A NOTE ON THE DOUBLE DOMINATION NUMBER IN MAXIMAL OUTERPLANAR AND PLANAR GRAPHS

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Abstract. In a graph, a vertex dominates itself and its neighbors. A subset $S$ of vertices of a graph $G$ is a double dominating set of $G$ if $S$ dominates every vertex of $G$ at least twice. The double domination number $\gamma_{\times2}(G)$ of $G$ is the minimum cardinality of a double dominating set of $G$. In this paper, we prove that the double domination number of a maximal outerplanar graph $G$ of order $n$ is bounded above by $n + \frac{k}{2}$, where $k$ is the number of pairs of consecutive vertices of degree two and with distance at least 3 on the outer cycle. We also prove that $\gamma_{\times2}(G) \leq \frac{5n}{8}$ for a Hamiltonian maximal planar graph $G$ of order $n \geq 7$.

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1. Introduction

For graph theory notation and terminology not given here we refer to [6]. We consider finite, undirected and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices of $G$ is called the order of $G$ and is denoted by $n = n(G)$. The open neighborhood of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is $N[v] = N_G[v] = N(v) \cup \{v\}$. The degree of a vertex $v$, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to $G$), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of $G$, respectively. A plane graph $G$ is said to be a triangulated disc if all of its faces except the infinite face are triangles. A graph $G$ is outerplanar if it has an embedding in the plane such that all vertices belong to the boundary of its outer face. A planar (resp. outerplanar) graph $G$ is maximal if $G + uv$ is not planar (resp. outerplanar) for any two nonadjacent vertices $u$ and $v$ of $G$. An inner face of a maximal outerplanar graph $G$ is said to be an internal triangle if it is not adjacent to the outer face. A maximal outerplanar graph $G$ is called striped if it has no internal triangles. A subset $S \subseteq V$ is a dominating set of $G$ if every vertex in $V - S$ has a neighbor in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. For a comprehensive survey on the subject of domination parameters in graphs the reader can refer to the two books [6,7].

Harary and Haynes [5] defined a generalization of domination, namely $k$-tuple domination. For a positive integer $k$, a subset $S$ of vertices of a graph $G$ is a $k$-tuple dominating set of $G$ if for every vertex $v \in V(G)$,
The $k$-tuple domination number $\gamma_{\times k}(G)$ is the minimum cardinality of a $k$-tuple dominating set of $G$, if such a set exists. A $k$-tuple dominating set where $k = 2$ is called a double dominating set. A double dominating set of cardinality $\gamma_{\times 2}(G)$ is referred to as a $\gamma_{\times 2}(G)$-set. The concept of double domination in graph was further studied in, for example, [1, 2, 4, 8]. Bidia et al. [1] showed that $\gamma_{\times 2}(G) \leq \frac{11n}{13}$ if $G$ is a graph of order $n$ with $\delta(G) \geq 2$. Henning [8] proved that $\gamma_{\times 2}(G) \leq \frac{3n}{4}$ provided that $G$ is not a 5-cycle.

Domination in maximal planar graphs and outer-planar graphs has received great attention and several domination parameters for these classes of graph have been studied. See, for example, Dorfling et al. [3], Henning and Kaemawichanurat [9], Lemanska et al. [11], Li et al. [12], King and Pelsmajer [10], Matheson and Tarjan [14], Tokunaga [16] and Liu [13]. Recently, Zhuang [18] studied double domination in maximal outerplanar graphs, and proved the following.

**Theorem 1.1** (Zhuang [18]). Let $G$ be a maximal outerplanar graph of order $n \geq 3$. Then $\gamma_{\times 2}(G) \leq \lfloor \frac{2n}{3} \rfloor$.

**Theorem 1.2** (Zhuang [18]). Let $G$ be a maximal outerplanar graph of order $n \geq 3$ and $t$ be the number of vertices of degree 2 in $G$. Then $\gamma_{\times 2}(G) \leq \frac{n+t}{2}$.

In this paper, we first improve Theorem 1.2 by showing that $\gamma_{\times 2}(G) \leq \frac{n+k}{2}$, where $k$ is the number of pairs of consecutive vertices of degree two with distance at least 3 on the outer cycle. We also prove that $\gamma_{\times 2}(G) \leq \frac{2n}{3}$ for a Hamiltonian maximal planar graph $G$ of order $n \geq 7$, which improves Theorem 1.1 and all previous bounds.

We follow the notations and method given in [12]. For a Hamiltonian maximal planar graph $G$ with a Hamilton cycle $C$, let $G_{in}^C$ be the maximal outerplanar graph consists of $C$ and all edges inside of $C$ and $G_{out}^C$ be the maximal outerplanar graph consists of $C$ and all edges outside of $C$. Let $v_1, \ldots, v_t$ be all the vertices of degree 2 which appear in the clockwise direction on $C$. A vertex $v_i$ is called a bad vertex if the distance between $v_i$ and $v_{i+1}$ on $C$ is at least 3, for $i = 1, 2, \ldots, t$, where the subscript is taken modulo $t$. We make use of the following.

**Theorem 1.3** (Li et al. [12]). For a Hamiltonian maximal planar graph $G$ of order $n$, there exists a Hamilton cycle $C$ of $G$ such that $G_{in}^C$ or $G_{out}^C$ has at most $\frac{n}{4}$ bad vertices.

**Theorem 1.4** (Whitney [17]). Every 4-connected maximal planar graph is Hamiltonian.

## 2. Main Results

Let $G$ be a maximal outerplanar graph. There is an embedding of $G$ in the plane such that all of its vertices are on the outer cycle $C$ which is the boundary of the outer face and each inner face is a triangle. Let $v_1, \ldots, v_t$ be all the vertices of degree 2 which appear in the clockwise direction on $C$. We will prove the following.

**Theorem 2.1.** Let $G$ be a maximal outerplanar graph of order $n \geq 4$. If $G$ has $k \geq 0$ bad vertices, then $\gamma_{\times 2}(G) \leq \frac{n+k}{2}$.

As a consequence of Theorems 2.1 and 1.3 we obtain the following.

**Theorem 2.2.** Let $G$ be a Hamiltonian maximal planar graph of order $n \geq 7$. Then $\gamma_{\times 2}(G) \leq \frac{5n}{8}$.

As another immediate consequence of Theorems 2.2 and 1.4 we have the following.

**Corollary 2.3.** If $G$ is a 4-connected maximal planar graph of order $n \geq 7$, then $\gamma_{\times 2}(G) \leq \frac{5n}{8}$.
3. Proof of Theorem 2.1

The proof is by induction on \( n + k \). The result is obvious if \( 4 \leq n \leq 5 \). Let \( H_i \) be graphs shown in Figure 1 for \( i = 1, 2, 3, 4, 5, 6, 7 \).

Assume that \( n = 6 \). If \( k = 0 \), then \( G = H_1 \) in which \( \gamma_{x2}(G) = 3 = \frac{n + 0}{2} \). If \( k = 1 \), then \( G = H_2 \) in which \( \gamma_{x2}(G) = 3 < \frac{n + 1}{2} \). Thus assume that \( k = 2 \). Then \( G = H_3 \) in which \( \gamma_{x2}(G) = 4 = \frac{n + 2}{2} \). Next assume that \( n \geq 7 \). Clearly \( 1 \leq k \leq 2 \). If \( k = 1 \), then \( G \in \{H_4, H_5\} \) in which \( \gamma_{x2}(G) = 4 = \frac{n + 1}{2} \). Thus assume that \( k = 2 \).

Then \( G \in \{H_6, H_7\} \) in which \( \gamma_{x2}(G) = 4 < \frac{n + 2}{2} \). These are enough for the basic step of the induction. Assume the result holds for all maximal outerplanar graphs of order \( n' \) with \( k' \) bad vertices, where \( n' + k' < n + k \).

Now consider the maximal outerplanar graph \( G \) of order \( n \geq 7 \) and with \( k \) bad vertices. If \( n = 7 \) then either \( t = k = 2 \) or \( t = 3 \) and \( k = 1 \), and in both cases \( \gamma_{x2}(G) = 4 \leq \frac{n + k}{2} \). Thus assume that \( n \geq 8 \). First assume that \( k = 0 \). Let \( C \) be the outer cycle of \( G \) and \( v_1, v_2, \ldots, v_t \) be a cyclic clockwise order of its \( t \) vertices of degree 2. Since \( G \) has no bad vertices, the distance between each \( v_i \) and \( v_{i+1} \) on \( C \) is exactly two, for \( i = 1, 2, \ldots, t \). Thus, \( n = 2t \). Then \( V(G) - \{v_1, \ldots, v_t\} \) is a double dominating set for \( G \), implying that \( \gamma_{x2}(G) \leq n - t = \frac{n}{2} = \frac{n + 0}{2} \).

Assume first that \( \deg_G(u) = 3 \). Then there exists exactly one vertex \( v \in N_G(u) \) with \( \deg_G(v) = 2 \). Let \( N_G(u) = \{v, u_1, u_2\} \), where \( u_1 \in N_G(v) \cap N_G(u) \). Since \( G \) is a maximal outerplanar graph, from \( \deg_G(u) = 3 \) we obtain that \( u_1 u_2 \in E(G) \). We may assume without loss of generality that \( u \) is after \( v \) in the cyclic clockwise order on \( C \). Thus \( v \) is a bad vertex in \( G \). Let \( u_3 \in N_G(u_2) - \{u\} \) be the vertex just after \( u_2 \) in the cyclic clockwise order on \( C \).

Assume that \( \deg_G(u_3) = 2 \). Let \( G' = (G - u) + uv_2 \). Then \( G' \) is a maximal outerplanar graph of order \( n - 1 \) with the hamiltonian cycle \((C - \{u_2, w\}) \cup \{v_2\}\). Note that \( v \) is not a bad vertex of \( G' \). Thus \( G' \) has \( k' = k - 1 \) bad vertices. Applying the inductive hypothesis, \( \gamma_{x2}(G') \leq \frac{n' + k'}{2} = \frac{n + k - 2}{2} \). Let \( D' \) be a \( \gamma_{x2}(G') \)-set. If \( v \notin D' \) then \( \{u_1, u_2\} \subseteq D' \) and so \( D' \cup \{u\} \) is a double dominating set for \( G \), implying that \( \gamma_{x2}(G) \leq \frac{n + k - 2}{2} + 1 = \frac{n + k}{2} \). Thus assume that \( v \in D' \). Then clearly we may assume that \( |D' \cap \{u_1, u_2\}| = 1 \). Then \( (D' - \{v\}) \cup \{u, u_1, u_2\} \) is a double dominating set for \( G \) of cardinality \( |D'| + 1 \), implying that \( \gamma_{x2}(G) \leq \frac{n + k - 2}{2} + 1 = \frac{n + k}{2} \). Thus, \( \deg_G(u_3) \geq 3 \).

Assume that \( u_3 u_1 \in E(G) \). Let \( u_4 \in N_G(u_3) - \{u_1, u_2\} \) be the vertex just after \( u_3 \) in the cyclic clockwise order on \( C \), and let \( G' = (G - \{u, u_2\}) + uv_3 \). Then \( G' \) is a maximal outerplanar graph of order \( n - 2 \) with the hamiltonian
cycle \((C - \{u_2u_3, uw_2, uv\}) \cup \{uv_3\}\). Note that \(G'\) has \(k - 1\) bad vertices if \(\deg_G(u_v) = 2\) and \(k\) bad vertices if \(\deg_G(u_4) > 2\). Applying the inductive hypothesis, \(\gamma_{x_2}(G') \leq \frac{n + k'}{2} = \frac{n + k - 2}{2}\). Let \(D'\) be a \(\gamma_{x_2}(G')\)-set. If \(v \notin D'\) then \(\{u_1, u_3\} \subseteq D'\) and so \(D' \cup \{u\}\) is a double dominating set for \(G\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k - 2}{2} + 1 = \frac{n + k}{2}\). Thus assume that \(v \in D'\). Then clearly we may assume that \(|D' \cap \{u_1, u_3\}| = 1\). Then \(|D' - \{v\}| \cup \{u, u_1, u_3\}\) is a double dominating set for \(G\) of cardinality \(|D'| + 1\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k - 2}{2} + 1 = \frac{n + k}{2}\). Thus \(u_3u_1 \notin E(G)\).

Let \(u_0 \in N_G(u_1) - \{v, u_2\}\) be the vertex just before \(u_1\) in the cyclic clockwise order on \(C\), and let \(G' = G - \{u, v\}\). Then \(G'\) is a maximal outerplanar graph of order \(n - 2\) with the hamiltonian cycle \((C - \{v_1u_1, uw_2, uv\}) \cup \{u_1v_2\}\). Assume that \(\deg_G(u_0) = 2\). Then \(G'\) has at most \(k\) bad vertices. Applying the inductive hypothesis, \(\gamma_{x_2}(G') \leq \frac{n + k'}{2} = \frac{n + k - 2}{2}\). Let \(D'\) be a \(\gamma_{x_2}(G')\)-set. If \(u_0 \notin D'\) then \(u_1 \in D'\) and so \(D' \cup \{u\}\) is a double dominating set for \(G\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k - 2}{2} + 1 = \frac{n + k}{2}\). Thus assume that \(u_0 \in D'\). Then clearly we may assume that \(|D' \cap N_G(u_0)| = 1\). Then \(|D' - \{u_0\}| \cup N_G(u_0) \cup \{u\}\) is a double dominating set for \(G\) of cardinality \(|D'| + 1\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k - 2}{2} + 1 = \frac{n + k}{2}\). Thus we may assume that \(\deg_G(u_0) \geq 3\). Note that \(\deg_G(u_1) \geq 5\). Let \(G'\) be the graph obtained from \(G\) by removing the vertices \(u\) and \(v\) and then contracting the edge \(u_1u_2\). Then \(G'\) has \(k - 1\) bad vertices. Let \(u^*\) be the vertex in \(G'\) forming by contracting the edge \(u_1u_2\). Applying the inductive hypothesis, \(\gamma_{x_2}(G') \leq \frac{n' + k'}{2} = \frac{n + 3 + k - 1}{2}\). Let \(D'\) be a \(\gamma_{x_2}(G')\)-set. If \(u^* \notin D'\) then \(D' \cup \{u_1, u_2\}\) is a double dominating set for \(G\), implying that \(\gamma_{x_2}(G) \leq \frac{n + 2 + k - 1}{2} = \frac{n + k}{2}\). Thus assume that \(u^* \notin D'\). Then each of \(u_1\) and \(u_2\) is dominated by a vertex of \(D'\) in \(G\), and so \(D' \cup \{v, u\}\) is a double dominating set for \(G\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k}{2} + 1 = \frac{n + k}{2}\).

Next assume that \(\deg_G(u_1) = 4\). Then there exist two vertices \(v_1, v_2 \in N_G(u)\) such that \(\deg_G(u_1) = \deg_G(v_2) = 2\). Let \(N_G(u) = \{v_1, v_2, u_1, u_2\}\), where in the cyclic clockwise order on \(C\), \(u_1\) is before than \(v_1\), \(v_1\) is before than \(u\), \(u\) is before than \(v_2\) and \(v_2\) is before than \(u_2\). By the choice of \(u\), \(u_1u_2 \in E(G)\). Let \(u_3 \in N_G(u_2)\) be the vertex after \(u_2\) in the cyclic clockwise order on \(C\).

Assume that \(\deg_G(u_3) = 2\). Let \(G' = G - \{v_1, v_2\}\). Then \(G'\) is a maximal outerplanar graph of order \(n - 2\) with the hamiltonian cycle \((C - \{v_2u_2, u_2v, u_1v_1\}) \cup \{u_1u, u_2u_2\}\). Note that \(G'\) has \(k\) bad vertices. Applying the inductive hypothesis, \(\gamma_{x_2}(G') \leq \frac{n' + k'}{2} = \frac{n + k - 2}{2}\). Let \(D'\) be a \(\gamma_{x_2}(G')\)-set. If \(u \notin D'\) then \(u_1, u_2 \in D'\) and so \(D' \cup \{u\}\) is a double dominating set for \(G\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k - 2}{2} + 1 = \frac{n + k}{2}\). Thus assume that \(u \in D'\). Then we may assume that \(|D' \cap \{v_1, v_2\}| = 1\). Then \(|D' \cup \{u_1, u_2\}|\) is a double dominating set for \(G\) of cardinality \(|D'| + 1\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k}{2} + 1 = \frac{n + k}{2}\). Thus, \(\deg_G(u_3) \geq 3\).

Thus \(v_2\) is a bad vertex of \(G\). Let \(G' = G - \{v_1, v_2\}\). Then \(G'\) is a maximal outerplanar graph of order \(n - 2\) with the hamiltonian cycle \((C - \{v_2u_2, u_2v, u_1v_1\}) \cup \{u_1u, u_2u_2\}\). Since \(v_2\) is a bad vertex of \(G\), \(u\) is a bad vertex of \(G'\), and \(G'\) has \(k\) bad vertices. Applying the inductive hypothesis, \(\gamma_{x_2}(G') \leq \frac{n' + k'}{2} = \frac{n + k - 2}{2}\). Let \(D'\) be a \(\gamma_{x_2}(G')\)-set. If \(u \notin D'\) then \(u_1, u_2 \in D'\) and so \(D' \cup \{u\}\) is a double dominating set for \(G\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k}{2} + 1 = \frac{n + k}{2}\). Thus assume that \(u \in D'\). Then we may assume that \(|D' \cap \{v_1, u_2\}| = 1\). Then \(|D' \cup \{u_1, u_2\}|\) is a double dominating set for \(G\) of cardinality \(|D'| + 1\), implying that \(\gamma_{x_2}(G) \leq \frac{n + k}{2} + 1 = \frac{n + k}{2}\). Thus, \(u_1u_3 \notin E(G)\). Let \(G' = G - \{u, v_2\}\). Then \(G'\) has \(k\) bad vertices. Applying the inductive hypothesis, \(\gamma_{x_2}(G') \leq \frac{n' + k'}{2} = \frac{n + k - 2}{2}\). Let \(D'\) be a \(\gamma_{x_2}(G')\)-set. Now as before, we obtain that \(\gamma_{x_2}(G) \leq \frac{n + k}{2} + 1 = \frac{n + k}{2}\).

4. Proof of Theorem 2.2

Let \(G\) be a Hamiltonian maximal planar graph of order \(n \geq 7\). Let \(C\) be a Hamilton cycle of \(G\), and without loss of generality, assume that \(G_n^G\) has at most \(\frac{n}{4}\) bad vertices according to Theorem 1.3. Then \(k \leq \frac{n}{4}\) and by Theorem 2.1, \(\gamma_{x_2}(G) \leq \frac{n + k}{2} \leq \frac{5n}{8}\).

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References


