_{r1}}{\partial u^2} < 0$ holds. This implies that $TP_{r1}\left(p, N, T, u\right)$ is strictly concave function of$u$.

Appendix C.

Let,

$$TP_m(u, T) = \frac{\phi(T)}{\psi(T)}, \tag{C.1}$$

where

$$\psi\left(T\right) = T > 0. \tag{C.2}$$

$$\phi\left(T\right) = \begin{bmatrix} wQe^{-\chi M} - cPt_p - A_m \\ -\frac{h_m P}{\theta_m(1-f(u))}\left\{t_p + \frac{1}{\theta_m(1-f(u))}\left\{e^{-\theta_\mathrm{m} t_p(1-f(u))} - 1\right\}\right\} - s\tau P\mu t_p^2 - wi_m MQ - (1-\alpha)guT \end{bmatrix}. \tag{C.3}$$

Also, we have

$$t_p = \frac{1}{\theta_m(1 - f(u))}\log\left[\frac{P}{P - Q\theta_\mathrm{m}(1 - f(u))}\right], \tag{C.4}$$

and

$$Q = \frac{D(p, N)}{\theta_r(1 - f(u))}\left[e^{\theta_r T(1-f(u))} - 1\right]. \tag{C.5}$$

Taking Taylor series expansion of $\left(e^{-\theta_\mathrm{m} t_p(1-f(u))} - 1\right)$ and $\log\left[\frac{P}{P - Q\theta_\mathrm{m}(1-f(u))}\right]$ and neglecting the higher order terms, the expression of $\phi\left(T\right)$ becomes

$$\phi\left(T\right) = wQe^{-\chi M} - cPt_p - A_m - \frac{h_m P t_p^2}{2} - s\tau P\mu t_p^2 - wi_m MQ - (1 - \alpha)guT. \tag{C.6}$$

Now,

$$\frac{d\phi(T)}{dT} = -\left(wi_m M - we^{-\chi M}\right)\frac{dQ}{dT} - cP\frac{dt_p}{dT} - (h_m P + 2s\tau\mu P)t_p\frac{dt_p}{dT} - (1-\alpha)gu, \tag{C.7}$$

and

$$\frac{d^2\phi(T)}{dT^2} = -\left(wi_m M - we^{-\chi M}\right)\frac{d^2Q}{dT^2} - cP\frac{d^2t_p}{dT^2} - (h_m P + 2s\tau\mu P)\left(t_p\frac{d^2t_p}{dT^2} + \left(\frac{dt_p}{dT}\right)^2\right) < 0, \tag{C.8}$$

where,

$$\frac{dQ}{dT} = D\left(p,N\right)e^{\theta_r T(1-f(u))} > 0; \frac{d^2Q}{dT^2} = D\left(p,N\right)\theta_r\left(1-f\left(u\right)\right)e^{\theta_r T(1-f(u))} > 0, \tag{C.9}$$

$$\frac{dt_p}{dT} = \left[\frac{Q\theta_m(1-f(u))}{(P-Q\theta_m(1-f(u)))^2}\frac{dQ}{dT} + \frac{\frac{dQ}{dT}}{(P-Q\theta_m(1-f(u)))}\right] > 0, \tag{C.10}$$

$$\begin{aligned}\frac{d^2t_p}{dT^2} = \theta_m\left(1-f\left(u\right)\right)&\left[\begin{array}{c}\frac{Q}{(P-Q\theta_m(1-f(u)))^2}\frac{d^2Q}{dT^2} + 2Q\left(\frac{dQ}{dT}\right)^2\frac{\theta_m(1-f(u))}{(P-Q\theta_m(1-f(u)))^3}\\ +\frac{\left(\frac{dQ}{dT}\right)^2}{(P-Q\theta_m(1-f(u)))^3}\end{array}\right]\\ &+\left[\left(\frac{dQ}{dT}\right)^2\frac{\theta_m(1-f(u))}{(P-Q\theta_m(1-f(u)))^2} + \frac{\frac{d^2Q}{dT^2}}{(P-Q\theta_m(1-f(u)))}\right] > 0.\end{aligned} \tag{C.11}$$

Therefore, by applying [3] theorem $TP_m\left(u,T\right)$ is a strictly pseudo-concave function in T.

Appendix D.

Taking Taylor series expansion of $\left(e^{-\theta_m t_p(1-f(u))} - 1\right)$, and $\log\left[\frac{P}{P-Q\theta_m(1-f(u))}\right]$ and neglecting the higher order terms, the expression of
$TP_m\left(u,T\right)$ becomes

$$TP_m\left(u,T\right) = \frac{1}{T}\left[wQe^{-\chi M} - cPt_p - A_m - h_m P\frac{t_p^2}{2} - s\tau P\mu t_p^2 - wi_m MQ - (1-\alpha)guT\right]. \tag{D.1}$$

Now,

$$\frac{\partial TP_m(u,T)}{\partial u} = \frac{1}{T}\left[-\left(wi_m M - we^{-\chi M}\right)\frac{dQ}{du} - cP\frac{dt_p}{du} - (h_m P + 2s\tau\mu P)t_p\frac{dt_p}{du} - (1-\alpha)gT\right], \tag{D.2}$$

$$\frac{\partial^2 TP_m(u,T)}{\partial u^2} = \frac{1}{T}\left[-\left(wi_m M - we^{-\chi M}\right)\frac{d^2Q}{du^2} - cP\frac{d^2t_p}{du^2} - (h_m P + 2s\tau\mu P)\left(t_p\frac{d^2t_p}{du^2} + \left(\frac{dt_p}{du}\right)^2\right)\right], \tag{D.3}$$

where

$$t_p = \frac{1}{\theta_m(1-f(u))}\log\left[\frac{P}{P-Q\theta_m(1-f(u))}\right] \tag{D.4}$$

$$Q = \frac{D(p,N)}{\theta_r(1-f(u))}\left[e^{\theta_r T(1-f(u))} - 1\right] \tag{D.5}$$

Neglecting the higher order terms of the above expression (D4) and (D5) and differentiating we have

$$\frac{dQ}{du} = -\frac{D(p,N)\theta_r T^2 f^{'}(u)}{2} < 0; \quad \frac{d^2Q}{du^2} = -\frac{D\left(p,N\right)\theta_r T^2 f^{''}\left(u\right)}{2} > 0, \tag{D.6}$$

$$\frac{dt_p}{du} = \frac{Q\frac{dQ}{du}\theta_m(1-f(u))}{(P-Q\theta_m(1-f(u)))^2} - \frac{Q^2\theta_m f^{'}\left(u\right)}{(P-Q\theta_m(1-f(u)))^2} + \frac{dQ}{du}\frac{1}{(P-Q\theta_m\left(1-f\left(u\right)\right))} < 0. \tag{D.7}$$

$$\frac{d^2t_p}{du^2} = \left[\begin{array}{c} \frac{\left(\frac{dQ}{du}\right)^2\theta_m(1-f(u))}{(P-Q\theta_m(1-f(u)))^2} + \frac{Q\frac{d^2Q}{du^2}\theta_m(1-f(u))}{(P-Q\theta_m(1-f(u)))^2} - \frac{Q\theta_m\frac{dQ}{du}f^{'}(u)}{(P-Q\theta_m(1-f(u)))^2} \\ +\frac{2Q\left(\frac{dQ}{du}\right)^2[\theta_m(1-f(u))]^2}{(P-Q\theta_m(1-f(u)))^3} - \frac{2Q^2\theta_m^2(1-f(u))f^{'}(u)\frac{dQ}{du}}{(P-Q\theta_m(1-f(u)))^3} \\ -\frac{2Q\frac{dQ}{du}\theta_m f^{'}(u)}{(P-Q\theta_m(1-f(u)))^2} - \frac{Q^2\theta_m f^{''}(u)}{(P-Q\theta_m(1-f(u)))^2} - \frac{2Q^2\theta_m f^{'}(u)\frac{dQ}{du}\theta_m(1-f(u))}{(P-Q\theta_m(1-f(u)))^3} + \frac{2Q^3\theta_m^2\left(f^{'}(u)\right)^2}{(P-Q\theta_m(1-f(u)))^3} \\ +\frac{\frac{d^2Q}{du^2}}{(P-Q\theta_m(1-f(u)))^2} + \frac{\left(\frac{dQ}{du}\right)^2\theta_m(1-f(u))}{(P-Q\theta_m(1-f(u)))^2} - \frac{Q\frac{dQ}{du}\theta_m f^{'}(u)}{(P-Q\theta_m(1-f(u)))^2} \end{array}\right]. \tag{D.8}$$

Clearly, $\frac{d^2t_p}{du^2} > 0$ as $f^{'}\left(u\right) > 0$ and $f^{''}\left(u\right) < 0$ with $0 < f(u) < 1$.

Consequently, $\frac{\partial^2 TP_m(u,M,T)}{\partial u^2} < 0$. This implies $TP_m\left(u,T\right)$ is a strictly pseudo-concave function in u.

Acknowledgements. The authors are thankful to the anonymous reviewers and honorable Associate Editor for their valuable comments and suggestions to improve the quality of this article.

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