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SOLVING GEOMETRIC PROGRAMMING PROBLEMS WITH TRIANGULAR AND TRAPEZOIDAL UNCERTAINTY DISTRIBUTIONS

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Abstract. The geometric programming problem is an important optimization technique that is often used to solve different nonlinear optimization problems and engineering problems. The geometric programming models that are commonly used are generally based on deterministic and accurate parameters. However, it is observed that in real-world geometric programming problems, the parameters are frequently inaccurate and ambiguous. In this paper, we consider chance-constrained geometric programming problems with uncertain coefficients and with geometric programming techniques in the uncertain-based framework. We show that the associated chance-constrained uncertain geometric programming problem can be converted into a crisp geometric programming problem by using triangular and trapezoidal uncertainty distributions for the uncertain variables. The main aim of this paper is to provide the solution procedures for geometric programming problems under triangular and trapezoidal uncertainty distributions. To show how well the procedures and algorithms work, two numerical examples and an application in the inventory model are given.

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1. Introduction

Geometric programming (GP) is a powerful optimization tool to solve various nonlinear optimization problems under posynomial constraints. In 1967, Duffin *et al.* [8] developed the fundamental theories of the GP technique. Subsequently, this technique is used by many researchers to solve various engineering design problems such as circuit design, inventory modelling, production planning, risk management, chemical processing, information theory, and structural design [2–6, 10–12, 15, 24, 25, 33].

In the literature, many researchers have worked on the GP problems with exact and deterministic coefficients of the objective and constraint functions [2, 3, 7, 9, 13, 14, 19, 21–23, 34, 35].

For the traditional GP problems, the coefficients of cost and constraint functions are precisely known. However, it is observed that the values of the parameters in real-world GP problems may be imprecise and ambiguous. As a result, numerous strategies in connection with inaccurate and inconclusive data in GP models are proposed.

Stochastic GP was established by Avriel and Wilde [1], with deterministic exponents along with the coefficients, which are nonnegative random variables. Peykani *et al.* [20] introduced a novel Fuzzy Data Envelopment

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Analysis (FDEA) model by the use of possibility, necessity, and credibility approaches. They displayed comparative research work between the FDEA model and the Robust Fuzzy Data Envelopment Analysis (RFDEA) model. Shiraz *et al.* [29] studied a chance-constrained Data Envelopment Analysis (DEA) model with random-rough data inputs and outputs and investigated the stability and robustness through sensitivity analysis. The authors considered a chance-constrained DEA model with random and rough input and output data. They proposed a deterministic equivalent model with quadratic constraints to solve the model.

Liu [16] developed a strategy to find out the range of the values of the posynomial objective function for the GP problems with interval coefficients and exponents. Subsequently, the same author [17] solved the GP problems with fuzzy coefficients and exponents.

Liu [18] worked on uncertainty theory, which is a new area of mathematics. Solano-Charris *et al.* [32] solved the bi-objective robust vehicle routing problem with uncertain costs and demands by nontraditional methods such as the Non-dominated Sorting Genetic Algorithm (NSGAII) and the Multiobjective Evolutionary Algorithm (MOEA). The authors considered it a robust optimization problem with uncertain demands and travel times. They used the min-max optimization criterion to address the uncertainties in the objective function. Shiraz *et al.* [27] constructed the GP problems based on an uncertainty frame model with the cost and constraint coefficients which follow normal, linear, and zigzag uncertainty distributions. Subsequently, Shiraz *et al.* [28] solved the fuzzy chance-constrained GP problems with possibility, necessity, and credibility approaches. Shiraz and Fukuyama [26] later developed the GP problems using the rough set theory. Recently, Shiraz *et al.* [30] used the copula theory approach to solve the stochastic GP problems. The authors considered a joint chance-constrained GP problem with copula and converted it into deterministic convex programming. They used sequential convex approximation and piecewise tangent approximation to find the lower and upper limits of the objective function.

From the literature survey, we see that much research has been done on the traditional GP problems with precise coefficients of posynomial objective and constraint functions. However, in real-world GP problems, the coefficients may be imprecise. The GP problems with uncertainty have been solved under normal, linear, zigzag uncertainty distributions, and L-R fuzzy and rough fuzzy environments. However, the GP problems may have triangular and trapezoidal uncertainty in the coefficients of posynomial objective and constraint functions. There is no previous research work on the GP problems when the coefficients of cost and constraint functions follow triangular and trapezoidal uncertainty distributions. Thus, we make an attempt to study the GP problems with triangular and trapezoidal uncertainty distributions. The main contributions to this paper are the followings.

- We introduce the uncertain GP problems along with the coefficients of cost and constraint functions, which are independent uncertain variables.
- The equivalent chance-constrained GP problem is developed under triangular and trapezoidal uncertainty distributions.
- We show how the uncertain chance-constrained GP models can be converted into crisp GP models and, consequently, the crisp GP models can be solved in a conventional way.
- Two numerical examples and one application in an inventory model are incorporated to validate our proposed method.

The organization of the rest of the paper is as follows.

The preliminaries are discussed in Section 2. In Section 3, we consider the chance-constrained GP problems with triangular and trapezoidal uncertain coefficients and show how the chance-constrained uncertain GP models can be converted into crisp GP models. In Section 4, two numerical examples are given to demonstrate the efficiency of the procedures. Furthermore, an application based problem in an inventory model is solved under triangular and trapezoidal uncertainty distributions in Section 5. Finally, a conclusion on the work is incorporated in Section 6.

2. Preliminaries

In this section, we discuss some basic concepts of uncertainty theory and credibility theory referred to [18].

Definition 2.1. Let $\mathcal{L}$ be a σ-algebra defined on a nonempty set Ω. The set function $\mathcal{M} : \mathcal{L} \to [0,1]$ is said to be *uncertain measure* if and only if it satisfies the following properties.

Property 1. (Normality) $\mathcal{M}\{\Omega\} = 1$.
Property 2. (Monotonicity) $\mathcal{M}\{\Upsilon_1\} \leq \mathcal{M}\{\Upsilon_2\}$ whenever $\Upsilon_1 \subset \Upsilon_2$.
Property 3. (Self-Duality) $\mathcal{M}\{\Upsilon\} + \mathcal{M}\{\Upsilon^c\} = 1$ for any event Υ.
Property 4. (Countable Subadditivity) $\mathcal{M}\{\bigcup\limits_{i=1}^{\infty} \Upsilon_i\} \leq \sum\limits_{i=1}^{\infty} \mathcal{M}\{\Upsilon_i\}$ for every countable sequence of events $\{\Upsilon_i\}$.

Definition 2.2. Let $\mathcal{P}$ be the power set of a nonempty set Θ. The set function $Cr : \mathcal{P} \to [0,1]$ is said to be *credibility measure* if and only if it satisfies the following properties.

Property 1. (Normality) $Cr\{\Theta\} = 1$.
Property 2. (Monotonicity) $Cr\{\Upsilon_1\} \leq Cr\{\Upsilon_2\}$ whenever $\Upsilon_1 \subset \Upsilon_2$.
Property 3. (Self-Duality) $Cr\{\Upsilon\} + Cr\{\Upsilon^c\} = 1$ for any event Υ.
Property 4. (Maximality) $Cr\{\bigcup\limits_{i=1}^{\infty} \Upsilon_i\} = \operatorname*{Sup}\limits_i Cr\{\Upsilon_i\}$ for every countable sequence of events $\{\Upsilon_i\}$ with $\operatorname*{Sup}\limits_i Cr\{\Upsilon_i\} < 0.5$.

Note: The credibility measure is also countably subadditive *i.e.,*
$Cr\{\bigcup\limits_{i=1}^{\infty} \Upsilon_i\} \leq \sum\limits_{i=1}^{\infty} Cr\{\Upsilon_i\}$ for every countable sequence of events $\{\Upsilon_i\}$.

Theorem 2.3. *An uncertain measure is a credibility measure if and only if it satisfies the maximality property.*

Definition 2.4. If Ω is a nonempty set, $\mathcal{L}$ is a σ-algebra over Ω and $\mathcal{M}$ is an uncertain measure, then the triplet $(\Omega, \mathcal{L}, \mathcal{M})$ is labeled as *uncertainty space*.

Definition 2.5. If Θ is a nonempty set, $\mathcal{P}$ is a power set of Θ and Cr is a credibility measure, then the triplet $(\Theta, \mathcal{P}, Cr)$ is labeled as *credibility space*.

Definition 2.6. A measurable function $\eta : (\Omega, \mathcal{L}, \mathcal{M}) \to \mathbb{R}$ is called *uncertain variable* (UV), where $(\Omega, \mathcal{L}, \mathcal{M})$ is the uncertainty space and $\mathbb{R}$ is the set of real numbers.

Definition 2.7. A measurable function $\xi : (\Theta, \mathcal{P}, Cr) \to \mathbb{R}$ is called *fuzzy variable* (FV), where $(\Theta, \mathcal{P}, Cr)$ is the credibility space and $\mathbb{R}$ is the set of real numbers.

Theorem 2.8. *A UV is an FV if and only if the uncertain measure is a credibility measure.*

Definition 2.9. A UV η is *positive* if and only if $\mathcal{M}\{\eta \leq 0\} = 0$ and *nonnegative* if and only if $\mathcal{M}\{\eta < 0\} = 0$.

Definition 2.10. An FV ξ is *positive* if and only if $Cr\{\xi \leq 0\} = 0$ and *nonnegative* if and only if $Cr\{\xi < 0\} = 0$.

Definition 2.11. The *uncertainty distribution* (UD) function $\Phi_\eta : \mathbb{R} \to [0,1]$ corresponding to the UV η is defined as $\Phi_\eta(x) = \mathcal{M}\{\eta \leq x\}$ for every $x \in \mathbb{R}$.

Definition 2.12. The *credibility distribution* (CD) function $\Phi_\xi : \mathbb{R} \to [0,1]$ corresponding to the FV ξ is defined as $\Phi_\xi(x) = Cr\{\xi \leq x\}$ for every $x \in \mathbb{R}$.

Definition 2.13. A UV η is said to be *triangular* if and only if it has UD of the form

$$\begin{cases} 0, & x \leq a; \\ \frac{(x-a)^2}{(b-a)(c-a)}, & a \leq x \leq b; \\ 1 - \frac{(c-x)^2}{(c-a)(c-b)}, & b \leq x \leq c; \\ 1, & x \geq c; \end{cases}$$

where $a, b, c \in \mathbb{R}, a < b < c$. In notation, it can be written as $\eta \sim \nabla(a, b, c)$.

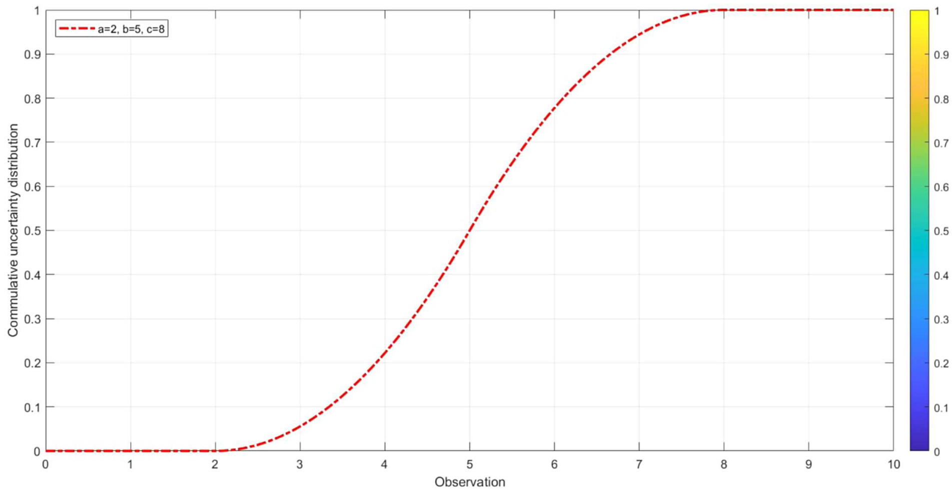

FIGURE 1. Triangular UD.

The graphical representation of triangular UD is given in Figure 1.

Definition 2.14. A UV η is said to be *trapezoidal* if and only if it has UD of the form

$$\begin{cases} 0, & x \leq a; \\ \frac{(x-a)^2}{(d+c-a-b)(b-a)}, & a \leq x \leq b; \\ \frac{(2x-a-b)}{(d+c-a-b)}, & b \leq x \leq c; \\ 1-\frac{(d-x)^2}{(d+c-a-b)(d-c)}, & c \leq x \leq d; \\ 1, & x \geq d; \end{cases}$$

where $a, b, c, d \in \mathbb{R}, a < b < c < d$. In notation, it can be written as $\eta \sim T(a, b, c, d)$.

The graphical representation of trapezoidal UD is given in Figure 2.

Definition 2.15. If η is a UV, then the *mean value* of η is

$$E[\eta] = \int_0^\infty \mathcal{M}\{\eta \geq \tau\} d\tau - \int_{-\infty}^0 \mathcal{M}\{\eta \leq \tau\} d\tau,$$

provided at least one of the two integral is finite.

Proposition 2.16. *If f is a real valued measurable function of $\eta_1, \eta_2, \ldots, \eta_n$, where $\eta_1, \eta_2, \ldots, \eta_n$ are UVs, then $f(\eta_1, \eta_2, \ldots, \eta_n)$ is a UV.*

Theorem 2.17. *If $\eta_1, \eta_2, \ldots, \eta_n$ are independent UVs with UDs $\Phi_{\eta_1}, \Phi_{\eta_2}, \ldots, \Phi_{\eta_n}$ respectively, $h(x, \eta_1, \eta_2, \ldots, \eta_n)$ is a strictly increasing constraint function with respect to $\eta_1, \eta_2, \ldots, \eta_n$ and C is a constant, then for every $\gamma \in (0, 1)$,*

$$\mathcal{M}[h(x, \eta_1, \eta_2, \ldots, \eta_n) \leq C] \geq \gamma \text{ is equivalent to } h(x, \Phi_{\eta_1}^{-1}(\gamma), \Phi_{\eta_2}^{-1}(\gamma), \ldots, \Phi_{\eta_n}^{-1}(\gamma)) \leq C.$$

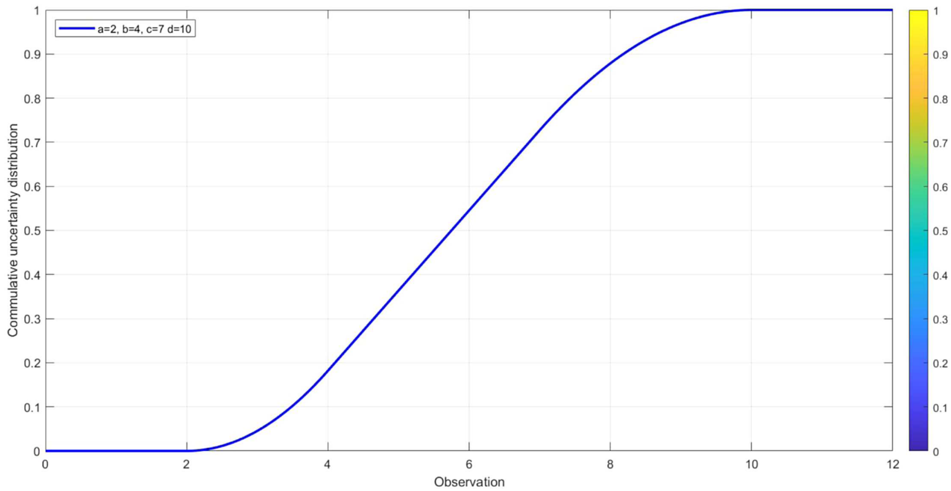

Figure 2. Trapezoidal UD.

3. Chance-constrained geometric programming model with uncertain coefficients

A traditional GP problem permits solving a minimization problem where the objective function as well as the constraints are posynomials. GP problems whose coefficients and decision variables are all positive except the exponents, are called posynomial problems. A classical primal GP model and its dual form are defined as follows.

Primal problem.

$$\begin{aligned} &\min \quad f_0(\mathbf{x}) = \sum_{i=1}^{N_0} \beta_{i0} \prod_{j=1}^{n} x_j^{\alpha_{0ij}} \\ &\text{s.t.} \\ &\qquad f_k(\mathbf{x}) = \sum_{i=1}^{N_k} \beta_{ik} \prod_{j=1}^{n} x_j^{\alpha_{kij}} \leq 1, k = 1, 2, \ldots K, \end{aligned} \tag{3.1}$$

where $\beta_{ik} > 0, x_j > 0, \alpha_{kij} \in \mathbb{R}, \forall i, j, k$. Here, N_0 and N_k are the total number of terms present in the objective function and k^{th} constraints respectively.

To define the dual problem of the above primal problem (3.1), let $N = \sum_{k=0}^{K} N_k$ be the total numbers of terms presents in the primal GP problem (3.1) and δ_{ik} be the dual variables such that $\lambda_k = \sum_{i=1}^{N_k} \delta_{ik}, k = 0, 1, 2, \ldots, K$. Then

Dual problem.

$$
\begin{aligned}
&\max \quad V(\delta) = \prod_{i=1}^{N} \left(\frac{\beta_{ik}}{\delta_{ik}}\right)^{\delta_{ik}} \left(\lambda_k\right)^{\lambda_k} \\
&\text{s.t.} \\
&\quad \sum_{i=1}^{N_0} \delta_{i0} = \lambda_0 = 1, \quad \text{(Normality condition)} \\
&\quad \sum_{i=1}^{N} \delta_{ik}\alpha_{kij} = 0, j = 1, 2, \ldots, n. \quad \text{(Orthogonality conditions)}
\end{aligned}
\tag{3.2}
$$

Here is the relationship between primal and dual problems.

Primal-dual relationship. The relationship between primal and dual GP problems due to strong duality theorem [7,8] is as follows.

$$
\begin{aligned}
&\sum_{j=1}^{n} \alpha_{0ij} \ln(x_j) = \ln\left(\frac{\delta_{i0} f_0(\mathbf{x})}{\beta_{i0}}\right), i = 1, 2, \ldots, N_0, \\
&\sum_{j=1}^{n} \alpha_{kij} \ln(x_j) = \ln\left(\frac{\delta_{ik}}{\lambda_k \beta_{ik}}\right), i = 1, 2, \ldots, N_k, k = 1, 2, \ldots, K.
\end{aligned}
\tag{3.3}
$$

Now, depending on N, we have the following two cases.

Case I. If $N \geq n + 1$, then the dual problem given in equation (3.2) is feasible as the number of equations is less than or equal to the number of dual variables, which guarantees the existence of solution for dual variables (Beightler and Philips [3]).

Case II. If $N < n+1$, then the dual problem given in equation (3.2) is inconsistent as the number of equations is greater than the number of dual variables. It guarantees that there is no analytical solution for dual variables. However, an approximate solution can be found by least square or linear programming method (Sinha *et al.* [31]).

Now, we expand a chance-constrained GP model with uncertain coefficients. First, we convert the standard GP model given in equation (3.1) in an uncertain GP model as

$$
\begin{aligned}
&\min \quad f_0(\mathbf{x}) = \sum_{i=1}^{N_0} \widetilde{\beta_{i0}} \prod_{j=1}^{n} x_j^{\alpha_{0ij}} \\
&\text{s.t.} \\
&\quad f_k(\mathbf{x}) = \sum_{i=1}^{N_k} \widetilde{\beta_{ik}} \prod_{j=1}^{n} x_j^{\alpha_{kij}} \leq 1, k = 1, 2, \ldots K,
\end{aligned}
\tag{3.4}
$$

where $\widetilde{\beta_{ik}}$ are UVs, $x_j > 0, \alpha_{kij} \in \mathbb{R}, \forall i, j, k$. The chance-constrained uncertain GP model can be formulated based on equation (3.4) as

$$
\begin{aligned}
&\min \quad E\left[\sum_{i=1}^{N_0} \widetilde{\beta_{i0}} \prod_{j=1}^{n} x_j^{\alpha_{0ij}}\right] \\
&\text{s.t.} \\
&\quad \mathcal{M}\left(\sum_{i=1}^{N_k} \widetilde{\beta_{ik}} \prod_{j=1}^{n} x_j^{\alpha_{kij}} \leq 1\right) \geq \gamma, k = 1, 2, \ldots K,
\end{aligned}
\tag{3.5}
$$

where $\gamma \in (0,1)$ is predefined uncertainty level, $x_j > 0, j = 1, 2, \ldots, n$. In the following two subsections, we develop the equivalent crisp primal-dual GP model of the above uncertain model (3.5) under triangular and trapezoidal UDs.

3.1. Uncertain chance-constrained geometric programming problem with triangular uncertainty distribution

We propose a solving technique for the chance-constrained uncertain GP problem given in equation (3.5) considering the coefficients to be UVs with triangular UDs. Let the coefficients $\widetilde{\beta_{ik}}$ given in equation (3.5) be independent triangular UVs *i.e.*, $\widetilde{\beta_{ik}} \sim \nabla(a_{ik}, b_{ik}, c_{ik})$. The following lemmas are helpful to transform the chance-constrained uncertain GP model given in equation (3.5) to a crisp or deterministic model. Accordingly, we solve it by converting into the dual form.

Lemma 3.1. *([18]) The expected value of a triangular UV $\widetilde{\eta} \sim \nabla(a,b,c)$ is*

$$E[\widetilde{\eta}] = \frac{a+b+c}{3}\cdot$$

Lemma 3.2. *Let $\widetilde{\eta_p} \sim \nabla(a_p, b_p, c_p), p = 1, 2, \ldots, P$, be independent, and $U_p, p = 1, 2, \ldots, P$, be nonnegative variables, then for every $\gamma \in (0,1)$,*

$$\mathcal{M}\left(\sum_{p=1}^{P} \widetilde{\eta_p} U_p \leq 1\right) \geq \gamma$$

is equivalent to

$$\begin{cases} \sum\limits_{p=1}^{P} U_p\left(a_p + \sqrt{(c_p - a_p)(b_p - a_p)\gamma}\right) \leq 1, & \text{if } 0 < \gamma \leq \frac{b_p - a_p}{c_p - a_p}; \\ \sum\limits_{p=1}^{P} U_p\left(c_p - \sqrt{(c_p - a_p)(c_p - b_p)(1-\gamma)}\right) \leq 1, & \text{if } \frac{b_p - a_p}{c_p - a_p} \leq \gamma < 1. \end{cases}$$

By Lemma (3.1), the objective function given in equation (3.5) becomes

$$E\left[\sum_{i=1}^{N_0} \widetilde{\beta_{i0}} \prod_{j=1}^{n} x_j^{\alpha_{0ij}}\right] = \sum_{i=1}^{N_0} E[\widetilde{\beta_{i0}}] \prod_{j=1}^{n} x_j^{\alpha_{0ij}}, \tag{3.6}$$

where $E[\widetilde{\beta_{i0}}] = \frac{a_{i0}+b_{i0}+c_{i0}}{3}$. Subsequently, by Lemma (3.2), the constraints given in equation (3.5) can be written as

$$\sum_{i=1}^{N_k} \zeta_{ik} \prod_{j=1}^{n} x_j^{\alpha_{kij}} \leq 1, k = 1, 2, \ldots K, \tag{3.7}$$

where

$$\zeta_{ik} = \begin{cases} a_{ik} + \sqrt{(c_{ik} - a_{ik})(b_{ik} - a_{ik})\gamma}, & \text{if } 0 < \gamma \leq \frac{b_{ik} - a_{ik}}{c_{ik} - a_{ik}}; \\ c_{ik} - \sqrt{(c_{ik} - a_{ik})(c_{ik} - b_{ik})(1-\gamma)}, & \text{if } \frac{b_{ik} - a_{ik}}{c_{ik} - a_{ik}} \leq \gamma < 1. \end{cases}$$

Equations (3.6) and (3.7) are the objective function and the constraints of the transformed crisp primal GP model, respectively. Therefore, the equivalent crisp GP model of (3.5) is

$$\begin{aligned} \min \quad & \sum_{i=1}^{N_0} E[\widetilde{\beta_{i0}}] \prod_{j=1}^{n} x_j^{\alpha_{0ij}} \\ \text{s.t.} \quad & \\ & \sum_{i=1}^{N_k} \zeta_{ik} \prod_{j=1}^{n} x_j^{\alpha_{kij}} \leq 1, k = 1, 2, \ldots K, \end{aligned} \tag{3.8}$$

where $E[\widetilde{\beta_{i0}}], \zeta_{ik}$ are given in (3.6),(3.7) respectively. Consequently, the dual problem of the primal model (3.8) is

$$\begin{aligned} \max \quad & \prod_{i=1}^{N_0} \left(\frac{E[\widetilde{\beta_{i0}}]}{\delta_{i0}} \right)^{\delta_{i0}} \prod_{k=1}^{K} \prod_{i=1}^{N_k} \left(\frac{\zeta_{ik}}{\delta_{ik}} \right)^{\delta_{ik}} \left(\lambda_k \right)^{\lambda_k} \\ \text{s.t.} \quad & \\ & \sum_{i=1}^{N_0} \delta_{i0} = \lambda_0 = 1, \quad \text{(Normality condition)} \\ & \sum_{i=1}^{N} \delta_{ik} \alpha_{kij} = 0, j = 1, 2, \ldots, n, \quad \text{(Orthogonality conditions)} \end{aligned} \tag{3.9}$$

where $N = \sum_{k=0}^{K} N_k, \lambda_k = \sum_{i=1}^{N_k} \delta_{ik}, k = 0, 1, 2, \ldots, K.$

3.2. Uncertain chance-constrained geometric programming problem with trapezoidal uncertainty distribution

We propose a solving technique for the chance-constrained uncertain GP problem given in equation (3.5). We consider the coefficients to be UVs with trapezoidal UDs. Let the coefficients $\widetilde{\beta_{ik}}$ given in equation (3.5) be independent trapezoidal UVs *i.e.,* $\widetilde{\beta_{ik}} \sim T(a_{ik}, b_{ik}, c_{ik}, d_{ik})$. Similar to the previous discussion, we present the following two lemmas that transform the chance-constrained uncertain GP model given in equation (3.5) to a crisp or deterministic model and, hence, we solve it by converting into the dual form as earlier.

Lemma 3.3. *([18]) The expected value of a trapezoidal UV* $\widetilde{\eta} \sim T(a, b, c, d)$ *is*

$$E[\widetilde{\eta}] = \frac{1}{3(d + c - a - b)} \left(\frac{d^3 - c^3}{d - c} - \frac{b^3 - a^3}{b - a} \right).$$

Lemma 3.4. *Let* $\widetilde{\eta_p} \sim T(a_p, b_p, c_p, d_p)$ *be independent for* $p = 1, 2, \ldots, P$ *and* U_p *be a nonnegative variables for* $p \in \{1, 2, \ldots, P\}$*. Then for every* $\gamma \in (0, 1)$,

$$\mathcal{M}\left(\sum_{p=1}^{P} \widetilde{\eta_p} U_p \leq 1 \right) \geq \gamma$$

is equivalent to

$$\begin{cases} \sum_{p=1}^{P} U_p\left(a_p + \sqrt{(d_p + c_p - a_p - b_p)(b_p - a_p)\gamma}\right) \le 1, & \textit{if } 0 < \gamma \le \frac{b_p - a_p}{d_p + c_p - a_p - b_p}; \\ \sum_{p=1}^{P} U_p\left(\frac{1}{2}\Big[(a_p + b_p) + \gamma(d_p + c_p - a_p - b_p)\Big]\right) \le 1, & \textit{if } \frac{b_p - a_p}{d_p + c_p - a_p - b_p} \le \gamma \le \frac{2c_p - b_p - a_p}{d_p + c_p - a_p - b_p}; \\ \sum_{p=1}^{P} U_p\left(d_p - \sqrt{(d_p + c_p - a_p - b_p)(d_p - c_p)(1-\gamma)}\right) \le 1, & \textit{if } \frac{2c_p - b_p - a_p}{d_p + c_p - a_p - b_p} \le \gamma < 1. \end{cases}$$

By Lemma (3.3), the objective function given in equation (3.5) becomes

$$E\Big[\sum_{i=1}^{N_0} \widetilde{\beta_{i0}} \prod_{j=1}^{n} x_j^{\alpha_{0ij}}\Big] = \sum_{i=1}^{N_0} E[\widetilde{\beta_{i0}}] \prod_{j=1}^{n} x_j^{\alpha_{0ij}}, \tag{3.10}$$

where $E[\widetilde{\beta_{i0}}] = \frac{1}{3(d_{i0}+c_{i0}-a_{i0}-b_{i0})}\left(\frac{d_{i0}^3 - c_{i0}^3}{d_{i0}-c_{i0}} - \frac{b_{i0}^3 - a_{i0}^3}{b_{i0}-a_{i0}}\right)$. Moreover, by Lemma (3.4), the constraints given in equation (3.5) can be written as

$$\sum_{i=1}^{N_k} \zeta_{ik} \prod_{j=1}^{n} x_j^{\alpha_{kij}} \le 1, k = 1, 2, \dots K, \tag{3.11}$$

where

$$\zeta_{ik} = \begin{cases} a_{ik} + \sqrt{(d_{ik} + c_{ik} - a_{ik} - b_{ik})(b_{ik} - a_{ik})\gamma}, & \text{if } 0 < \gamma \le \frac{b_{ik} - a_{ik}}{d_{ik} + c_{ik} - a_{ik} - b_{ik}}; \\ \frac{1}{2}\big[(a_{ik} + b_{ik}) + \gamma(d_{ik} + c_{ik} - a_{ik} - b_{ik})\big], & \text{if } \frac{b_{ik} - a_{ik}}{d_{ik} + c_{ik} - a_{ik} - b_{ik}} \le \gamma \le \frac{2c_{ik} - b_{ik} - a_{ik}}{d_{ik} + c_{ik} - a_{ik} - b_{ik}}; \\ d_{ik} - \sqrt{(d_{ik} + c_{ik} - a_{ik} - b_{ik})(d_{ik} - c_{ik})(1-\gamma)}, & \text{if } \frac{2c_{ik} - b_{ik} - a_{ik}}{d_{ik} + c_{ik} - a_{ik} - b_{ik}} \le \gamma < 1. \end{cases}$$

Equation (3.10) and the equation (3.11) are the objective function and the constraints of the transformed crisp primal GP model, respectively. Therefore, the equivalent crisp GP model of (3.5) is

$$\begin{aligned} \min \quad & \sum_{i=1}^{N_0} E[\widetilde{\beta_{i0}}] \prod_{j=1}^{n} x_j^{\alpha_{0ij}} \\ \text{s.t.} \quad & \\ & \sum_{i=1}^{N_k} \zeta_{ik} \prod_{j=1}^{n} x_j^{\alpha_{kij}} \le 1, k = 1, 2, \dots K, \end{aligned} \tag{3.12}$$

where $E[\widetilde{\beta_{i0}}], \zeta_{ik}$ are given in (3.10),(3.11) respectively. Consequently, the dual problem of the primal model (3.12) is

$$\max \quad \prod_{i=1}^{N_0} \left(\frac{E[\widetilde{\beta_{i0}}]}{\delta_{i0}}\right)^{\delta_{i0}} \prod_{k=1}^{K} \prod_{i=1}^{N_k} \left(\frac{\zeta_{ik}}{\delta_{ik}}\right)^{\delta_{ik}} \Big(\lambda_k\Big)^{\lambda_k}$$

$$
\begin{aligned}
\text{s.t.}\quad & \\
& \sum_{i=1}^{N_0} \delta_{i0} = \lambda_0 = 1, \quad \text{(Normality condition)} \\
& \sum_{i=1}^{N} \delta_{ik}\alpha_{kij} = 0, j = 1, 2, \dots, n, \quad \text{(Orthogonality conditions)}
\end{aligned} \tag{3.13}
$$

where $N = \sum_{k=0}^{K} N_k, \lambda_k = \sum_{i=1}^{N_k} \delta_{ik}, k = 0, 1, 2, \dots, K$. Now, we provide the numerical illustration of our proposed method in the following section.

4. Numerical examples

Here, we present two numerical examples to demonstrate the efficiency of the triangular and trapezoidal uncertain GP models.

Example 4.1. Consider the following GP problem with uncertain coefficients

$$
\begin{aligned}
\min \quad & f_0(\mathbf{x}) = \widetilde{\beta_{10}} x_1 x_3 + \widetilde{\beta_{20}} x_2 x_3 + \widetilde{\beta_{30}} x_1 x_2 \\
\text{s.t.} \quad & \\
& f_1(\mathbf{x}) = \frac{\widetilde{\beta_{11}}}{x_1 x_2 x_3} \le 1,
\end{aligned} \tag{4.1}
$$

where all the coefficients are treated as UVs and $x_1, x_2, x_3 > 0$.

Triangular case. Let $\widetilde{\beta_{10}} \sim \nabla(5, 8, 10)$, $\widetilde{\beta_{20}} \sim \nabla(8, 12, 14)$, $\widetilde{\beta_{30}} \sim \nabla(6, 9, 11)$ and $\widetilde{\beta_{11}} \sim \nabla(2, 4, 5)$. Then, using Lemma (3.1), we get $E[\widetilde{\beta_{10}}] = 7.67, E[\widetilde{\beta_{20}}] = 11.33, E[\widetilde{\beta_{30}}] = 8.67$. Therefore, the equivalent crisp model of uncertain primal GP problem given in equation (4.1) is

$$
\begin{aligned}
\min \quad & 7.67 x_1 x_3 + 11.33 x_2 x_3 + 8.67 x_1 x_2 \\
\text{s.t.} \quad & \\
& \frac{\zeta_{11}}{x_1 x_2 x_3} \le 1, \\
& x_1, x_2, x_3 > 0.
\end{aligned} \tag{4.2}
$$

The equivalent dual problem is

$$
\begin{aligned}
\max \quad & \left(\frac{7.67}{\delta_{10}}\right)^{\delta_{10}} \left(\frac{11.33}{\delta_{20}}\right)^{\delta_{20}} \left(\frac{8.67}{\delta_{30}}\right)^{\delta_{30}} \left(\frac{\zeta_{11}}{\delta_{11}}\right)^{\delta_{11}} (\delta_{11})^{\delta_{11}} \\
\text{s.t.} \quad & \\
& \delta_{10} + \delta_{20} + \delta_{30} = 1, \\
& \delta_{10} + \delta_{30} - \delta_{11} = 0, \\
& \delta_{20} + \delta_{30} - \delta_{11} = 0, \\
& \delta_{10} + \delta_{20} - \delta_{11} = 0.
\end{aligned} \tag{4.3}
$$

Using (3.7), we find that ζ_{11} has the following form.

$$
\zeta_{11} = \begin{cases} 2 + \sqrt{6\gamma}, & \text{if } 0 < \gamma \le \frac{2}{3}; \\ 5 - \sqrt{3(1-\gamma)}, & \text{if } \frac{2}{3} \le \gamma < 1. \end{cases}
$$

TABLE 1. Optimal solutions under triangular UD.

γ	ζ_{11}	x_1^*	x_2^*	x_3^*	δ_{10}^*	δ_{20}^*	δ_{30}^*	δ_{11}^*	f^*
0.1	2.775	1.750	1.184	1.339	0.333	0.333	0.334	0.667	53.907
0.2	3.095	1.815	1.228	1.389	0.333	0.333	0.334	0.667	57.975
0.3	3.342	1.862	1.260	1.425	0.333	0.333	0.334	0.667	61.020
0.4	3.549	1.899	1.286	1.453	0.333	0.333	0.334	0.667	63.515
0.5	3.732	1.931	1.307	1.478	0.333	0.333	0.334	0.667	65.680
0.6	3.897	1.959	1.326	1.499	0.333	0.333	0.334	0.667	67.602
0.7	4.051	1.985	1.344	1.519	0.333	0.333	0.334	0.667	69.371
0.8	4.225	2.013	1.363	1.540	0.333	0.333	0.334	0.667	71.343
0.9	4.452	2.048	1.387	1.567	0.333	0.333	0.334	0.667	73.876

Consequently, we solve the dual problem (4.3) for different uncertainty levels $\gamma \in (0,1)$ and provide the primal-dual solutions corresponding to different uncertainty levels in Table 1.

Trapezoidal case. We follow the similar approach for this case. Let $\widetilde{\beta_{10}} \sim T(5,6,9,10)$, $\widetilde{\beta_{20}} \sim T(8,10,12,13)$, $\widetilde{\beta_{30}} \sim T(6,8,9,10)$ and $\widetilde{\beta_{11}} \sim T(1,2,4,5)$. Then, by Lemma (3.3), we have $E[\widetilde{\beta_{10}}] = 7.5, E[\widetilde{\beta_{20}}] = 10.71, E[\widetilde{\beta_{30}}] = 8.2$. Therefore, the equivalent crisp model of uncertain primal GP problem given in equation (4.1) is

$$\begin{aligned}
\min \quad & 7.5x_1x_3 + 10.71x_2x_3 + 8.2x_1x_2 \\
\text{s.t.} \quad & \\
& \frac{\zeta_{11}}{x_1x_2x_3} \leq 1, \\
& x_1, x_2, x_3 > 0.
\end{aligned} \tag{4.4}$$

The equivalent dual problem is

$$\begin{aligned}
\max \quad & \left(\frac{7.5}{\delta_{10}}\right)^{\delta_{10}} \left(\frac{10.71}{\delta_{20}}\right)^{\delta_{20}} \left(\frac{8.2}{\delta_{30}}\right)^{\delta_{30}} \left(\frac{\zeta_{11}}{\delta_{11}}\right)^{\delta_{11}} (\delta_{11})^{\delta_{11}} \\
\text{s.t.} \quad & \\
& \delta_{10} + \delta_{20} + \delta_{30} = 1, \\
& \delta_{10} + \delta_{30} - \delta_{11} = 0, \\
& \delta_{20} + \delta_{30} - \delta_{11} = 0, \\
& \delta_{10} + \delta_{20} - \delta_{11} = 0.
\end{aligned} \tag{4.5}$$

Using (3.11), we find that ζ_{11} has the following form.

$$\zeta_{11} = \begin{cases} 1 + \sqrt{6\gamma}, & \text{if } 0 < \gamma \leq \frac{1}{6}; \\ \frac{1}{2}(3 + 6\gamma), & \text{if } \frac{1}{6} \leq \gamma \leq \frac{5}{6}; \\ 5 - \sqrt{6(1-\gamma)}, & \text{if } \frac{5}{6} \leq \gamma < 1. \end{cases}$$

Consequently, we solve the dual problem (4.5) for different uncertainty levels $\gamma \in (0,1)$ and provide the primal-dual solutions corresponding to different uncertainty levels in Table 2.

The expected values of the objective function under triangular and trapezoidal cases for different uncertainty levels $\gamma \in (0,1)$ are shown in Figure 3.

TABLE 2. Optimal solutions under trapezoidal UD.

γ	ζ_{11}	x_1^*	x_2^*	x_3^*	δ_{10}^*	δ_{20}^*	δ_{30}^*	δ_{11}^*	f^*
0.1	1.775	1.490	1.044	1.141	0.333	0.333	0.334	0.667	38.266
0.2	2.100	1.576	1.104	1.207	0.333	0.333	0.334	0.667	42.804
0.3	2.400	1.648	1.154	1.262	0.333	0.333	0.334	0.667	46.790
0.4	2.700	1.714	1.200	1.312	0.333	0.333	0.334	0.667	50.612
0.5	3.000	1.775	1.243	1.359	0.333	0.333	0.334	0.667	54.295
0.6	3.300	1.833	1.283	1.403	0.333	0.333	0.334	0.667	57.856
0.7	3.600	1.887	1.321	1.444	0.333	0.333	0.334	0.667	61.312
0.8	3.900	1.938	1.357	1.483	0.333	0.333	0.334	0.667	64.672
0.9	4.225	1.990	1.394	1.524	0.333	0.333	0.334	0.667	68.217

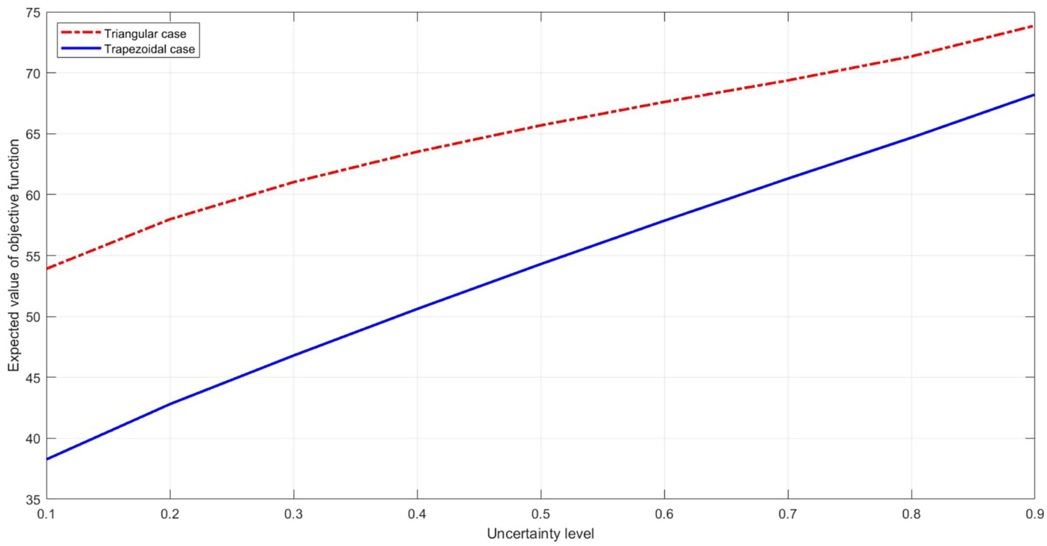

FIGURE 3. Expected values of objective function under triangular and trapezoidal cases.

Example 4.2. Let us consider the GP problem with uncertain coefficients as

$$\begin{aligned} &\min \quad f_0(\mathbf{x}) = \widetilde{\beta_{10}} x_1^{-1} x_2^{-\frac{1}{2}} x_3^{-1} + \widetilde{\beta_{20}} x_1 x_3 + \widetilde{\beta_{30}} x_1 x_2 x_3 \\ &\text{s.t.} \\ &\qquad f_1(\mathbf{x}) = \widetilde{\beta_{11}} x_1^{-2} x_2^{-2} + \widetilde{\beta_{21}} x_2^{\frac{1}{2}} x_3^{-1} \leq 1, \end{aligned} \tag{4.6}$$

where all the coefficients are treated as UVs and $x_1, x_2, x_3 > 0$.

Triangular case.

Let $\widetilde{\beta_{10}} \sim \nabla(10, 20, 40)$, $\widetilde{\beta_{20}} \sim \nabla(15, 25, 35)$, $\widetilde{\beta_{30}} \sim \nabla(35, 65, 75)$, $\widetilde{\beta_{11}} \sim \nabla(\frac{1}{3}, \frac{2}{3}, 1)$ and $\widetilde{\beta_{21}} \sim \nabla(\frac{2}{3}, 1, \frac{4}{3})$. Then, by Lemma (3.1), we have $E[\widetilde{\beta_{10}}] = 23.33$, $E[\widetilde{\beta_{20}}] = 25$, $E[\widetilde{\beta_{30}}] = 58.33$. Therefore, the equivalent crisp

TABLE 3. Optimal solutions under triangular UD.

γ	ζ_{11}	ζ_{21}	x_1^*	x_2^*	x_3^*	δ_{10}^*	δ_{20}^*	δ_{30}^*	δ_{11}^*	δ_{21}^*	f^*
0.1	0.482	0.816	2.806	0.429	0.801	0.124	0.438	0.438	0.376	0.753	128.262
0.2	0.544	0.877	2.981	0.429	0.861	0.098	0.451	0.451	0.402	0.805	142.233
0.3	0.592	0.925	3.109	0.429	0.908	0.082	0.459	0.459	0.418	0.836	153.839
0.4	0.631	0.965	3.210	0.429	0.948	0.072	0.464	0.464	0.429	0.857	163.818
0.5	0.667	1.000	3.300	0.429	0.982	0.064	0.468	0.468	0.436	0.879	173.050
0.6	0.702	1.035	3.386	0.429	1.016	0.056	0.472	0.472	0.443	0.886	182.426
0.7	0.742	1.075	3.481	0.429	1.056	0.050	0.475	0.475	0.450	0.900	193.439
0.8	0.789	1.123	3.590	0.429	1.103	0.044	0.478	0.478	0.456	0.913	206.934
0.9	0.851	1.184	3.728	0.429	1.163	0.036	0.482	0.482	0.463	0.927	224.949

model of uncertain primal GP problem given in equation (4.6) is

$$\begin{aligned} &\min \quad 23.33x_1^{-1}x_2^{-\frac{1}{2}}x_3^{-1} + 25x_1x_3 + 58.33x_1x_2x_3 \\ &\text{s.t.} \\ &\qquad \zeta_{11}x_1^{-2}x_2^{-2} + \zeta_{21}x_2^{\frac{1}{2}}x_3^{-1} \le 1, \\ &\qquad x_1, x_2, x_3 > 0. \end{aligned} \tag{4.7}$$

The equivalent dual problem is

$$\begin{aligned} &\max \quad \left(\frac{23.33}{\delta_{10}}\right)^{\delta_{10}} \left(\frac{25}{\delta_{20}}\right)^{\delta_{20}} \left(\frac{58.33}{\delta_{30}}\right)^{\delta_{30}} \left(\frac{\zeta_{11}}{\delta_{11}}\right)^{\delta_{11}} \left(\frac{\zeta_{21}}{\delta_{21}}\right)^{\delta_{21}} (\delta_{11}+\delta_{21})^{(\delta_{11}+\delta_{21})} \\ &\text{s.t.} \\ &\qquad \delta_{10} + \delta_{20} + \delta_{30} = 1, \\ &\qquad -\delta_{10} + \delta_{20} + \delta_{30} - 2\delta_{11} = 0, \\ &\qquad -\frac{1}{2}\delta_{10} + \delta_{30} - 2\delta_{11} + \frac{1}{2}\delta_{21} = 0, \\ &\qquad -\delta_{10} + \delta_{20} + \delta_{30} - \delta_{21} = 0. \end{aligned} \tag{4.8}$$

We determine the values of ζ_{11}, ζ_{21} by (3.7) as

$$\zeta_{11} = \begin{cases} \frac{1+\sqrt{2\gamma}}{3}, & \text{if } 0 < \gamma \le \frac{1}{2}; \\ \frac{3-\sqrt{2(1-\gamma)}}{3}, & \text{if } \frac{1}{2} \le \gamma < 1. \end{cases}$$

$$\zeta_{21} = \begin{cases} \frac{2+\sqrt{2\gamma}}{3}, & \text{if } 0 < \gamma \le \frac{1}{2}; \\ \frac{4-\sqrt{2(1-\gamma)}}{3}, & \text{if } \frac{1}{2} \le \gamma < 1. \end{cases}$$

Subsequently, solve the dual problem (4.8) for different uncertainty levels $\gamma \in (0,1)$ and provide the primal-dual solutions corresponding to different uncertainty levels in Table 3.

Trapezoidal case.

We follow the similar approach for this case. Let $\widetilde{\beta_{10}} \sim T(10,25,35,50)$, $\widetilde{\beta_{20}} \sim T(12,20,30,40)$, $\widetilde{\beta_{30}} \sim T(30,45,60,70)$, $\widetilde{\beta_{11}} \sim T(\frac{1}{3},\frac{2}{3},1,\frac{4}{3})$ and $\widetilde{\beta_{21}} \sim T(\frac{2}{3},1,\frac{4}{3},2)$. Then, by Lemma (3.3), we get $E[\widetilde{\beta_{10}}] = 30, E[\widetilde{\beta_{20}}] =$

$25.58, E[\widetilde{\beta_{30}}] = 51.06$. Therefore, the equivalent crisp model of uncertain primal GP problem given in equation (4.6) is

$$\begin{aligned} \min \quad & 30x_1^{-1}x_2^{-\frac{1}{2}}x_3^{-1} + 25.58x_1x_3 + 51.06x_1x_2x_3 \\ \text{s.t.} \quad & \\ & \zeta_{11}x_1^{-2}x_2^{-2} + \zeta_{21}x_2^{\frac{1}{2}}x_3^{-1} \leq 1, \\ & x_1, x_2, x_3 > 0. \end{aligned} \tag{4.9}$$

The equivalent dual problem is

$$\begin{aligned} \max \quad & \left(\frac{30}{\delta_{10}}\right)^{\delta_{10}} \left(\frac{25.58}{\delta_{20}}\right)^{\delta_{20}} \left(\frac{51.06}{\delta_{30}}\right)^{\delta_{30}} \left(\frac{\zeta_{11}}{\delta_{11}}\right)^{\delta_{11}} \left(\frac{\zeta_{21}}{\delta_{21}}\right)^{\delta_{21}} (\delta_{11}+\delta_{21})^{(\delta_{11}+\delta_{21})} \\ \text{s.t.} \quad & \\ & \delta_{10} + \delta_{20} + \delta_{30} = 1, \\ & -\delta_{10} + \delta_{20} + \delta_{30} - 2\delta_{11} = 0, \\ & -\frac{1}{2}\delta_{10} + \delta_{30} - 2\delta_{11} + \frac{1}{2}\delta_{21} = 0, \\ & -\delta_{10} + \delta_{20} + \delta_{30} - \delta_{21} = 0. \end{aligned} \tag{4.10}$$

Using (3.11), we determine the values of ζ_{11}, ζ_{21} as

$$\zeta_{11} = \begin{cases} \frac{1+2\sqrt{\gamma}}{3}, & \text{if } 0 < \gamma \leq \frac{1}{4}; \\ \frac{3+4\gamma}{6}, & \text{if } \frac{1}{4} \leq \gamma \leq \frac{3}{4}; \\ \frac{4-2\sqrt{(1-\gamma)}}{3}, & \text{if } \frac{3}{4} \leq \gamma < 1. \end{cases}$$

$$\zeta_{21} = \begin{cases} \frac{2+\sqrt{5\gamma}}{3}, & \text{if } 0 < \gamma \leq \frac{1}{5}; \\ \frac{5+5\gamma}{6}, & \text{if } \frac{1}{5} \leq \gamma \leq \frac{3}{5}; \\ \frac{6-\sqrt{10(1-\gamma)}}{3}, & \text{if } \frac{3}{5} \leq \gamma < 1. \end{cases}$$

Subsequently, solve the dual problem (4.10) for different uncertainty levels $\gamma \in (0,1)$ and provide the primal-dual solutions corresponding to different uncertainty levels in Table 4.

The expected values of the objective function under triangular and trapezoidal cases for different uncertainty levels $\gamma \in (0,1)$ are shown in Figure 4.

5. Application in inventory model

We solve a real application based problem under the uncertainty based framework using GP technique applied in an inventory model. Shiraz *et al.* [27] solved an uncertain inventory model under normal, linear, and zigzag UDs. We consider similar kind of inventory model under triangular and trapezoidal UDs. In this model, our assumptions are the followings.

– Demand is affected by selling price.
– Replenishment is instantaneous.
– No shortage is allowed.
– p, price per unit in Rs.
– q, order quantity in units.

TABLE 4. Optimal solutions under trapezoidal UD.

γ	ζ_{11}	ζ_{21}	x_1^*	x_2^*	x_3^*	δ_{10}^*	δ_{20}^*	δ_{30}^*	δ_{11}^*	δ_{21}^*	f^*
0.1	0.544	0.902	2.550	0.501	0.958	0.122	0.439	0.439	0.378	0.756	142.290
0.2	0.631	1.000	2.746	0.501	1.062	0.088	0.456	0.456	0.411	0.822	163.708
0.3	0.700	1.083	2.893	0.501	1.150	0.070	0.465	0.465	0.430	0.861	182.901
0.4	0.767	1.167	3.028	0.501	1.239	0.056	0.472	0.472	0.444	0.889	203.227
0.5	0.833	1.250	3.155	0.501	1.327	0.046	0.477	0.477	0.455	0.910	224.364
0.6	0.900	1.333	3.280	0.501	1.415	0.038	0.481	0.481	0.463	0.926	246.609
0.7	0.967	1.423	3.400	0.501	1.511	0.030	0.485	0.485	0.470	0.939	271.031
0.8	1.035	1.529	3.517	0.501	1.623	0.024	0.488	0.488	0.475	0.950	299.536
0.9	1.123	1.667	3.664	0.501	1.770	0.020	0.490	0.490	0.481	0.961	338.277

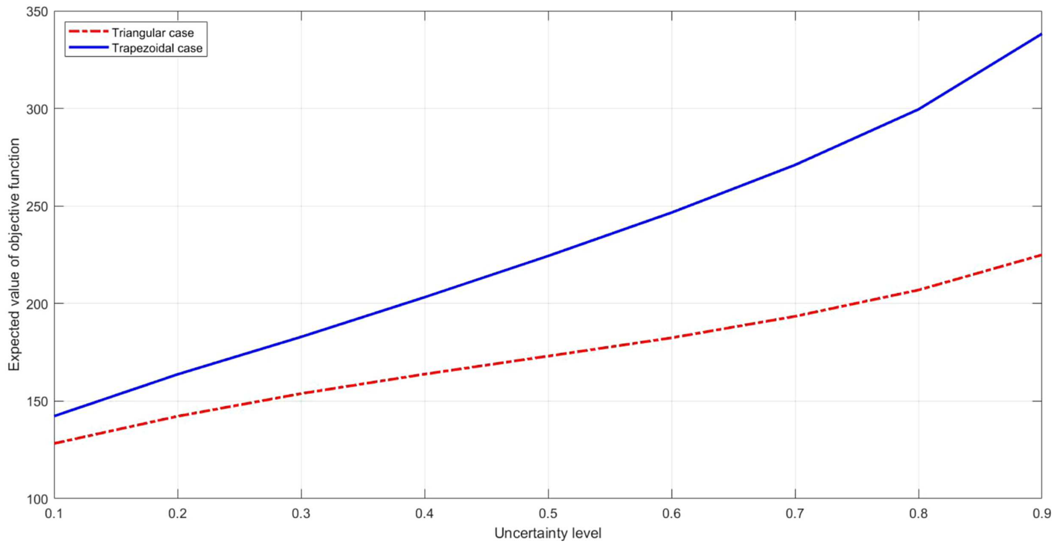

FIGURE 4. Expected values of objective function under triangular and trapezoidal cases.

– The demand per unit $D(p)$ is a decreasing function of price per unit.
  i.e., $D(p) = \beta_1 p^{-\alpha_1}, \beta_1 > 0$ is constant, $\alpha_1 > 1$ is the price elasticity of demand.
– The ordering cost is $\frac{\beta_2 D(p)}{q}$, where β_2 is the ordering cost per batch.
– The purchasing cost per unit is $C(q) = \beta_3 q^{-\alpha_3}, \beta_3 > 0, \alpha_3 \in (0, 1)$ is the cost elasticity of product.
– The holding cost is $\frac{1}{2}\beta_4 q C(q)$, where $\beta_4 > 0$ is the inventory carrying cost rate per unit time.

Therefore, the profit function $P(p, q)$ is given by

$$\begin{aligned} \max \quad & P(p,q) = \text{Revenue} - \text{Ordering cost} - \text{Holding cost} - \text{Purchasing cost} \\ &= pD(p) - \frac{\beta_2 D(p)}{q} - \frac{1}{2}\beta_4 q C(q) - C(q)D(p) \\ &= \beta_1 p^{-\alpha_1+1} - \beta_2\beta_1 p^{-\alpha_1} q^{-1} - \frac{1}{2}\beta_4\beta_3 q^{-\alpha_3+1} - \beta_3\beta_1 p^{-\alpha_1} q^{-\alpha_3}. \end{aligned} \tag{5.1}$$

So, the uncertain inventory model is formulated as

$$\begin{aligned} &\max \quad T \\ &\text{s.t.} \\ &\beta_1 p^{-\alpha_1+1} - \beta_2\beta_1 p^{-\alpha_1} q^{-1} - \frac{1}{2}\beta_4\beta_3 q^{-\alpha_3+1} - \beta_3\beta_1 p^{-\alpha_1} q^{-\alpha_3} \geq T. \end{aligned} \tag{5.2}$$

In standard form, the model (5.2) is of the form

$$\begin{aligned} &\min \quad T^{-1} \\ &\text{s.t.} \\ &\beta_2 p^{-1} q^{-1} + \frac{1}{2}\beta_1^{-1}\beta_3\beta_4 p^{\alpha_1-1} q^{-\alpha_3+1} + \beta_3 p^{-1} q^{-\alpha_3} + \beta_1^{-1} p^{\alpha_1-1} T \leq 1. \end{aligned} \tag{5.3}$$

Let us assume that $\beta_1, \beta_2, \beta_3, \beta_4$ be independent UVs. We choose the price elasticity of demand $\alpha_1 = 1.5$ and the cost elasticity of product $\alpha_3 = 0.02$. Therefore, the equivalent deterministic primal GP model is

$$\begin{aligned} &\min \quad T^{-1} \\ &\text{s.t.} \\ &\zeta_2 p^{-1} q^{-1} + \frac{\zeta_3\zeta_4}{2\zeta_1} p^{0.5} q^{0.98} + \zeta_3 p^{-1} q^{-0.02} + \frac{1}{\zeta_1} p^{0.5} T \leq 1. \end{aligned} \tag{5.4}$$

Consequently, the dual problem of model (5.4) is

$$\begin{aligned} &\max \quad \left(\frac{1}{\delta_{10}}\right)^{\delta_{10}} \left(\frac{\zeta_2}{\delta_{11}}\right)^{\delta_{11}} \left(\frac{\zeta_3\zeta_4}{2\zeta_1\delta_{21}}\right)^{\delta_{21}} \left(\frac{\zeta_3}{\delta_{31}}\right)^{\delta_{31}} \left(\frac{1}{\zeta_1\delta_{41}}\right)^{\delta_{41}} (\lambda)^{\lambda} \\ &\text{s.t.} \\ &\delta_{10} = 1, \\ &-\delta_{10} + \delta_{41} = 0, \\ &-\delta_{11} + 0.5\delta_{21} - \delta_{31} + 0.5\delta_{41} = 0, \\ &-\delta_{11} + 0.98\delta_{21} - 0.02\delta_{31} = 0, \\ &\lambda = (\delta_{11} + \delta_{21} + \delta_{31} + \delta_{41}), \end{aligned} \tag{5.5}$$

where $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ are crisp values of $\beta_1, \beta_2, \beta_3, \beta_4$ respectively.

Triangular case. Let $\beta_1 \sim \nabla(15000, 18000, 21000)$, $\beta_2 \sim \nabla(60, 90, 110)$, $\beta_3 \sim \nabla(4, 6, 9)$ and $\beta_4 \sim \nabla(0.2, 0.4, 0.6)$. Therefore, using (3.7), the crisp values $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ are determined as

$$\zeta_1 = \begin{cases} 15000 + 3000\sqrt{2\gamma}, & \text{if } 0 < \gamma \leq \frac{1}{2}; \\ 21000 - 3000\sqrt{2(1-\gamma)}, & \text{if } \frac{1}{2} \leq \gamma < 1. \end{cases}$$

$$\zeta_2 = \begin{cases} 60 + 10\sqrt{15\gamma}, & \text{if } 0 < \gamma \leq \frac{3}{5}; \\ 110 - 10\sqrt{10(1-\gamma)}, & \text{if } \frac{3}{5} \leq \gamma < 1. \end{cases}$$

$$\zeta_3 = \begin{cases} 4 + \sqrt{10\gamma}, & \text{if } 0 < \gamma \leq \frac{2}{5}; \\ 9 - \sqrt{15(1-\gamma)}, & \text{if } \frac{2}{5} \leq \gamma < 1. \end{cases}$$

Table 5. Optimal solutions under triangular UD.

Uncertainty level (γ)	Price per unit in Rs. (p^*)	Order quantity in units (q^*)	Profit (P^*)
0.25	16	172	2664
0.50	19	143	2605
0.75	22	119	2530

$$\zeta_4 = \begin{cases} 0.2 + \sqrt{0.08\gamma}, & \text{if } 0 < \gamma \le \frac{1}{2}; \\ 0.6 - \sqrt{0.08(1-\gamma)}, & \text{if } \frac{1}{2} \le \gamma < 1. \end{cases}$$

Hence, we solve the dual problem (5.5) for different uncertainty levels $\gamma \in (0,1)$ for triangular UDs. In particular, for the uncertainty levels $\gamma = 0.25$, $\gamma = 0.50$, and $\gamma = 0.75$, we find the optimal price per unit, optimal order quantity in units and maximum profit which are given in Table 5.

Trapezoidal case.

Let $\beta_1 \sim T(14000, 18000, 20000, 22000), \beta_2 \sim T(50, 80, 100, 120), \beta_3 \sim (4, 6, 7, 10), \beta_4 \sim (0.2, 0.4, 0.6, 0.8)$. Therefore, using (3.11), the crisp values $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ are determined as

$$\zeta_1 = \begin{cases} 14000 + 2000\sqrt{10\gamma}, & \text{if } 0 < \gamma \le \frac{2}{5}; \\ 16000 + 5000\gamma, & \text{if } \frac{2}{5} \le \gamma \le \frac{4}{5}; \\ 22000 - 2000\sqrt{5(1-\gamma)}, & \text{if } \frac{4}{5} \le \gamma < 1. \end{cases}$$

$$\zeta_2 = \begin{cases} 50 + 30\sqrt{3\gamma}, & \text{if } 0 < \gamma \le \frac{1}{3}; \\ 65 + 45\gamma, & \text{if } \frac{1}{3} \le \gamma \le \frac{7}{9}; \\ 120 - 30\sqrt{2(1-\gamma)}, & \text{if } \frac{7}{9} \le \gamma < 1. \end{cases}$$

$$\zeta_3 = \begin{cases} 4 + \sqrt{14\gamma}, & \text{if } 0 < \gamma \le \frac{2}{7}; \\ \frac{10+7\gamma}{2}, & \text{if } \frac{2}{7} \le \gamma \le \frac{4}{7}; \\ 10 - \sqrt{21(1-\gamma)}, & \text{if } \frac{4}{7} \le \gamma < 1. \end{cases}$$

$$\zeta_4 = \begin{cases} 0.2 + 0.4\sqrt{\gamma}, & \text{if } 0 < \gamma \le \frac{1}{4}; \\ 0.3 + 0.4\gamma, & \text{if } \frac{1}{4} \le \gamma \le \frac{3}{4}; \\ 0.8 - 0.4\sqrt{(1-\gamma)}, & \text{if } \frac{3}{4} \le \gamma < 1. \end{cases}$$

Hence, as earlier, we solve the dual problem (5.5) for different uncertainty levels $\gamma \in (0,1)$ for trapezoidal UDs. In particular, for the uncertainty levels $\gamma = 0.25$, $\gamma = 0.50$, and $\gamma = 0.75$, we find the optimal price per unit, optimal order quantity in units and maximum profit which are given in the Table 6.

One obvious question from the preceding two cases is what happens if we change the uncertainty level γ. In that situation, we get a different solution due to the changes in feasible space. Therefore, the decision maker has to be very careful about uncertainty levels and what kind of uncertainty is present in that model.

Table 6. Optimal solutions under trapezoidal UD.

Uncertainty level (γ)	Price per unit in Rs. (p^*)	Order quantity in units (q^*)	Profit (P^*)
0.25	18	143	2583
0.50	21	115	2537
0.75	24	95	2456

6. Conclusion

Geometric programming (GP) is a powerful optimization tool for solving a wide range of optimization and engineering design problems. Traditional GP problems assume that the parameters of GP problems are deterministic and specific. However, in real-world GP problems, the parameters may be uncertain and imprecise. Recently, uncertainty theory has been developed to deal with such kinds of uncertain problems. In this article, we solve an uncertain GP problem under triangular and trapezoidal uncertainty distributions. Here, we develop the equivalent chance-constrained GP models. We solve it conventionally by transforming the uncertain GP models into crisp GP models. Two numerical examples are given to demonstrate the effectiveness of the techniques and algorithms. We solve two numerical GP problems under triangular and trapezoidal uncertainty distributions for different uncertainty levels γ and find the corresponding expected values of the cost function. Following that, we plot the expected cost function in relation to the uncertainty level γ. Lastly, we add an example of how our proposed method could be used in an inventory model to make the most profit.

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Data availability. Data sharing not applicable to this article as no datasets were generated. A random data set is taken for the examples and the inventory model given in this paper.

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