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NORDHAUS–GADDUM TYPE INEQUALITIES ON THE TOTAL ITALIAN DOMINATION NUMBER IN GRAPHS

Seyed Mahmoud Sheikholeslami[1,*] and Lutz Volkmann[2]

Abstract. Let G be a graph with vertex set $V(G)$. A total Italian dominating function (TIDF) on a graph G is a function $f : V(G) \longrightarrow \{0,1,2\}$ such that (i) every vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$, and (ii) every vertex v with $f(v) \geq 1$ is adjacent to a vertex u with $f(u) \geq 1$. The total Italian domination number $\gamma_{tI}(G)$ on a graph G is the minimum weight of a total Italian dominating function. In this paper, we present Nordhaus–Gaddum type inequalities for the total Italian domination number.

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1. Introduction

For definitions and notations not given here we refer to [15]. We consider simple and finite graphs G without isolated vertices with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. The *neighborhood* of a vertex v is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$. The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The *complement* of a graph G is denoted by $\overline{G}$. For a subset D of vertices in a graph G, we denote by $G[D]$ the subgraph of G induced by D. The *diameter* of a connected graph G, denoted by $\operatorname{diam}(G)$, is the greatest distance between two vertices of G. We write P_n for the *path* of order n, C_n for the *cycle* of length n and K_n for the *complete graph* of order n. The *corona* $G = F \circ K_1$ of a graph F is that graph obtained from F by adding a pendant edge to each vertex of F.

A subset $D \subseteq V$ is a *(total) dominating set* of G if every vertex in $V - D$ (V) has a neighbor in D. The *(total) domination number* $\gamma(G)$ $(\gamma_t(G))$ is the minimum cardinality of a (total) dominating set of G.

In this paper we continue the study of Roman and Italian dominating functions in graphs (see, *e.g.*, the survey articles [9–11]). If $f : V(G) \longrightarrow \{0,1,2\}$ is a function, then let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0,1,2\}$. There is a 1–1 correspondence between the function f and the ordered partition (V_0, V_1, V_2). So we also write $f = (V_0, V_1, V_2)$. A *Roman dominating function* (RDF) on a graph G is defined in [12] as a function $f : V(G) \longrightarrow \{0,1,2\}$ such that every vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) = 2$. The weight of an RDF f is the value

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[1] Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran.

[2] Lehrstuhl II fur Mathematik, RWTH Aachen University, 52056 Aachen, Germany.

[*]Corresponding author: s.m.sheikholeslami@azaruniv.ac.ir

$\omega(f) = f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G.

A *total Roman dominating function* (TRDF) on a graph G without isolated vertices is defined in [17] as a Roman dominating function f with the property that the subgraph induced by $V_1 \cup V_2$ has no isolated vertex. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a TRDF on G. A TRDF on G with weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$-function. Total Roman domination is studied in [1, 4–7, 18].

The concept of Italian domination has been introduced in 2016 by Chellali *et al.* [8] as a new variation of Roman domination but called differently, Roman {2}-domination. An *Italian dominating function* (IDF, for short) on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ having the property that $f(N(u)) \geq 2$ for each vertex u with $f(u) = 0$. The weight of an IDF f is the value $\omega(f) = f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Italian domination number* $\gamma_I(G)$ is the minimum weight of an IDF on G. A *total Italian dominating function* (TIDF) on a graph G without isolated vertices is an Italian dominating function f with the property that every vertex v with $f(v) \geq 1$ has a neighbor u with $f(u) \geq 1$. The *total Italian domination number* $\gamma_{tI}(G)$ is the minimum weight of a TIDF on G. A TIDF on G with weight $\gamma_{tI}(G)$ is called a $\gamma_{tI}(G)$-function. The (total) Italian domination number has been studied by several authors [2, 3, 7, 14, 16, 19, 21].

If G is a graph without isolated vertices, then the definitions lead to $\gamma_I(G) \leq \gamma_{tI}(G) \leq \gamma_{tR}(G)$.

In this paper, we present Nordhaus–Gaddum type inequalities for total Italian domination. In particular, we prove $7 \leq \gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n + 4$ and $\gamma_{tI}(G) \cdot \gamma_{tI}(\overline{G}) \leq 6n - 8$, if G and $\overline{G}$ are graphs of order $n \geq 4$ without isolated vertices. If G and $\overline{G}$ are graphs of order $n \geq 12$ without isolated vertices, then we even show that $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n + 3$.

The following results will be useful in the rest of the paper.

Observation 1.1. Let G be a graph without isolated vertices. If f is a TIDF on G, then $f(N[u]) \geq 2$ for each vertex $u \in V(G)$.

Proof. If $f(u) = 0$, then the definition leads to $f(N[u]) = f(N(u)) \geq 2$. If $f(u) \geq 1$, then u has a neighbor v with $f(v) \geq 1$ and therefore $f(N[u]) \geq 2$. □

Observation 1.2. Let G be a nontrivial connected graph of order n. If $\mathrm{diam}(G) \geq 3$, then $\gamma_{tI}(\overline{G}) \leq 4$.

Proof. Let x, y be two vertices of G at distance $\mathrm{diam}(G)$ and define $g : V(\overline{G}) \to \{0, 1, 2\}$ by $g(x) = g(y) = 2$ and $f(u) = 0$ for the remaining vertices. Obviously, g is a TIDF of $\overline{G}$ and hence $\gamma_{tI}(\overline{G}) \leq 4$. □

Observation 1.3. Let G be a graph of order n with $\delta \geq 2$. Then $\gamma_{tI}(G) \leq n - \delta + 1$.

Proof. Let $x_1, x_2, \ldots, x_{\delta-1}$ be arbitrary vertices of G and define $g : V(G) \to \{0, 1, 2\}$ by $g(x_i) = 0$ for $1 \leq i \leq \delta-1$ and $f(u) = 1$ for the remaining vertices. Obviously, g is a TIDF of G and hence $\gamma_{tI}(\overline{G}) \leq n - \delta + 1$. □

Theorem 1.4 ([3, 14])**.** *Let G be an nontrivial connected graph of order n. Then*

(1) $\gamma_{tI}(G) = 2$ *if and only if G has two vertices of degree $n - 1$.*
(2) $\gamma_{tI}(G) \leq n$, *with equality if and only if $G \in \{K_2, K_{1,2}\}$ or every vertex of G is either a leaf or a weak support vertex.*

Theorem 1.5. (1) *[14] For any graph G of order n and $\delta(G) \geq 2$, $\gamma_{tI}(G) \leq \left\lfloor \frac{n+\gamma_t(G)}{2} \right\rfloor$.*
(2) *[14] For any graph G of order n with $\delta(G) \geq 3$, $\gamma_{tI}(G) \leq \frac{3n}{4}$.*
(3) *For any graph G of order n with $\delta(G) \geq 4$, $\gamma_{tI}(G) \leq \frac{5n}{7}$.*
(4) *For any graph G of order n with $\delta(G) \geq 5$, $\gamma_{tI}(G) \leq \frac{61n}{88}$.*

Proof. It is enough to prove (3) and (4). It is proved that $\gamma_t(G) \leq 3n/7$ for any graph G of order n with $\delta(G) \geq 4$ (see [20]) and that $\gamma_t(G) \leq 17n/44$ for any graph G of order n with $\delta(G) \geq 5$ (see [13]). Applying Item (1), we obtain $\gamma_{tI}(G) \leq \frac{5n}{7}$ if $\delta(G) \geq 4$ and $\gamma_{tI}(G) \leq \frac{61n}{88}$ if $\delta(G) \geq 5$. □

2. Nordhaus–Gaddum bounds for total Italian domination

In this section, we present Nordhaus–Gaddum type results for the total Italian domination number. We first provide upper bounds on the total Italian domination number.

A set $S \subseteq V(G)$ is a *packing* of a graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The *packing number* $\rho(G)$ is defined by

$$\rho(G) = \max\{|S| : S \text{ is a packing of } G\}.$$

Theorem 2.1. *If G is a graph without isolated vertices, then $\gamma_{tI}(G) \geq 2\rho(G)$.*

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ be a packing of G, and let f be a $\gamma_{tI}(G)$-function. If we define the set $A = \bigcup_{i=1}^{\rho(G)} N[v_i]$, then, since $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ is a packing, it follows from Lemma 1.1 that

$$\begin{aligned}\gamma_{tI}(G) &= \sum_{x \in V(G)} f(x) = \sum_{x \in A} f(x) + \sum_{x \in V(G)\setminus A} f(x)\\ &= \sum_{i=1}^{\rho(G)} f(N[v_i]) + \sum_{x \in V(G)\setminus A} f(x)\\ &\geq \sum_{i=1}^{\rho(G)} 2 + \sum_{x \in V(G)\setminus A} f(x) \geq 2\rho(G).\end{aligned}$$

□

Example 2.2. Let K_p be a complete graph, and let $X_1, X_2, \ldots, X_t$ be a partition of $V(K_p)$ with $X_1 \cup X_2 \cup \ldots \cup X_t = V(K_p)$ and $|X_i| \geq 1$ for $1 \leq i \leq t$. Now let F be the graph consisting of K_p and t further vertices $v_1, v_2, \ldots, v_t$ such that v_i is adjacent to all vertices of X_i for $1 \leq i \leq t$. We observe that $v_1, v_2, \ldots, v_t$ is a packing of F, and therefore $\gamma_{tI}(F) \geq 2t$ according to Theorem 2.1.

Let next $x_i \in X_i$ for $1 \leq i \leq t$. Then the function f defined by $f(v_i) = f(x_i) = 1$ for $1 \leq i \leq t$ and $f(x) = 0$ otherwise, is a TIDF on F. Therefore $\gamma_{tI}(F) \leq 2t$ and thus $\gamma_{tI}(F) = 2t$.

Example 2.2 shows that Theorem 2.1 is sharp. Let $\delta^* = \delta^*(G) = \min\{\delta(G), \delta(\overline{G})\}$.

Theorem 2.3. *Let G be a graph of order n with* $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 2$. *If $\delta^*(G) = \delta(G)$, then* $\gamma_{tI}(G) \leq \min\left\{2\delta^*(G) + 1, \frac{n+\delta^*+1}{2}\right\}$ *and* $\gamma_{tI}(\overline{G}) \leq \delta^*(G) + 3$.

Proof. We deduce from $\operatorname{diam}(G) = \operatorname{diam}(\overline{G}) = 2$ that $\delta^*(G) \geq 2$. Let x be a vertex of minimum degree $\delta^*(G)$ in G and let $Y = V(G) \setminus N_G[x]$. Since $\operatorname{diam}(G) = 2$, any vertex in $y \in Y$ has at least one neighbor in $N_G(x)$ and since $\delta^*(G) \geq 2$, each isolated vertex in the induced subgraph $G[Y]$ has at least two neighbors in $N_G(x)$. Let I be the set of isolated vertices of $G[Y]$ and let S be a minimum dominating set of $G[Y - I]$. Then the function $f : V(G) \longrightarrow \{0, 1, 2\}$ defined by $f(u) = 1$ if $u \in N_G[x] \cup S$ and $f(z) = 0$ for the remaining vertices, is a TIDF of G. Using Ore's Theorem we obtain $\gamma_{tI}(G) \leq w(f) = |N_G[x]| + |Y - I|/2 = \delta^* + 1 + \frac{n-\delta^*-1}{2} = \frac{n+\delta^*+1}{2}$. On the other hand, the function $g : V(G) \longrightarrow \{0, 1, 2\}$ defined by $g(x) = 1, g(u) = 2$ if $u \in N_G(x)$ and $g(z) = 0$ for the remaining vertices, is a TIDF of G. It follows that $\gamma_{tI}(G) \leq w(g) = 2|N_G(x)| + 1 = 2\delta^*(G) + 1$.

Next we show that $\gamma_{tI}(\overline{G}) \leq \delta^*(G) + 3$. Let I be the set of isolated vertices of $\overline{G}[N_G(x)]$. If $I = \emptyset$, then the function $f : V(\overline{G}) \longrightarrow \{0, 1, 2\}$ defined by $f(z) = 1$ for $z \in N_G(x)$, $f(x) = 2$, $f(y) = 1$ for some $y \in Y$ and $f(u) = 0$ otherwise, is a TIDF of $\overline{G}$ and so $\gamma_{tI}(\overline{G}) \leq w(g) = |N_G(x)| + 3 = \delta^*(G) + 3$. Assume that $I \neq \emptyset$ and let $I = \{v_1, \ldots, v_t\}$. It follows from $\operatorname{diam}(\overline{G}) = 2$ that $d_{\overline{G}}(x, v_i) = 2$ and so x and v_i have a common neighbor $u_i \in Y$ in $\overline{G}$ for each $1 \leq i \leq t$. First let t be even. Then $d_{\overline{G}}(v_{2i-1}, v_{2i}) = 2$ and we may assume, without loss of generality, that $u_{2i-1} = u_{2i}$ for each $1 \le i \le t/2$. Then the function $g : V(\overline{G}) \longrightarrow \{0,1,2\}$ defined by $g(x) = 2, g(u_{2i}) = 2$ for $1 \le i \le t/2$, $g(z) = 1$ for each $z \in N_G(x) - I$ and $g(z) = 0$ otherwise, is a TIDF of $\overline{G}$ and so $\gamma_{tI}(\overline{G}) \le w(g) = 2 + \delta^*(G)$. Now let t be odd. As before, we may assume that $u_{2i-1} = u_{2i}$ for each $1 \le i \le (t-1)/2$. Then the function $g : V(\overline{G}) \longrightarrow \{0,1,2\}$ defined by $g(x) = g(u_t) = 2, g(u_{2i}) = 2$ for $1 \le i \le (t-1)/2$, $g(z) = 1$ for each $z \in N_G(x) - I$ and $g(z) = 0$ for the remaining vertices, is a TIDF of $\overline{G}$ and so $\gamma_{tI}(\overline{G}) \le w(g) = 3 + \delta^*(G)$. □

Using a similar argument as in the proof of Theorem 2.3 we obtain the next result.

Corollary 2.4. *If G is a graph of order n with $\mathrm{diam}(G) = 2$ and $2 \le \delta(G) \le \frac{n-1}{2}$, then*

$$\gamma_{tI}(G) \le \frac{3n+1}{4}\cdot$$

This bound is sharp for C_5.

Proof. Let x be a vertex of minimum degree $\delta(G)$ in G and let $Y = V(G) \setminus N_G[x]$. Since $\mathrm{diam}(G) = 2$, any vertex in $y \in Y$ has at least one neighbor in $N_G(x)$ and since $\delta(G) \ge 2$, each isolated vertex in the induced subgraph $G[Y]$ has at least two neighbors in $N_G(x)$. Let I be the set of isolated vertices of $G[Y]$ and let S be a minimum dominating set of $G[Y - I]$. Then the function $f : V(G) \longrightarrow \{0,1,2\}$ defined by $f(u) = 1$ if $u \in N_G[x] \cup S$ and $f(z) = 0$ for the remaining vertices, is a TIDF of G. Using Ore's theorem we obtain $\gamma_{tI}(G) \le w(f) = |N_G[x]| + |Y - I|/2 = \delta(G) + 1 + \frac{n-\delta(G)-1}{2} = \frac{n+\delta(G)+1}{2}$. Since $\delta(G) \le \frac{n-1}{2}$, we obtain $\gamma_{tI}(G) \le \frac{3n+1}{4}$. □

Theorem 2.5. *Let G be a graph of order n with $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. If $\delta^* = \frac{n-1}{2}$, then $\gamma_{tI}(G) \le \max\{\delta^*+1, 5\}$ and $\gamma_{tI}(\overline{G}) \le \max\{\delta^*+1, 5\}$.*

Proof. We deduce from $\delta^* = \frac{n-1}{2}$ that both G and $\overline{G}$ are δ^*-regular. Hence it is enough to show that $\gamma_{tI}(G) \le \max\{\delta^*+1, 5\}$. Let v be a vertex of minimum degree δ^* in G and let $Y = V(G) \setminus N_G[v]$. Since $\mathrm{diam}(G) = 2$, each vertex in Y has at least one neighbor in $N_G(v)$. If each vertex in Y has at least two neighbors in $N_G(v)$, then the function f defined on G by $f(x) = 1$ for $x \in N[v]$ and $f(x) = 0$ otherwise, is a TIDF of G and so $\gamma_{tI}(G) \le \delta^* + 1$. Hence we assume that there exists a vertex $y \in Y$ such that y has exactly one neighbor w in $N_G(v)$. Then y is adjacent to all vertices in $Y - \{y\}$ and the function f defined on G by $f(v) = f(y) = 2$, $f(w) = 1$ and $f(x) = 0$ otherwise, is a TIDF of G of weight 5 and so $\gamma_{tI}(G) \le 5$. Thus $\gamma_{tI}(G) \le \max\{\delta^*+1, 5\}$. □

Theorem 2.6. *Let G be a graph of order n with $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. If $\delta^* = \frac{n-2}{2}$, then one of the following hold.*

(1) $\gamma_{tI}(G) \le \max\{\delta^*+1, 7\}$ *and* $\gamma_{tI}(\overline{G}) \le \max\{\delta^*+2, 5\}$.
(2) $\gamma_{tI}(G) \le \max\{\delta^*+2, 5\}$ *and* $\gamma_{tI}(\overline{G}) \le \max\{\delta^*+1, 7\}$.

Proof. Assume that $\delta(G) = \delta^* = \frac{n-2}{2}$ (the case $\delta(\overline{G}) = \delta^* = \frac{n-2}{2}$ is similar). Let v be a vertex of minimum degree δ^* in G and let $Y = V(G) \setminus N_G[v]$. Since $\mathrm{diam}(G) = 2$, each vertex in Y has at least one neighbor in $N_G(v)$. If each vertex in Y has at least two neighbors in $N_G(v)$, then as in the proof of Theorem 2.5 we have $\gamma_{tI}(G) \le \delta^* + 1$. Hence we assume that there exists a vertex $y \in Y$ such that y has exactly one neighbor w in $N_G(v)$. Then y is adjacent to all vertices in $Y - \{y\}$ but at most one. If y is adjacent to all vertices in $Y - \{y\}$, then as in the proof of Theorem 2.5 we have $\gamma_{tI}(G) \le 5$. Hence, suppose y is not adjacent to a vertex $y' \in Y - \{y\}$. Let $w_1 \in N(v) \cap N(y)$ and $w_2 \in N(v) \cap N(y')$ and define the function f on G by $f(v) = f(y) = f(w_2) = 2$, $f(w_1) = 1$ and $f(x) = 0$ otherwise. Clearly f is a TIDF of G of weight 7 and so $\gamma_{tI}(G) \le 7$. Thus $\gamma_{tI}(G) \le \max\{\delta^*+1, 7\}$.

Now we show that $\gamma_{tI}(\overline{G}) \leq \max\{\delta^*(G)+2, 5\}$. We deduce from $\delta(G) = \delta^* = \frac{n-2}{2}$ that $\Delta(\overline{G}) = \frac{n}{2}$. Let v be a vertex of maximum degree $\frac{n}{2}$ in $\overline{G}$ and let $Y = V(\overline{G}) \setminus N_{\overline{G}}[v]$. Since $\operatorname{diam}(\overline{G}) = 2$, each vertex in Y has at least one neighbor in $N_{\overline{G}}(v)$. If each vertex in Y has at least two neighbors in $N_{\overline{G}}(v)$, then as in the proof of Theorem 2.5 and using the fact that $\Delta(\overline{G}) = \delta^* + 1$ we have $\gamma_{tI}(\overline{G}) \leq \delta^* + 2$. Hence we assume that there exists a vertex $y \in Y$ such that y has exactly one neighbor w in $N_{\overline{G}}(v)$. Then y is adjacent to all vertices in $Y - \{y\}$ and as in the proof of Theorem 2.5 we have $\gamma_{tI}(\overline{G}) \leq 5$. Thus $\gamma_{tI}(\overline{G}) \leq \max\{\delta^* + 2, 5\}$. □

In the next result we present a sharp lower bound on $\gamma_{tI}(G) + \gamma_{tI}(\overline{G})$.

Theorem 2.7. *If G and $\overline{G}$ are graphs of order $n \geq 4$ without isolated vertices, then $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \geq 7$.*

Proof. Since G and $\overline{G}$ are without isolated vertices, we observe that $\Delta(G) \leq n-2$ and $\Delta(\overline{G}) \leq n-2$. Hence Theorem 1.4-(1) implies $\gamma_{tI}(G), \gamma_{tI}(\overline{G}) \geq 3$. Next let $\gamma_{tI}(G) = 3$. We will show that $\gamma_{tI}(\overline{G}) \geq 4$. If f is a $\gamma_{tI}(G)$-function, then $|V_2| = |V_1| = 1$ or $|V_2| = 0$ and $|V_1| = 3$. If $|V_2| = |V_1| = 1$, then it is easy to see that $\Delta(G) = n-1$, a contradiction. Thus assume that $|V_1| = 3$, and let v_1, v_2 and v_3 be the vertices with $f(v_1) = f(v_2) = f(v_3) = 1$, and assume, without loss of generality, that v_2 is adjacent to v_1 and v_3. Then every vertex $x \in V(G) \setminus \{v_1, v_2, v_3\}$ is adjacent to least two vertices of the set $\{v_1, v_2, v_3\}$. Let $X_{i,j}$ be the set of vertices exactly adjacent to v_i and v_j for $1 \leq i < j \leq 3$ and $X_{1,2,3}$ be the set of vertices adjacent to v_1, v_2 and v_3.

If $X_{1,3} = \emptyset$, then $\Delta(G) = n-1$, a contradiction. Therefore we assume in the following that $X_{1,3} \neq \emptyset$. Next we distinguish the cases $v_1v_3 \in E(G)$ or $v_1v_3 \in E(\overline{G})$.

Case 1. Let $v_1v_3 \in E(G)$.
If $X_{1,2} \neq \emptyset$ and $X_{2,3} = \emptyset$ or $X_{2,3} \neq \emptyset$ and $X_{1,2} = \emptyset$ or $X_{1,2} = X_{2,3} = \emptyset$, then $\Delta(G) = n-1$, a contradiction. Thus we assume now that $X_{1,2} \neq \emptyset$ and $X_{2,3} \neq \emptyset$. We observe that $\{v_1, v_2, v_3\}$ is a packing of $\overline{G}$, and hence we deduce from Theorem 2.1 that $\gamma_{tI}(\overline{G}) \geq 6$.

Case 2. Let $v_1v_3 \in E(\overline{G})$. In this case note that $\{v_1, v_2\}$ is a packing of $\overline{G}$, and hence Theorem 2.1 implies that $\gamma_{tI}(\overline{G}) \geq 4$.

□

Example 2.8. Let H be the graph consisting of a path $z_1z_2z_3$ and the vertex sets $A = \{u_1, u_2, \ldots, u_p\}$, $B = \{v_1, v_2, \ldots, v_q\}$ and $C = \{w_1, w_2, \ldots, w_r\}$ with $p, q, r \geq 2$ such that all vertices of A are adjacent to z_1 and z_2, all vertices of B are adjacent to z_1 and z_3 and all vertices of C are adjacent to z_2 and z_3. Clearly, $\gamma_{tI}(H) = 3$. If we define the function g by $g(z_1) = g(z_3) = 1$, $g(v_1) = g(v_2) = 1$ and $g(x) = 0$ for $x \in V(\overline{H}) \setminus \{v_1, v_2, z_1, z_3\}$, then g is a TIDF on $\overline{H}$. Therefore $\gamma_{tI}(H) + \gamma_{tI}(\overline{H}) \leq 7$ and thus $\gamma_{tI}(H) + \gamma_{tI}(\overline{H}) = 7$, according to Theorem 2.7.

Example 2.8 demonstarates that Theorem 2.7 is sharp. The proof of the following theorem can be found in [5].

Theorem 2.9. *Let G and $\overline{G}$ be connected graphs of order n. Then the following holds.*

(1) $(\gamma_{tR}(G) - 4)(\gamma_{tR}(\overline{G}) - 4) \leq 4\delta^*(G) - 4$.
(2) $\gamma_{tR}(G) + \gamma_{tR}(\overline{G}) \leq 2\delta^*(G) + 8 - \frac{(\gamma_{tR}(G)-6)(\gamma_{tR}(\overline{G})-6)}{2}$.
(3) *If $\gamma_{tR}(G) \geq 8$ and $\gamma_{tR}(\overline{G}) \geq 8$, then $\gamma_{tR}(G) + \gamma_{tR}(\overline{G}) \leq 2\delta^*(G) + 5$.*

Theorem 2.10. *If G and $\overline{G}$ are graphs of order n without isolated vertices, then*

$$\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n + 4.$$

The bound is sharp for P_4.

Proof. Let G be a graph of order n such that neither G nor $\overline{G}$ has an isolated vertex. We observe then that $n \geq 4$. If G is disconnected, then clearly $\gamma_{tI}(\overline{G}) \leq 4$ and the result is immediate. Hence we assume that G is connected. We assume likewise that $\overline{G}$ is connected. If $\operatorname{diam}(G) \geq 3$, then the result is immediate by Observation 1.2 and Theorem 1.4-(2). Thus we assume that $\operatorname{diam}(G) = 2$. Similarly, we can assume that $\operatorname{diam}(\overline{G}) = 2$.

By symmetry, we can assume that $\gamma_{tI}(\overline{G}) \geq \gamma_{tI}(G)$. If $\gamma_{tI}(G) = 3$ or 4, then the result is true since $\gamma_{tI}(\overline{G}) \leq n$. If $\gamma_{tI}(G) = 5$, then we conclude from Theorem 1.4-(2) that $\gamma_{tI}(\overline{G}) \leq n-1$ yielding $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+4$. Assume that $\gamma_{tI}(G) = 6$. We deduce from Theorem 2.3 that $\delta^* \geq 3$. Now for any edge $uv \in E(\overline{G})$, the function f defined on $\overline{G}$ by $f(u) = f(v) = 0$ and $f(x) = 1$ otherwise, is an IDF of $\overline{G}$ of weight $n-2$ implying that $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+4$. If $\gamma_{tI}(G) \geq 8$, then by Theorem 2.9-(3) we have $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq 2\delta^*(G) + 5 \leq 2\frac{n-1}{2} + 5 = n+4$. Let $\gamma_{tI}(G) = 7$. If $\gamma_{tI}(\overline{G}) \geq 11$, then as above by Theorem 2.9-(2) we have $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq 2\delta^*(G) + 5 \leq 2\frac{n-1}{2} + 5 = n+4$. Assume that $\gamma_{tI}(\overline{G}) \leq 10$. If $\gamma_{tI}(\overline{G}) = 8$ and $\delta^* = \delta(G)$, then by Theorem 2.3 we have $\delta^* + 3 \geq 8$ which leads to $n \geq 11$, and if $\delta^* = \delta(\overline{G})$, then using Corollary 2.4 for $\overline{G}$, we have $8 \leq \frac{3n+1}{4}$ and so $n \geq 11$. Therefore, $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+4$. If $\gamma_{tI}(\overline{G}) = 9$ and $\delta^* = \delta(G)$, then by Theorem 2.3 we have $\delta^* + 3 \geq 9$ which leads to $n \geq 12$, and if $\delta^* = \delta(\overline{G})$, then using Corollary 2.4 for $\overline{G}$, we have $9 \leq \frac{3n+1}{4}$ and so $n \geq 12$. Therefore, $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+4$. If $\gamma_{tI}(\overline{G}) = 10$ and $\delta^* = \delta(G)$, then by Theorem 2.3 we have $\delta^* + 3 \geq 10$ which leads to $n \geq 15$, and if $\delta^* = \delta(\overline{G})$, then using Corollary 2.4 for $\overline{G}$, we have $10 \leq \frac{3n+1}{4}$ and so $n \geq 13$. Therefore, $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+4$. Finally let $\gamma_{tI}(\overline{G}) = 7$. Then $\delta^* \geq 4$ and so $n \geq 9$. If $n \geq 10$, then the result is immediate. Let $n = 9$. Thus G and $\overline{G}$ are 4-regular. This leads to $\gamma_{tI}(\overline{G}) \leq 5$ which contradicts the assumption $\gamma_{tI}(\overline{G}) = 7$. This completes the proof. □

Using Theorem 1.4-(2), one can improve the bound of Theorem 2.10 slightly.

Theorem 2.11. *Let G and $\overline{G}$ be graphs of order $n \geq 6$ without isolated vertices. If* $\operatorname{diam}(G) \geq 3$ *or* $\operatorname{diam}(\overline{G}) \geq 3$*, then*

$$\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+3.$$

Proof. Assume, without loss of generality, that $\operatorname{diam}(G) \geq 3$. If $\gamma_{tI}(G) \leq n-1$, then we deduce from Observation 1.2 that $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+3$. Hence we assume that $\gamma_{tI}(G) = n$. It follows from Theorem 1.4-(2) that the components of G are isomorphic to $K_2, K_{1,2}$ or the corona $H = F \circ K_1$ for a connected graph F with $n(F) \geq 2$. Assume first that G has a component $H = F \circ K_1$ for a connected graph F with $n(F) \geq 3$. Let u_1, u_2 and u_3 be three leaves of H. Then the function f defined on $\overline{G}$ by $f(u_1) = f(u_2) = f(u_3) = 1$ and $f(x) = 0$ otherwise, is a TIDF of $\overline{G}$ and this implies that $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+3$.

Assume second that G has a component $H = K_2 \circ K_1$, and let u_1 and u_2 be the leaves of H. Since $n \geq 6$, there exists a further component. If v is a vertex of a further component, then define the function f on $\overline{G}$ by $f(u_1) = f(u_2) = f(v) = 1$ and $f(x) = 0$ otherwise. Then f is a TIDF on $\overline{G}$ and thus $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+3$.

Assume third that G has a component $K_{1,2} = u_1u_2u_3$. Since $n \geq 6$, there exists a further component. If v is a vertex of a further component, then define the function f on $\overline{G}$ by $f(u_1) = f(u_2) = f(v) = 1$ and $f(x) = 0$ otherwise. Then f is a TIDF on $\overline{G}$ and thus $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+3$.

Finally, assume that all components are isomorphic to K_2. Let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the vertices of two such components. Then the function f defined on $\overline{G}$ by $f(u_1) = f(u_2) = f(v_1) = 1$ and $f(x) = 0$ otherwise is a TIDF on $\overline{G}$ and thus $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+3$. □

If H is the corona $F \circ K_1$ of a connected graph F with $n(F) \geq 3$, then we have equality in the above theorem.

Theorem 2.12. *If G and $\overline{G}$ are graphs of order $n \geq 12$ without isolated vertices, then*

$$\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq n+3.$$

Proof. Considering Theorem 2.11, we may assume that $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. It follows from Theorem 1.4-(2) that $\gamma_{tI}(G) \le n-1$ and $\gamma_{tI}(\overline{G}) \le n-1$, since $n \ge 12$. By symmetry, we can assume that $\gamma_{tI}(\overline{G}) \ge \gamma_{tI}(G)$. If $\gamma_{tI}(G) = 3$ or 4, then the result is immediate. If $\gamma_{tI}(G) \ge 9$, then by Theorem 2.9-(2), we have $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \le 2\delta^*(G) + 3 \le 2\frac{n-1}{2} + 3 = n+2$. Assume that $\gamma_{tI}(G) \in \{5,6,7,8\}$. Let $\gamma_{tI}(G) = 5$. Then the result is clear if $\gamma_{tI}(\overline{G}) \le 10$, because $n \ge 12$. Let $\gamma_{tI}(\overline{G}) \ge 11$. If $\delta^* = \delta(G)$, then it follows from Theorem 2.3 that $\delta^* \ge \gamma_{tI}(\overline{G}) - 3 = 8$ and if $\delta^* = \delta(\overline{G})$, then it follows from Theorem 2.3 that $\delta^* \ge \frac{\gamma_{tI}(\overline{G})-1}{2} = 5$. By Observation 1.3 we get $\gamma_{tI}(\overline{G}) \le n-4$, which leads to $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \le n+1$.

Assume now that $\gamma_{tI}(G) = 6$. If $\gamma_{tI}(\overline{G}) \le 9$, then the result is immediate since $n \ge 12$. Let $\gamma_{tI}(\overline{G}) \ge 10$. Using the argument above, we obtain $\delta^* \ge 5$ and we deduce from Observation 1.3 that $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \le n+2$.

Assume next that $\gamma_{tI}(G) = 7$. If $\gamma_{tI}(\overline{G}) \in \{7,8\}$, then the result is immediate since $n \ge 12$. Let $\gamma_{tI}(\overline{G}) \ge 9$. If $\gamma_{tI}(\overline{G}) \ge 10$, then as above, we have $\gamma_{tI}(\overline{G}) \le n-4$ and so $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \le n+3$. Hence, let $\gamma_{tI}(\overline{G}) = 9$. Using the argument above, we obtain $\delta^* \ge 4$, Applying Theorem 1.5-(3) to $\overline{G}$ we observe that $\gamma_{tI}(\overline{G}) = 9 \le \frac{5n}{7}$ and so $n \ge 13$, yielding $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \le n+3$.

Finally let $\gamma_{tI}(G) = 8$. If $\gamma_{tI}(\overline{G}) \ge 12$, then Theorem 2.3 implies that $\delta^* \ge 6$ and by Observation 1.3 we have $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \le 8 + (n - \delta^* + 1) \le n+3$. Hence we assume that $\gamma_{tI}(\overline{G}) \in \{8,9,10,11\}$. If $\delta^* = \delta(G)$, then by Theorem 8 we have $\delta^* + 3 \ge \gamma_{tI}(\overline{G}) \ge 8$ and if $\delta^* = \delta(\overline{G})$, then by Theorem 8 we have $\delta^* + 3 \ge \gamma_{tI}(G) = 8$. Thus $\delta^*(G) \ge 5$. If $\gamma_{tI}(\overline{G}) = 11$, then it follows from Theorem 1.5-(4) that $n \ge 16$, yielding $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) = 19 \le n+3$. If $\gamma_{tI}(\overline{G}) = 10$, then as above, we obtain $n \ge 15$ which leads to $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) = 18 \le n+3$. Assume that $\gamma_{tI}(\overline{G}) = 9$. Then as above, we obtain $n \ge 13$. If $n \ge 14$, then the result is immediate. Let $n = 13$. It follows that $\delta(\overline{G}) = \delta^* = 5$. Therefore $\Delta(G) = 7$. Let $x \in V(G)$ be a vertex of maximum degree 7 and let $N(x) = \{x_1, \ldots, x_7\}$. Assume that $X = V(G) - N[x] = \{y_1, \ldots, y_5\}$ and let y_i' be a common neighbor of x and y_i. If y_i is adjacent to all vertices in $X - \{y_i\}$ for some i, then the function f defined on G by $f(x) = f(y_i) = 2$, $f(y_i') = 1$ and $f(z) = 0$ otherwise, is a TIDF of G which contradicts the assumption $\gamma_{tI}(G) = 8$. If y_i is adjacent to all vertices in $X - \{y_i\}$ but one, say y_j, for some i, then the function f defined on G by $f(x) = f(y_i) = f(y_j') = 2$, $f(y_i') = 1$ and $f(z) = 0$ otherwise, is a TIDF of G, which contradicts the assumption $\gamma_{tI}(G) = 8$ again. Thus each y_i is adjacent to at most two vertices in $X - \{y_i\}$ and so it is adjacent to at least three vertices in $N(x)$. If x_i is adjacent to all vertices in X for some i, then the function f defined on G by $f(x) = f(x_i) = 2$, and $f(z) = 0$ otherwise, is a TIDF of G which contradicts the assumption $\gamma_{tI}(G) = 8$. If x_i is adjacent to all vertices in X but one, say y_j, for some i, then the function f defined on G by $f(x) = f(x_i) = f(y_j') = 2$ and $f(z) = 0$ otherwise, is a TIDF of G which contradicts the assumption $\gamma_{tI}(G) = 8$ again. Thus each vertex x_i is adjacent to at most three vertices in X and so it is adjacent to at least one vertex in $N(x)$. Then the function f defined on G by $f(x) = f(x_i) = 1$ for $1 \le i \le 6$ and $f(z) = 0$ otherwise, is a TIDF of G of weight 7 which leads to a contradiction.

Assume next that $\gamma_{tI}(\overline{G}) = 8$. If $n \ge 13$, then the result is immediate. Let $n = 12$. Without loss of generality, we may assume that $\Delta(G) = 6$. Let $x \in V(G)$ be a vertex of maximum degree 6, $N(x) = \{x_1, \ldots, x_6\}$ and $X = V(G) - N[x] = \{y_1, \ldots, y_5\}$. If some y_i is adjacent to all vertices in $X - \{y_i\}$, then the function f defined by $f(x) = f(y_i) = 2$, $f(w) = 1$ for some $w \in N(x) \cap N(y_i)$ and $f(z) = 0$ otherwise, is a TIDF of G which contradicts the assumption $\gamma_{tI}(G) = 8$. Thus each y_i is adjacent to at most three vertices in X and so each y_i is adjacent to at least two vertices in $N(x)$. But then the function f defined by $f(z) = 1$ for $z \in N[x]$ and $f(z) = 0$ otherwise, is a TIDF of G of weight 7 which contradicts the assumption $\gamma_{tI}(G) = 8$. This completes the proof. □

Theorem 2.13. *If G and $\overline{G}$ are graphs of order n without isolated vertices, then*

$$\gamma_{tI}(G)\gamma_{tI}(\overline{G}) \le 6n - 8.$$

The bound is sharp for P_4.

Proof. We observe that $n \geq 4$. If G is disconnected, then clearly $\gamma_{tI}(\overline{G}) \leq 4$ and the result is immediate. Hence we assume that G is connected. We assume likewise that $\overline{G}$ is connected. If $\mathrm{diam}(G) \geq 3$, then the result is immediate by Observation 1.2 and Theorem 1.4-(2). Thus we assume that $\mathrm{diam}(G) = 2$. Similarly, we can assume that $\mathrm{diam}(\overline{G}) = 2$.

By symmetry, we can assume that $\gamma_{tI}(G) \geq \gamma_{tI}(\overline{G}) \geq 3$. If $\gamma_{tI}(\overline{G}) = 3$ or 4, then the bound is immediate. If $\gamma_{tI}(\overline{G}) = 5$, then we deduce from $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$ and Theorem 1.4-(2) that $\gamma_{tI}(G) \leq n-1$. Thus $\gamma_{tI}(G)\gamma_{tI}(\overline{G}) \leq 5(n-1) \leq 6n-8$. Hence we assume that $\gamma_{tI}(G) \geq \gamma_{tI}(\overline{G}) \geq 6$. Hence by Theorem 2.3, $\delta^* \geq 3$ and so $n \geq 7$. If $\gamma_{tI}(\overline{G}) = 6$, then by Theorem 1.5-(2) we have $\gamma_{tI}(G)\gamma_{tI}(\overline{G}) \leq 6\frac{3n}{4} < 6n-8$. If $\gamma_{tI}(\overline{G}) = 7$, then $\delta^* \geq 4$ and by Theorem 1.5-(3) we have $\gamma_{tI}(G)\gamma_{tI}(\overline{G}) \leq 7\frac{5n}{7} \leq 6n-8$. If $\gamma_{tI}(\overline{G}) \geq 9$, then using first Theorem 2.9-(1), then Theorem 2.9-(2) and the that fact $\delta^*(G) \leq \frac{n-1}{2}$, we obtain

$$\begin{aligned}
\gamma_{tI}(G)\gamma_{tI}(\overline{G}) \leq \gamma_{tR}(G)\gamma_{tR}(\overline{G}) &\leq 4\delta^*(G) - 20 + 4(\gamma_{tR}(G) + \gamma_{tR}(\overline{G})) \\
&\leq 4\delta^*(G) - 20 + 4(2\delta^*(G) + 3) \\
&\leq 12\frac{n-1}{2} - 8 \\
&= 6n - 14.
\end{aligned}$$

Now assume that $\gamma_{tI}(\overline{G}) = 8$. Theorem 2.3 implies that $\delta^* \geq 5$ and so $n \geq 11$. If $\gamma_{tI}(G) \geq 10$, then by Theorem 2.9 we have $\gamma_{tI}(G) + \gamma_{tI}(\overline{G}) \leq 2\delta^* + 4$, and as above we get the desired result. Hence we assume that $\gamma_{tI}(G) \in \{8, 9\}$. First let $\gamma_{tI}(G) = 8$. If $n \geq 12$, then the result is immediate. Let $n \leq 11$. Then $n = 11$ and both G and $\overline{G}$ must be 5-regular which is impossible. Now let $\gamma_{tI}(G) = 9$. By Theorem 1.5-(3) we have $n \geq 13$. If $n \geq 14$, then the result is immediate. Let $n = 13$ and so $\delta^* \leq 6$. If $\delta^* = 6$, then Theorem 2.5 implies that $\gamma_{tI}(G) \leq 7$, contradicting the assumption $\gamma_{tI}(G) = 9$. Thus $\delta^* = 5$ and so $\Delta^* = \max\{\Delta(G), \Delta(\overline{G})\} = 7$. Without loss of generality, assume that $\Delta(\overline{G}) = 7$ and let u be a vertex with degree 7 in $\overline{G}$. If some vertex v in $N_{\overline{G}}(u)$ is adjacent to all vertices of $V(\overline{G}) - N_{\overline{G}}[u]$, then the function f defined by $f(u) = f(v) = 2$ and $f(x) = 0$ otherwise, is a TIDF of $\overline{G}$ which contradicts the assumption $\gamma_{tI}(\overline{G}) = 8$. Hence we assume that no vertex of $N_{\overline{G}}(u)$ is adjacent to all vertices of $Y = V(\overline{G}) - N_{\overline{G}}[u]$ and so the induced subgraph of $\overline{G}$ by $N_{\overline{G}}(u)$ has no isolated vertex because $\delta^* \geq 5$. If some vertex $v \in Y$ is adjacent to all vertices $Y - \{v\}$, then function f defined by $f(u) = f(v) = 2$, $f(u') = 1$ and $f(x) = 0$ otherwise, where u' is a common neighbor of u and v, is a TIDF of $\overline{G}$ which leads to a contradiction again. Thus we assume that each vertex in Y has at most three neighbors in Y and so each vertex in Y has at least two neighbors in $N_{\overline{G}}(u)$. Then the function f defined by $f(v) = 1$ for $v \in N_{\overline{G}}(u)$ and $f(x) = 0$ otherwise, is a TIDF of $\overline{G}$ of weight 7 which is a contradiction. This completes the proof. □

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