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A NOTE ON THE DOUBLE DOMINATION NUMBER IN MAXIMAL OUTERPLANAR AND PLANAR GRAPHS

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Abstract. In a graph, a vertex dominates itself and its neighbors. A subset S of vertices of a graph G is a double dominating set of G if S dominates every vertex of G at least twice. The double domination number $\gamma_{\times 2}(G)$ of G is the minimum cardinality of a double dominating set of G. In this paper, we prove that the double domination number of a maximal outerplanar graph G of order n is bounded above by $\frac{n+k}{2}$, where k is the number of pairs of consecutive vertices of degree two and with distance at least 3 on the outer cycle. We also prove that $\gamma_{\times 2}(G) \leq \frac{5n}{8}$ for a Hamiltonian maximal planar graph G of order $n \geq 7$.

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1. Introduction

For graph theory notation and terminology not given here we refer to [6]. We consider finite, undirected and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of G, respectively. A plane graph G is said to be a *triangulated disc* if all of its faces except the infinite face are triangles. A graph G is *outerplanar* if it has an embedding in the plane such that all vertices belong to the boundary of its outer face. A planar (resp. outerplanar) graph G is *maximal* if $G + uv$ is not planar (resp. outerplanar) for any two nonadjacent vertices u and v of G. An inner face of a maximal outerplanar graph G is said to be an *internal triangle* if it is not adjacent to the outer face. A maximal outerplanar graph G is called *striped* if it has no internal triangles. A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For a comprehensive survey on the subject of domination parameters in graphs the reader can refer to the two books [6,7].

Harary and Haynes [5] defined a generalization of domination, namely k-tuple domination. For a positive integer k, a subset S of vertices of a graph G is a *k-tuple dominating set* of G if for every vertex $v \in V(G)$,

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$|N[v] \cap S| \geq k$. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of a k-tuple dominating set of G, if such a set exists. A k-tuple dominating set where $k = 2$ is called a *double dominating set*. A double dominating set of cardinality $\gamma_{\times 2}(G)$ is referred to as a $\gamma_{\times 2}(G)$-set. The concept of double domination in graph was further studied in, for example, [1,2,4,8]. Blidia *et al.* [1] showed that $\gamma_{\times 2}(G) \leq \frac{11n}{13}$ if G is a graph of order n with $\delta(G) \geq 2$. Henning [8] proved that $\gamma_{\times 2}(G) \leq \frac{3n}{4}$ provided that G is not a 5-cycle.

Domination in maximal planar graphs and outer-planar graphs has recieved great attention and several domination parameters for these classes of graph have been studied. See, for example, Dorfling *et al.* [3], Henning and Kaemawichanurat [9], Lemanska *et al.* [11], Li *et al.* [12], King and Pelsmajer [10], Matheson and Tarjan [14], Tokunaga [16] and Liu [13]. Recently, Zhuang [18] studied double domination in maximal outerplanar graphs, and proved the following.

Theorem 1.1 (Zhuang [18])**.** *Let G be a maximal outerplanar graph of order $n \geq 3$. Then $\gamma_{\times 2}(G) \leq \lfloor \frac{2n}{3} \rfloor$.*

Theorem 1.2 (Zhuang [18])**.** *Let G be a maximal outerplanar graph of order $n \geq 3$ and t be the number of vertices of degree 2 in G. Then $\gamma_{\times 2}(G) \leq \frac{n+t}{2}$.*

In this paper, we first improve Theorem 1.2 by showing that $\gamma_{\times 2}(G) \leq \frac{n+k}{2}$, where k is the number of pairs of consecutive vertices of degree two with distance at least 3 on the outer cycle. We also prove that $\gamma_{\times 2}(G) \leq \frac{5n}{8}$ for a Hamiltonian maximal planar graph G of order $n \geq 7$, which improves Theorem 1.1 and all previous bounds.

We follow the notations and method given in [12]. For a Hamiltonian maximal planar graph G with a Hamilton cycle C, let G^C_{in} be the maximal outerplanar graph consists of C and all edges inside of C and G^C_{out} be the maximal outerplanar graph consists of C and all edges outside of C. Let $v_1, \ldots, v_t$ be all the vertices of degree 2 which appear in the clockwise direction on C. A vertex v_i is called a *bad* vertex if the distance between v_i and v_{i+1} on C is at least 3, for $i = 1, 2, \ldots, t$, where the subscript is taken modulo t. We make use of the following.

Theorem 1.3 (Li *et al.* [12])**.** *For a Hamiltonian maximal planar graph G of order n, there exists a Hamilton cycle C of G such that G^C_{in} or G^C_{out} has at most $\frac{n}{4}$ bad vertices.*

Theorem 1.4 (Whitney [17])**.** *Every 4-connected maximal planar graph is Hamiltonian.*

2. Main results

Let G be a maximal outerplanar graph. There is an embedding of G in the plane such that all of its vertices are on the outer cycle C which is the boundary of the outer face and each inner face is a triangle. Let $v_1, \ldots, v_t$ be all the vertices of degree 2 which appear in the clockwise direction on C. We will prove the following.

Theorem 2.1. *Let G be a maximal outerplanar graph of order $n \geq 4$. If G has $k \geq 0$ bad vertices, then $\gamma_{\times 2}(G) \leq \frac{n+k}{2}$.*

As a consequence of Theorems 2.1 and 1.3 we obtain the following.

Theorem 2.2. *Let G be a Hamiltonian maximal planar graph of order $n \geq 7$. Then $\gamma_{\times 2}(G) \leq \frac{5n}{8}$.*

As another immediate consequence of Theorems 2.2 and 1.4 we have the following.

Corollary 2.3. *If G is a 4-connected maximal planar graph of order $n \geq 7$, then $\gamma_{\times 2}(G) \leq \frac{5n}{8}$.*

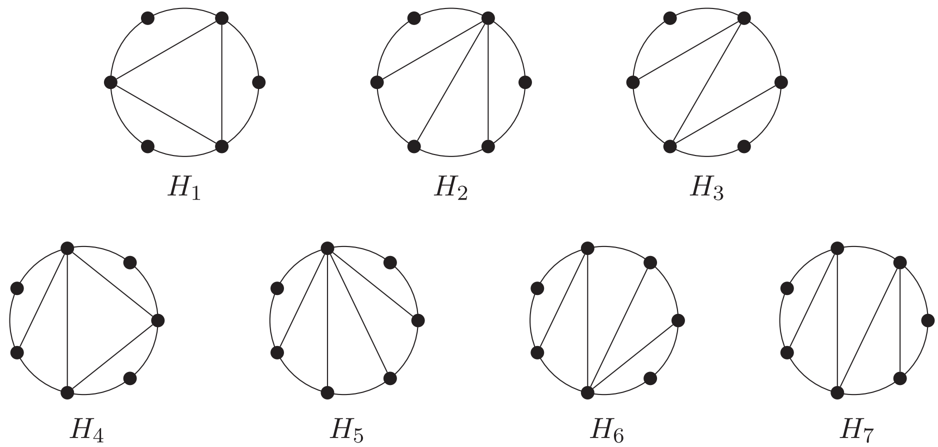

FIGURE 1. Graphs H_i, $i = 1, 2, \dots, 7$.

3. PROOF OF THEOREM 2.1

The proof is by induction on $n + k$. The result is obvious if $4 \le n \le 5$. Let H_i be graphs shown in Figure 1 for $i = 1, 2, 3, 4, 5, 6, 7$.

Assume that $n = 6$. If $k = 0$, then $G = H_1$ in which $\gamma_{\times 2}(G) = 3 = \frac{6+0}{2}$. If $k = 1$, then $G = H_2$ in which $\gamma_{\times 2}(G) = 3 < \frac{6+1}{2}$. Thus assume that $k = 2$. Then $G = H_3$ in which $\gamma_{\times 2}(G) = 4 = \frac{6+2}{2}$. Next assume that $n = 7$. Clearly $1 \le k \le 2$. If $k = 1$, then $G \in \{H_4, H_5\}$ in which $\gamma_{\times 2}(G) = 4 = \frac{7+1}{2}$. Thus assume that $k = 2$. Then $G \in \{H_6, H_7\}$ in which $\gamma_{\times 2}(G) = 4 < \frac{7+2}{2}$. These are enough for the basic step of the induction. Assume the result holds for all maximal outerplanar graphs of order n' with k' bad vertices, where $n' + k' < n + k$. Now consider the maximal outerplanar graph G of order $n \ge 7$ and with k bad vertices. If $n = 7$ then either $t = k = 2$ or $t = 3$ and $k = 1$, and in both cases $\gamma_{\times 2}(G) = 4 \le \frac{n+k}{2}$. Thus assume that $n \ge 8$. First assume that $k = 0$. Let C be the outer cycle of G and $v_1, v_2, \dots, v_t$ be a cyclic clockwise order of its t vertices of degree 2. Since G has no bad vertices, the distance between each v_i and v_{i+1} on C is exactly two, for $i = 1, 2, \dots, t$. Thus, $n = 2t$. Then $V(G) - \{v_1, \dots, v_t\}$ is a double dominating set for G, implying that $\gamma_{\times 2}(G) \le n - t = \frac{n}{2} = \frac{n+0}{2}$. Thus assume that $k > 0$. Then there is an integer $i \in \{1, 2, \dots, t\}$ such that the distance between v_i and v_{i+1} on C is at least 3. Let $G_1 = G - \{v_1, v_2, \dots, v_t\}$. Clearly G_1 is also a maximal outerplanar graph. Let u be a vertex of degree 2 in G_1. Then $3 \le \deg_G(u) \le 4$.

Assume first that $\deg_G(u) = 3$. Then there exists exactly one vertex $v \in N_G(u)$ with $\deg_G(v) = 2$. Let $N_G(u) = \{v, u_1, u_2\}$, where $u_1 \in N_G(v) \cap N_G(u)$. Since G is a maximal outerplanar graph, from $\deg_G(u) = 3$ we obtain that $u_1u_2 \in E(G)$. We may assume without loss of generality that u is after v in the cyclic clockwise order on C. Thus v is a bad vertex in G. Let $u_3 \in N_G(u_2) - \{u\}$ be the vertex just after u_2 in the cyclic clockwise order on C.

Assume that $\deg_G(u_3) = 2$. Let $G' = (G - u) + vu_2$. Then G' is a maximal outerplanar graph of order $n - 1$ with the hamiltonian cycle $(C - \{uu_2, uv\}) \cup \{vu_2\}$. Note that v is not a bad vertex of G'. Thus G' has $k' = k - 1$ bad vertices. Applying the inductive hypothesis, $\gamma_{\times 2}(G') \le \frac{n'+k'}{2} = \frac{n+k-2}{2}$. Let D' be a $\gamma_{\times 2}(G')$-set. If $v \notin D'$ then $\{u_1, u_2\} \subseteq D'$ and so $D' \cup \{u\}$ is a double dominating set for G, implying that $\gamma_{\times 2}(G) \le \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus assume that $v \in D'$. Then clearly we may assume that $|D' \cap \{u_1, u_2\}| = 1$. Then $(D' - \{v\}) \cup \{u, u_1, u_2\}$ is a double dominating set for G of cardinality $|D'| + 1$, implying that $\gamma_{\times 2}(G) \le \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus, $\deg_G(u_3) \ge 3$.

Assume that $u_3u_1 \in E(G)$. Let $u_4 \in N_G(u_3) - \{u_1, u_2\}$ be the vertex just after u_3 in the cyclic clockwise order on C, and let $G' = (G - \{u, u_2\}) + vu_3$. Then G' is a maximal outerplanar graph of order $n - 2$ with the hamiltonian cycle $(C - \{u_2u_3, uu_2, uv\}) \cup \{vu_3\}$. Note that G' has $k-1$ bad vertices if $\deg_G(u_4) = 2$ and k bad vertices if $\deg_G(u_4) > 2$. Applying the inductive hypothesis, $\gamma_{\times 2}(G') \leq \frac{n'+k'}{2} = \frac{n+k-2}{2}$. Let D' be a $\gamma_{\times 2}(G')$-set. If $v \notin D'$ then $\{u_1, u_3\} \subseteq D'$ and so $D' \cup \{u\}$ is a double dominating set for G, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus assume that $v \in D'$. Then clearly we may assume that $|D' \cap \{u_1, u_3\}| = 1$. Then $(D' - \{v\}) \cup \{u, u_1, u_3\}$ is a double dominating set for G of cardinality $|D'| + 1$, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus $u_3u_1 \notin E(G)$.

Let $u_0 \in N_G(u_1) - \{v, u, u_2\}$ be the vertex just before u_1 in the cyclic clockwise order on C, and let $G' = G - \{u, v\}$. Then G' is a maximal outerplanar graph of order $n-2$ with the hamiltonian cycle $(C - \{vu_1, uv, uu_2\}) \cup \{u_1u_2\}$. Assume that $\deg_G(u_0) = 2$. Then G' has at most k bad vertices. Applying the inductive hypothesis, $\gamma_{\times 2}(G') \leq \frac{n'+k'}{2} = \frac{n+k-2}{2}$. Let D' be a $\gamma_{\times 2}(G')$-set. If $u_0 \notin D'$ then $u_1 \in D'$ and so $D' \cup \{u\}$ is a double dominating set for G, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus assume that $u_0 \in D'$. Then clearly we may assume that $|D' \cap N_G(u_0)| = 1$. Then $(D' - \{u_0\}) \cup N_G(u_0) \cup \{u\}$ is a double dominating set for G of cardinality $|D'| + 1$, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus we may assume that $\deg_G(u_0) \geq 3$. Note that $\deg_G(u_1) \geq 5$. Let G' be the graph obtained from G by removing the vertices u and v and then contracting the edge u_1u_2. Then G' has $k-1$ bad vertices. Let u^* be the vertex in G' forming by contracting the edge u_1u_2. Applying the inductive hypothesis, $\gamma_{\times 2}(G') \leq \frac{n'+k'}{2} = \frac{n-3+k-1}{2}$. Let D' be a $\gamma_{\times 2}(G')$-set. If $u^* \in D'$ then $(D' - \{u^*\}) \cup \{u, u_1, u_2\}$ is a double dominating set for G, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-4}{2} + 2 = \frac{n+k}{2}$. Thus assume that $u^* \notin D'$. Then each of u_1 and u_2 is dominated by a vertex of D' in G, and so $D' \cup \{u, v\}$ is a double dominating set for G, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-4}{2} + 2 = \frac{n+k}{2}$.

Next assume that $\deg_G(u) = 4$. Then there exist two vertices $v_1, v_2 \in N_G(u)$ such that $\deg_G(v_1) = \deg_G(v_2) = 2$. Let $N_G(u) = \{v_1, v_2, u_1, u_2\}$, where in the cyclic clockwise order on C, u_1 is before than v_1, v_1 is before than u, u is before than v_2 and v_2 is before than u_2. By the choice of u, $u_1u_2 \in E(G)$. Let $u_3 \in N_G(u_2)$ be the vertex after u_2 in the cyclic clockwise order on C.

Assume that $\deg_G(u_3) = 2$. Let $G' = G - \{v_1, v_2\}$. Then G' is a maximal outerplanar graph of order $n-2$ with the hamiltonian cycle $(C - \{v_2u_2, v_2u, uv_1, v_1u_1\}) \cup \{u_1u, uu_2\}$. Note that G' has k bad vertices. Applying the inductive hypothesis, $\gamma_{\times 2}(G') \leq \frac{n'+k'}{2} = \frac{n+k-2}{2}$. Let D' be a $\gamma_{\times 2}(G')$-set. If $u \notin D'$ then $u_1, u_2 \in D'$ and so $D' \cup \{u\}$ is a double dominating set for G, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus assume that $u \in D'$. Then we may assume that $|D' \cap \{u_1, u_2\}| = 1$. Then $D' \cup \{u_1, u_2\}$ is a double dominating set for G of cardinality $|D'| + 1$, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus, $\deg_G(u_3) \geq 3$.

Thus v_2 is a bad vertex of G. Let $G' = G - \{v_1, v_2\}$. Then G' is a maximal outerplanar graph of order $n-2$ with the hamiltonian cycle $(C - \{v_2u_2, v_2u, uv_1, v_1u_1\}) \cup \{u_1u, uu_2\}$. Since v_2 is a bad vertex of G, u is a bad vertex of G', and G' has k bad vertices. Applying the inductive hypothesis, $\gamma_{\times 2}(G') \leq \frac{n'+k'}{2} = \frac{n+k-2}{2}$. Let D' be a $\gamma_{\times 2}(G')$-set. If $u \notin D'$ then $u_1, u_2 \in D'$ and so $D' \cup \{u\}$ is a double dominating set for G, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus assume that $u \in D'$. Then we may assume that $|D' \cap \{u_1, u_2\}| = 1$. Then $D' \cup \{u_1, u_2\}$ is a double dominating set for G of cardinality $|D'| + 1$, implying that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$. Thus, $u_1u_3 \notin E(G)$. Let $G' = G - \{u, v_2\}$. Then G' has k bad vertices. Applying the inductive hypothesis, $\gamma_{\times 2}(G') \leq \frac{n'+k'}{2} = \frac{n+k-2}{2}$. Let D' be a $\gamma_{\times 2}(G')$-set. Now as before, we obtain that $\gamma_{\times 2}(G) \leq \frac{n+k-2}{2} + 1 = \frac{n+k}{2}$.

4. Proof of Theorem 2.2

Let G be a Hamiltonian maximal planar graph of order $n \geq 7$. Let C be a Hamilton cycle of G, and without loss of generality, assume that G_{in}^C has at most $\frac{n}{4}$ bad vertices according to Theorem 1.3. Then $k \leq \frac{n}{4}$ and by Theorem 2.1, $\gamma_{\times 2}(G) \leq \frac{n+k}{2} \leq \frac{5n}{8}$.

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